J. Sci. Hiroshima Univ. Ser. A-I
29 (1965), 181-215

Composition of some Series of Association Algebras

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(Received September 21, 1965)

1. Introduction

The concept of association schemes was introduced first by Bose and Shimamoto [4]. It was investigated in relation to the definition of the partially balanced incomplete block (PBIB) designs introduced first by Bose and Nair [3]. This concept, however, has recently been treated without referring to the definition of the PBIB designs. An association scheme with m associate classes, which is defined among v objects, usually called treatments, is a relation of association defined among those satisfying the following three conditions:

(i) Any two treatments are either 1st, 2nd, ..., or m-th associates, the relation of association being symmetrical. Each treatment is the zeroth associate of itself.

(ii) Each treatment α has n_i i-th associates, the number n_i being independent of α.

(iii) If any two treatments α and β are i-th associates, then the number of treatments which are j-th associates of α and k-th associates of β is p^i_{jk} and is independent of the pair of i-th associates α and β.

Matrix representation of the relationship of association along the concept of relationship algebra by James [8] was immediately followed by the definition of the association algebra by Bose and Mesner [2]. The structure of the association algebras was studied by Ogawa [14], [15] in some detail. Further steps were taken by Yamamoto and Fujii [23].

An association algebra with m associate classes is a semi-simple commutative matrix algebra generated by the association matrices $A_0, A_1, \cdots, A_m$ over the real field. It is completely reducible and its minimum two sided ideals are linear. The principal idempotents $A_0^*, A_1^*, \cdots, A_m^*$ of those ideals and the association matrices are mutually linked by the linear combinations of the others. That is, $A_i=\sum_{j=0}^{m} z_{ji}A_j^*$ and $A_i^*=\sum_{j=0}^{m} z^{ij}A_j$ where $z^{ij}=\alpha_i z_{ij}/vn_j$ and $\alpha_i=$ rank (A_i^*). In many cases it happens that all of the idempotent matrices are rational. In such cases we consider the association algebras to be defined over the rational field.

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In his previous paper [22], Yamamoto, one of the present authors, developed some theoretical aspects of the composition of the relationship algebras of experimental designs using his notions of the similar and the partially similar mappings of the semi-simple matrix algebras. He pointed out that an association algebra defined by an association scheme was considered as a sort of relationship algebra defined among a set of parameters, and that the structure of the association algebra determined uniquely the decomposition of the parameter sum of squares. Taking a step further, he suggested that an association algebra could be considered a sort of relationship algebra defined among a set of apparent parameters and could be composed of one or more primitive relationships with which the primary objects of the experimenter were concerned. From this point of view, he suggested with some examples that, without referring to the PBIB designs, most of the association algebras introduced by the association schemes of the PBIB designs might well be composed of one or more primitive relationship algebras, each of which was generated by the identity relationship matrix I, (the unit matrix), and the universal relationship matrix G, (a matrix whose elements are all unity).

The purpose of this paper is to realize the ideas of the above mentioned paper and to deal with systematic compositions of the several series of the association algebras, using series of similar or partially similar mappings and the orthogonal composition of the algebras [22]. These results cover almost all the association schemes and their algebras treated hitherto by many workers. The latent structures of the associated parameters are also indicated with reference to each of the ways of composition. These considerations may be regarded as throwing a new light on the nature of the association schemes.

As Yamamoto [22] pointed out, even in the case of an incomplete block design, an association scheme characterizing the latent structure of the relevant parameters is not necessarily the same as the association scheme of the block design. The existence of a PBIB design which admits a certain association scheme, however, is one of the traditional problems in the theory of experimental designs. Much work has been done with the necessary conditions for the existence of the regular and symmetrical BIB and PBIB designs in connection with Hasse's p-invariant [1], [19], [21], [13], [15], [17], [20], [6], [18], [10], [12], [16]. In such cases, the evaluation of the Gramian is required with respect to each set of independent vectors which span an invariant subspace of the parameter space determined by corresponding idempotent matrix, provided all idempotent matrices are rational [15]. As a specific application of the systematic composition of the series of association algebras, a straightforward method of evaluating these Gramians is given. Hasse's p-invariant may be calculated easily with respect to each of the specific cases which have been treated by many workers.

The following notation is used throughout this paper:

I_s: The unit matrix of order s.

G_s: An $s \times s$ matrix whose elements are all unity.
$\boldsymbol{j}_s$: An s dimensional column vector whose components are all unity.
A': Transpose of the matrix A.
$A \otimes B$: Kronecker product of the matrices $A = ||a_{ij}||$ and B, *i.e.*, $A \otimes B = ||a_{ij}B||$.
$^\sharp$: The superscript $^\sharp$ indicates that the matrix is an idempotent matrix.
$\mathfrak{A} = [A_i; i = 0, 1, \cdots, m]$: An algebra generated by the linear closure of those matrices indicated in the [ ].
$\mathfrak{A}_1 \cup \mathfrak{A}_2$: The smallest algebra containing both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ as its subalgebras.
$\mathfrak{A}_1 \cap \mathfrak{A}_2$: The largest subalgebra contained in both $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

2. Triangular series of association schemes

(a) *Definition of* T_m *type association schemes*

Suppose that there are $v_m = \binom{s}{m}$ objects or treatments $\phi(\alpha_1, \alpha_2, \cdots, \alpha_m)$ indexed by the combinations or subsets of m integers $(\alpha_1, \alpha_2, \cdots, \alpha_m)$ out of the set of s integers $(1, 2, \cdots, s)$. Among those v_m treatments, an association of triangular type or T_m type with m associate classes is defined as follows:

DEFINITION: Two treatments $\phi(\alpha_1, \alpha_2, \cdots, \alpha_m)$ and $\phi(\beta_1, \beta_2, \cdots, \beta_m)$ are i-th associates if their indices $(\alpha_1, \alpha_2, \cdots, \alpha_m)$ and $(\beta_1, \beta_2, \cdots, \beta_m)$ have $m-i$ integers in common. Each treatment is the 0-th associate of itself.

In this section, m is assumed to be not greater than $s/2$. Otherwise, i can assume at most an integral value not exceeding $s/2$. The latter case, however, requires only a slight modification in the descriptions of this section.

The association defined above satisfies three conditions of the association scheme with m associate classes, *i.e.*;

(i) Any two treatments are either 1st, 2nd, $\cdots$, or m-th associates, the relation of association being symmetrical.

(ii) Each treatment has

$$n_i^{(m)} = \binom{m}{i}\binom{s-m}{i} \tag{2.1}$$

i-th associates, the number $n_i^{(m)}$ being independent of $\phi(\alpha_1, \alpha_2, \cdots, \alpha_m)$.

(iii) If any two treatments $\phi(\alpha_1, \alpha_2, \cdots, \alpha_m)$ and $\phi(\beta_1, \beta_2, \cdots, \beta_m)$ are i-th associates, then the number of treatments which are j-th associates of $\phi(\alpha_1, \alpha_2, \cdots, \alpha_m)$ and at the same time k-th associates of $\phi(\beta_1, \beta_2, \cdots, \beta_m)$ is

$$p_{jk}^{i(m)} = \sum_{l=0}^{m-i} \binom{m-i}{l}\binom{i}{m-j-l}\binom{i}{m-k-l}\binom{s-m-i}{j+k+l-m} \tag{2.2}$$

and this is independent of the pair of i-th associates $\phi(\alpha_1, \alpha_2, \cdots, \alpha_m)$ and

$\phi(\beta_1, \beta_2, \cdots, \beta_m)$.

This scheme is called a triangular type association scheme with m associate classes, or briefly, T_m type association scheme. It is a generalization of what is called a triangular association scheme with two associate classes [4] and an association scheme of T_3 type defined by Kusumoto [10]. Ogasawara [12] has defined the scheme and studied it in some detail.

(b) *Association matrices and association algebras.*

Consider a T_m type association scheme. After numbering v_m indices in some way but once for all, association matrices can be expressed as follows:

$$\overset{(m)}{A}_i = \| \mathrm{a}^{(\beta_1, \beta_2, \cdots, \beta_m)}_{(\alpha_1, \alpha_2, \cdots, \alpha_m) i} \| \tag{2.3}$$

where $\mathrm{a}^{(\beta_1, \beta_2, \cdots, \beta_m)}_{(\alpha_1, \alpha_2, \cdots, \alpha_m) i} = \begin{cases} 1 & \text{if the treatments } \phi(\alpha_1, \alpha_2, \cdots, \alpha_m) \text{ and} \\ & \phi(\beta_1, \beta_2, \cdots, \beta_m) \text{ are } i\text{-th associates} \\ 0 & \text{otherwise} \end{cases}$

for $i = 0, 1, \cdots, m$. $\overset{(m)}{A}_i$'s are symmetric $v_m \times v_m$ matrices and satisfy the following relations:

$$\begin{aligned} &\overset{(m)}{A}_0 = I_{v_m}, \quad \sum_{i=0}^{m} \overset{(m)}{A}_i = G_{v_m} \\ &\overset{(m)}{A}_i \overset{(m)}{A}_j = \overset{(m)}{A}_j \overset{(m)}{A}_i = \sum_{k=0}^{m} p^{k}_{ij} \overset{(m)}{A}_k. \end{aligned} \tag{2.4}$$

The linear closure of the association matrices $\overset{(m)}{A}_0, \overset{(m)}{A}_1, \cdots, \overset{(m)}{A}_m$ over the rational field is called a T_m type association algebra and denoted by $\mathfrak{A}(T_m)$ or $[\overset{(m)}{A}_i; i = 0, 1, \cdots, m]$.

The following linear relations are known between the association matrices $\overset{(m)}{A}_j$ $(j = 0, 1, \cdots, m)$ and the principal idempotent matrices $\overset{(m)}{A}^{\sharp}_i$ $(i = 0, 1, \cdots, m)$ of the two sided ideals of $\mathfrak{A}(T_m)$:

$$\overset{(m)}{A}_j = \sum_{i=0}^{m} z^{(m)}_{ij} \overset{(m)}{A}^{\sharp}_i \tag{2.5}$$

or

$$\overset{(m)}{A}^{\sharp}_i = \sum_{j=0}^{m} z^{ij}_{(m)} \overset{(m)}{A}_j = \sum_{j=0}^{m} \frac{\alpha_i z^{(m)}_{ij}}{v_m n^{(m)}_j} \overset{(m)}{A}_j, \tag{2.5'}$$

where $\| z^{(m)}_{ij} \|$ is an $(m+1) \times (m+1)$ non-singular matrix and $\alpha_i = \mathrm{rank}\, (\overset{(m)}{A}^{\sharp}_i)$.

In the case $m = 1$, we have

$$\overset{(1)}{A}_0 = I_s, \qquad \overset{(1)}{A}_1 = G_s - I_s$$

$$\overset{(1)}{A}{}^{\#}_0 = \frac{1}{s} G_s, \qquad \overset{(1)}{A}{}^{\#}_1 = I_s - \frac{1}{s} G_s$$

$$\mathfrak{A}(\mathrm{T}_1) = [\overset{(1)}{A}_0, \overset{(1)}{A}_1] = [\overset{(1)}{A}{}^{\#}_0, \overset{(1)}{A}{}^{\#}_1] = [I_s, G_s]$$

$$\|z^{(1)}_{ij}\| = \begin{bmatrix} 1 & s-1 \\ 1 & -1 \end{bmatrix} \quad \|z^{ij}_{(1)}\| = \frac{1}{s} \begin{bmatrix} 1 & 1 \\ s-1 & -1 \end{bmatrix}.$$

(c) *Composition of the series* $\{\mathfrak{A}(\mathrm{T}_m)\}$

Now we show the way of composing step by step the series of the association algebras $\{\mathfrak{A}(\mathrm{T}_m)\}$ starting from the primitive association algebra $\mathfrak{A}(\mathrm{T}_1) = [I_s, G_s]$ with the aid of the series of the partially similar mappings $\{\sigma_k\}$ as,

$$\mathfrak{A}(\mathrm{T}_1) = [I_s, G_s] \overset{\sigma_1}{\to} \mathfrak{A}(\mathrm{T}_2) \overset{\sigma_2}{\to} \mathfrak{A}(\mathrm{T}_3) \overset{\sigma_3}{\to} \cdots\cdots \overset{\sigma_{m-2}}{\to} \mathfrak{A}(\mathrm{T}_{m-1}) \overset{\sigma_{m-1}}{\to} \mathfrak{A}(\mathrm{T}_m) \overset{\sigma_m}{\to} \cdots.$$

Taking into account the geometric structure of the T_m type association scheme, we define a $v_m \times v_{m-1}$ matrix F_{m-1} giving a linear mapping from the $v_{m-1} = \binom{s}{m-1}$ dimensional vector space to the $v_m = \binom{s}{m}$ dimensional vector space as

$$F_{m-1} = \frac{1}{m} \| f^{(\mu_1, \mu_2, \cdots, \mu_{m-1})}_{(\alpha_1, \alpha_2, \cdots, \alpha_m)} \|, \tag{2.6}$$

where
$$f^{(\mu_1, \mu_2, \cdots, \mu_{m-1})}_{(\alpha_1, \alpha_2, \cdots, \alpha_m)} = \begin{cases} 1 & \text{if } (\mu_1, \mu_2, \cdots, \mu_{m-1}) \subset (\alpha_1, \alpha_2, \cdots, \alpha_m) \\ 0 & \text{otherwise.} \end{cases}$$

Using F_{m-1} we define a linear mapping σ_{m-1} of $\mathfrak{A}(\mathrm{T}_{m-1})$ as

$$\sigma_{m-1}: \quad \mathfrak{A}(\mathrm{T}_{m-1}) \ni A \to F_{m-1} A F'_{m-1}. \tag{2.7}$$

In this connection, we have the theorem:

THEOREM 2.1 *The linear mapping* σ_{m-1} *defined by the matrix* F_{m-1} *is a partially similar mapping of* $\mathfrak{A}(\mathrm{T}_{m-1})$ *for any positive integer satisfying* $2 \leq m \leq \frac{s}{2}$.

