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On C^∞ Maps which admit Transposed Image of every Distribution

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Introduction

Let Ω be a non-empty open subset of an N-dimensional Euclidean space R^N. For any $\phi \in \mathcal{D}(R^N)$ such that $\phi \geqq 0$ and $\int \phi(x)dx=1$ we put $\phi_\lambda(x)=\frac{1}{\lambda^N}\phi\left(\frac{x}{\lambda}\right)$, where λ is a positive real number. For given S, $T \in \mathcal{D}'(\Omega)$ we understand the product ST by $\lim_{\lambda\to 0}(S*\phi_\lambda)T$ in $\mathcal{D}'(\Omega)$ if it exists for every $\phi \in \mathcal{D}(\Omega)$ (Notation $S \bigcirc T$ was used in [5]). The multiplication thus defined is invariant under diffeomorphism.

A distribution S in R^n may be considered as the distribution $S\otimes 1_y$ in $R^n_x \times R^m_y$. Similarly $T \in \mathcal{D}'(R^m)$ is identified with $1_x \otimes T$ in $R^n_x \times R^m_y$. The product $(S\otimes 1_y)(1_x\otimes T)$ exists and is equal to $S\otimes T$ (Proposition 3). If $\varkappa$ is any diffeomorphism of $R^n \times R^m$ onto itself, then $(S\otimes 1_y)(\varkappa)(1_x\otimes T)(\varkappa)$ exists and is equal to $((S\otimes 1_y)(1_x\otimes T))(\varkappa)$ [5, p. 163]. Now a question arises as to its converse: if ξ and η are C^∞ maps of $R^n \times R^m$ into R^n and R^m respectively, what is a necessary and sufficient condition in order that the product $S(\xi)T(\eta)$ may be defined for every $S \in \mathcal{D}'(R^n)$ and $T \in \mathcal{D}'(R^m)$? To answer this question, first we have to clear up the meaning of the notation such as $S(\xi)$. To do so, we shall introduce the concept of an admissible map (see Definitions 1, 2 below), which allows us to make an extension of a notion of "a function of functions". Roughly speaking, a C^∞ map $\varkappa$ of an open set Ω into another Ω' is admissible whenever the transposed image ξ^*S of every $S \in \mathcal{D}'(\Omega')$ exists. The question is then answered: If ξ and η are admissible, a necessary and sufficient condition is that the map (ξ, η) of $R^n \times R^m$ into itself has no critical point (Corollary to Theorem 2). From a different approach, though mainly designed for practical purpose, I. M. Gel'fand and G. E. Shilov have developed the detailed discussions on "a distribution of functions" [2, Chapter III].

It is probable that a C^∞ map is admissible if and only if it has no critical point. We did not succeed in deciding whether it is true or not. In certain special instances we can show that the conjecture holds true (Example 2, Proposition 8).

Section 1 is mainly devoted to the preliminary discussions on transposed images and admissible maps. Owing to these notions we can study the properties of multiplication between distributions closely connected with tensor product (Proposition 3). In Section 2 we show that a C^∞ map of an open subset of R^N into another subset of R^N is admissible if and only if the map has no critical point (Theorem 2). Basing on this result we show that an admissible C^∞ map ξ into a domain is a diffeomorphism if and only if the transposed map ξ^* is onto. And another characterization is also given (Theorem 3). In the final section we consider admissible map into a one dimensional space and special attention is paid to the case $N=2$, $n=1$ where the conjecture mentioned before is true. The section is closed with the consideration of admissible maps into a closed half line.

For the sake of simplicity, in this paper we shall use the word "map" to mean C^∞ map unless otherwise stated.

§1. Transposed images and admissible maps

Let ξ be a continuous map of a non-empty open subset $\Omega \subset R^N$, an N-dimensional Euclidean space, into another open subset $\Omega' \subset R^n$. For any $\phi \in \mathcal{D}(\Omega)$ the map $\alpha \rightarrow <\alpha \circ \xi, \phi>$ is a continuous linear form on $\mathcal{D}(\Omega')$. There exists therefore $\xi_\phi \in \mathcal{D}'(\Omega')$ such that

$$<\alpha \circ \xi, \phi> = <\alpha, \xi_\phi>.$$

Evidently ξ_ϕ is a Radon measure on Ω' with support $\subset \xi$ (supp ϕ).

DEFINITION 1. ξ is *admissible* if $\xi_\phi \in \mathcal{D}(\Omega')$ for any $\phi \in \mathcal{D}(\Omega)$.

This definition is equivalent to

DEFINITION 2. ξ is *admissible* if the map $\xi^* : \mathcal{D}(\Omega') \ni \alpha \rightarrow \alpha \circ \xi \in \mathcal{D}'(\Omega)$ is continuously extended to the map of $\mathcal{D}'(\Omega')$ (or equivalently of $\mathcal{E}'(\Omega')$) into $\mathcal{D}'(\Omega)$.

The definition means that for any $S \in \mathcal{D}'(\Omega')$ there exists a unique distribution $W \in \mathcal{D}'(\Omega)$ such that

$$< W, \phi> = < S, \xi_\phi> \qquad \text{for any} \quad \phi \in \mathcal{D}(\Omega).$$

W is called the transposed image of S and will be denoted by $\xi^* S$ or $S(\xi)$. Then, to any $x' \in \Omega'$ the map $\phi \rightarrow \xi_\phi(x')$ is obviously a positive form on $\mathcal{D}(\Omega)$, and so there exists a positive Radon measure $\mu_{x'}$ on Ω, by which we can write

$$\xi_\phi(x') = \int \phi(x) d\mu_{x'}(x),$$

where supp $\mu_{x'} \subset \xi^{-1}(x')$.

DEFINITION 3. ξ is *admissible at a point* $x_0 \in \Omega$ if there is a neighbourhood U of x_0 such that the restriction $\xi | U$ is admissible.

Owing to the principle of localization, it is easy to see that ξ is admissible if and only if ξ is admissible at each point of Ω.

