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On the Convolutions of Currents in R^n

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Recently, in his paper [3], F. Norguet has developed the theory of convolution of currents by introducing the two types of convolution through the notion of the direct image of a differentiable mapping f of an oriented manifold into another, and has shown a number of formulae about these convolutions and their mutual relation. The theory involves in virtue of the presence of the mapping f a natural extension of the notion of convolution even when applied to the distributions in R^n, where we may specify the mapping f so as to reach the usual notion of convolution. As for the extent of its applicability, however, the definition is more restrictive than usual as the theory is not designed to deal with the currents with arbitrary supports.

On the other hand, the various approaches for defining the convolution of distributions in R^n have been discussed in our previous papers [2, 5, 6], where we have shown the equivalence of the definitions resulting from these different approaches, and made a detailed study on $\mathscr{S}'$-convolution which plays an important rôle in discussing the convolution of tempered distributions.

The purpose of the present paper is to generalize by the modification of Norguet's ideas the notion of convolution of distributions which, when applied to the distributions, will lead to the same results as established in our papers cited above. In this paper a distribution will be understood to be a current of degree 0.

As we shall confine ourselves with the considerations of currents defined in R^n, we can speak of a summable current as a generalization of a summable distribution. This allows us to introduce the notion of convolution of currents in the reminiscence of the notion for distributions. In Section 1 we shall define the two kinds of convolution which are adjoint to each other and discuss the equivalent conditions for the existence of these convolutions. In Section 2 some fundamental properties of these convolutions will be discussed. Section 3 will be devoted to the characterization of convolution maps, which is a generalization of the result of [6] established in the case where distributions are concerned. Finally, in Section 4, we shall consider two kinds of Fourier transform which are adjoint to each other in a certain sense, and show the exchange formulae, an analogue to the formula obtained in [1], which asserts that the Fourier transform of the $\mathscr{S}'$-convolution of two tempered distributions is the multiplicative product of the respective Fourier transforms.

§1. The definition of convolutions for currents

Let R^n be an n-dimensional Euclidean space. Let us denote by $\mathcal{D}$ the space of all C^∞-forms with compact supports in R^n, equipped with the usual topology, and by $\overset{p}{\mathcal{D}}$, $0 \leqq p \leqq n$, the subspace of p-forms $\in \mathcal{D}$. We denote by $\mathcal{D}'$ and $\overset{p}{\mathcal{D}}'$ the strong duals of $\mathcal{D}$ and $\overset{n-p}{\mathcal{D}}$ respectively. $\mathcal{D}'$ is the space of currents in R^n and $\overset{p}{\mathcal{D}}'$ the space of homogeneous currents of degree p (of dimension n-p) in R^n. In what follows a distribution is understood to be a current of degree 0. A current $S \in \overset{p}{\mathcal{D}}'$ is considered to be a form whose coefficients are distributions in R^n, that is, we can write for $p>0$

$$S = \sum_I S_I dx_I, \qquad S_I \in \overset{0}{\mathcal{D}}',$$

where $\sum$ means that the summation is performed only over strictly increasing multi-indices $I = \{i_1, i_2, \cdots, i_p\}$, $1 \leqq i_1 < i_2 < \cdots < i_p \leqq n$, and we have written

$$dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}.$$

The same notation will be used even for $p=0$ with the understanding that $dx_I = 1$.

Let $\mathcal{D}'_{L^1}$ be the space of currents whose coefficients are summable distributions. $\overset{p}{\mathcal{D}}'_{L^1}$ will be defined in an obvious fashion.

Modifying the idea of F. Norguet [3], we shall introduce two kinds of convolution for currents. In what follows we assume that S and T are two homogeneous currents of degree p and q respectively.

DEFINITION 1 (the convolution of the 1st kind). *S and T are called to be $*_1$-composable if the condition*

$$(*_1) \qquad S(x) \wedge T(y) \wedge \phi(x+y) \in (\overset{2n}{\mathcal{D}}'_{L^1})_{x,y} \qquad \text{for every} \quad \phi \in \overset{2n-p-q}{\mathcal{D}}$$

is satisfied. Then, in virtue of the closed graph theorem, the map $\phi \to S(x) \wedge T(y) \wedge \phi(x+y)$ of $\overset{2n-p-q}{\mathcal{D}}$ into $(\overset{2n}{\mathcal{D}}'_{L^1})_{x,y}$ is continuous. The condition $(*_1)$ allows us to define the convolution of the 1st kind $S*_1 T$ as follows:

$$< S*_1 T, \phi > = (-1)^{(n-p)q} \int\int S(x) \wedge T(y) \wedge \phi(x+y),$$

in other words, $(-1)^{(n-p)q}(S*_1 T)$ is defined as the direct image of the current $S(x) \wedge T(y)$ on $R^n_x \times R^n_y$ under the map $f : (x, y) \to x+y$ of $R^n_x \times R^n_y$ into R^n.