The algebra $\mathfrak{A}(\mathrm{T}_m)$ *is composed by the mapping*:

$$\mathfrak{A}(\mathrm{T}_m) = \sigma_{m-1}(\mathfrak{A}(\mathrm{T}_{m-1})) \cup [I_{v_m}]. \tag{2.8}$$

Since $\mathfrak{A}(\mathrm{T}_1)=[I_s, G_s]$ is well defined, we can compose the series of triangular type association algebras $\left\{\mathfrak{A}(\mathrm{T}_m);\ m=2, 3, \cdots, \left[\frac{s}{2}\right]\right\}$ step by step with the aid of the series of mappings $\{\sigma_k\}$. ($\left[\frac{s}{2}\right]$ denotes the greatest integer not exceeding $s/2$.)

PROOF. It is sufficient to show that σ_{m-1} is a partially similar mapping of $\mathfrak{A}(\mathrm{T}_{m-1})$ and that

$$\sigma_{m-1}(\mathfrak{A}(\mathrm{T}_{m-1}))\cup[I_{v_m}]=\mathfrak{A}(\mathrm{T}_m)$$

under the assumption that $\mathfrak{A}(\mathrm{T}_{m-1})=[\overset{(m-1)}{A}_i;\ i=0, 1, \cdots, m-1]=[\overset{(m-1)}{A}{}^{\sharp}_j;\ j=0, 1, \cdots, m-1]$ has been composed.

From the definition of $\overset{(m-1)}{A}_l$ and F_{m-1}, we have

$$F_{m-1}\overset{(m-1)}{A}_l F'_{m-1}=\frac{(m-l)(m+1-l)}{m^2}\overset{(m)}{A}_{l-1}+\frac{(2l+1)(m-l)}{m^2}\overset{(m)}{A}_l + \frac{(l+1)^2}{m^2}\overset{(m)}{A}_{l+1} \tag{2.9}$$

for $l=0, 1, \cdots, m-1$, where $\overset{(m)}{A}_{-1}$ is assumed to be a zero matrix. From (2.4) and (2.9), we have

$$\begin{pmatrix} I_{v_m} \\ F_{m-1}\overset{(m-1)}{A}_0 F'_{m-1} \\ F_{m-1}\overset{(m-1)}{A}_1 F'_{m-1} \\ \vdots \\ F_{m-1}\overset{(m-1)}{A}_l F'_{m-1} \\ \vdots \\ F_{m-1}\overset{(m-1)}{A}_{m-1} F'_{m-1} \end{pmatrix} = \frac{1}{m^2}\begin{pmatrix} m^2 & & & & & 0 \\ m & 1^2 & & & & \\ m(m-1) & 3(m-1) & 2^2 & & & \\ & \ddots & & \ddots & & \ddots \\ (m+1-l)(m-l) & & (2l+1)(m-l) & & (l+1)^2 & \\ & \ddots & & \ddots & & \ddots \\ 0 & & 2 & & (2m-1) & m^2 \end{pmatrix}\begin{pmatrix} \overset{(m)}{A}_0 \\ \overset{(m)}{A}_1 \\ \overset{(m)}{A}_2 \\ \vdots \\ \overset{(m)}{A}_{l+1} \\ \vdots \\ \overset{(m)}{A}_m \end{pmatrix}. \tag{2.10}$$

Since the $(m+1)\times(m+1)$ matrix of the right hand side of (2.10) is non-singular, we have

$$[I_{v_m}, F_{m-1}\overset{(m-1)}{A}_l F'_{m-1};\ l=0, 1, \cdots, m-1]=[\overset{(m)}{A}_i;\ i=0, 1, \cdots, m].$$

As the rank of F_{m-1} is v_{m-1}, $F'_{m-1}F_{m-1}$ is a non-singular positive definite matrix. It also satisfies the relation

$$F'_{m-1}F_{m-1}=\frac{(s-m+1)}{m_2}\overset{(m-1)}{A}_0+\frac{1}{m^2}\overset{(m-1)}{A}_1\in\mathfrak{A}(\mathrm{T}_{m-1}). \tag{2.11}$$

Thus we can write

$$F'_{m-1}F_{m-1}=\sum_{i=0}^{m-1}c_i^{(m-1)}\overset{(m-1)}{A}_i^{\#} \tag{2.12}$$

where $c_i^{(m-1)}$ $(i=0, 1, \cdots, m-1)$ are positive constants. The linear mapping σ_{m-1} of $\mathfrak{A}(\mathrm{T}_{m-1})$ defined by F_{m-1} is G-preserving partially similar [22]. The algebra $\mathfrak{A}(\mathrm{T}_m)$ is, therefore, an algebra composed of $\mathfrak{A}(\mathrm{T}_{m-1})$ by the partially similar mapping σ_{m-1}, *i.e.*,

$$\mathfrak{A}(\mathrm{T}_m)=\sigma_{m-1}(\mathfrak{A}(\mathrm{T}_{m-1}))\cup[I_{v_m}].$$

(d) *Some applications*

The structure of the series of triangular type association algebras might be useful in determing recurrently the properties of the T_m type association scheme and its algebra. We treat some of them in the following.

From the definition of the partially similar mapping [22], $\frac{1}{c_i^{(m-1)}}F_{m-1}\overset{(m-1)}{A}_i^{\#}F'_{m-1}$ $(i=0, 1, \cdots, m-1)$ are the mutually orthogonal idempotents of $\mathfrak{A}(\mathrm{T}_m)$. It is well known that the $(m+1)$ mutually orthogonal idempotents of $\mathfrak{A}(\mathrm{T}_m)$, $\overset{(m)}{A}_0^{\#}, \overset{(m)}{A}_1^{\#}, \cdots, \overset{(m)}{A}_m^{\#}$, are uniquely determined apart from their order. We therefore number them step by step through the following correspondence:

$$\begin{aligned}&\overset{(m)}{A}_i^{\#}=\frac{1}{c_i^{(m-1)}}F_{m-1}\overset{(m-1)}{A}_i^{\#}F'_{m-1} \qquad (i=0, 1, \cdots, m-1)\\ &\overset{(m)}{A}_m^{\#}=I_{v_m}-\sum_{i=0}^{m-1}\overset{(m)}{A}_i^{\#}.\end{aligned} \tag{2.13}$$

From (2.11), (2.9), (2.4) and (2.13), we have

$$\begin{aligned}F'_mF_m&=\frac{(s-m)}{(m+1)^2}\overset{(m)}{A}_0+\frac{1}{(m+1)^2}\overset{(m)}{A}_1=\frac{(s-2m)}{(m+1)^2}I_{v_m}+\frac{m^2}{(m+1)^2}F_{m-1}I_{v_{m-1}}F'_{m-1}\\ &=\frac{(s-2m)}{(m+1)^2}\sum_{i=0}^{m}\overset{(m)}{A}_i^{\#}+\frac{m^2}{(m+1)^2}\sum_{i=0}^{m-1}F_{m-1}\overset{(m-1)}{A}_i^{\#}F'_{m-1}\\ &=\sum_{i=0}^{m-1}\left\{\frac{m^2}{(m+1)^2}c_i^{(m-1)}+\frac{(s-2m)}{(m+1)^2}\right\}\overset{(m)}{A}_i^{\#}+\frac{(s-2m)}{(m+1)^2}\overset{(m)}{A}_m^{\#}.\end{aligned} \tag{2.14}$$

On the other hand, from (2.12) we have

$$F'_m F_m = \sum_{i=0}^{m} c_i^{(m)} \overset{(m)}{A}{}_i^{\#}. \tag{2.15}$$

Solving the recurrence relations obtained by comparing the corresponding coefficients of $\overset{(m)}{A}{}_i^{\#}$ in (2.14) and (2.15), we have

$$c_i^{(m)} = \frac{(m+1-i)(s-m-i)}{(m+1)^2} \tag{2.16}$$

for $i=0, 1, \cdots, m$ and $m=1, 2, \cdots, \left[\frac{s}{2}\right]-1$. Thus we have the following recurrence relations for mutually orthogonal idempotents.

$$\overset{(m)}{A}{}_i^{\#} = \frac{m^2}{(m-i)(s-m-i+1)} F_{m-1} \overset{(m-1)}{A}{}_i^{\#} F'_{m-1} \tag{2.17}$$

for $i=0, 1, \cdots, m-1$ and

$$\overset{(m)}{A}{}_m^{\#} = I_{v_m} - \sum_{i=0}^{m-1} \overset{(m)}{A}{}_i^{\#}. \tag{2.18}$$

Next we derive two explicit formulas for $z_{ij}^{(m)}$ in (2.5) and (2.5′), *i.e.*,

$$z_{ij}^{(m)} = \frac{\binom{s-m}{j}}{\binom{s-m}{i}} \sum_{a=0}^{i} (-1)^{i-a} \binom{m-a}{j} \binom{m-a}{m-i} \binom{s-i+1}{a} \tag{2.19}$$

or

$$z_{ij}^{(m)} = \sum_{a=0}^{j} (-1)^{j-a} \binom{m-i}{a} \binom{m-a}{m-j} \binom{s-m-i+a}{a} \quad (i, j=0, 1, \cdots, m). \tag{2.19′}$$

Ogasawara [12] obtained (2.19′) by an ingenious method which was a generalization of an idea of Corsten's [5]. The method, however, was very involved. Our method of derivation presented in the following is based on the structure of the series of triangular type association schemes and is accordingly analytical.

We define the vectors, $\overset{(m)}{\boldsymbol{A}}$, $\boldsymbol{z}_{(m)}^{i}$ as follows:

$$\begin{aligned} &\overset{(m)}{\boldsymbol{A}}{}' = (\overset{(m)}{A}_0, \overset{(m)}{A}_1, \cdots, \overset{(m)}{A}_j, \cdots, \overset{(m)}{A}_m) \\ &\boldsymbol{z}_{(m)}^{i\prime} = (z_{(m)}^{i0}, z_{(m)}^{i1}, \cdots, z_{(m)}^{ij}, \cdots, z_{(m)}^{im}) \qquad (i=0, 1, \cdots, m). \end{aligned} \tag{2.20}$$

Then we can write (2.5′) as

$$\overset{(m)}{A}_k{}^{\sharp}=\overset{(m)}{A'}\boldsymbol{z}^k_{(m)} \qquad (k=0, 1, \cdots, m). \tag{2.21}$$

On the other hand, we have from (2.13), (2.21) and (2.10)

$$\begin{aligned}\overset{(m)}{A}_k{}^{\sharp}&=\frac{1}{c_k^{(m-1)}}\sigma_{m-1}(\overset{(m-1)}{A}_k{}^{\sharp})=\frac{1}{c_k^{(m-1)}}\sigma_{m-1}(\overset{(m-1)}{A'}\boldsymbol{z}^k_{(m-1)})\\&=\frac{1}{c_k^{(m-1)}}\sigma_{m-1}(\overset{(m-1)}{A'})\boldsymbol{z}^k_{(m-1)}=\frac{1}{c_k^{(m-1)}}\overset{(m)}{A'}H_m\boldsymbol{z}^k_{(m-1)}\\&\qquad (k=0, 1, \cdots, m-1),\end{aligned} \tag{2.22}$$

where

$$H_m=\frac{1}{m^2}\begin{pmatrix} m & m(m-1) & & & & 0\\ 1^2 & 3(m-1) & (m-1)(m-2) & & & \\ \ddots & \ddots & \ddots & & & \\ & j^2 & (2j+1)(m-j) & (m-j)(m-j-1) & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & 2\\ & 0 & & & & 2m-1\\ & & & & & m^2\end{pmatrix}. \tag{2.23}$$

Comparing (2.21) with (2.22), we have

$$c_k^{(m-1)}\boldsymbol{z}^k_{(m)}=H_m\boldsymbol{z}^k_{(m-1)} \qquad (k=0, 1, \cdots, m-1). \tag{2.24}$$

Adding (2.21) over k from 0 to m and using (2.4) and (2.18), we have

$$\sum_{k=0}^{m}\boldsymbol{z}^k_{(m)}=\boldsymbol{l}_m \qquad \left(m=1, 2, \cdots, \left[\frac{s}{2}\right]\right), \tag{2.25}$$

where

$$\boldsymbol{l}_m'=(1, 0, \cdots, 0, \cdots, 0). \tag{2.26}$$

Using (2.24) repeatedly and adding over k from 0 to i, we have

$$\sum_{k=0}^{i}c_k^{(m-1)}c_k^{(m-2)}\cdots c_k^{(i)}\boldsymbol{z}^k_{(m)}=H_mH_{m-1}\cdots H_{i+1}\left(\sum_{k=0}^{i}\boldsymbol{z}^k_{(i)}\right). \tag{2.27}$$

Substituting (2.16) and (2.25) into (2.27), we have

$$\sum_{k=0}^{i}\binom{m-k}{m-i}\binom{s-i-k}{m-i}z^{kj}_{(m)}=\binom{m-j}{i} \qquad (i, j=0, 1, \cdots, m). \tag{2.28}$$

From the above relations, we have, after some calculations,

$$z_{(m)}^{ij}=\sum_{a=0}^{i}(-1)^{i-a}\frac{(s-2i+1)\binom{m-a}{m-i}}{(s-i-a+1)\binom{s-i-a}{m-a}}\binom{m-j}{a}$$

$$=\frac{\alpha_i}{v_m n_j^{(m)}}\frac{\binom{s-m}{j}}{\binom{s-m}{i}}\sum_{a=0}^{i}(-1)^{i-a}\binom{m-a}{j}\binom{m-a}{m-i}\binom{s-i+1}{a}. \tag{2.29}$$

Hence, we obtain (2.19).