If ξ is admissible and $\Omega' \subset \Omega''$ (open subset of R^n), the map $\xi : \Omega \to \Omega''$ is also admissible. If a map ξ of Ω into Ω' and a map η of Ω' into $\Omega'' \subset R^p$ are both admissible, then $\eta \circ \xi$ becomes admissible. In fact, $\eta^* \alpha \in C(\Omega')$ for any $\alpha \in \mathcal{D}(\Omega'')$ and the map ξ^* of $\mathcal{D}(\Omega')$ into $\mathcal{D}'(\Omega)$ is always continuously extended to the map of $C(\Omega')$ into $\mathcal{D}'(\Omega)$, and $(\eta \circ \xi)^* \alpha = \xi^*(\eta^* \alpha)$ for any $\alpha \in \mathcal{D}(\Omega'')$. Consequently the map $(\eta \circ \xi)^*$ of $\mathcal{D}(\Omega'')$ into $\mathcal{D}'(\Omega)$ is continuously extended to the map of $\mathcal{D}'(\Omega'')$ into $\mathcal{D}'(\Omega)$, and so we have $(\eta \circ \xi)^* S = \xi^*(\eta^* S)$ for every $S \in \mathcal{D}'(\Omega'')$.

In what follows, without specific mention a map is meant to be differentiable of class C^∞.

DEFINITION 4. Let $S \in \mathcal{D}'(\Omega')$. If, for any sequence $\{\alpha^j\}$, $\alpha^j \in \mathcal{D}(\Omega')$, which converges to S in $\mathcal{D}'(\Omega')$ and to 0 uniformly on every compact subset of $\Omega' \setminus \operatorname{supp} S$ $\{\xi^* \alpha^j\}$ converges to an element of $\mathcal{D}'(\Omega)$, then the limit is called *transposed image* of S under the map ξ and will be denoted by $\xi^* S$ or $S(\xi)$.

The transposed image $\xi^* S$ exists if and only if for each point of Ω there is a neighbourhood U such that $(\xi | U)^* S$ exists.

In practice a further modification of this definition is sometimes needed [2, Chapter III].

PROPOSITION 1. *$\xi^* S$ exists for every $S \in \mathcal{D}'(\Omega')$ if and only if ξ is admissible.*

PROOF. We have only to prove the "only if" part. Let $\{\rho_j\}$, $\rho_j \in \mathcal{D}(\Omega')$, be a sequence of regularization. If we consider the maps $u^j : \mathcal{E}'(\Omega') \ni S \to \xi^*(S * \rho_j) \in \mathcal{D}'(\Omega)$, then $\{S * \rho_j\}$ converges to S in $\mathcal{D}'(\Omega')$ and to 0 uniformly on every compact subset of $\Omega' \setminus \operatorname{supp} S$, and so $\{u_j(S)\}$ converges to $\xi^* S$ in $\mathcal{D}'(\Omega)$ as $j \to \infty$. It follows therefore from the Banach-Steinhaus theorem that the map $\mathcal{E}'(\Omega') \ni S \to \xi^* S \in \mathcal{D}'(\Omega)$ is continuous, which completes the proof.

PROPOSITION 2. *If ξ is an admissible map of $\Omega \subset R^N$ into $\Omega' \subset R^n$, then $N \geqq n$.*

PROOF. Suppose the contrary. For any compact ball B contained in Ω, the image $\xi(B)$ is a null set owing to a theorem of Sard. If we let θ be the characteristic function of $\xi(B)$, $\theta = 0$ in $\mathcal{D}'(\Omega')$. We choose a sequence $\{\alpha^j\}$ from $\mathcal{D}(\Omega')$ such that $\alpha^j \downarrow \theta$ and $\alpha^j = 1$ on $\xi(B)$. Take a $\phi \in \mathcal{D}_B$ with $\int \phi(x) dx = 1$. Then $< \alpha^j \circ \xi, \phi > = 1$. On the other hand $< \alpha^j \circ \xi, \phi > = < \alpha^j, \xi_\phi > \to 0$ as $j \to \infty$, which is a contradiction.

REMARK 1. Let ξ be a map of $\Omega\subset R^N$ into $\Omega'\subset R^n$ and rank $d\xi_{x_0}=n$, $x_0\in\Omega$. There exist a neighbourhood U of x_0 and a map η of U into R^{N-n} such that the map $\chi=(\xi, \eta)$ is a diffeomorphism of U onto an open subset $\chi(U)\subset R^N$. Then we can write for any $\phi\in\mathcal{D}(U)$

$$\text{(a)}\qquad \xi_\phi(x')=\int_{\eta(U)}\frac{1}{|J_\chi(x)|}(\phi\circ\chi^{-1})(x', y')dy',$$

where J_χ is the Jacobian of the map χ. This expression does not depend on the choice of η. Actually we can write for any $\phi\in\mathcal{D}(\Omega)$ with support in a neighbourhood of x_0

$$\text{(b)}\qquad \xi_\phi(x')=\int_{x'=\xi(x)}\frac{\phi d\omega}{\sqrt{\sum_{i_1<\cdots<i_n}\left|\frac{\partial(\xi_1, \cdots, \xi_n)}{\partial(x_{i_1}, \cdots, x_{i_n})}\right|^2}},$$

where $d\omega$ is the surface element of the surface given by $\xi=x'$. If $n=1$, the denominator of the integrand becomes $|\text{grad}\,\xi|$. We note that if $N=n$, the formula (b) means that

$$\text{(c)}\qquad \xi_\phi(x')=\sum_{x\in\xi^{-1}(x')}\frac{1}{|J_\xi(x)|}\phi(x).$$

Therefore if rank $d\xi_{x_0}=n$, then ξ is admissible in a neighbourhood of x_0. We also see that if x' is not a critical value, the formulas (b), (c) hold true of any $\phi\in\mathcal{D}(\Omega)$.