Let $|J|$ stand for the number of the components of J. We shall introduce the following notations: if $|J|>0$, $\epsilon^J_{J_1, J_2}$ is a number which is 0 unless $\{J_1, J_2\}$ and J are derangements of the same $|J|$ distinct integers lying be-

tween 1 and n, in which case $\epsilon^{J}_{J_1,J_2}$ is the sign of the permutation $\{{}^{J}_{J_1,J_2}\}$. If $|J|=0$, we shall agree that $\epsilon^{J}_{J_1,J_2}=1$.

We put $(-1)^{\rho(I,CI)}=\epsilon^{\{1,2,\cdots,n\}}_{I,CI}$, where CI is the complementary set of indices, and

$$\epsilon_{I,J,K}=\begin{cases}(-1)^{\rho(I,CI)+\rho(J,CJ)+\rho(K,CI,CJ)} & \text{for } K=I\cap J \text{ such that} \\ & |K|=|I|+|J|-n, \\ 0, \quad \text{otherwise.}\end{cases}$$

Then the $*$-operation is defined by

$$*dx_I=(-1)^{\rho(I,CI)}dx_{CI} \quad \text{and} \quad *^{-1}dx_I=(-1)^{\rho(CI,I)}dx_{CI}.$$

Owing to L. Schwartz [9], we shall say that a distribution $K(x,\ y)\in\overset{0}{\mathcal{D}}'_{x,y}$ is partially summable with respect to y if $K(x,\ y)\in(\overset{0}{\mathcal{D}}'_{L^1})_y(\overset{0}{\mathcal{D}}'_x)$, that is, $<K(x,\ \hat{y}),\ \phi(x)>\in\overset{0}{\mathcal{D}}'_{L^1}$ for every $\phi\in\overset{n}{\mathcal{D}}$. The integral $\int K(x,\ y)dy$ is defined by the relation

$$<\int K(x,\ y)dy,\ \phi(x)>=\int<K(x,\ y),\ \phi(x)>dy$$

for every $\phi\in\overset{n}{\mathcal{D}}$.

Proposition 1. *S and T are $*_1$-composable when and only when each $\sum_{I,J}\epsilon_{I,J,K}S_I(y)T_J(x-y)$, $|K|=p+q-n$, is partially summable with respect to y. If this is the case, we can write*

$$S*_1T=\sum_K\Big(\int\sum_{I,J}\epsilon_{I,J,K}S_I(y)T_J(x-y)dy\Big)dx_K.$$

Proof. From the relations

$$S(x)\wedge T(y)\wedge\phi(x+y)=\sum S_I(x)T_J(y)\phi_{CK}(x+y)dx_I\wedge dy_J\wedge d(x+y)_{CK}$$

$$=(-1)^{(n-p)q}\sum(-1)^{\rho(I,CI)+\rho(J,CJ)}\epsilon^{CK}_{CI,CJ}S_I(x)T_J(y)\phi_{CK}(x+y)dx\wedge dy,$$

where $K=I\cap J$, $|K|=p+q-n$, $dx=dx_1\wedge\cdots\wedge dx_n$ and $dy=dy_1\wedge\cdots\wedge dy_n$, it follows by a change of variables that the condition $(*_1)$ is tantamount to saying that each $\sum_{I,J}\epsilon_{I,J,K}S_I(y)T_J(x-y)$ is partially summable with respect to y. If this is the case, then

$$\iint S(x)\wedge T(y)\wedge\phi(x+y)=(-1)^{(n-p)q}\int\sum_K\Big(\int\sum_{I,J}\epsilon_{I,J,K}S_I(y)T_J(x-y)dy\Big)dx_K\wedge\phi_{CK}dx_{CK}$$

$$=(-1)^{(n-p)q}<\sum_K\Big(\int\sum_{I,J}\epsilon_{I,J,K}S_I(y)T_J(x-y)dy\Big)dx_K,\ \phi>.$$

Consequently if S and T are $*_1$-composable, we can write

$$S*_1 T = \sum_K \Big(\int \sum_{I,J} \epsilon^{\ }_{I,J,K} S_I(y) T_J(x-y) dy \Big) dx_K.$$

Thus the proof is complete.