The formula (2.19′) may be obtained as follows. Multiplying (2.28) by $z_{tj}^{(m)}$ and adding over j from 0 to m, we have

$$\sum_{j=0}^{m}\binom{m-j}{i}z_{tj}^{(m)}=\binom{m-t}{m-i}\binom{s-i-t}{m-i} \qquad (i, t=0, 1, \cdots, m). \tag{2.30}$$

Solving (2.30), we have

$$z_{ij}^{(m)}=\sum_{k=0}^{m}(-1)^{k-m+j}\binom{k}{m-j}\binom{m-i}{m-k}\binom{s-k-i}{m-k}$$

$$=\sum_{a=0}^{j}(-1)^{j-a}\binom{m-a}{m-j}\binom{m-i}{a}\binom{s-m-i+a}{a}. \tag{2.31}$$

(e) *Parameter models*

In each stage of the composition of a T_m type association algebra, the method of composition induces a natural parameter vector associated with the corresponding triangular type algebra. We show the structure of the composed parameter vectors inductively. The decomposition of the sum of squares of the composed parameters, which is determined uniquely by the composed algebra, is also given.

Let $\boldsymbol{\tau}_1$ be an s dimensional parameter vector, the elements of which represent the effects of s sub-factors or levels, respectively. Within these elements, a primitive relationship algebra $\mathfrak{A}(T_1)=[I_s, G_s]$ is assumed to be defined. The associated decomposition of the parameter sum of squares is

$$\boldsymbol{\tau}_1'\boldsymbol{\tau}_1=\boldsymbol{\tau}_1'\overset{(1)}{A}_0{}^{\sharp}\boldsymbol{\tau}_1+\boldsymbol{\tau}_1'\overset{(1)}{A}_1{}^{\sharp}\boldsymbol{\tau}_1=s\bar{\tau}_1^2+\sum_{j=1}^{s}(\tau_{1j}-\bar{\tau}_1)^2, \tag{2.32}$$

where $\boldsymbol{\tau}_1'=(\tau_{11}, \cdots, \tau_{1s})$ and $\bar{\tau}_1=\frac{1}{s}\sum_{j=1}^{s}\tau_{1j}$. The first component of the decomposition is due to the general mean and the second component is due to the

main effect of the level effects.

If we map $\mathfrak{A}(T_1)$ by σ_1 and compose $\mathfrak{A}(T_2)$, the composed parameter vector may naturally be defined as

$$\boldsymbol{\xi}_2 = F_1\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2, \tag{2.33}$$

where $F_1\boldsymbol{\tau}_1$ is a partially similar image of the parameter vector of the level effects and $\boldsymbol{\tau}_2$ is a residual parameter vector subject to the condition

$$F_1'\boldsymbol{\tau}_2 = 0 \tag{2.34}$$

The first component vector $F_1\boldsymbol{\tau}_1$ of the composed parameter is a vector which consists of all combination of two different level effects, *i.e.*, $\frac{1}{2}(\tau_{1i}+\tau_{1j})$ $(i \neq j)$. The composed parameter, therefore, can be represented by the sum of such a parameter vector and a parameter vector which cannot be explained by such an image vector. The second component vector $\boldsymbol{\tau}_2$ may naturally be explained by the vector representing the first order interaction effects between two different levels.

The parameter sum of squares associated to the composed association algebra of T_2 type is

$$\boldsymbol{\xi}_2'\boldsymbol{\xi}_2 = \sum_{i=0}^{2} \boldsymbol{\xi}_2' \overset{(2)}{A}_i{}^{\sharp}\boldsymbol{\xi}_2$$

and is reduced to

$$\begin{aligned}\boldsymbol{\xi}_2'\boldsymbol{\xi}_2 &= -\frac{s-1}{2}\boldsymbol{\tau}_1'\overset{(1)}{A}_0{}^{\sharp}\boldsymbol{\tau}_1 + \frac{s-2}{4}\boldsymbol{\tau}_1'\overset{(1)}{A}_1{}^{\sharp}\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2'\overset{(2)}{A}_2{}^{\sharp}\boldsymbol{\tau}_2 \\ &= -\frac{s(s-1)}{2}\bar{\tau}_1^2 + \frac{s-2}{4}\sum_{i=1}^{s}(\tau_{1i}-\bar{\tau}_1)^2 + \boldsymbol{\tau}_2'\boldsymbol{\tau}_2. \end{aligned} \tag{2.35}$$

In the same way, if we map $\mathfrak{A}(T_2)$ by σ_2 and define the composed parameter vector for $\mathfrak{A}(T_3)$ as

$$\boldsymbol{\xi}_3 = F_2\boldsymbol{\xi}_2 + \boldsymbol{\tau}_3 = F_2F_1\boldsymbol{\tau}_1 + F_2\boldsymbol{\tau}_2 + \boldsymbol{\tau}_3, \tag{2.36}$$

then the residual parameter vector $\boldsymbol{\tau}_3$ is subject to the restriction

$$F_2'\boldsymbol{\tau}_3 = 0 \tag{2.37}$$

and is explained by the vector representing the second order interaction effects between three different levels.

In general, the associated parameter vector for a T_m type association algebra $\mathfrak{A}(T_m)$, which may naturally be defined along the line of the series of composition, is

$$\boldsymbol{\xi}_m = F_{m-1}F_{m-2}\cdots F_1\boldsymbol{\tau}_1 + F_{m-1}F_{m-2}\cdots F_2\boldsymbol{\tau}_2 + \cdots\cdots + F_{m-1}\boldsymbol{\tau}_{m-1} + \boldsymbol{\tau}_m, \tag{2.38}$$

where $\boldsymbol{\tau}_2, \boldsymbol{\tau}_3, \cdots, \boldsymbol{\tau}_m$ represent the effects of the 1st order, 2nd order, $\cdots$ and $(m-1)$th order interaction within the level effects, respectively. They are subject to the restrictions

$$F'_{l-1}\boldsymbol{\tau}_l = 0 \tag{2.39}$$

for $l=2, 3, \cdots, m$.

The decomposition of the parameter sum of squares uniquely determined by the composed association algebra $\mathfrak{A}(T_m)$ is

$$\boldsymbol{\xi}'_m\boldsymbol{\xi}_m = \sum_{i=0}^{m} \boldsymbol{\xi}'_m \overset{(m)}{A}_i{}^{\sharp}\, \boldsymbol{\xi}_m. \tag{2.40}$$

Using (2.13), we can reduce the components as follows:

$$\boldsymbol{\xi}'_m \overset{(m)}{A}_0{}^{\sharp}\, \boldsymbol{\xi}_m = c_0^{(m-1)} c_0^{(m-2)} \cdots c_0^{(1)} \boldsymbol{\tau}'_1 \overset{(1)}{A}_0{}^{\sharp}\, \boldsymbol{\tau}_1 = \binom{s}{m}\bar{\tau}_1^2$$

$$\boldsymbol{\xi}'_m \overset{(m)}{A}_1{}^{\sharp}\, \boldsymbol{\xi}_m = c_1^{(m-1)} \cdots c_1^{(1)} \boldsymbol{\tau}'_1 \overset{(1)}{A}_1{}^{\sharp}\, \boldsymbol{\tau}_1 = \frac{\binom{s-2}{m-1}}{m^2} \sum_{i=1}^{s} (\tau_{1i} - \bar{\tau}_1)^2$$

$$\boldsymbol{\xi}'_m \overset{(m)}{A}_i{}^{\sharp}\, \boldsymbol{\xi}_m = c_i^{(m-1)} \cdots c_i^{(i)} \boldsymbol{\tau}'_i \overset{(i)}{A}_i{}^{\sharp}\, \boldsymbol{\tau}_i = \frac{\binom{s-2i}{m-i}}{\binom{m}{i}^2} \boldsymbol{\tau}'_i\boldsymbol{\tau}_i \qquad (i=2, 3, \cdots, m-1)$$

$$\boldsymbol{\xi}'_m \overset{(m)}{A}_m{}^{\sharp}\, \boldsymbol{\xi}_m = \boldsymbol{\tau}'_m\boldsymbol{\tau}_m.$$

Thus we have

$$\boldsymbol{\xi}'_m\boldsymbol{\xi}_m = \binom{s}{m}\bar{\tau}_1^2 + \frac{\binom{s-2}{m-1}}{m^2} \sum_{i=1}^{s} (\tau_{1i} - \bar{\tau}_1)^2 + \sum_{i=2}^{m} \frac{\binom{s-2i}{m-i}}{\binom{m}{i}^2} \boldsymbol{\tau}'_i\boldsymbol{\tau}_i. \tag{2.41}$$

The degree of freedom of these components are

$$\text{d.f. } (\boldsymbol{\xi}'_m \overset{(m)}{A}_i{}^{\sharp}\, \boldsymbol{\xi}_m) = \text{rank } (\overset{(m)}{A}_i{}^{\sharp}) = \binom{s}{i} - \binom{s}{i-1}$$

for $i=0, 1, \cdots, m$.

Note that some of the interaction parameters $\tau_2, \cdots, \tau_m$ are assumed to be zero, and the corresponding degrees of freedom are assigned to the estimation of error.

3. Nested series of association schemes

(a) *Definition of* N_m *type association schemes*

Suppose that there are $v_m=s_1s_2\cdots s_m$ treatments $\phi(\alpha_1, \alpha_2, \cdots, \alpha_m)$ indexed by m-tuples $(\alpha_1, \alpha_2, \cdots, \alpha_m)$ where $\alpha_i=1, 2, \cdots, s_i$ and $i=1, 2, \cdots, m$. Among these treatments, we define a relation of m-fold nested type or N_m type association as follows:

DEFINITION. A pair of treatments $\phi(\alpha_1, \alpha_2, \cdots, \alpha_m)$ and $\phi(\beta_1, \beta_2, \cdots, \beta_m)$ are i-th associates if $\alpha_j=\beta_j$ for all $j=1, 2, \cdots, m-i$ and $\alpha_{m-i+1}\neq\beta_{m-i+1}$. Each treatment is 0-th associate of itself.

We can easily verify that the association defined above satisfies three conditions of the association scheme with m associate classes. For the case $m=2$, Bose and Shimamoto [4] called it a group divisible association scheme. Raghavarao [17] called the association a generalized group divisible association scheme with m associate classes. Following traditional nomenclature, we prefer to call the association an m-fold nested type association scheme, or an N_m type association scheme.

(b) *Association matrices and association algebras*

Consider an N_m type association scheme. After numbering v_m treatments in dictionary-wise, we can express the i-th association matrices as

$$\overset{(m)}{A}_i = I_{v_{m-i}} \otimes (G_{s_{m-i+1}} - I_{s_{m-i+1}}) \otimes G_{s_{m-i+2}} \otimes \cdots \otimes G_{s_m} \tag{3.1}$$

$$(i=0, 1, 2, \cdots, m),$$

where $v_j=s_1s_2\cdots s_j$. Some of the special cases are

$$\overset{(m)}{A}_0 = I_{v_m} \tag{3.2}$$

$$\overset{(m)}{A}_1 = I_{s_1} \otimes \cdots \otimes I_{s_{m-1}} \otimes (G_{s_m} - I_{s_m}) \tag{3.3}$$

$$\overset{(m)}{A}_m = (G_{s_1} - I_{s_1}) \otimes G_{s_2} \otimes \cdots \otimes G_{s_m}. \tag{3.4}$$

We call the linear closure of the association matrices $\overset{(m)}{A}_i$ over the rational field an m-fold nested type association algebra or an N_m type association

algebra and denote it as $\mathfrak{A}(\mathrm{N_m})$ or $[\overset{(m)}{A_i};\ i=0, 1, \cdots, m]$.

The mutually orthogonal idempotents of the algebra $\mathfrak{A}(\mathrm{N_m})$ are denoted by $\overset{(m)}{A_0}{}^{\#}, \overset{(m)}{A_1}{}^{\#}, \cdots, \overset{(m)}{A_m}{}^{\#}$. For $\mathfrak{A}(\mathrm{N_1})=[I_{s_1}, G_{s_1}]$, we write

$$\overset{(1)}{A_0}{}^{\#}=\frac{1}{s_1}G_{s_1} \qquad \text{and} \qquad \overset{(1)}{A_1}{}^{\#}=I_{s_1}-\frac{1}{s_1}G_{s_1}. \tag{3.5}$$

(c) *Composition of the series* $\{\mathfrak{A}(\mathrm{N_m})\}$

Now we show the way of composing step by step the series of the association algebra $\{\mathfrak{A}(\mathrm{N_m})\}$ starting from the primitive association algebra $\mathfrak{A}(\mathrm{N_1})=[I_{s_1}, G_{s_1}]$ with the aid of the series of the similar mappings $\{\sigma_k\}$ as,

$$\mathfrak{A}(\mathrm{N_1})=[I_{s_1}, G_{s_1}]\overset{\sigma_1}{\to}\mathfrak{A}(\mathrm{N_2})\overset{\sigma_2}{\to}\cdots\cdots\overset{\sigma_{m-2}}{\to}\mathfrak{A}(\mathrm{N_{m-1}})\overset{\sigma_{m-1}}{\to}\mathfrak{A}(\mathrm{N_m})\overset{\sigma_m}{\to}\cdots \tag{3.6}$$

As $\mathfrak{A}(\mathrm{N_1})=[I_{s_1}, G_{s_1}]$ is well defined, it is sufficient to show how to compose $\mathfrak{A}(\mathrm{N_m})$ under the assumption that $\mathfrak{A}(\mathrm{N_{m-1}})$ has been composed.