Let $R^N=R^n_x\times R^m_y$, where x and y denote the generic points of R^n_x and R^m_y respectively. Consider non-empty open subsets $\Omega_1\subset R^n_x$ and $\Omega_2\subset R^m_y$. Then the map $\xi:\Omega_1\times\Omega_2\ni(x, y)\rightarrow x\in R^n_x$ is admissible as $d\xi$ has the rank n at each point of $\Omega_1\times\Omega_2$. Let $S\in\mathcal{D}'(R^n_x)$. ξ^*S is given by

$$<\xi^*S, \phi(x, y)>=<S, \int\phi(x, y)dy>, \qquad \phi\in\mathcal{D}(\Omega_1\times\Omega_2).$$

That is, $\xi^*S=S\otimes 1_y$. We shall also use the symbol $S(x)$ to denote ξ^*S. Similarly the map $\eta:\Omega_1\times\Omega_2\ni(x, y)\rightarrow y\in R^m_y$ is admissible and we may write $\eta^*T=T(y)$ for any $T\in\mathcal{D}'(R^m_y)$.

We shall consider the relationship between the tensor product of distributions and the multiplicative product between them in the sense of [5].

PROPOSITION 3. *For any $S\in\mathcal{D}'(R^n_x)$ and $T\in\mathcal{D}'(R^m_y)$, the multiplicative product $S(x)T(y)$ exists and is equal to $S\otimes T$.*

PROOF. Owing to Proposition 2 of [5] (p. 167), since $S1_x=S$, $1_yT=T$, we have

$$S\otimes T=(S1_x)\otimes(1_yT)=(S\otimes 1_y)(1_x\otimes T)=S(x)T(y).$$

The multiplication used here is normal in the sense of [5]. Especially it is invariant under diffeomorphism. Namely, let $x=f(x', y')$, $y=g(x', y')$ be a diffeomorphism and put $W(x, y)=S(x)T(y)$. Then we have

$$S(f(x', y'))\, T(g(x', y'))=W(f(x', y'), \quad g(x', y')).$$

The same is true if we only assume that the Jacobian $\dfrac{\partial(f, g)}{\partial(x, y)}\neq 0$ at any point of Ω. This follows from the fact the multiplicative product is determined by its local character [5, p. 162].

Let ξ be a map of $\Omega\subset R^N$ into $\Omega_1'\subset R^p$ and η of Ω into $\Omega_2'\subset R^q$ such that the map $\chi=(\xi, \eta)$ of Ω into $\Omega_1'\times\Omega_2'$ has no critical point. Then the multiplicative product $(\xi^*S)(\eta^*T)$ exists for every $S\in\mathcal{D}'(\Omega_1')$ and $T\in\mathcal{D}'(\Omega_2')$. In fact, suppose $\operatorname{rank} d\chi_{x_0}=p+q$. We can find a map ζ of a neighbourhood of x_0 into R^{N-p-q} such that the map (ξ, η, ζ) is a diffeomorphism of a neighbourhood of x_0 onto an open subset of R^N. The consideration made just before will show that the multiplicative product $(\xi^*S)(\eta^*T)$ exists in a neighbourhood of x_0 for any $S\in\mathcal{D}'(\Omega_1')$ and $T\in\mathcal{D}'(\Omega_2')$.

Theorem 1. *Let ξ be an admissible map of Ω into Ω_1' and η of Ω into Ω_2'. If the multiplicative product $(\xi^*S)(\eta^*T)$ exists for every $S\in\mathcal{D}'(\Omega_1')$ and $T\in\mathcal{D}'(\Omega_2')$, then the map $\chi=(\xi, \eta)$ of Ω into $\Omega_1'\times\Omega_2'$ is also admissible.*

Proof. By assumption the multiplicative product $(\xi^*S)(\eta^*T)$ exists for any $S\in\mathcal{E}'(\Omega_1')$, $T\in\mathcal{E}'(\Omega_2')$. Let $\beta\in\mathcal{D}(\Omega)$ and let x_0 be an arbitrary point of Ω. Now consider for sufficiently small positive λ the map

$$\mathcal{E}'(\Omega_1')\times\mathcal{E}'(\Omega_2')\ni(S, T)\to((\beta\xi^*S)*\phi_\lambda)(\eta^*T)\in\mathcal{D}'(\Omega),$$

where we put $\phi_\lambda=\dfrac{1}{\lambda^N}\phi\left(\dfrac{x}{\lambda}\right)$, $\phi\in\mathcal{D}(R^N)$ being chosen so that $\phi\geqq 0$ and $\int\phi(x)dx=1$. It is a separately continuous bilinear map depending on λ and $((\beta\xi^*S)*\phi_\lambda)(\eta^*T)$ converges to $\beta(\xi^*S)(\eta^*T)$ in $\mathcal{D}'(\Omega)$ as $\lambda\to 0$. As a consequence the bilinear map

$$\mathcal{E}'(\Omega_1')\times\mathcal{E}'(\Omega_2')\ni(S, T)\to(\xi^*S)(\eta^*T)\in\mathcal{D}'(\Omega)$$

will be hypocontinuous since $\mathcal{E}'(\Omega_1')$ and $\mathcal{E}'(\Omega_2')$ are barrelled spaces [1, p. 40]. Owing to the theorem of Grothendieck [3, p. 66], since $\mathcal{E}'(\Omega_1')$ and $\mathcal{E}'(\Omega_2')$ are (**DF**)-spaces, the map is continuous. Consequently it can be continuously extended to the map of $\mathcal{E}'(\Omega_1')\hat{\otimes}_\pi\mathcal{E}'(\Omega_2')=\mathcal{E}'(\Omega_1'\times\Omega_2')$ into $\mathcal{D}'(\Omega)$. This means that the map $\chi=(\xi, \eta)$ is admissible, completing the proof.