REMARK 1. We can also write

$$S*_1 T = \sum_K \Big(\int \sum_{I,J} \epsilon^{\ }_{I,J,K} S_I(x-y) T_J(y) dy \Big) dx_K.$$

REMARK 2. If the convolution $S_I * T_J$ exists for every I and J in a usual sense [5, 10], it is clear from our definition that $S*_1 T$ is well defined, and we can also write

$$S*_1 T = \sum_{I,J} (S_I * T_J) *^{-1} (*dx_I \wedge *dx_J)$$

as a consequence of the relations

$$\begin{aligned} *^{-1}(*dx_I \wedge *dx_J) &= *^{-1}\big((-1)^{\rho(I,CI)+\rho(J,CJ)} dx_{CI} \wedge dx_{CJ}\big) \\ &= (-1)^{\rho(I,CI)+\rho(J,CJ)+\rho(K,CI,CJ)} dx_K \\ &= \epsilon^{\ }_{I,J,K} dx_K, \end{aligned}$$

where $K = I \cap J$ and $|K| = |I| + |J| - n$.

REMARK 3. When $p+q<n$, $S*_1 T$ is well defined and equals 0.

Next we shall define the convolution of the 2nd kind for currents. For any current $S = \sum_I S_I dx_I$, we put $*S = \sum_I \bar{S}_I (*dx_I)$ and $*^{-1}S = \sum_I \bar{S}_I (*^{-1} dx_I)$, where $\bar{S}_I$ denotes the complex conjugate of S_I.

DEFINITION 2 (the convolution of the 2nd kind). *S and T are called to be $*_2$-composable if $*S$ and $*T$ are $*_1$-composable. Then we shall define the convolution of the 2nd kind $S*_2 T$ as follows*:

$$S*_2 T = *^{-1}\big((*S)*_1(*T)\big).$$

PROPOSITION 2. *S and T are $*_2$-composable when and only when $\sum_{I,J} \epsilon^{K}_{I,J} S_I(y) T_J(x-y)$ for every K with $|K| = p+q \leqq n$ is partially summable with respect to y. Then we have*

$$S*_2 T = \sum_K \Big(\int \sum_{I+J=K} \epsilon^{K}_{I,J} S_I(y) T_J(x-y) dy \Big) dx_K,$$

where $I+J=K$ denotes that $K = I \cup J$ and $I \cap J = \emptyset$.

PROOF. Since $*S = \sum (-1)^{\rho(I,CI)} \bar{S}_I dx_{CI}$ and $*T = \sum (-1)^{\rho(J,CJ)} \bar{T}_J dx_{CJ}$, it

follows from Prop. 1 that $*S$, $*T$ are $*_1$-composable if and only if each $\sum_{I,J}(-1)^{\rho(I,CI)+\rho(J,CJ)}\epsilon_{CI,CJ,CK}S_I(y)T_J(x-y)$, $|K|=p+q$, is partially summable with respect to y. And we have

$$(*S)*_1(*T)=\sum_K\Big(\int\sum_{I+J=K}(-1)^{\rho(I,CI)+\rho(J,CJ)}\epsilon_{CI,CJ,CK}\bar{S}_I(y)\bar{T}_J(x-y)dy\Big)dx_{CK}.$$

Using the relation

$$(-1)^{\rho(I,CI)+\rho(J,CJ)+\rho(K,CK)}\epsilon_{CI,CJ,CK}=\epsilon^K_{I,J},$$

we can infer that S, T are $*_2$-composable if and only if each $\sum_{I,J}\epsilon^K_{I,J}S_I(y)T_J(x-y)$ is partially summable with respect to y. It follows then that

$$S*_2T=\sum_K\Big(\int\sum_{I+J=K}\epsilon^K_{I,J}S_I(y)T_J(x-y)dy\Big)dx_K,$$

which completes the proof.

We can also define $S*_2T$ by the equivalent relation

$$S*_2T=*\big((*^{-1}S)*_1(*^{-1}T)\big).$$

REMARK 4. If S_I*T_J exists for every I and J in a sense described before, then $S*_2T$ is well defined, and we can write

$$S*_2T=\sum_{I,J}(S_I*T_J)dx_I\wedge dx_J.$$

REMARK 5. When $p+q>n$, S and T are always $*_2$-composable and $S*_2T$ equals 0.

PROPOSITION 3. *Each of the following conditions is equivalent to* $(*_1)$.

(i) $S\wedge(\check{T}*_1\phi)\in\overset{n}{\mathcal{D}}'_{L^1}$ *for every* $\phi\in\overset{2n-p-q}{\mathcal{D}}$;

(ii) $(\check{S}*_1\phi)\wedge T\in\overset{n}{\mathcal{D}}'_{L^1}$ *for every* $\phi\in\overset{2n-p-q}{\mathcal{D}}$;

where $\check{T}=\sum\check{T}_Jdx_J$ *and* $\check{S}=\sum\check{S}_Idx_I$.