Let F_{m-1} be a $v_m \times v_{m-1}$ matrix defined by

$$F_{m-1}=I_{v_{m-1}}\otimes \boldsymbol{j}_{s_m} \qquad (m=1, 2, \cdots). \tag{3.7}$$

Let σ_{m-1} be a linear mapping of $\mathfrak{A}(\mathrm{N_{m-1}})$ defined by

$$\sigma_{m-1}:\ \mathfrak{A}(\mathrm{N_{m-1}})\ni A\to F_{m-1}AF'_{m-1}. \tag{3.8}$$

From (3.7) we have

$$F'_{m-1}F_{m-1}=s_m I_{v_{m-1}}\in\mathfrak{A}(\mathrm{N_{m-1}}). \tag{3.9}$$

The linear mapping σ_{m-1} of $\mathfrak{A}(\mathrm{N_{m-1}})$ defined by F_{m-1} is, therefore, similar. Thus the composed full rank algebra is

$$F_{m-1}\mathfrak{A}(\mathrm{N_{m-1}})F'_{m-1}\cup[I_{v_m}]. \tag{3.10}$$

On the other hand, as it is easy to prove that

$$F_{m-1}\overset{(m-1)}{A}_{i-1}F'_{m-1}=\overset{(m)}{A}_i \qquad (i=2, \cdots, m) \tag{3.11}$$

$$I_{v_m}=\overset{(m)}{A}_0, \qquad F_{m-1}\overset{(m-1)}{A}_0F'_{m-1}=\overset{(m)}{A}_0+\overset{(m)}{A}_1$$

and we have the theorem:

THEOREM 3.1 *For every integer* $m(\geq 2)$ *the mapping* σ_{m-1} *of* $(m-1)$*-fold nested type association algebra defined in* (3.8) *gives a similar mapping of*

$\mathfrak{A}(\mathrm{N}_{m-1})$. *The image algebra with* I_{v_m} *gives an* m*-fold nested association algebra* $\mathfrak{A}(\mathrm{N}_m)$, *i.e.*,

$$\mathfrak{A}(\mathrm{N}_m)=\sigma_{m-1}(\mathfrak{A}(\mathrm{N}_{m-1}))\cup[I_{v_m}]. \tag{3.12}$$

The similarity of the mapping σ_{m-1} defines a natural correspondence of the mutually orthogonal idempotents of $\mathfrak{A}(\mathrm{N}_{m-1})$ and $\mathfrak{A}(\mathrm{N}_m)$, *i.e.*,

$$\overset{(m)}{A}{}_i^{\#}=\frac{1}{s_m}F_{m-1}\overset{(m-1)}{A}{}_i^{\#}F'_{m-1} \qquad (i=0, 1, \cdots, m-1) \tag{3.13}$$

$$\overset{(m)}{A}{}_m^{\#}=I_{v_m}-\sum_{i=0}^{m-1}\overset{(m)}{A}{}_i^{\#} \tag{3.14}$$

for $m=2, 3, \cdots$.

Starting from (3.5) for $\mathfrak{A}(\mathrm{N}_1)$, we have

$$\overset{(m)}{A}{}_i^{\#}=I_{v_{i-1}}\otimes\left(I_{s_i}-\frac{1}{s_i}G_{s_i}\right)\otimes\frac{1}{s_{i+1}}G_{s_{i+1}}\otimes\cdots\otimes\frac{1}{s_m}G_{s_m} \tag{3.15}$$

for $i=0, 1, \cdots, m$. In particular,

$$\overset{(m)}{A}{}_0^{\#}=\frac{1}{v_m}G_{v_m}$$

$$\overset{(m)}{A}{}_1^{\#}=\left(I_{s_1}-\frac{1}{s_1}G_{s_1}\right)\otimes\frac{1}{s_2}G_{s_2}\otimes\cdots\otimes\frac{1}{s_m}G_{s_m}$$

$$\overset{(m)}{A}{}_m^{\#}=I_{s_1}\otimes I_{s_2}\otimes\cdots\otimes I_{s_{m-1}}\otimes\left(I_{s_m}-\frac{1}{s_m}G_{s_m}\right).$$

(d) *Parameter models*

In each stage of the composition of an m-fold nested type association algebra, the method of composition induces a natural parameter vector associated to the corresponding nested type association algebra. We show the structure of composed parameter vectors inductively. The results correspond to the usual parameter models of the designs of nested type. The decomposition of the sum of squares of the composed parameters, which is determined uniquely by the composed algebra, is also given.

Let $\boldsymbol{\tau}_1$ be an s_1 dimensional parameter vector, the elements of which represent the class effects of the first classification. Among s_1 elements of $\boldsymbol{\tau}_1$, a primitive relationship algebra, $\mathfrak{A}(\mathrm{N}_1)=[I_{s_1}, G_{s_1}]$, is assumed to be defined. The associated decomposition of the parameter sum of squares is

$$\boldsymbol{\tau}_1'\boldsymbol{\tau}_1=\boldsymbol{\tau}_1'\overset{(1)}{A}_0{}^{\#}\boldsymbol{\tau}_1+\boldsymbol{\tau}_1'\overset{(1)}{A}_1{}^{\#}\boldsymbol{\tau}_1=s_1\bar{\tau}_1^2+\sum_{j=1}^{s_1}(\tau_{1j}-\bar{\tau}_1)^2 \tag{3.16}$$

where $\boldsymbol{\tau}_1'=(\tau_{11}, \tau_{12}, \cdots, \tau_{1s_1})$ and $\bar{\tau}_1=\frac{1}{s_1}\sum_{j=1}^{s_1}\tau_{1j}$. The first component of the decomposition is due to the general mean and the second component of it is due to the main effects of the first classification.

If we map $\mathfrak{A}(\mathrm{N}_1)$ by σ_1 and compose $\mathfrak{A}(\mathrm{N}_2)$, the composed parameter vector may naturally be defined as

$$\boldsymbol{\xi}_2=F_1\boldsymbol{\tau}_1+\boldsymbol{\tau}_2, \tag{3.17}$$

where $F_1\boldsymbol{\tau}_1$ is a similar image of the parameter of the first classification and $\boldsymbol{\tau}_2$ is a residual parameter vector subject to the condition

$$F_1'\boldsymbol{\tau}_2=0. \tag{3.18}$$

The residual parameter vector may be accounted for by the parameter vector representing the class effects of the second classification. The restriction (3.18) means that the sum of the effects of the second classification within each of the same classes of the first classification is zero. The parameter model introduced here coincides with the usual parameter model associated with a two-fold nested classification design.

In the same way, if we map $\mathfrak{A}(\mathrm{N}_2)$ by σ_2 and define the composed parameter vector for $\mathfrak{A}(\mathrm{N}_3)$ as

$$\boldsymbol{\xi}_3=F_2\boldsymbol{\xi}_2+\boldsymbol{\tau}_3=F_2F_1\boldsymbol{\tau}_1+F_2\boldsymbol{\tau}_2+\boldsymbol{\tau}_3, \tag{3.19}$$

then the residual parameter vector $\boldsymbol{\tau}_3$ is subject to the restriction

$$F_2'\boldsymbol{\tau}_3=0 \tag{3.20}$$

and is explained by the parameter vector representing the effects of the third classification.

In general, the associated parameter vector for an m-fold nested association algebra $\mathfrak{A}(\mathrm{N}_m)$, which may naturally be defined along the lines of the series of composition, is

$$\begin{aligned}\boldsymbol{\xi}_m=&F_{m-1}F_{m-2}\cdots F_1\boldsymbol{\tau}_1+F_{m-1}F_{m-2}\cdots F_2\boldsymbol{\tau}_2+\cdots+F_{m-1}\cdots F_l\boldsymbol{\tau}_l+\\&\cdots+F_{m-1}\boldsymbol{\tau}_{m-1}+\boldsymbol{\tau}_m,\end{aligned} \tag{3.21}$$

where $\boldsymbol{\tau}_2, \boldsymbol{\tau}_3, \cdots, \boldsymbol{\tau}_m$ represent the effects of the second, the third, $\cdots$ and the m-th classification, respectively. They are subject to the restrictions

$$F'_{l-1}\boldsymbol{\tau}_l=0 \tag{3.22}$$

for $l=2, 3, \cdots, m$.

The decomposition of the parameter sum of squares uniquely determined by the composed association algebra $\mathfrak{A}(\mathrm{N}_m)$ is

$$\boldsymbol{\xi}'_m\boldsymbol{\xi}_m=\sum_{i=0}^{m}\boldsymbol{\xi}'_m\overset{(m)}{A}{}_i^{\sharp}\boldsymbol{\xi}_m \tag{3.23}$$

and can easily be reduced to the following form by using (3.13):

$$\begin{aligned}\boldsymbol{\xi}'_m\boldsymbol{\xi}_m&=\frac{v_m}{v_1}\boldsymbol{\tau}'_1\overset{(1)}{A}{}_0^{\sharp}\boldsymbol{\tau}_1+\sum_{i=1}^{m}\frac{v_m}{v_i}\boldsymbol{\tau}'_i\overset{(i)}{A}{}_i^{\sharp}\boldsymbol{\tau}_i\\&=v_m\bar{\tau}_1^2+\frac{v_m}{s_1}\sum_{j=1}^{s_1}(\tau_{1j}-\bar{\tau}_1)^2+\sum_{i=2}^{m}\frac{v_m}{v_i}\boldsymbol{\tau}'_i\boldsymbol{\tau}_i.\end{aligned} \tag{3.24}$$

The degrees of freedom of these components are

$$\begin{aligned}&\text{d.f. }(\boldsymbol{\xi}'_m\overset{(m)}{A}{}_i^{\sharp}\boldsymbol{\xi}_m)=\text{rank }(\overset{(m)}{A}{}_i^{\sharp})=s_1s_2\cdots s_{i-1}(s_i-1)\\&\qquad\qquad(i=1, 2, \cdots, m)\\&\text{d.f. }(\boldsymbol{\xi}'_m\overset{(m)}{A}{}_0^{\sharp}\boldsymbol{\xi}_m)=\text{rank }(\overset{(m)}{A}{}_0^{\sharp})=1.\end{aligned} \tag{3.25}$$

4. Factorial series of association schemes

(a) *Definition of* F_p *type association schemes*

Suppose that there are $v_p=s_1s_2\cdots s_p$ treatments $\phi(\alpha_1, \alpha_2, \cdots, \alpha_p)$ indexed by p-tuples $(\alpha_1, \alpha_2, \cdots, \alpha_p)$ where $\alpha_i=1, 2, \cdots, s_i$ and $i=1, 2, \cdots, p$. Among those treatments, an association of factorial type or F_p type is defined as follows:

Definition: Two treatments $\phi(\alpha_1, \alpha_2, \cdots, \alpha_p)$ and $\phi(\beta_1, \beta_2, \cdots, \beta_p)$ are $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p)$-th associates if $(\varepsilon(\alpha_1-\beta_1), \varepsilon(\alpha_2-\beta_2), \cdots, \varepsilon(\alpha_p-\beta_p))=(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p)$, where $\varepsilon(x)$ is a function of x which assumes either the value zero or one according as x is zero or not. Each treatment is $(0, 0, \cdots, 0)$-th associate of itself.

Note that the number of associate classes is expressed not decimally but in binary notation.

It is easy to verify that the association defined above satisfies three conditions of the association scheme with 2^p-1 associate classes. Hinkelmann and Kemphone [7] called the association an extended group divisible (EGD) association. Hinkelmann [6] investigated the EGD(2^p-1)-PBIBD in some

detail. Following traditional nomenclature, we prefer to call the association a p-way factorial association scheme or an F_p type association scheme.

(b) *Association matrices and association algebras*

Consider an F_p type association scheme. After numbering v_p treatments $\phi(\alpha_1, \alpha_2, \cdots, \alpha_p)$ in dictionary-wise, we can express the $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p)$-th association matrices as

$$A_{\varepsilon_1\varepsilon_2\cdots\varepsilon_p}=A_{\varepsilon_1}\otimes A_{\varepsilon_2}\otimes\cdots\otimes A_{\varepsilon_p}, \tag{4.1}$$

where

$$A_{\varepsilon_i}=(1-\varepsilon_i)\,I_{s_i}+\varepsilon_i(G_{s_i}-I_{s_i}),$$

$$\varepsilon_i=0 \quad \text{or} \quad 1, \quad i=1, 2, \cdots, p.$$

It is known that the linear closure of these association matrices $A_{\varepsilon_1\cdots\varepsilon_p}$ over the rational field forms a linear associative algebra generated by these association matrices. We call the algebra an F_p type association algebra and denote it as $\mathfrak{A}(F_p)$ or $[A_{\varepsilon_1\cdots\varepsilon_p};\ \varepsilon_i=0, 1,\ i=1, \cdots, p]$.