Example 1. Let P and Q be C^∞ functions defined in an open subset $\Omega\subset R^N$

having no critical point. We denote by δ the Dirac measure concentrated at the origin $0 \in R^1$. If the surfaces $P=0$ and $Q=0$ have no point in common, we have in accordance with Definition 4

$$\delta(PQ)=\frac{\delta(P)}{|Q|}+\frac{\delta(Q)}{|P|}.$$

We only need to show the equation in a neighbourhood of the inverse image $(PQ)^{-1}(0)$. We have $|\mathrm{grad}\, PQ|=|Q|\,|\mathrm{grad}\, P|$ on $P=0$ and $|\mathrm{grad}\, PQ|=|P|\,|\mathrm{grad}\, Q|$ on $Q=0$. Let U be a neighbourhood of $(PQ)^{-1}(0)$ of which the map $x \rightarrow PQ(x)$ has no critical point. We have for any $\phi \in \mathcal{D}(U)$

$$<\delta(PQ), \phi>=\int_{P=0}\frac{\phi\, d\omega}{|Q|\,|\mathrm{grad}\, P|}+\int_{Q=0}\frac{\phi\, d\omega}{|P|\,|\mathrm{grad}\, Q|}$$

$$=<\delta(P), \frac{\phi}{|Q|}>+<\delta(Q), \frac{\phi}{|P|}>=<\frac{\delta(P)}{|Q|}+\frac{\delta(Q)}{|P|}, \phi>.$$

Next suppose the surfaces $P=0$ and $Q=0$ intersect at x_0, that is, the Jacobian matrix of (P, Q) has rank 2 at x_0. Therefore in a neighbourhood U of x_0 the map $x \rightarrow (P(x), Q(x)) \in R^2$ has no critical point. Consider the map $\eta:(u, v) \rightarrow uv$ of R^2 into R^1. $(0, 0)$ is the only critical point of η. And we have for any $c \neq 0$

$$\delta(uv-c)=(\delta\otimes\delta)\log\frac{1}{c^2}+\delta\otimes\frac{1}{|v|}+\frac{1}{|u|}\otimes\delta+o(1)$$

as $c \rightarrow 0$. Then we have

$$\delta(PQ-c)=\delta(P)\delta(Q)\log\frac{1}{c^2}+\delta(P)\frac{1}{|Q|}+\frac{1}{|P|}\delta(Q)+o(1),$$

where the multiplicative products $\delta(P)\delta(Q)$, $\delta(P)\frac{1}{|Q|}$ and $\frac{1}{|P|}\delta(Q)$ are well defined by Proposition 3. Accoring to Gel'fand and Shilov [2, p. 319], if we define $\delta(PQ)$ as the finite part of the limit of $\delta(PQ-c)$ when c tends to 0, we again obtain the equation

$$\delta(PQ)=\frac{\delta(P)}{|Q|}+\frac{\delta(Q)}{|P|}$$

in a neighbourhood of x_0.

Now let us recall the definition of the simultaneous product of the distributions S, T and $W \in \mathcal{D}'(\Omega')$. If, for any ϕ, ψ and $\rho \in \mathcal{D}(R^n)$ such that $\phi \geqq 0$, $\psi \geqq 0$, $\rho \geqq 0$ and $\int\phi(x)dx=\int\psi(x)dx=\int\rho(x)dx=1$, the distributional limit

$$\Xi = \lim_{\lambda, \lambda', \lambda'' \to +0} (S*\phi_\lambda)(T*\psi_{\lambda'})(W*\rho_{\lambda''})$$

exists and does not depend on the choice of ϕ, ψ and ρ, then the limit will be called the multiplicative product of S, T and W and benoted by STW [5, p. 172].

The definition means that the limit

$$\lim_{\lambda, \lambda', \lambda'' \to -0} <(S\otimes T\otimes W)(x-\lambda y,\ x-\lambda' z,\ x-\lambda'' w),\ \phi(x)\phi^1(y)\phi^2(z)\phi^3(w)>$$

exists for any ϕ^1, ϕ^2, $\phi^3 \in \mathcal{D}(R^n)$ and $\phi \in \mathcal{D}(\Omega')$, and is equal to

$$<\Xi, \phi> \int \phi^1(y)dy \int \phi^2(z)dz \int \phi^3(w)dw.$$

With necessary modifications we can prove the analogue of Theorem 1 for simultaneous multiplication. For our later purpose we shall show

Proposition 4. *Let ξ be a diffeomorphism of Ω onto Ω'. If the multiplicative product Ξ of S, T and $W \in \mathcal{D}'(\Omega')$ exists, then the product of ξ^*S, ξ^*T and ξ^*W exists and $\xi^*\Xi = (\xi^*S)(\xi^*T)(\xi^*W)$.*

Proof. There is no loss of generality in assuming S, T, $W \in \mathcal{E}'(\Omega')$. Let $\tilde\phi^1$, $\tilde\phi^2$, $\tilde\phi^3 \in \mathcal{D}(R^n)$ and $\tilde\phi \in \mathcal{D}(\Omega)$. Put $\eta = \xi^{-1}$. Then we have for sufficiently small positive λ, λ' and λ''

$$\begin{aligned} &f(\lambda, \lambda', \lambda'') \\ &= <\big((\xi^*S)\otimes(\xi^*T)\otimes(\xi^*W)\big)(\tilde x-\lambda\tilde y,\ \tilde x-\lambda'\tilde z,\ \tilde x-\lambda''\tilde w),\ \tilde\phi(\tilde x)\tilde\phi^1(\tilde y)\tilde\phi^2(\tilde z)\tilde\phi^3(\tilde w)> \\ &= <\big((\xi^*S)\otimes(\xi^*T)\otimes(\xi^*W)\big)(\tilde y,\ \tilde z,\ \tilde w),\ \tilde\phi(\tilde x)\tilde\phi^1_\lambda(\tilde x-\tilde y)\tilde\phi^2_{\lambda'}(\tilde x-\tilde z)\tilde\phi^3_{\lambda''}(\tilde x-\tilde w)> \\ &= <(S\otimes T\otimes W)(y, z, w), \\ &\qquad \frac{1}{|J_\xi(x)J_\xi(y)J_\xi(z)J_\xi(w)|}\tilde\phi(\eta(x))\tilde\phi^1_\lambda(\eta(x)-\eta(y))\cdots\tilde\phi^3_{\lambda''}(\eta(x)-\eta(w))>. \end{aligned}$$