Moreover, in any case the following relations hold:

$$\begin{aligned}<S*_1T,\phi>&=<S,\ \check{T}*_1\phi>\\&=(-1)^{(n-p)(n-q)}<T,\ \check{S}*_1\phi>.\end{aligned}$$

PROOF. Let $S=\sum S_Idx_I$ and $T=\sum T_Jdx_J$. As shown in Prop. 1, the condition $(*_1)$ is equivalent to the conditions

$$\sum_{I,J}\epsilon_{I,J,K}S_I(y)T_J(x-y)\in\overset{0}{\mathcal{D}}'_x(\overset{0}{\mathcal{D}}'_{L^1})_y\qquad\text{for every}\quad K=I\cap J.$$

$(*_1)\Rightarrow$(i): Putting $\phi=\sum\phi_{CK}dx_{CK}\in\overset{2n-p-q}{\mathcal{D}}$, we have

$$<\sum \varepsilon_{I,J,K} S_I(y) T_J(x-y), \phi_{CK}(x)dx> = \sum \varepsilon_{I,J,K} S_I(\check{T}_J * \phi_{CK}) \in \overset{0}{\mathcal{D}}'_{L^1}.$$

Since the relations $(-1)^{\rho(K,CK)+\rho(I,CI)} \varepsilon_{I,J,K} \varepsilon_{J,CK,CI} = 1$ hold, it follows that

$$\begin{aligned}\sum_K (-1)^{\rho(K,CK)} \sum_{I,J} \varepsilon_{I,J,K} S_I(\check{T}_J * \phi_{CK})dx &= \sum_K \sum_{I,J} S_I dx_I \wedge \varepsilon_{J,CK,CI} (\check{T}_J * \phi_{CK}) dx_{CI} \\ &= \sum S_I dx_I \wedge (\check{T}_{*_1} \phi)_{CI} dx_{CI} \\ &= S \wedge (\check{T}_{*_1} \phi) \in \overset{n}{\mathcal{D}}'_{L^1}.\end{aligned}$$

(i)$\Rightarrow$($*_1$): Choose $\phi = \phi_{CK} dx_{CK} \in \overset{2n-p-q}{\mathcal{D}}$, $K = I \cap J$. By (i) we have

$$S \wedge (\check{T}_{*_1} \phi) = (-1)^{\rho(K,CK)} < \sum_{I,J} \varepsilon_{I,J,K} S_I(y) T_J(x-y), \phi_{CK}(x)dx > dy \in \overset{n}{\mathcal{D}}'_{L^1},$$

which shows that $\sum_{I,J} \varepsilon_{I,J,K} S_I(y) T_J(x-y) \in \overset{0}{\mathcal{D}}'_x (\overset{0}{\mathcal{D}}'_{L^1})_y$, as desired.

The implications ($*_1$)$\Rightarrow$(ii)$\Rightarrow$($*_1$) can be proved in the same way as in the case ($*_1$)$\Rightarrow$(i)$\Rightarrow$($*_1$).

Finally, suppose S and T are $*_1$-composable. For every $\phi \in \overset{2n-p-q}{\mathcal{D}}$ we have

$$\begin{aligned}<S *_1 T, \phi> &= <\sum_K \int \sum_{I,J} \varepsilon_{I,J,K} S_I(y) T_J(x-y) dy\, dx_K, \phi_{CK}(x) dx_{CK}> \\ &= \sum_K (-1)^{\rho(K,CK)} \int \sum_{I,J} \varepsilon_{I,J,K} S_I(\check{T}_J * \phi_{CK}) dx \\ &= <S, \check{T} *_1 \phi>.\end{aligned}$$

Similarly

$$<S *_1 T, \phi> = (-1)^{(n-p)(n-q)} < T, \check{S} *_1 \phi>.$$

Thus the proof is complete.

As a consequence of the preceding proposition and the definition 2 we have

PROPOSITION 4. *A necessary and sufficient condition for $S *_2 T$ to exist is that one of the following equivalent conditions holds*:

(i) $S \wedge (\check{T} *_2 \phi) \in \overset{n}{\mathcal{D}}'_{L^1}$ *for every* $\phi \in \overset{n-p-q}{\mathcal{D}}$;

(ii) $(\check{S} *_2 \phi) \wedge T \in \overset{n}{\mathcal{D}}'_{L^1}$ *for every* $\phi \in \overset{n-p-q}{\mathcal{D}}$.