The mutually orthogonal principal idempotents of 2^p two-sided ideals of the algebra $\mathfrak{A}(F_p)$ are

$$A^{\sharp}_{\varepsilon_1\varepsilon_2\cdots\varepsilon_p}=A^{\sharp}_{\varepsilon_1}\otimes A^{\sharp}_{\varepsilon_2}\otimes\cdots\otimes A^{\sharp}_{\varepsilon_p}, \tag{4.2}$$

where

$$A^{\sharp}_{\varepsilon_i}=\varepsilon_i\left(I_{s_i}-\frac{1}{s_i}G_{s_i}\right)+(1-\varepsilon_i)\frac{1}{s_i}G_{s_i},$$

$$(\varepsilon_i=0, 1;\ i=1, 2, \cdots, p).$$

In the case $p=1$,

$$A_0=I_s, \quad A_1=G_s-I_s; \quad A^{\sharp}_0=\frac{1}{s}G_s, \quad A^{\sharp}_1=I_s-\frac{1}{s}G_s.$$

In the case $p=2$,

$$A_{00}=I_{s_1}\otimes I_{s_2} \qquad A_{01}=I_{s_1}\otimes(G_{s_2}-I_{s_2})$$

$$A_{10}=(G_{s_1}-I_{s_1})\otimes I_{s_2} \qquad A_{11}=(G_{s_1}-I_{s_1})\otimes(G_{s_2}-I_{s_2});$$

$$A^{\sharp}_{00}=\frac{1}{s_1s_2}G_{s_1}\otimes G_{s_2} \qquad A^{\sharp}_{01}=\frac{1}{s_1}G_{s_1}\otimes\left(I_{s_2}-\frac{1}{s_2}G_{s_2}\right)$$

$$A_{10}^{\sharp}=\left(I_{s_1}-\frac{1}{s_1}G_{s_1}\right)\otimes\frac{1}{s_2}G_{s_2}\qquad A_{11}^{\sharp}=\left(I_{s_1}-\frac{1}{s_1}G_{s_1}\right)\otimes\left(I_{s_2}-\frac{1}{s_2}G_{s_2}\right).$$

(c) *Composition of the series* $\{\mathfrak{A}(\mathrm{F}_p)\}$

Now we show the way of composing step by step the series of the association algebras $\{\mathfrak{A}(\mathrm{F}_p)\}$ starting from p primitive association algebras $\mathfrak{A}_i(\mathrm{F}_1)=[I_{s_i}, G_{s_i}]$ $(i=1, 2, \cdots, p)$, defined respectively over the s_i levels of the i-th factor.

It was already shown in [22] that the orthogonal composition of two primitive algebra, say $\mathfrak{A}_1(\mathrm{F}_1)=[I_{s_1}, G_{s_1}]$ and $\mathfrak{A}_2(\mathrm{F}_1)=[I_{s_2}, G_{s_2}]$, yield the two way factorial association algebra $\mathfrak{A}_{12}(\mathrm{F}_2)$. We therefore, show the way of yielding $\mathfrak{A}(\mathrm{F}_p)$ by composing orthogonally p $\mathfrak{A}(\mathrm{F}_{p-1})$'s simultaneously.

Consider p factors $S_1, \cdots, S_p$ with $s_1, \cdots, s_p$ levels respectively and suppose that among $p-1$ factors $S_1, \cdots, S_{l-1}S_{l+1}, \cdots, S_p$, the $(p-1)$-way factorial association algebra $\mathfrak{A}_{1\cdots l-1\ l+1\cdots p}(\mathrm{F}_{p-1})$ has been composed for every $l=1, 2, \cdots, p$.

Let

$$\boldsymbol{\Phi}_{b_1b_2\cdots b_l}^{(a_1a_2\cdots a_k)}=H_1\otimes H_2\otimes\cdots\otimes H_k, \tag{4.3}$$

where

$$H_i=\begin{cases} I_{s_{a_i}} & a_i=b_1, b_2, \cdots, \text{ or } b_l, \\ \boldsymbol{j}_{s_{a_i}} & \text{otherwise} \end{cases}$$

for $i=1, 2, \cdots, k$. Here $a_1<a_2<\cdots<a_k$ are k different integers and $b_1<b_2<\cdots<b_l$ are l different integer selected from them. For example, some of the special cases are illustrated below.

$$\boldsymbol{\Phi}_1^{(12)}=I_{s_1}\otimes\boldsymbol{j}_{s_2},\quad \boldsymbol{\Phi}_3^{(234)}=\boldsymbol{j}_{s_2}\otimes I_{s_3}\otimes\boldsymbol{j}_{s_4}$$

$$\boldsymbol{\Phi}_{1\cdots l-1\ l+1\cdots p}^{(12\cdots p)}=I_{s_1}\otimes\cdots\otimes I_{s_{l-1}}\otimes\boldsymbol{j}_{s_l}\otimes I_{s_{l+1}}\otimes\cdots\otimes I_{s_p}.$$

Let $\sigma_{1\cdots(l)\cdots p}$ be a linear mapping of $\mathfrak{A}_{1\cdots l-1\ l+1\cdots p}(\mathrm{F}_{p-1})$ defined by

$$\sigma_{1\cdots(l)\cdots p}:$$

$$\mathfrak{A}_{1\cdots l-1\ l-1\cdots p}(\mathrm{F}_{p-1})\ni A\to\boldsymbol{\Phi}_{1\cdots l-1\ l+l\cdots p}^{(12\cdots p)}A\boldsymbol{\Phi}_{1\cdots l-1\ l+1\cdots p}^{(12\cdots p)\prime} \tag{4.4}$$

for every $l=1, 2, \cdots, p$.

It is easy to see that each of the linear mappings $\sigma_{1\cdots(l)\cdots p}$ gives a similar mapping of $\mathfrak{A}_{1\cdots l-1\ l+1\cdots p}(\mathrm{F}_{p-1})$, respectively. Moreover, for each pair of l and $l'(l<l')$, we can see that the intersection of the image algebras

$$\mathfrak{B}_{1\cdots p}^{l\,l'}=\sigma_{1\cdots(l)\cdots p}\left(\mathfrak{A}_{1\cdots l-1\ l+1\cdots p}(\mathrm{F}_{p-1})\right)\cap\sigma_{1\cdots(l')\cdots p}\left(\mathfrak{A}_{1\cdots l'-1\ l'+1\cdots p}(\mathrm{F}_{p-1})\right)$$

$$= \Big[A^{\sharp}_{\varepsilon_1} \otimes \cdots \otimes A^{\sharp}_{\varepsilon_{l-1}} \otimes \frac{1}{s_l} G_{s_l} \otimes A^{\sharp}_{\varepsilon_{l+1}} \otimes \cdots \otimes A^{\sharp}_{\varepsilon_{l'-1}}$$

$$\otimes \frac{1}{s_{l'}} G_{s_{l'}} \otimes A^{\sharp}_{s_{l'+1}} \otimes \cdots \otimes A^{\sharp}_{s_p};\ \varepsilon_i = 0, 1,\ i \neq l, l' \Big] \tag{4.5}$$

is the maximal common two-sided ideal of those algebras. Hence the two mappings $\sigma_{1\cdots(l)\cdots p}$ of $\mathfrak{A}_{1\cdots l-1\ l+1\cdots p}(\mathrm{F}_{p-1})$ and $\sigma_{1\cdots(l')\cdots p}$ of $\mathfrak{A}_{1\cdots l'-1\ l'+1\cdots p}(\mathrm{F}_{p-1})$ are $\mathfrak{B}^{ll'}_{1\cdots p}$-orthogonal [22]. Then we have the theorem:

THEOREM 4.1 *Similar mappings $\sigma_{1\cdots(l)\cdots p}$'s of p $(p-1)$-way factorial association algebras $\mathfrak{A}_{1\cdots l-1\ l+1\cdots p}(\mathrm{F}_{p-1})$ defined by $\Phi^{(12\cdots p)}_{1\cdots l-1\ l+1\cdots p}$ give pairwise orthogonal mappings modulo $\mathfrak{B}^{ll'}_{1\cdots p}$ for every pair of l and l' and the union, or orthogonal composition of p image algebras for $l=1, 2, \cdots, p$ with I_{v_p}, yields a p-way factorial association algebra $\mathfrak{A}_{12\cdots p}(\mathrm{F}_p)$, i.e.,*

$$\mathfrak{A}_{12\cdots p}(\mathrm{F}_p) = \bigcup_{l=1}^{p} \sigma_{1\cdots(l)\cdots p}\big(\mathfrak{A}_{12\cdots l-1\,l+1\cdots p}(\mathrm{F}_{p-1})\big) \cup [I_{v_p}]. \tag{4.6}$$

(d) *Parameter models*

The composition of the series of factorial association algebras induces in each stage of the composition a natural parameter vector associated to the composed factorial association algebra. We show the structure of the composed parameter vectors inductively in the following. The results correspond to the usual parameter models of the designs of factorial type. The decomposition of the sum of squares of the composed parameters, which is determined uniquely by the corresponding composed algebra, is also given.

Let $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \cdots, \boldsymbol{\tau}_p$ be the parameter vectors representing the level-effects of the p factors $S_1, S_2, \cdots, S_p$ respectively.

As was indicated in [22], the orthogonal composition of two algebras, say $\mathfrak{A}_1(\mathrm{F}_1)=[I_{s_1}, G_{s_1}]$ and $\mathfrak{A}_2(\mathrm{F}_1)=[I_{s_2}, G_{s_2}]$, yields a two-way factorial association algebra $\mathfrak{A}_{12}(\mathrm{F}_2)$, the composed parameter vector being

$$\boldsymbol{\xi}_{12} = \Phi^{(12)}_1 \boldsymbol{\tau}_1 + \Phi^{(12)}_2 \boldsymbol{\tau}_2 + \boldsymbol{\tau}_{12}, \tag{4.7}$$

where $\boldsymbol{\tau}_{12}$ was subject to the conditions $\Phi^{(12)\prime}_1 \boldsymbol{\tau}_{12} = \Phi^{(12)\prime}_2 \boldsymbol{\tau}_{12} = 0$ and was explained by an interaction parameter vector of two factors S_1 and S_2. $\boldsymbol{\tau}_{12}=0$ or not according as the additivity of the level-effects of two factors was assumed or not. Associated decomposition of the composed parameter sum of squares was

$$\boldsymbol{\xi}'_{12}\boldsymbol{\xi}_{12} = \boldsymbol{\xi}'_{12}A^{\sharp}_{00}\boldsymbol{\xi}_{12} + \boldsymbol{\xi}'_{12}A^{\sharp}_{10}\boldsymbol{\xi}_{12} + \boldsymbol{\xi}'_{12}A^{\sharp}_{01}\boldsymbol{\xi}_{12} + \boldsymbol{\xi}'_{12}A^{\sharp}_{11}\boldsymbol{\xi}_{12}$$

$$= s_1 s_2 (\bar{\tau}_1 + \bar{\tau}_2)^2 + s_2 \boldsymbol{\tau}'_1 \Big(I_{s_1} - \frac{1}{s_1} G_{s_1} \Big) \boldsymbol{\tau}_1 + s_1 \boldsymbol{\tau}'_2 \Big(I_{s_2} - \frac{1}{s_2} G_{s_2} \Big) \boldsymbol{\tau}_2 + \boldsymbol{\tau}'_{12}\boldsymbol{\tau}_{12}, \tag{4.8}$$

where $\bar{\tau}_i = s_i^{-1}\tau_i' \boldsymbol{j}_{s_i}$ $(i=1, 2)$.

Now we show the composition of the parameter vectors for three way factorial association algebra composed of three factors, say S_1, S_2 and S_3, assuming that three parameter vector $\boldsymbol{\xi}_{12}$, $\boldsymbol{\xi}_{13}$ and $\boldsymbol{\xi}_{23}$ for $\mathfrak{A}_{12}(\mathrm{F}_2)$, $\mathfrak{A}_{13}(\mathrm{F}_2)$ and $\mathfrak{A}_{23}(\mathrm{F}_2)$ are already composed, *i.e.*,

$$\begin{aligned}
&\boldsymbol{\xi}_{12} = \boldsymbol{\Phi}_1^{(12)}\boldsymbol{\tau}_1 + \boldsymbol{\Phi}_2^{(12)}\boldsymbol{\tau}_2 + \boldsymbol{\tau}_{12} && (\boldsymbol{\Phi}_1^{(12)\prime}\boldsymbol{\tau}_{12} = \boldsymbol{\Phi}_2^{(12)\prime}\boldsymbol{\tau}_{12} = 0)\\
&\boldsymbol{\xi}_{13} = \boldsymbol{\Phi}_1^{(13)}\boldsymbol{\tau}_1 + \boldsymbol{\Phi}_3^{(13)}\boldsymbol{\tau}_3 + \boldsymbol{\tau}_{13} && (\boldsymbol{\Phi}_1^{(13)\prime}\boldsymbol{\tau}_{13} = \boldsymbol{\Phi}_3^{(13)\prime}\boldsymbol{\tau}_{13} = 0) \qquad (4.9)\\
&\boldsymbol{\xi}_{23} = \boldsymbol{\Phi}_2^{(23)}\boldsymbol{\tau}_2 + \boldsymbol{\Phi}_3^{(23)}\boldsymbol{\tau}_3 + \boldsymbol{\tau}_{23} && (\boldsymbol{\Phi}_2^{(23)\prime}\boldsymbol{\tau}_{23} = \boldsymbol{\Phi}_3^{(23)\prime}\boldsymbol{\tau}_{23} = 0).
\end{aligned}$$