After the change of variables:

$$y \to x-\lambda y, \quad z \to x-\lambda' z, \quad w \to x-\lambda'' w,$$

we have

$$\begin{aligned} &f(\lambda, \lambda', \lambda'') \\ &= <(S\otimes T\otimes W)(x-\lambda y,\ x-\lambda' z,\ x-\lambda'' w)\frac{1}{|J_\xi(x)J_\xi(x-\lambda y)\cdots J_\xi(x-\lambda'' w)|},\ g>, \end{aligned}$$

where

$$g=g_{\lambda,\lambda',\lambda''}(x,\ y,\ z,\ w)=\tilde{\phi}(\eta(x))\tilde{\phi}^1\Big(\frac{\eta(x)-\eta(x-\lambda y)}{\lambda}\Big)\cdots\tilde{\phi}^3\Big(\frac{\eta(x)-\eta(x-\lambda'' w)}{\lambda''}\Big).$$

We can easily show that the set $\{g_{\lambda,\lambda',\lambda''}(x,\ y,\ z,\ w)\}_{0<\lambda,\lambda',\lambda''\leqq 1}$ is bounded in $\mathcal{D}$. Passing to the limit λ, λ' and $\lambda''\to+0$, $f(\lambda,\ \lambda',\ \lambda'')$ converges to

$$<\Xi(x)\frac{1}{|J_\xi(x)|^4},\ \tilde{\phi}(\eta(x))\int\tilde{\phi}^1\Big(\sum_j\frac{\partial\eta}{\partial x_j}(x)y_j\Big)dy\cdots\int\tilde{\phi}^3\Big(\sum_j\frac{\partial\eta}{\partial x_j}(x)w_j\Big)dw>$$

$$=<\Xi(x),\ \frac{1}{|J_\xi(x)|}\ \tilde{\phi}(\eta(x))>\int\tilde{\phi}^1(y)dy\int\tilde{\phi}^2(z)dz\int\tilde{\phi}^3(w)dw$$

$$=<\xi^*\Xi,\ \tilde{\phi}(\tilde{x})>.$$

Thus the product $(\xi^*S)(\xi^*T)(\xi^*W)$ exists and is equal to $\xi^*\Xi$. The proof is completed.

§2. A characterization of diffeomorphism

We continue under the same notations as before. First we prove

THEOREM 2. *Let χ be a C^1 map of $\Omega\subset R^N$ into $\Omega'\subset R^n$. We assume that $N=n$. Then the following two conditions are equivalent to each other:*

(1) *χ is admissible.*

(2) *χ is a C^∞ map and the Jacobian J_χ does not vanish.*

PROOF. We only need to show the implication (1)$\Rightarrow$(2). First we show that J_χ does not vanish. Suppose the contrary and assume that $J_\chi=0$ at $x_0\in\Omega$. There exists a sequence $\{x^j\}$ in Ω converging to x_0 and such that x^j are regular points, that is, not critical points. In fact, let $U\subset\Omega$ be a relatively compact neighbourhood of x_0 and Γ_0 the set of the critical points in $\bar{U}$. Then Γ_0 is compact and $\chi(\Gamma_0)$ is a null set. The set

$$\Gamma=\chi^{-1}(\chi(\Gamma_0))\cap\bar{U}$$

is compact. In the same way as in Proposition 2 we can show that the interior Γ^0 of Γ is empty. Thus we can choose a sequence $\{x^j\}$ in U such that x^j is a regular point and $\{x^j\}$ converges to x_0. Let $\phi\in\mathcal{D}(\Omega)$ be chosen so that $\phi\geqq 0$ and $\phi(x_0)=1$. There exists a neighbourhood U_j of x^j, of which χ is a homeomorphism onto a neighbourhood of $x'^j=\chi(x^j)$. Let $\alpha^j\in\mathcal{D}(U_j)$ be such that $0\leqq\alpha^j\leqq 1$ and $\alpha^j=1$ in a neighbourhood of x^j. Then

$$\chi_\phi(x'^j)\geqq\chi_{\alpha^j\phi}(x'^j)=\frac{\phi(x^j)}{|J_\chi|_{x=x^j}}.$$

Putting $x_0'=\chi(x_0)$, $\chi_\phi(x'^j)$ tends to $\chi_\phi(x_0')$, $\phi(x^j)$ to 1 and $|J_\chi|_{x=x^j}$ to 0 as $x^j\to x_0$. Consequently $\chi_\phi(x_0')=+\infty$, which is a contradiction. Thus the Jacobian J_χ

is not 0 at every $x \in \Omega$.

Next we shall show that $\varkappa$ is a C^∞ map. Let U be a neighbourhood of x_0 such that $\varkappa$ is a homeomorphism of U onto a neighbourhood of $x_0'=\varkappa(x_0)$. Then we have for any $\phi \in \mathcal{D}(U)$

$$\varkappa_\phi(x')=\frac{\phi((\varkappa\,|\,U)^{-1}(x'))}{|J_\varkappa|}.$$

If we choose $\phi \in \mathcal{D}(U)$ equal to 1 in a neighbourhood of x_0, then

$$\varkappa_\phi(x')=\frac{1}{|J_\varkappa|}$$

for any x' sufficiently near x_0'. Therefore $J_\varkappa$ is a C^∞ function in a neighbourhood of x_0'. If we also take ϕ equal to $\phi^k:(x_1, \cdots, x_N)\to x_k$ near x_0, then we have in a neighbourhood of x_0'

$$\varkappa_\phi(x')=\frac{1}{|J_\varkappa|}((\varkappa\,|\,U)^{-1})_k(x'),$$

where $((\varkappa\,|\,U)^{-1})_k$ denotes the k-th component of the inverse of $\varkappa\,|\,U$. $\varkappa_\phi$, $|J_\varkappa|$ being C^∞ functions in a neighbourhood of x_0', $((\varkappa\,|\,U)^{-1})_k(x')$ is a C^∞ function for $k=1, 2, \cdots, N$. Thus $\varkappa$ is a C^∞ map of Ω into Ω', the proof of the theorem is now complete.

Remark 2. There is a continuous admissible map $\varkappa \notin C^\infty$. For example, let $N=n=1$ and $\Omega=\Omega'=R^1$. Then the map $\varkappa(x)=x^{\frac{1}{3}}$ is admissible.