Then we can write

$$\begin{aligned}<S *_2 T, \phi> &= <S, \check{T} *_2 \phi> \\ &= (-1)^{pq} < T, \check{S} *_2 \phi>.\end{aligned}$$

PROOF. We first note that $*W\wedge *\alpha = W\wedge\alpha$ for any $W\in \overset{r}{\mathcal{D}}'$ and $\alpha\in \overset{n-r}{\mathcal{D}}$. From Prop. 3 together with Def. 2, it follows that the condition for $S*_2T$ to exist is equivalent to the following condition

$$(*S)\wedge\big((*\check{T})*_1(*\phi)\big)\in \overset{n}{\mathcal{D}}'_{L^1} \qquad \text{for every} \quad \phi\in \overset{n-p-q}{\mathcal{D}},$$

which is also equivalent to the condition

$$S\wedge(\check{T}*_2\phi)\in \overset{n}{\mathcal{D}}'_{L^1} \qquad \text{for every} \quad \phi\in \overset{n-p-q}{\mathcal{D}},$$

since $$(*S)\wedge\big((*\check{T})*_1(*\phi)\big)=*S\wedge *(\check{T}*_2\phi)=S\wedge(\check{T}*_2\phi).$$

Consequently

$$\begin{aligned}
<S*_2T,\ \phi> &= <*^{-1}\big((*S)*_1(*T)\big),\ \phi> \\
&= <(*S)*_1(*T),\ *\phi> \\
&= <*S,\ (*\check{T})*_1(*\phi)> \\
&= <S,\ *^{-1}\big((*\check{T})*_1(*\phi)\big)> \\
&= <S,\ \check{T}*_2\phi>.
\end{aligned}$$

Similarly we can show that $S*_2T$ exists if and only if the condition (ii) holds and then

$$<S*_2T,\ \phi>=(-1)^{pq}<T,\ \check{S}*_2\phi>.$$

Thus the proof is complete.

Now we shall consider the simultaneous convolutions of three currents. Let U be a homogeneous current of degree r.

Suppose that S, T, U satisfy the condition:

$$S(x)\wedge T(y)\wedge U(z)\wedge\phi(x+y+z)\in(\overset{3n}{\mathcal{D}}'_{L^1})_{x,y,z} \qquad \text{for every} \quad \phi\in \overset{3n-p-q-r}{\mathcal{D}}.$$

Then we can show that the map $\phi\to S(x)\wedge T(y)\wedge U(z)\wedge\phi(x+y+z)$ of $\overset{3n-p-q-r}{\mathcal{D}}$ into $(\overset{3n}{\mathcal{D}}'_{L^1})_{x,y,z}$ is continuous, which leads us to define *the simultaneous convolution of the* 1*st kind* $S*_1T*_1U$ by the relations

$$\begin{aligned}
<S*_1T*_1U,\ \phi> &= (-1)^{(n-p)(q+r)+(n-q)r}\iiint S(x)\wedge T(y)\wedge U(z)\wedge\phi(x+y+z) \\
&= <\sum_L\Big(\iint\sum_{I,J,K}\epsilon_{I,J,K,L}S_I(x-y-z)T_J(y)U_K(z)\,dy\,dz\Big)dx_L,\ \phi>,
\end{aligned}$$

where we have put $\epsilon_{I,J,K,L}=(-1)^{\rho(I,CI)+\rho(J,CJ)+\rho(K,CK)+\rho(L,CI,CJ,CK)}$ if $L=I\cap J\cap K$ such that $|L|=|I|+|J|+|K|-2n$, and $\epsilon_{I,J,K,L}=0$, otherwise.

PROPOSITION 5. *Suppose $S*_1T*_1U$ and $S*_1T$ are defined. Then $S*_1T$ and U are $*_1$-composable and*

$$S*_1T*_1U=(S*_1T)*_1U.$$

PROOF. Put $W=\sum_M W_M dx_M=S*_1T$. Then, using the equalities

$$\epsilon_{I,J,K,L}=\epsilon_{I,J,M}\epsilon_{M,K,L} \qquad \text{for} \quad M=I\cap J$$

we obtain

$$\sum_{M,K}\epsilon_{M,K,L}W_M(x-z)U_K(z)=\sum_{M,K}\epsilon_{M,K,L}\Big(\int\sum_{I,J}\epsilon_{I,J,M}\,S_I(x-y-z)T_J(y)dy\Big)U_K(z)$$

$$=\int\sum_{I,J,K}\epsilon_{I,J,K,L}S_I(x-y-z)T_J(y)U_K(z)dy\,\epsilon\,\overset{0}{\mathcal{D}}'_x(\overset{0}{\mathcal{D}}'_{L^1})_z.$$

Consequently we can conclude that $(S*_1T)*_1U$ exists and that $S*_1T*_1U=(S*_1T)*_1U$. Thus the proof is complete.

As to *the simultaneous convolution of the 2nd kind*, we can define $S*_2T*_2U$ by the relation

$$S*_2T*_2U=*^{-1}((*S)*_1(*T)*_1(*U)),$$

whenever the right hand side exists in the preceding sense.

