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## Some Examples Related to Duality Theorem in Linear Programming

## Michio YOSHIDA (Received March 11, 1966)

The duality problems in linear programming may read as follows. Suppose an  $m \times n$  matrix  $A = (a_{ij})$ , a column vector  $\mathbf{b} = (b_1, \dots, b_m)$  and a row vector  $\mathbf{c} = (c_1, \dots, c_n)$  are given.

The primal problem: Find a column vector  $\mathbf{u} = (u_1, \dots, u_n)$  which maximizes the linear form  $\mathbf{cu}$  subject to the conditions  $A\mathbf{u} \leq \mathbf{b}$  and  $\mathbf{u} \geq 0$ .

The dual problem: Find a row vector  $\mathbf{v} = (v_1, ..., v_n)$  which minimizes the linear form  $\mathbf{vb}$  subject to the conditions  $\mathbf{v}A \ge \mathbf{c}$  and  $\mathbf{v} \ge 0$ .

In each problem a vector satisfying the required conditions is called feasible, and if it attains the maximum or minimum it is called optimal.

These problems can be represented by the following tableau:

(≧0)	$u_1$	 $u_j$		$u_n$	$\leq$
$v_1$	$a_{11}$	 $a_{1j}$		$a_{in}$	$b_1$
•	•			•	•
$v_i$	$a_{i1}$	 $a_{ij}$		$a_{in}$	$b_i$
		 •	• • •	•	•
$v_m$	$a_{m1}$	 $a_{mj}$	•••	$a_{mn}$	$b_m$
VI	$c_1$	 $c_j$		$c_n$	min max

By taking inner products of the row of u's with the rows of A and the row of c's, we obtain the constraints  $Au \leq b$  and the linear form cu of the primal; the inner products of the column of v's with the columns of A and the column of b's yield the dual constraints  $Av \geq c$  and the linear form vb.

Associated with these problems is the following well-known theorem:

The Duality Theorem. If the primal is feasible and if  $\sup \mathbf{cu} < \infty$ , then there exist optimal solutions in the dual as well as in the primal, and moreover the extremal values of the linear forms coincide, i.e.,  $\max \mathbf{cu} = \min \mathbf{vb}$ .

In the foregoing paper [1], M. Ohtsuka investigated the problems in a very general situation, and obtained extensions of the duality theorem. We refer necessary notions and notations to [1]. We shall show in the present paper that the conditions imposed in Ohtsuka's Theorems 2 and 3 are in a way necessary. Actually, even if  $\mathcal{M} \neq \emptyset$ ,  $-\infty < M < \infty$  and  $\emptyset$ , f and g are

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all continuous and non-negative, we have examples where respectively

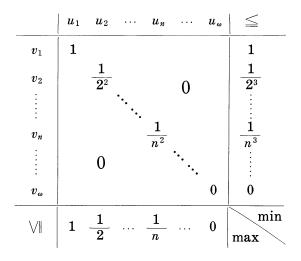
- 1. There exist no optimal measures in the primal.
- 2. Though an optimal measure exits in the primal,  $\mathcal{M}' = \emptyset$ .
- 3. There exists an optimal measure in the primal and  $\mathscr{M}' \neq \emptyset$ , M = M', nevertheless there exist no optimal in the dual.
- 4. Though there exist optimal measures both in the primal and the dual, nevertheless M < M'.

Examples shall be given by the tableaux which are so explanatory that further explanations will be superfluous.

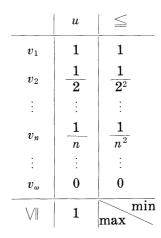
1.  $X = \{1\}, Y = \{N, \omega\}$ : the Alexandroff one point compactification of the discrete space N of all natural numbers.

	$u_1$	$u_2$		$u_n$		$u_{\omega}$	$\leq$
v	1	$\frac{1}{2}$		$\frac{1}{n}$		0	1
$\vee$	$\left  \begin{array}{c} \frac{1}{1} \left( 1 - \frac{1}{1} \right) \right.$	$\frac{1}{2}\left(1-\frac{1}{2}\right)$	) –	$\frac{1}{n}\left(1-\frac{1}{n}\right)$	)	0	min max

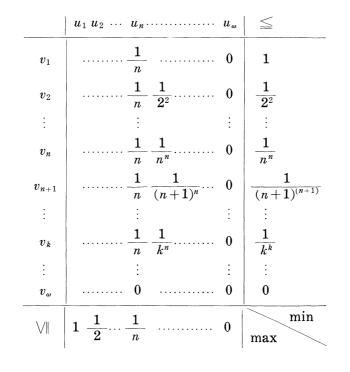
$$2. \quad X = Y = \{N, \omega\}$$



3. 
$$X = \{N, \omega\}, Y = \{1\}$$



4.  $X = Y = \{N, \omega\}$ 



## Reference

[1] M. Ohtsuka: A generalization of duality theorem in the theory of linear programming, this journal.

Department of Mathematics, Faculty of Science, Hiroshima University