As an immediate consequence of Theorems 1, 2 we have

Corollary. *Let ξ be an admissible map of $\Omega\subset R^N$ into $\Omega_1'\subset R^p$ and η of Ω into $\Omega_2'\subset R^q$. We assume that $N=p+q$. Then the following two conditions are equivalent to each other*:

(1) *The multiplicative product $(\xi^*S)(\eta^*T)$ exists for every $S \in \mathcal{D}'(\Omega_1')$, $T \in \mathcal{D}'(\Omega_2')$.*

(2) *The map $\varkappa=(\xi, \eta)$ of Ω into $\Omega_1'\times\Omega_2'$ has no critical point.*

It also follows from Proposition 4 we see that the analogue of the preceding corollary for the simultaneous multiplication remains valid.

Lemma 1. *Let ξ be a map of $\Omega\subset R^N$ onto $\Omega'\subset R^n$. If ξ has no critical point, then the map $\mathcal{D}(\Omega)\ni\phi\to\xi_\phi \in \mathcal{D}(\Omega')$ is surjective and therefore ξ^* is injective.*

Proof. If $N=n$, the lemma is trivial since then the map is locally a diffeomorphism. Therefore we consider the case $N>n$. Let $x_0 \in \Omega$ and put $x_0'=\xi(x_0)$. There exists a map $y'=\eta(x)$ of a neighbourhood W of x_0 into R^{N-n} such that $\varkappa=(\xi, \eta)$ is a diffeomorphism of W onto a neighbourhood of $(x_0', \eta(x_0))$.

Without loss of generality we may assume that $x_0=0$, $x'_0=0$, $y'_j=x_{n+j}$, $j=1, 2, \cdots, N-n$ and that there exists an open 0-neighbourhood $U=\{(x_1, \cdots, x_N) : |x_j|<\frac{1}{2}$ for $j=1, 2, \cdots, N\}\subset\subset W$.

Given $\alpha\in\mathcal{D}(\xi(U))$, we choose a compact set $K\subset U$ with $\xi(K)\supset \operatorname{supp}\alpha$ and take a $\psi\in\mathcal{D}(U)$ equal to 1 in a neighbourhood of K. It follows then from the formula (a) that we have

$$\xi_{\alpha(\xi)\psi|J\chi|}=\alpha.$$

This means that the map $\phi\to\xi_\phi$ of $\mathcal{D}(U)$ into $\mathcal{D}(\xi(U))$ is onto. An application of a decomposition of unity will allow to conclude the statement of the lemma.

We note that if ξ is an admissible map of $\Omega\subset R^N$ into $\Omega'\subset R^n$, then $\xi^*(\mathcal{E}'(\Omega'))\subset\mathcal{E}'(\Omega)$ if and only if ξ is proper, that is, the inverse image $\xi^{-1}(K')$ of any compact subset K' of Ω' is also compact.

THEOREM 3. *Let ξ be an admissible map of $\Omega\subset R^N$ into $\Omega'\subset R^n$. We assume that Ω' is connected. Then the following three conditions are equivalent to each other*:

(1) $\xi^*(\mathcal{D}'(\Omega'))=\mathcal{D}'(\Omega)$.

(2) $\xi^*(\mathcal{E}'(\Omega'))=\mathcal{E}'(\Omega)$.

(3) *ξ is a diffeomorphism of Ω onto Ω'.*

PROOF. The implications (3)$\Rightarrow$(1) and (3)$\Rightarrow$(2) are trivial. Each of the conditions (1) and (2) implies the map $\mathcal{D}(\Omega)\ni\phi\to\xi_\phi\in\mathcal{D}(\Omega')$ is injective, and each in turn implies $N=n$. For otherwise, let $x_0\in\Omega$ be a regular point. Then the surface $\xi(x)=\xi(x_0)$ would contain a regular point $x^1\neq x_0$ near x_0, so that we could find respective disjoint neighbourhoods U_0 and U_1 of x_0 and x^1 such that ξ has no critical point in U_0 and U_1 and $\xi(U_0)=\xi(U_1)$. Then owing to Lemma 1 we can find $\phi_0\in\mathcal{D}(U_0)$ and $\phi^1\in\mathcal{D}(U_1)$ with $\phi_0\neq\phi^1$ such that $\xi_{\phi_0}=\xi_{\phi^1}$, which contradicts the fact that the map $\mathcal{D}(\Omega)\ni\phi\to\xi_\phi\in\mathcal{D}(\Omega')$ is injective. It follows therefore that there is no loss of generality in assuming $N=n$ for the proof of the theorem.

As a consequence, owing to Theorem 2 ξ is open and we can infer by a similar reasoning made just before that ξ is injective, in other words ξ is a diffeomorphism of Ω onto $\xi(\Omega)$.

(1)$\Rightarrow$(3). Let $\Omega'_0=\xi(\Omega)$. Since Ω' is connected, if $\Omega'_0\neq\Omega'$, then we could find a distribution $S\in\mathcal{D}'(\Omega'_0)$ which can not be extended to a distribution on Ω'. Let $T\in\mathcal{D}'(\Omega')$ such that $\xi^*T=\xi^*S$. Then $\xi^*(T|\Omega'_0)=\xi^*S$. It follows then from Lemma 1 that $T|\Omega'_0=S$, which is a contradiction.

(2)$\Rightarrow$(3). We use the same notations as before. We shall show that Ω'_0 is closed. For otherwise, there would exist $x'_0\in\partial\Omega'_0\cap\Omega'$. Let $\{x'^j\}$ be a sequence from Ω'_0 which converges to x'_0. Since ξ is proper, $\xi^{-1}(\{x'^j, x'_0\})$ is compact, which would imply $x'_0 \in \Omega'_0$. But this is a contradiction. Ω' being connected, we must have $\Omega'_0=\Omega'$.

This completes the proof.

PROPOSITION 5. *Let χ be an admissible map of $\Omega\subset R^N$ into $\Omega'\subset R^n$. We assume that Ω is connected, Ω' is simply connected and $N=n$. If χ is proper, then it is a diffeomorphism of Ω onto Ω'.*

PROOF. Since $J_\chi\neq 0$ for every $x\in\Omega$, the set $\Omega'_0=\chi(\Omega)$ is open. On the other hand, in the same way as in the proof of Theorem 3 we can show that $\Omega'_0=\Omega'$. For any $x'_0\in\Omega'$, the set $\chi^{-1}(x'_0)$ is finite. We can show that $\chi^{-1}(x')$ consists of the same number of points for every $x'\in\Omega'$, so that Ω is a covering space of Ω'. As Ω' is simply connected, ξ must be a homeomorphism, and so a diffeomorphism, which completes the proof.