PROPOSITION 6. *Let W be any current of degree r with compact support.*

(i) *If $S*_1T$ exists, then $(S*_1T)*_1W$, $S*_1(T*_1W)$ and $(S*_1W)*_1T$ exist, and we have*

$$(S*_1T)*_1W=S*_1(T*_1W)$$

$$=(-1)^{(n-q)(n-r)}(S*_1W)*_1T$$

and

$$<S*_1T,\,\phi>=(S*_1T*_1\check{\phi})(0) \qquad \text{for every} \quad \phi\,\epsilon\,\overset{2n-p-q}{\mathcal{D}}.$$

(ii) *If $S*_2T$ exists, then $(S*_2T)*_2W$, $S*_2(T*_2W)$ and $(S*_2W)*_2T$ exist, and we can write*

$$(S*_2T)*_2W=S*_2(T*_2W)=(-1)^{qr}(S*_2W)*_2T,$$

and

$$<S*_2T,\,\phi>=(*(S*_2T*_2\check{\phi}))(0) \qquad \text{for every} \quad \phi\,\epsilon\,\overset{n-p-q}{\mathcal{D}}.$$

PROOF. (i): Let S, T be $*_1$-composable. Since $\check{W}*_1\phi\,\epsilon\,\overset{2n-p-q}{\mathcal{D}}$, it follows from Prop. 3 that

$$S\dot{\wedge}(\check{T}*_1(\check{W}*_1\phi))=S\wedge((T*_1W)\check{}*_1\phi)\in\overset{n}{\mathcal{D}}'_{L^1} \qquad \text{for every} \quad \phi\in\overset{3n-p-q-r}{\mathcal{D}}.$$

Therefore S and $T*_1W$ are $*_1$-composable and we have for any $\phi\in\overset{3n-p-q-r}{\mathcal{D}}$

$$\begin{aligned}<(S*_1T)*_1W,\ \phi>&=<S*_1T,\ \check{W}*_1\phi>\\&=<S,\ (T*_1W)\check{}*_1\phi>\\&=<S*_1(T*_1W),\ \phi>.\end{aligned}$$

Consequently, $(S*_1T)*_1W=S*_1(T*_1W)$. Similarly we can show that $S*_1W$, T are $*_1$-composable and the $(S*_1T)*_1W=(-1)^{(n-q)(n-r)}(S*_1W)*_1T$.

If $U=\sum U_I dx_I\in\overset{r}{\mathcal{D}}'$ and $\phi=\sum\phi_{CI}dx_{CI}\in\overset{n-r}{\mathcal{D}}$, we can write

$$<U,\ \phi>=(U*_1\check{\phi})(0).$$

In fact, this follows from the following relations

$$\begin{aligned}<U,\ \phi>&=\sum(-1)^{\rho(I,CI)}\int U_I(x)\phi_{CI}(x)dx\\&=\sum(-1)^{\rho(I,CI)}(U_I*\check{\phi}_{CI})(0),\\(U*_1\check{\phi})(0)&=\sum(U_I*\check{\phi}_{CI})(0)*^{-1}(*dx_I\wedge *dx_{CI})\\&=\sum(-1)^{\rho(I,CI)}(U_1*\check{\phi}_{CI})(0).\end{aligned}$$

Now putting $U=S*_1T$, we have then

$$<S*_1T,\ \phi>=(S*_1T*_1\check{\phi})(0)$$

as was required.

(ii): In view of the definition of the convolution of the 2nd kind, (i) together with Prop. 4 will lead us to the conclusions of the case (ii), and so we shall omit the proof.

§2. Properties of convolutions

This section will be devoted to the further investigation of the properties of convolutions defined in the preceding section. As before, we assume that S and T are homogeneous currents of degree p and q respectively.

PROPOSITION 7. (i) *If S has a compact support, then $S*_1T$ and $S*_2T$ exist.*

(ii) *If $S*_1T$ exists, then $S*_1T$ is a homogeneous current of degree $p+q-n$. This means that the dimension of $S*_1T$ is equal to the sum of dimensions of S and T.*

(iii) *If $S*_2T$ exists, then $S*_2T$ is a homogeneous current of degree $p+q$.*

(iv) *If $S*_1T$ exists, then $T*_1S$ exists and*

$$S*_1T=(-1)^{(n-p)(n-q)}T*_1S.$$

(v) *If* $S*_2T$ *exists, then* $T*_2S$ *exists and*

$$S*_2T=(-1)^{pq}T*_2S.$$

PROOF. (i), (ii) and (iii) are obvious from the definitions of convolutions. (iv) and (v) are valid from the following relations

$$\begin{aligned}S*_1T&=(-1)^{(n-p)q}f(S(x)\wedge T(y))\\&=(-1)^{(n-p)q}f((-1)^{pq}T(y)\wedge S(x))\\&=(-1)^{(n-p)q+pq+n+(n-q)p}T*_1S\\&=(-1)^{(n-p)(n-q)}T*_1S,\end{aligned}$$

$$\begin{aligned}S*_2T&=*^{-1}((*S)*_1(*T))\\&=(-1)^{pq}*^{-1}((*T)*_1(*S))\\&=(-1)^{pq}T*_2S\end{aligned}$$

as was required.