Three matrices $\boldsymbol{\Phi}_{12}^{(123)}$, $\boldsymbol{\Phi}_{13}^{(123)}$ and $\boldsymbol{\Phi}_{23}^{(123)}$ which define $\sigma_{12(3)}$, $\sigma_{1(2)3}$, and $\sigma_{(1)23}$ and yield an orthogonal composition $\mathfrak{A}_{123}(\mathrm{F}_3)$ of $\mathfrak{A}_{12}(\mathrm{F}_2)$, $\mathfrak{A}_{13}(\mathrm{F}_2)$ and $\mathfrak{A}_{23}(\mathrm{F}_2)$, map those parameters to the v_3 dimensional space over which the algebra $\mathfrak{A}_{123}(\mathrm{F}_3)$ is composed. Paying attention to the fact that the composition is $\mathfrak{B}$-orthogonal, *i.e.*, that the images of those parameters, $\boldsymbol{\Phi}_{12}^{(123)}\boldsymbol{\xi}_{12}$, $\boldsymbol{\Phi}_{13}^{(123)}\boldsymbol{\xi}_{13}$ and $\boldsymbol{\Phi}_{23}^{(123)}\boldsymbol{\xi}_{23}$ have common elements each other as, for example,

$$\boldsymbol{\Phi}_{12}^{(123)}\boldsymbol{\Phi}_1^{(12)}\boldsymbol{\tau}_1 = \boldsymbol{\Phi}_{13}^{(123)}\boldsymbol{\Phi}_1^{(13)}\boldsymbol{\tau}_1 = \boldsymbol{\Phi}_1^{(123)}\boldsymbol{\tau}_1,$$

we can naturally define a composed parameter as

$$\begin{aligned}
\boldsymbol{\xi}_{123} &= \{\boldsymbol{\Phi}_{12}^{(123)}\boldsymbol{\xi}_{12}\} \cup \{\boldsymbol{\Phi}_{13}^{(123)}\boldsymbol{\xi}_{13}\} \cup \{\boldsymbol{\Phi}_{23}^{(123)}\boldsymbol{\xi}_{23}\} + \boldsymbol{\tau}_{123}\\
&= \boldsymbol{\Phi}_1^{(123)}\boldsymbol{\tau}_1 + \boldsymbol{\Phi}_2^{(123)}\boldsymbol{\tau}_2 + \boldsymbol{\Phi}_3^{(123)}\boldsymbol{\tau}_3 \qquad (4.10)\\
&\quad + \boldsymbol{\Phi}_{12}^{(123)}\boldsymbol{\tau}_{12} + \boldsymbol{\Phi}_{13}^{(123)}\boldsymbol{\tau}_{13} + \boldsymbol{\Phi}_{23}^{(123)}\boldsymbol{\tau}_{23} + \boldsymbol{\tau}_{123}
\end{aligned}$$

where $\boldsymbol{\tau}_{123}$ is a residual vector orthogonal to the image algebras $\sigma_{12(3)}(\mathfrak{A}_{12}(\mathrm{F}_2))$, $\sigma_{1(2)3}(\mathfrak{A}_{13}(\mathrm{F}_2))$ and $\sigma_{(1)23}(\mathfrak{A}_{23}(\mathrm{F}_2))$, simultaneously. It can be explained by the second order interaction parameter vector of the three factors S_1, S_2 and S_3, subject to the restriction

$$\boldsymbol{\Phi}_{12}^{(123)\prime}\boldsymbol{\tau}_{123} = \boldsymbol{\Phi}_{13}^{(123)\prime}\boldsymbol{\tau}_{123} = \boldsymbol{\Phi}_{23}^{(123)\prime}\boldsymbol{\tau}_{123} = 0.$$

$\boldsymbol{\tau}_{123}$ is zero or not according as three composed parameter effects are assumed to be additive or not.

Inductive application of the above arguments gives us a natural composed parameter vector for p-way factorial association algebra $\mathfrak{A}_{12\cdots p}(\mathrm{F}_p)$, *i.e.*,

$$\begin{aligned}
\boldsymbol{\xi}_{12\cdots p} &= \sum_{(l_1)}{}' \boldsymbol{\Phi}_{l_1}^{(12\cdots p)}\boldsymbol{\tau}_{l_1} + \sum_{(l_1 l_2)}{}' \boldsymbol{\Phi}_{l_1 l_2}^{(12\cdots p)}\boldsymbol{\tau}_{l_1 l_2}\\
&\quad + \cdots + \sum_{(l_1\cdots l_k)}{}' \boldsymbol{\Phi}_{l_1\cdots l_k}^{(12\cdots p)}\boldsymbol{\tau}_{l_1\cdots l_k} + \cdots + \boldsymbol{\tau}_{12\cdots p},
\end{aligned} \qquad (4.11)$$

where $\sum_{(l_1\cdots l_k)}{}'$ means that the summation extends over all subsets of k elements

of the set $\{1, 2, \cdots, p\}$ and $\boldsymbol{\tau}_{12\cdots p}$ are subject to the restrictions $\boldsymbol{\Phi}^{(12\cdots p)'}_{12\cdots l-1\,l+1\cdots p}\boldsymbol{\tau}_{12\cdots p}=0$ for all $l=1, 2, \cdots, p$. $\boldsymbol{\tau}_{12\cdots p}$ is zero or not according as p composed parameter effects are assumed to be additive or not. The parameter vector $\boldsymbol{\tau}_{l_1\cdots l_k}$ is the $(k-1)$th order interaction between k factors $S_{l_1}, S_{l_2}, \cdots, S_{l_k}$.

The unique decomposition of the parameter sum of squares determined by the p-way factorial association algebra $\mathfrak{A}_{12\cdots p}(\mathrm{F}_p)$ is

$$\boldsymbol{\xi}'_{12\cdots p}\boldsymbol{\xi}_{12\cdots p}=\sum_{\varepsilon_1\cdots\varepsilon_p}\boldsymbol{\xi}'_{12\cdots p}A^{\#}_{\varepsilon_1\varepsilon_2\cdots\varepsilon_p}\boldsymbol{\xi}_{12\cdots p} \tag{4.12}$$

For those idempotents $A^{\#}_{\varepsilon_1\varepsilon_2\cdots\varepsilon_p}$ satisfying $\sum_{i=1}^{p}\varepsilon_i=k(>1)$, each member of the right hand side of (4.12) can be reduced to

$$\boldsymbol{\xi}'_{12\cdots p}A^{\#}_{\varepsilon_1\varepsilon_2\cdots\varepsilon_p}\boldsymbol{\xi}_{12\cdots p}=\frac{v_p}{s_{l_1}\cdots s_{l_k}}\boldsymbol{\tau}'_{l_1\cdots l_k}\boldsymbol{\tau}_{l_1\cdots l_k} \tag{4.13}$$

for $\varepsilon_{l_1}=\varepsilon_{l_2}=\cdots=\varepsilon_{l_k}=1$. It can be explained by the $(k-1)$-th order interaction sum of squares between k factors $S_{l_1}, S_{l_2}, \cdots, S_{l_k}$.

For those idempontents $A^{\#}_{\varepsilon_1\varepsilon_2\cdots\varepsilon_p}$ satisfying $\sum_{i=1}^{p}\varepsilon_i=1$, each member can be reduced to

$$\boldsymbol{\xi}'_{12\cdots p}A^{\#}_{0\cdots 0\,1\,0\cdots 0}\boldsymbol{\xi}_{12\cdots p}=\frac{v_p}{s_l}\boldsymbol{\tau}'_l\left(I_{s_l}-\frac{1}{s_l}G_{s_l}\right)\boldsymbol{\tau}_l \tag{4.14}$$

for $\varepsilon_l=1$. This is the sum of squares for the main effect of the l-th factor S_l.

For the idempotent $A^{\#}_{0\,0\cdots 0}$, it can reduce to

$$\boldsymbol{\xi}'_{12\cdots p}A^{\#}_{0\,0\cdots 0}\boldsymbol{\xi}_{12\cdots p}=v_p(\bar{\tau}_1+\bar{\tau}_2+\cdots+\bar{\tau}_p)^2 \tag{4.15}$$

and is explained by the sum of squares for grand mean.

The degrees of freedom of these components are:

$$\text{d.f. }(\boldsymbol{\xi}'_{1\cdots p}A^{\#}_{\varepsilon_1\varepsilon_2\cdots\varepsilon_p}\boldsymbol{\xi}_{1\cdots p})=\text{rank }(A^{\#}_{\varepsilon_1\varepsilon_2\cdots\varepsilon_p})=\prod_{i=1}^{p}(s_i-1)^{\varepsilon_i}$$
$$(\varepsilon_i=0, 1;\ i=1, 2, \cdots, p). \tag{4.16}$$

Note that some of the interaction parameter vectors are assumed to be zero, and the corresponding degrees of freedom are assigned for the estimation of error.

(e) *Hyper-cubic series of association schemes*

Suppose that there are $v_p=s^p$ treatments $\phi(\alpha_1, \cdots, \alpha_p)$ indexed by p-tuples

$(\alpha_1, \cdots, \alpha_p)$, $(\alpha_i = 1, 2, \cdots, s;\ i=1, 2, \cdots, p)$. Among these treatments, a p-dimensional hyper-cubic association scheme is defined as follows:

DEFINITION: Two treatments $\phi(\alpha_1, \cdots, \alpha_p)$ and $\phi(\beta_1, \cdots, \beta_p)$ are i-th associates if and only if $\sum_{k=1}^{p}{}'\varepsilon(\alpha_k-\beta_k)=i$. Each treatment is 0-th associate of itself.

For the case $p=2$, the association is called L_2 type [4]. For the case $p=3$, Raghavarao and Chandrasekharao [18] called the association a cubic type and studied it in some detail. For the case $p>3$, Kusumoto [11] called the association a hyper-cubic type and studied it in some detail. We call the series of association schemes for $p=1, 2, \cdots$ a hyper-cubic series of association schemes, or a C_p type series of association schemes.

It can easily be verified that a C_p type association scheme is a reduced association scheme of an F_p type association scheme where $s_1=s_2=\cdots=s_p=s$. The association matrices $\overset{(p)}{C}_i$ of C_p type can be expressed by using the association matrices $A_{\varepsilon_1\cdots\varepsilon_p}$ of F_p type as

$$\overset{(p)}{C}_i = \sum_{\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_p=i}{}' A_{\varepsilon_1\varepsilon_2\cdots\varepsilon_p}. \tag{4.17}$$

An association algebra of C_p type, $\mathfrak{A}(C_p)$, is a subalgebra of the algebra $\mathfrak{A}(F_p)$. The mutually orthogonal principal idempotents $\overset{(p)}{C}{}^{\#}_i$ of $\mathfrak{A}(C_p)$ are

$$\overset{(p)}{C}{}^{\#}_i = \sum_{\varepsilon_1+\cdots+\varepsilon_p=i}{}' A^{\#}_{\varepsilon_1}\otimes A^{\#}_{\varepsilon_2}\otimes\cdots\otimes A^{\#}_{\varepsilon_p}, \tag{4.18}$$

where

$$A^{\#}_{\varepsilon_j}=\varepsilon_j\left(I_s-\frac{1}{s}G_s\right)+(1-\varepsilon_j)\frac{1}{s}G_s$$

$$(\varepsilon_j=0, 1;\ j=1, 2, \cdots, p).$$

The relation (4.18) shows that those 2^p mutually orthogonal idempotents of $\mathfrak{A}(F_p)$ are pooled together into $p+1$ mutually orthogonal idempotents, $\overset{(p)}{C}{}^{\#}_i$.

One of the parameter models naturally defined for $\mathfrak{A}(C_p)$ is the same as that of $\mathfrak{A}(F_p)$, where the number of levels of all p factors are equal. In this case, the sum of squares of the parameters may be decomposed as

$$\boldsymbol{\xi}'_{12\cdots p}\boldsymbol{\xi}_{12\cdots p} = \sum_{i=0}^{p}{}'\boldsymbol{\xi}'_{12\cdots p}\overset{(p)}{C}{}^{\#}_i\boldsymbol{\xi}_{12\cdots p}$$

$$= \sum_{i=0}^{p}{}'\Big(\sum_{\varepsilon_1+\cdots+\varepsilon_p=i}{}'\boldsymbol{\xi}'_{12\cdots p}A^{\#}_{\varepsilon_1\varepsilon_2\cdots\varepsilon_p}\boldsymbol{\xi}_{12\cdots p}\Big). \tag{4.19}$$

For $i=0$,

$$\boldsymbol{\xi}'_{12\cdots p}\overset{(p)}{C}{}^{\sharp}_{0}\boldsymbol{\xi}_{12\cdots p}=\boldsymbol{\xi}'_{12\cdots p}A^{\sharp}_{00\cdots 0}\boldsymbol{\xi}_{12\cdots p}=s^{p}(\bar{\tau}_1+\cdots+\bar{\tau}_p)^2 \tag{4.20}$$

and is explained by the sum of squares for grand mean.
For $i=1$,

$$\boldsymbol{\xi}'_{12\cdots p}\overset{(p)}{C}{}^{\sharp}_{1}\boldsymbol{\xi}_{12\cdots p}=s^{p-1}\sum_{l=1}^{p}\boldsymbol{\tau}'_l\left(I_s-\frac{1}{s}G_s\right)\boldsymbol{\tau}_l \tag{4.21}$$

and is explained by the pooled sums of squares due to main effects of all p factors.
For $2\leq i\leq p$,

$$\boldsymbol{\xi}'_{12\cdots p}\overset{(p)}{C}{}^{\sharp}_{i}\boldsymbol{\xi}_{12\cdots p}=s^{p-i}\sum_{(l_1\cdots l_i)}\boldsymbol{\tau}'_{l_1\cdots l_i}\boldsymbol{\tau}_{l_1\cdots l_i} \tag{4.22}$$

and is explained by the pooled sum of squares of all the $(i-1)$-th order interaction sum of squares between i factors.