REMARK 3. The theorem is not true without the assumption $N=n$. For example, the map $\xi=\log(x_1^2+\cdots+x_N^2)$, $N>1$ of $R^N\setminus\{0\}$ into R^1 is proper but not a diffeomorphism.

§3. Admissible maps into R^1

Consider a map ξ of $\Omega\subset R^N$ into $\Omega'\subset R^n$. If ξ has no critical point, it is admissible. As to its converse, though we have plausible reasons to infer the truth, we did not succeed in proving it besides some special instances. We note here that if the conjecture holds for the case where $n=1$ and N is arbitrary, then we can show that it is always true. In fact, if ξ is admissible with $\operatorname{rank} d\xi<n$ at an $x_0\in\Omega$, we can find real numbers $a_1, a_2, \cdots, a_N$, not all 0, such that $\sum_i a_i \dfrac{\partial\xi_i}{\partial x_j}(x_0)=0$, $j=1, 2, \cdots, N$. Then the map $\Omega\ni x\rightarrow\sum_i a_i\xi_i(x)\in R^1$ must be admissible with x_0 as a critical point.

PROPOSITION 6. *Let ξ be an admissible map of $\Omega\subset R^N$ into R^1. Assume that $\xi(x_0)=0$. Then* $\operatorname{supp}\xi^*\delta=\xi^{-1}(0)$.

PROOF. It suffices to show that $\operatorname{supp}\xi^*\delta\supset\xi^{-1}(0)$. If $x_0\in\xi^{-1}(0)$ is a regular point, then it follows from the expression (b) that x_0 belongs to $\operatorname{supp}\xi^*\delta$. Now let $x_0\in\xi^{-1}(0)$ be a critical point. We may assume that $x_0=0$. Let Y be the Heaviside function. In a small neighbourhood U of x_0 we have

$$\frac{\partial}{\partial x_i}Y(\xi)=\delta(\xi)\frac{\partial\xi}{\partial x_i}=0,\ \frac{\partial}{\partial x_i}(1-Y(\xi))=0,\qquad i=1, 2, \cdots, N,$$

so that ξ admits no values of opposite signs, say $\xi(U)\subset(-\infty, 0)$. Then we would have $\xi^*\delta^{(k)}=0$, $k=0, 1, 2, \cdots$ in U. On the other hand, if m is a positive integer, then

$$\frac{1}{t_+^m}=\frac{1}{t_\varepsilon^m}-\frac{1}{m-1}\frac{\delta}{\varepsilon^{m-1}}+\frac{1}{m-2}\frac{\delta'}{\varepsilon^{m-2}}+\cdots+\frac{(-1)^m}{(m-1)!}\delta^{(m-1)}\log\varepsilon+o(1)$$

as $\varepsilon\to+0$, where $t_\varepsilon=\max(t,\varepsilon)$. This together with $\xi^*\delta^{(k)}=0$ implies that $\frac{1}{|\xi|^m}=\frac{1}{|\xi|_\varepsilon^m}+o(1)$ as $\varepsilon\to0$ which shows that $\frac{1}{|\xi|^m}$ is locally integrable in U. But this is a contradiction since we can take m to be arbitrarily large.

PROPOSITION 7. *If a map ξ of $\Omega\subset R^N$ into R^1 is not open, then ξ is not admissible.*

PROOF. There is a point $x_0\in\Omega$ such that ξ is not open in any neighbourhood of x_0. We may assume that $\xi(x_0)=0$, and $\xi(x)\geqq0$ in a neighbourhood U of x_0. Then we have $\xi^*\delta=0$ in U. It follows from the preceding Proposition 6 that ξ is not admissible.

EXAMPLE 2. The map $P(x, y)=x_1^2+\cdots+x_n^2-y_1^2-\cdots-y_m^2$ of $R_x^n\times R_y^m$ into R^1 is not admissible at $(0, 0)\in R_x^n\times R_y^m$. It is followed by such a more general consideration as follows: Let σ be a map of $\Omega_1\subset R_x^n$ into R_s^1, τ a map of $\Omega_2\subset R_y^m$ into R_t^1 and ζ the map of $R_s^1\times R_t^1$ into R_w^1 defined by $w=s-t$. Since $\zeta_{u\otimes v}=u*\check{v}$ for any $u\in\mathcal{E}'(R_s^1)$, $v\in\mathcal{E}'(R_t^1)$, the map $u=\sigma(x)-\tau(y)$ of $\Omega_1\times\Omega_2$ into R_u^1 is admissible if and only if $\sigma_\phi*\check{\tau}_\psi\in\mathcal{D}(R_u^1)$ for any $u\in\mathcal{D}(R_s^1)$, $v\in\mathcal{D}(R_t^1)$. By a support theorem [4, p. 279], if the singular support of either σ_ϕ or τ_ψ is finite, then the condition $\sigma_\phi*\tau_\psi\in\mathcal{D}(R_u^1)$ implies that at least one of σ_ϕ and τ_ψ is a C^∞ function. Each of the maps $(x_1\cdots, x_n)\to\sum_i x_i^2$ and $(y_1, \cdots, y_m)\to\sum_j y_j^2$ is not admissible only at the origin. It follows therefore that the map P is not admissible at the origin.