Let us denote by δ Dirac's distribution at the origin and by $\overset{n}{\delta}$ Dirac's n-current, then $\overset{n}{\delta}=\delta\,dx$. We then obtain

$$\overset{n}{\delta}*_1S=S*_1\overset{n}{\delta}=S \quad\text{and}\quad \delta*_2S=S*_2\delta=S.$$

We now introduce the following linear operators in $\mathcal{D}'$. For any $S=\sum S_I dx_I$, we put

$$i_k(S)=\sum_I \epsilon^{\,I}_{k,I\cap C\{k\}}S_I\,dx_{I\cap C\{k\}}$$

and

$$e_k(S)=\sum_I S_I dx_k\wedge dx_I,$$

where $k=1, 2,\cdots, n$.

We can write

$$i_kS=S*_1(*\delta\,dx_k) \quad\text{and}\quad e_kS=\delta\,dx_k*_2S.$$

In fact, $\delta\,dx_k*_2S=\sum_I S_I dx_k\wedge dx_I=e_kS$ and

$$\begin{aligned}S*_1(*\delta\,dx_k)&=\sum_I S_I*(*^{-1}dx_I\wedge *^{-1}*dx_k)\\&=\sum_I(-1)^{\rho(CI,I)}S_I*(dx_{CI}\wedge dx_k)\\&=\sum_I(-1)^{\rho(CI,I)+\rho(CI,k,I\cap C\{k\})}S_I\,dx_{I\cap C\{k\}}\\&=\sum_I \epsilon^{\,I}_{k,I\cap C\{k\}}S_I\,dx_{I\cap C\{k\}}\\&=i_kS.\end{aligned}$$

These considerations together with Prop. 6 with W replaced by $*\delta dx_k$ or δdx_k $(k=1, 2, \cdots, n)$ yield the following

PROPOSITION 8. (i) *If $S*_1T$ exists, then $S*_1i_kT$ does exist and coincide with $i_k(S*_1T)$, where $k=1, 2, \cdots, n$.*

(ii) *If $S*_2T$ exists, then e_kS*_2T does exist and coincide with $e_k(S*_2T)$, where $k=1, 2, \cdots, n$.*

Now we shall consider the differential operator d and the adjoint differential operator ∂. We know that dS and ∂S are defined by

$$<bS, \phi>=<S, d\phi>, \quad dS=wbS \quad \text{and} \quad \partial S=*^{-1}d*wS,$$

where the linear operator w (resp. w^*) associates to S the current $(-1)^pS$ (resp. $(-1)^{n-p}S$). Then

$$dS=\sum_k dx_k\wedge\frac{\partial S}{\partial x_k}=\sum_k e_k\frac{\partial S}{\partial x_k}$$

and

$$\partial S=-\sum_k i_k\frac{\partial S}{\partial x_k},$$

where $\dfrac{\partial S}{\partial x_k}=\sum_I\dfrac{\partial S_I}{\partial x_k}dx_I$. Indeed, we can write

$$\begin{aligned}\partial S&=*^{-1}d*wS=*^{-1}\sum_k e_k\frac{\partial(*wS)}{\partial x_k}\\&=\sum_k *^{-1}\Big((\delta dx_k)*_2\Big(*\frac{\partial(wS)}{\partial x_k}\Big)\Big)\\&=\sum_k(*^{-1}\delta dx_k)*_1\Big(\frac{\partial(wS)}{\partial x_k}\Big)\\&=(-1)^{p+(n-p)+n-1}\sum_k\frac{\partial S}{\partial x_k}*_1(*\delta dx_k)\\&=-\sum_k i_k\frac{\partial S}{\partial x_k}.\end{aligned}$$