The degrees of freedom of these sum of squares are

$$\text{d.f. }(\boldsymbol{\xi}'_{12\cdots p}\overset{(p)}{C}{}^{\sharp}_{i}\boldsymbol{\xi}_{12\cdots p})=\text{rank }(\overset{(p)}{C}{}^{\sharp}_{i})=\binom{p}{i}(s-1)^i \tag{4.23}$$

$$(\text{i}=1,\ 2,\ \cdots,\ p).$$

In a factorial scheme, the main effects of different factors and the interaction effects of the sets of different factors are separated from each other in the decomposition of the parameter sum of squares by the association algebra. While, in a hyper-cubic scheme, they are pooled in $p+1$ groups and the main effects of the different factors cannot be separated from each other. The interaction effects of the same order cannot be separated from each other. Those considerations lead to the another important model of the parameters associated with the hyper-cubic schemes. The model is related to the case where all factors $S_1,\ S_2,\ \cdots,\ S_p$ are the same, *i.e.*,

$$\begin{aligned}\boldsymbol{\xi}_{12\cdots p}=&\sum_{l_1}\boldsymbol{\Phi}^{(12\cdots p)}_{l_1}\boldsymbol{\tau}_1+\sum_{(l_1l_2)}\boldsymbol{\Phi}^{(12\cdots p)}_{l_1l_2}\boldsymbol{\tau}_2\\&+\cdots+\sum_{(l_1\cdots l_k)}\boldsymbol{\Phi}^{(12\cdots p)}_{l_1\cdots l_k}\boldsymbol{\tau}_k+\cdots+\boldsymbol{\tau}_p,\end{aligned} \tag{4.24}$$

where $\boldsymbol{\tau}_1$ is a vector of the level effects of a factor S, and $\boldsymbol{\tau}_k$, $(k=2,\ \cdots,\ p)$ is explained by a vector of $(k-1)$-th order interaction effects between these s level effects as was considered in the case of $\mathfrak{A}(\mathrm{T_m})$.

5. Orthogonal Latin square series of association schemes

(a) *Definition of* OL_r *type association scheme*

Suppose that there are $v=k^2$ treatments indexed respectively by $1, 2, \cdots, k^2$ and they are set forth in a square $\mathfrak{B}$ so that the $\{(i-1)k+j\}$-th treatment lies in the j-th column of the i-th row. Suppose, further, there exist $r-2$ mutually orthogonal Latin squares, $\mathfrak{B}_3, \cdots, \mathfrak{B}_r$, of order $k (r \leq k+1)$.

Among those treatments, we define an association of orthogonal Latin square type or OL_r type with m associate classes as follows:

DEFINITION: Two treatments α and β are 1st associates if they occur in the same row, 2nd associates if they occur in the same column, and i-th associates if they correspond to the same letter of i-th Latin square $\mathfrak{B}_i$ $(i=3, \cdots, r)$. Otherwise they are $(r+1)$-th associates. Each treatment is 0-th associate of itself. Note that if $r=k+1$, there is no pair of treatments which are neither 1st, 2nd, $\cdots$, nor r-th associates. The number of associate classes is therefore $m=\min(r+1, k+1)$.

Let F_1 be the $v \times k$ incidence matrix for treatments vs. rows and F_2 be that for trements vs. columns. Let F_i be the $v \times k$ incidence matrix for treatments vs. letters of the i-th Latin square $(i=3, \cdots, r)$. Then we have the following relations:

$$F_1 = I_k \otimes \boldsymbol{j}_k, \qquad F_2 = \boldsymbol{j}_k \otimes I_k \tag{5.1}$$

$$F_i' F_i = k I_k \qquad (i=1, 2, \cdots, r) \tag{5.2}$$

$$F_i' F_j = G_k \qquad (i \neq j \quad i, j=1, \cdots, r). \tag{5.3}$$

Note that the existence of the matrices $F_1, \cdots, F_r$, whose elements are either zero or one and satisfying (5.2) and (5.3), is equivalent to the existence of $r-2$ mutually orthogonal Latin squares of order k.

Now we prove that the ralation of association above defined satisfies three conditions of the association scheme. The first condition of the association scheme is an immediate consequence of the definition. In order to verify the second and the third condition of the association scheme, it is convenient to use the association matrices:

$$A_0 = I_v, \qquad A_i = F_i F_i' - I_v \qquad (i=1, \cdots, r)$$

$$A_{r+1} = G_v - \sum_{i=0}^{r} A_i, \tag{5.4}$$

where A_0, A_i and A_{r+1} are 0-th, i-th and, if $r \leq k$, $(r+1)$-th association matrices, respectively.

As it is easy to verify that

$$A_0\boldsymbol{j}_v=\boldsymbol{j}_v, \qquad A_i\boldsymbol{j}_v=(k-1)\boldsymbol{j}_v, \qquad (i=1, \cdots, r)$$
$$A_{r+1}\boldsymbol{j}_v=(k-1)\,(k-r+1)\boldsymbol{j}_v, \tag{5.5}$$

every treatment has n_j j-th associates. The number is independent of the treatment, *i.e.*, $n_0=1$, $n_1=n_2=\cdots=n_r=k-1$ and, if $r\leq k$, $n_{r+1}=(k-1)(k-r+1)$. Thus the second condition of the association scheme is satisfied.

As we have after some easy calculation,

$$\begin{aligned} &A_0A_i=A_iA_0=A_i && (i=0, 1, \cdots, r+1)\\ &A_i^2=(k-1)A_0+(k-2)A_i && (i=1, 2, \cdots, r)\\ &A_iA_j=A_jA_i=\sum_{k=1}^{r+1}A_k-A_i-A_j && (i\neq j;\ i, j=1, \cdots, r) \end{aligned} \tag{5.6}$$

and when $r\leq k$,

$$A_iA_{r+1}=A_{r+1}A_i=(k-r+1)\sum_{j=1}^{r}A_j-(k-r+1)A_i+(k-r)A_{r+1}$$

$$A_{r+1}A_{r+1}=(k-1)(k-r+1)A_0+(k-r)(k-r+1)\sum_{i=1}^{r}A_i+\{(k-r)^2+r-2\}A_{r+1}, \tag{5.7}$$

the number of the treatments which are j-th associates of α and, at the same time, k-th associates of β is seen to be independent of the pair of treatments α and β, which are i-th associates. Thus the third condition of the association scheme is also satisfied.

(b) OL_r *type association algebra*

We call the linear closure of the association matrices $A_0, A_1, \cdots, A_m$ $(m=\min(r+1, k+1))$ over the rational field an association algebra of orthogonal Latin square type or OL_r type and denote it $\mathfrak{A}(\mathrm{OL}_r)$ or $[A_i;\ i=0, 1, \cdots, m]$.

The mutually orthogonal idempotents of the algebra $\mathfrak{A}(\mathrm{OL}_r)$ are

$$A_0^{\sharp}=\frac{1}{v}G_v$$
$$A_i^{\sharp}=\frac{1}{k}F_i\left(I_k-\frac{1}{k}G_k\right)F_i'=\frac{1}{k}F_iF_i'-\frac{1}{v}G_v \tag{5.8}$$
$$(i=1, 2, \cdots, r)$$

and for $r\leq k$

$$A_{r+1}^{\sharp}=I_v-A_0^{\sharp}-\sum_{i=1}^{r}A_i^{\sharp}. \tag{5.8'}$$

(c) *Composition of the series* $\mathfrak{A}(\mathrm{OL}_r)$

Now we show the way of composing $\mathfrak{A}(\mathrm{OL}_r)$ $(r=2, 3, \cdots, k+1)$ orthogonally with r primitive algebras $\mathfrak{A}_i(\mathrm{OL}_1)=[I_k, G_k]$ $i=1, 2, \cdots, r$ defined over the k levels of the i-th factor, respectively.

Let σ_i be a linear mapping of $\mathfrak{A}_i(\mathrm{OL}_1)$ defined by F_i for every $i=1, 2, \cdots, r$, i.e.

$$\sigma_i : \mathfrak{A}_i(\mathrm{OL}_1) \ni A \to F_i A F_i'. \tag{5.9}$$

Then, from (5.2) σ_i is G-preserving similar and from (5.3) σ_i and σ_j give the G-orthogonal composition of $\mathfrak{A}_i(\mathrm{OL}_1)$ and $\mathfrak{A}_j(\mathrm{OL}_1)$ for all pairs of i and j. The full rank algebra orthogonally composed by $\sigma_1, \sigma_2, \cdots, \sigma_r$ is

$$\bigcup_{i=1}^{r} \sigma_i(\mathfrak{A}_i(\mathrm{OL}_1)) \cup [I_v]. \tag{5.10}$$

As $F_i I_k F_i' = A_i + A_0$, $F_i G_k F_i' = G_v$ for all $i=1, 2, \cdots, r$, we have

$$\bigcup_{i=1}^{r} \sigma_i(\mathfrak{A}_i(\mathrm{OL}_1)) \cup [I_v] = [A_i;\ i=0, 1, \cdots, m]. \tag{5.11}$$

Thus we have the theorem:

THEOREM 5.1 *The composed full-rank algebra defined in* (5.10) *is* $\mathfrak{A}(\mathrm{OL}_r)$.

The case $r=2$, $\mathfrak{A}(\mathrm{OL}_2)$ is the special case of the F_2 type association algebra with three associate classes where $s_1=s_2=k$. The generators of the algebra are

$$\begin{aligned} &F_1 I_k F_1' = I_k \otimes G_k, && F_2 I_k F_2' = G_k \otimes I_k \\ &F_1 G_k F_1' = F_2 G_k F_2' = G_k \otimes G_k, && I_v = I_k \otimes I_k. \end{aligned} \tag{5.12}$$

The mutually orthogonal idempotents are

$$A_0^{\sharp} = \frac{1}{v} G_v = \frac{1}{k} F_i \frac{1}{k} G_k F_i' \qquad (i=1, 2)$$

$$A_i^{\sharp} = \frac{1}{k} F_i \left(I_k - \frac{1}{k} G_k \right) F_i' \qquad (i=1, 2)$$

$$A_3^{\sharp} = I_{k^2} - A_0^{\sharp} - A_1^{\sharp} - A_2^{\sharp}. \tag{5.12'}$$

In the case $r \leq k$, the mutually orthogonal idempotents of $\mathfrak{A}(\mathrm{OL}_r)$ are

$$A_0^\sharp = \frac{1}{v} G_v, \qquad A_i^\sharp = \frac{1}{k} F_i \left(I_k - \frac{1}{k} G_k \right) F_i' \qquad (i=1, \cdots, r)$$

$$A_{r+1}^\sharp = I_v - \sum_{j=0}^{r} A_j^\sharp. \tag{5.13}$$

In the case $r=k+1$, the mutually orthogonal idempotents are

$$A_0^\sharp = \frac{1}{v} G_v, \qquad A_i^\sharp = \frac{1}{k} F_i \left(I_k - \frac{1}{k} G_k \right) F_i' \qquad (i=1, 2, \cdots, k+1). \tag{5.14}$$

(d) *Parameter models*

We now define a natural parameter model for $\mathfrak{A}(\mathrm{OL}_r)$. Let $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \cdots, \boldsymbol{\tau}_r$ be k-dimensional parameter vectors representing the level-effects of the r factors $S_1, S_2, \cdots, S_r$, respectively. Assume that in each of the factor $S_i (i=1, \cdots, r)$, a primitive relationship algebra $\mathfrak{A}_i(\mathrm{OL}_1)=[I_k, G_k]$ is defined among the level effects of the factor.

The orthogonal composition of these $\mathfrak{A}_i(\mathrm{OL}_1)$ by σ_i defined in (5.9) leads to a natural definition of parameter model $\boldsymbol{\xi}_r$ for the composed full-rank algebra $\mathfrak{A}(\mathrm{OL}_r)$, *i.e.*,

$$\boldsymbol{\xi}_r = \sum_{i=1}^{r} F_i \boldsymbol{\tau}_i + \boldsymbol{\delta}_r \tag{5.15}$$

where $F_i \boldsymbol{\tau}_i$ are the similar images of the effects of the factors S_i and $\boldsymbol{\delta}_r$ is a residual parameter subjected to the restrictions

$$F_i' \boldsymbol{\delta}_r = 0 \tag{5.16}$$

for $i=1, 2, \cdots, r$. The vector $\boldsymbol{\delta}_r$ may be explained by the non-additive part of the effects of those r factors. In some cases, it is assumed to be zero. In the case $r=k+1$, where complete set of orthogonal Latin squares exists, the vector $\boldsymbol{\delta}_r$ is forced to be zero.

The unique decomposition of the parameter sum of squares thus defined for $\mathfrak{A}(\mathrm{OL}_r)$ is

$$\boldsymbol{\xi}_r' \boldsymbol{\xi}_r = \sum_{i=0}^{m} \boldsymbol{\xi}_r' A_i^\sharp \boldsymbol{\xi}_r \tag{5.17}$$

and is reduced to

$$\boldsymbol{\xi}_r' \boldsymbol{\xi}_r = k^2 \left(\sum_{i=1}^{r} \bar{\tau}_i \right)^2 + \sum_{i=1}^{r} \left\{ k \sum_{l=1}^{k} (\tau_{il} - \bar{\tau}_i)^2 \right\} + \boldsymbol{\delta}_r' \boldsymbol{\delta}_r, \tag{5.18}$$

where $\boldsymbol{\tau}_i' = (\tau_{i1}, \cdots, \tau_{ik})$ and $\bar{\tau}_i = \frac{1}{k} \sum_{l=1}^{k} \tau_{il}$. Note that the last term of (5.18) is

zero when $r=k+1$. Otherwise it is assumed to be zero or not according to the assumption adopted.