PROPOSITION 8. *Let ξ be a map of $\Omega\subset R^2$ into R^1. Then ξ is admissible if and only if ξ has no critical point.*

PROOF. We have only to show the "only if" part. Suppose that $(0, 0)$ is a critical point and $\xi(0, 0)=0$. If $\frac{\partial^2\xi}{\partial x^2}(0, 0)=\frac{\partial^2\xi}{\partial x\partial y}(0, 0)=\frac{\partial^2\xi}{\partial y^2}(0, 0)=0$, then in a neighbourhood U of $(0, 0)$ we have $|\xi(x, y)|\leqq Mr^3$, where $r=\sqrt{x^2+y^2}$ and M is a constant, and therefore, for any $\phi\in\mathcal{D}(U)$ such that $\phi\geqq0$ and $\phi(0, 0)>0$,

$$\xi_\phi(0)=\lim_{\varepsilon\to+0}\frac{1}{\pi}\iint\frac{\varepsilon\phi}{\varepsilon^2+|\xi|^2}\,dx\,dy\geqq\lim_{\varepsilon\to+0}\frac{1}{\pi}\iint\frac{\varepsilon\phi}{\varepsilon^2+M^2r^6}\,r\,dr\,d\theta=\infty.$$

This is a contradiction. Consequently ξ can be written in a neighbourhood of $(0, 0)$ in one of the following forms:

(1) $\xi(x, y)=x^2+y^2+o(r^2)$

(2) $\xi(x, y)=x^2-y^2+o(r^2)$

(3) $\xi(x, y)=x^2+o(r^2)$,

if necessary, after a change of coordinates or replacing ξ by $-\xi$.

By Proposition 7 we see that (1) does not occur. If ξ takes the form (3), then we would have $|\xi(x, y)| \leqq 2x^2 + c\,y^2$ in a sufficiently small neighbourhood U of $(0, 0)$, where c is any assigned positive number. If ϕ is taken as before, then $\xi_\phi(0) \geqq \dfrac{\pi\phi(0, 0)}{2\sqrt{2c}}$. Now let ψ be any element of $\mathcal{D}(\Omega)$ such that $\psi \geqq 0$ and $\psi(0, 0) > 0$ and let $\alpha \in \mathcal{D}(U)$ be taken so that $0 \leqq \alpha \leqq 1$ and $\alpha(0, 0) = 1$. Then $\xi_\psi(0) \geqq \xi_{\alpha\psi}(0) \geqq \dfrac{\pi\psi(0, 0)}{2\sqrt{2c}}$. Consequently $\xi_\psi(0) = \infty$. But this is a contradiction. Finally we assume that ξ takes the form (2). By an elementary calculation we can show that ξ admits in a neighbourhood of $(0, 0)$ the solutions of the forms $y = u(x)x$ and $y = -v(x)x$, where u, v are C^∞ functions such that $u(0) = v(0) = 1$. Taking ϕ as before, we have with a constant $C > 0$

$$\xi_\phi(0) \geqq C\Big(\int_{y=u(x)x} + \int_{y=-v(x)x}\Big)\frac{\phi}{|x|}\,dx = \infty.$$

This is also a contradiction. The proof is thereby completed.

Finally we consider a map ξ of R^2 into $\bar{R}_+ = [0, \infty)$. In an obvious way we can extend the notion of admissibility to this instant. In this situation, if $\xi(x_0, y_0) = 0$, ξ is admissible at (x_0, y_0) if and only if $\operatorname{grad}\xi \neq 0$ at (x_0, y_0).

Proposition 9. *Let ξ be a map of Ω into $\bar{R}_+$ such that $\xi(x_0, y_0) = 0$ and $\operatorname{grad}\xi = 0$ at (x_0, y_0). ξ is admissible at (x_0, y_0) if and only if ξ can be written in the form*

$$\xi = a(x-x_0)^2 + 2b(x-x_0)(y-y_0) + c(y-y_0)^2 + o(|x-x_0|^2 + |y-y_0|^2),\ b^2 - ac < 0$$

in a neighbourhood of (x_0, y_0).

Proof. We may assume $(x_0, y_0) = (0, 0)$ and continue to use the same notations as in the proof of the preceding proposition. The argument there shows that if ξ is admissible, then it must be of the form (1)

$$\xi = x^2 + y^2 + o(r^2)$$

in a neighbourhood of $(0, 0)$ after a change of coordinates. It remains therefore to show the converse.

Let $t = \xi(x, y) = x^2 + y^2 + o(r^2)$ in a neighbourhood of $(0, 0)$. Put $t = s^2$. We shall find a function u in s and θ such that if we put $x = su\cos\theta$, $y = su\sin\theta$, then $\xi(su\cos\theta, su\sin\theta) = s^2$. For the function $h(s, \theta, u) = \xi(su\cos\theta, su\sin\theta)$ we have $h(0, \theta, u) = \dfrac{\partial h}{\partial s}(0, \theta, u) = 0$, hence the equation is reduced to $1 = \displaystyle\int_0^1 (1-\tau)\frac{\partial^2 h}{\partial s^2}(\tau s, \theta, u)\,d\tau$. If we put

$$F(s, \theta, u) = \int_0^1 (1-\tau)\frac{\partial^2 h}{\partial s^2}(\tau s, \theta, u)\,d\tau - 1,$$

then $F(0, \theta, 1)=0$ and $\frac{\partial F}{\partial u}(0, \theta, 1)=1$. The implicit function theorem allows us to determine $u=u(s, \theta)$ of class C^∞ for small $|s|$ under the condition $u(0, \theta)\equiv 1$, where u is periodic in θ with period 2π. Since $|\mathrm{grad}\,\xi|=2|s|(1+o(1))$ as $s\to 0$ and $dx^2+dy^2=s^2\left(\left(\frac{\partial u}{\partial \theta}\right)^2+u^2\right)d\theta^2$, we have for any $\phi\in\mathcal{D}(R^2)$ with support in a neighbourhood of (0, 0)

$$\xi_\phi(s^2)=\int_0^{2\pi}\phi(su\cos\theta, su\sin\theta)\frac{\sqrt{\left(\frac{\partial u}{\partial\theta}\right)^2+u^2}}{\left|\frac{1}{s}\mathrm{grad}\,\xi\right|}d\theta.$$

$\xi_\phi(s^2)$ is a C^∞ function in s in a deleted neighbourhood of 0. However, the function defined by the integral is of class C^∞ in a 0-neighbourhood of s. Consequently the function $\xi_\phi(t)$ must belong to $C^\infty([0, \infty))$, which completes the proof.

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