(e) L_r *association algebra*

A subalgebra of an OL_r association algebra for $r \leq k$ is known as an L_r association algebra [4]. We can reduce OL_r to L_r by pooling the association matrices of OL_r as

$$\begin{aligned} \tilde{A}_0 &= I_{k^2} \\ \tilde{A}_1 &= A_1 + A_2 + \cdots + A_r \\ \tilde{A}_2 &= A_{r+1}. \end{aligned} \tag{5.19}$$

It can easily be verified that $\tilde{A}_0$, $\tilde{A}_1$ and $\tilde{A}_2$ correspond to the association matrices for 0-th, 1st and 2nd associates of the familiar L_r association scheme.

It can also be seen that the mutually orthogonal idempotents of $\mathfrak{A}(OL_r)$ are pooled as follows:

$$\begin{aligned} \tilde{A}_0^{\sharp} &= A_0^{\sharp} \\ \tilde{A}_1^{\sharp} &= A_1^{\sharp} + \cdots + A_r^{\sharp} \\ \tilde{A}_2^{\sharp} &= A_{r+1}^{\sharp}. \end{aligned} \tag{5.20}$$

The parameter models which are naturally introduced into the L_r scheme may be derived in two fashions as have been introduced in a C_m type association algebra. The description is trivial and is omitted here.

6. Supplementary remarks

Each of these association schemes and their algebras treated so far has specific statistical implications in the structure of their associated parameters, as was indicated in each of the discussions.

There remain, however, many possibilities of composing the relationship algebras which indicate the structure of the associated parameter vectors. The composed relationship algebra may or may not be an association algebra in the sense of Bose.

Even in the case of the composition of the association algebras, the right angular association scheme due to Tharthare [20] and its extension are examples of other possibilities. The right angular association scheme can be composed of an $\mathfrak{A}(F_2)$ for $s_1=2$ by a similar mapping $\sigma_{12(3)}$ defined in (4.4). Immediate extension of the scheme may be obtained by the similar composition of $\mathfrak{A}(F_p)$ using a similar mapping $\sigma_{12\ldots p(p+1)}$. Those procedures are simple

and straightforward. Details are omitted here.

7. Evaluation of Gramians

It is well known that the Gramian of a set of independent column vectors of a rational symmetric matrix M is independent of the selection of the set, apart from a square factor of some rational number. Thus we call it the Gramian of M and denote it $g(M)$.

The following lemma is useful in evaluating the Gramians:

LEMMA 7.1 *Let A and B be two symmetric rational matrices of the same order and let C be a symmetric rational matrix. Then*:

(i) $AB=0$ *implies*

$$g(A+B)\sim g(A)\,g(B), \tag{7.1}$$

provided $A\neq 0$ *and* $B\neq 0$. *Where* $a\sim b$ *means that* a/b *is a square of some rational number.*

(ii)
$$g(A\otimes C)\sim[g(A)]^q[g(C)]^p \tag{7.2}$$

or

$$g(A\otimes C)\sim\begin{cases} 1: & \text{if both } p \text{ and } q \text{ are even}\\ g(A): & \text{if } p \text{ is even and } q \text{ is odd}\\ g(C): & \text{if } q \text{ is even and } p \text{ is odd}\\ g(A)\,g(C): & \text{if both } p \text{ and } q \text{ are odd}\end{cases}$$

where $p=\mathrm{rank}(A)$, $q=\mathrm{rank}(C)$.

(iii)
$$g(kA)\sim g(A) \tag{7.3}$$

where k is a rational number.

Let $\mathfrak{A}$ be a semi-simple algebra over the rational field, the mutually orthogonal principal idempotents of the minimum two-sided ideals being $E_1, E_2, \cdots, E_m$. Assume that these idempotents are rational symmetric matrices of order u. Let F be a $v\times u$ rational matrix which define a linear mapping σ of $\mathfrak{A}$, *i.e.*,

$$\sigma:\ \mathfrak{A}\ni A\to A^*=FAF' \tag{7.4}$$

and assume that F satisfies the sufficient condition that σ be partially similar [22], *i.e.*,

$$F'F=\sum_{i=1}^{m} c_i E_i \qquad (c_i \geq 0). \tag{7.5}$$

Note that the condition (7.5) is not always necessary because the field under consideration is rational.

We assume without loss of generality that $c_i>0$ for $i=1, \cdots, l$ and $c_j=0$ for $j=l+1, \cdots, m$. Then, the mutually orthogonal principal idempotents of the minimum two-sided ideals of the composed full-rank algebra [22], $\tilde{\mathfrak{A}}=\sigma(\mathfrak{A})\vee[I_v]$, are

$$\tilde{E}_i=\frac{1}{c_i}FE_iF' \qquad (i=1, 2, \cdots, l)$$

$$\tilde{E}_{l+1}=I_v-\sum_{i=1}^{l}\tilde{E}_i. \tag{7.6}$$

In this connection, the following theorem plays a fundamental role in evaluating the Gramians of those idempotents.

THEOREM 7.1

(i) $$g(\tilde{E}_i)\sim c_i^{\alpha_i}\, g(E_i) \tag{7.7}$$

where $\alpha_i=\mathrm{rank}\,(E_i)$

(ii) $$g(\tilde{E}_{l+1})\sim \prod_{i=1}^{l} g(\tilde{E}_i)\sim \prod_{i=1}^{l} c_i^{\alpha_i} g(E_i). \tag{7.8}$$

PROOF. (i) Since $\tilde{E}_i=\frac{1}{c_i}FE_iF'=\frac{1}{c_i}FE_i(FE_i)'$ and $\mathrm{rank}(\tilde{E}_i)=\mathrm{rank}(E_i)=\alpha_i$, we have

$$\begin{aligned} g(\tilde{E}_i)&\sim g(c_i\tilde{E}_i)\sim g(FE_i)\sim g(FE_i^*)\\ &=|E_i^{*\prime}F'FE_i^*|=c_i^{\alpha_i}|E_i^{*\prime}E_i^*|\sim c_i^{\alpha_i}\, g(E_i), \end{aligned}$$

where E_i^* is a matrix whose column vectors are a set of α_i independent column vectors of E_i. Hence we have (7.7).

(iii) Since $I_v=\sum_{i=1}^{l+1}\tilde{E}_i$, we have

$$g(\tilde{E}_{l+1})\prod_{i=1}^{l} g(\tilde{E}_i)\sim g(I_v)\sim 1.$$

Thus we have

$$g(\tilde{E}_{l+1}) \sim \prod_{i=1}^{l} g(\tilde{E}_i) \sim \prod_{i=1}^{l} c_i^{\alpha_i} g(E_i)$$

In the special case where $\mathfrak{A}$ is the rational field and $F=\boldsymbol{j}_v$, since $E_1=1$, $F'F=v$ and

$$\tilde{E}_1 = \frac{1}{v} FE_1F' = \frac{1}{v} G_v, \qquad \tilde{E}_2 = I_v - \frac{1}{v} G_v,$$

we have

$$g\left(\frac{1}{v} G_v\right) \sim v, \qquad g\left(I_v - \frac{1}{v} G_v\right) \sim v. \tag{7.9}$$

For the Gramians of the series of association algebras $\mathfrak{A}_m = [\overset{(m)}{A_i^\sharp}; i=0, 1, \cdots, m]$ which are composed of $[I, G]$ by the series of (partially) similar mappings σ_{m-1} defined by F_{m-1} satisfying

$$F'_{m-1}F_{m-1} = \sum_{i=0}^{m-1} c_i^{(m-1)} \overset{(m-1)}{A_i^\sharp}, \tag{7.10}$$

we have the recurrence relations

$$g(\overset{(m)}{A_i^\sharp}) \sim (c_i^{(m-1)})^{\alpha_i} g(\overset{(m-1)}{A_i^\sharp}) \tag{7.11}$$

for $i=0, 1, \cdots, m-1$ and

$$g(\overset{(m)}{A_m^\sharp}) \sim \prod_{l=0}^{m-1} g(\overset{(m)}{A_l^\sharp}). \tag{7.12}$$

There recurrence relations lead to the following formula:

$$g(\overset{(m)}{A_i^\sharp}) \sim \{\prod_{k=1}^{m-i} c_i^{(m-k)}\}^{\alpha_i} \prod_{l=0}^{i-1} \{c_l^{(i-1)}\}^{\alpha_l}. \tag{7.13}$$

Some special cases are as follows:

(i) For a T_m type association algebra, since

$$c_i^{(l)} = \frac{(l-i+1)(s-l-i)}{(l+1)^2} \sim (l-i+1)(s-l-i) \tag{7.14}$$

we have from (7.13)

$$g(\overset{(m)}{A_i^\sharp}) \sim \binom{s-2i}{m-i}^{\alpha_i} \prod_{l=0}^{i-1} \{(i-l)(s-i-l+1)\}^{\alpha_l} \tag{7.15}$$

where $\alpha_i=\binom{s}{i}-\binom{s}{i-1}$.

This coincides with the value calculated by M. Ogasawara [12].

(ii) For an N_m type association algebra, since

$$c_i^{(l)}=s_{l+1} \tag{7.16}$$

we have from (7.13)

$$g(\overset{(m)}{A_i^{\sharp}})\sim(s_m s_{m-1}\cdots s_{i+1})^{\alpha_i}s_i^{v_{i-1}} \tag{7.17}$$

where $\alpha_0=1$, $\alpha_i=\mathrm{rank}\,(\overset{(m)}{A_i^{\sharp}})=v_i-v_{i-1}=s_1s_2\cdots s_{i-1}(s_i-1)$.

(iii) For an OL_r type association algebra, since

$$F_i'F_i=kI_k,\qquad A_0^{\sharp}=\frac{1}{k}F_i\Big(\frac{1}{k}G_k\Big)F_i'=\frac{1}{v}G_v$$

$$A_i^{\sharp}=\frac{1}{k}F_i\Big(I_k-\frac{1}{k}G_k\Big)F_i',\quad \mathrm{rank}(A_i^{\sharp})=k-1\qquad (i=1,2,\cdots,r)$$

and if $r\leq k$,

$$A_{r+1}^{\sharp}=I_v-\sum_{i=0}^{r}A_i^{\sharp},$$

we have from (7.11) and (7.12),

$$g(A_0^{\sharp})\sim k^2\sim 1,\qquad g(A_i^{\sharp})\sim k^k \tag{7.18}$$

and if $r\leq k$,

$$g(A_{r+1}^{\sharp})\sim k^{rk}. \tag{7.19}$$

(iv) For an L_r type association algebra, we have

$$g(\tilde{A}_0^{\sharp})\sim 1,\qquad g(\tilde{A}_1^{\sharp})\sim k^{rk},\qquad g(\tilde{A}_2^{\sharp})\sim k^{rk}. \tag{7.20}$$

For the evaluation of the Gramians of the series of association algebras generated by the series of orthogonal compositions; Lemma 7.1 is useful. *i.e.*,

(v) For an F_p type association algebra, we have

$$g(A^{\sharp}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_p})\sim\prod_{i=1}^{p}g(A^{\sharp}_{\varepsilon i})^{\prod_{l\neq i}(s_l-1)^{\varepsilon_l}}\sim\prod_{i=1}^{p}s_i^{\prod_{l\neq i}(s_l-1)^{\varepsilon_l}} \tag{7.21}$$

(vi) For a C_p type association algebra, we have

$$g(C_i^{\sharp(p)}) \sim \prod_{\varepsilon_1+\cdots+\varepsilon_p=i} g(A^{\sharp}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_p}) \sim \{s^{(p-i)(s-1)^i+i(s-1)^{i-1}}\}^{\binom{p}{i}}. \tag{7.22}$$

This coincides with the value calculated by K. Kusumoto [11].

The necessary conditions for the existence of a regular symmetrical PBIB design admitting an association algebra

$$\mathfrak{A}=[A_0, A_1, \cdots, A_m]=[A_0^{\sharp}, A_1^{\sharp}, \cdots, A_m^{\sharp}]$$

over tha rational field may be derived as follows.

Let N be the incidence matrix of a PBIB design and assume that a pair of treatments which are i-th associates occur in λ_i blocks. Then, we have

$$NN'=\sum_{i=0}^{m}\lambda_i A_i=\sum_{j=0}^{m}\rho_j A_j^{\sharp}, \tag{7.23}$$

where $A_i=\sum_{j=0}^{m} z_{ji}A_j^{\sharp}$ and consequently

$$\rho_j=\sum_{i=0}^{m} z_{ji}\lambda_i$$

which are positive for all $j=0, 1, \cdots, m$ [23].

One of the necessary conditions for the existence of such a design derived from (7.23) by Hasse's theorem [9] is

$$\prod_{i=0}^{m}\rho_i^{\alpha_i}\sim 1 \tag{7.24}$$

by virtue of $\prod_{i=0}^{m} g(A_i^{\sharp})\sim 1$.

Another necessary condition derived from (7.23) by Hasse's theorem is

$$\prod_{i=1}^{m}(-1, \rho_i)_p^{\frac{\alpha_i(\alpha_i+1)}{2}}(\rho_i, g(A_i^{\sharp}))_p \prod_{1\le i\le j\le m}(\rho_i, \rho_j)_p^{\alpha_i\alpha_j}=1. \tag{7.25}$$

These conditions are the straightforward generalization of those due to Ogawa [15].

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