PROPOSITION 9. (i) *Assume that $S*_1T$ exists. Then $\partial S*_1T$, $S*_1\partial T$ exist and*

$$\partial(S*_1T)=\partial S*_1w^*T=S*_1\partial T. \tag{1}$$

(ii) *Assume that $S*_2T$ exists. Then $dS*_2T$, $S*_2dT$ exist and*

$$d(S*_2T)=dS*_2T=wS*_2dT. \tag{2}$$

PROOF. Under the assumptions it follows from Prop. 8 that $\partial S*_1T$ and

$S*_1\partial T$ (resp. $dS*_2T$ and $S*_2dT$) exist. Then the formulae (1) and (2) result from the equalities

$$\partial(S*_1T)=-\sum_k i_k\frac{\partial}{\partial x_k}(S*_1T)=-\sum_k S*_1\Big(i_k\frac{\partial T}{\partial x_k}\Big)=S*_1\partial T,$$

$$\partial(S*_1T)=(-1)^{(n-p)(n-q)}\partial(T*_1S)=(-1)^{(n-p)(n-q)}(T*_1\partial S)$$

$$=(\partial S)*_1(w^*T)$$

and

$$d(S*_2T)=\sum_k e_k\frac{\partial}{\partial x_k}(S*_2T)=\sum_k\Big(e_k\frac{\partial S}{\partial x_k}\Big)*_2T=dS*_2T,$$

$$d(S*_2T)=(-1)^{pq}d(T*_2S)=(-1)^{pq}(dT*_2S)=wS*_2dT.$$

Thus the proof is complete.

PROPOSITION 10. (i) *Assume that* $S*_1T$ *and* $(e_kS)*_1T$, $k=1, 2, \cdots, n$, *exist. Then* $(dS)*_1T$, $S*_1dT$ *exist and*

$$d(S*_1T)=(dS)*_1(w^*T)+S*_1dT.$$

(ii) *Assume that* $S*_2T$ *and* $(i_kS)*_2T$, $k=1, 2, \cdots, n$, *exist. Then* $(\partial S)*_2T$, $S*_2\partial T$ *exist and*

$$\partial(S*_2T)=(\partial S)*_2T+(wS)*_2\partial T.$$

PROOF. (i): Let $\phi\in\overset{2n-p-q-1}{\mathcal{D}}$. Since $e_k(S)*_1T$ exists for every k, we have

$$(dx_k\wedge S(x))\wedge T(y)\wedge\phi(x+y)\in(\overset{2n}{\mathcal{D}}'_{L^1})_{x,y},$$

whence

$$\Big(dx_k\wedge\frac{\partial S}{\partial x_k}\Big)\wedge T(y)\wedge\phi(x+y)\in(\overset{2n}{\mathcal{D}}'_{L^1})_{x,y}.$$

Thus we obtain

$$\big(dS(x)\big)\wedge T(y)\wedge\phi(x+y)\in(\overset{2n}{\mathcal{D}}'_{L^1})_{x,y},$$

which means that $(dS)*_1T$ exists.

Next it follows from our assumption that for every k

$$S(x)\wedge T(y)\wedge d(x+y)_k\wedge\phi(x+y)\in(\overset{2n}{\mathcal{D}}'_{L^1})_{x,y}$$

and

$$\big(dx_k\wedge S(x)\big)\wedge T(y)\wedge\phi(x+y)\in(\overset{2n}{\mathcal{D}}'_{L^1})_{x,y},$$

whence

$$S(x)\wedge(dy_k\wedge T(y))\wedge\phi(x+y)\in(\overset{2n}{\mathcal{D}}'_{L^1})_{x,y}.$$

We can therefore conclude in the same way as before that $S*_1(dT)$ exists.

We note that if $U\in\overset{n-1}{\mathcal{D}}'_{L^1}$, then $dU\in\overset{n}{\mathcal{D}}'_{L^1}$ and $\int dU=0$. This follows as a consequence of the relations

$$\int dU=\lim_{k\to\infty}<dU,\ \alpha_k>=(-1)^n\lim_{k\to\infty}<U,\ d\alpha_k>=0,$$

where $\{\alpha_k\}$ is a sequence of multiplicators. Now the following equality is valid.

$$\begin{aligned}&(-1)^{p+q}(S(x)\wedge T(y)\wedge d\phi(x+y))\\&=d(S(x)\wedge T(y)\wedge\phi(x+y))-(dS(x)\wedge T(y)\wedge\phi(x+y))\\&\quad-(-1)^p(S(x)\wedge dT(y)\wedge\phi(x+y)).\end{aligned}$$

Since $S(x)\wedge T(y)\wedge\phi(x+y)\in(\overset{2n-1}{\mathcal{D}}'_{L^1})_{x,y}$, so by the above remark

$$\int d(S(x)\wedge T(y)\wedge\phi(x+y))=0.$$

Hence it follows that

$$\begin{aligned}(-1)^{p+q+(n-p)q}<S*_1T,\ d\phi>&=-(-1)^{(n-p-1)q}<dS*_1T,\ \phi>\\&\quad-(-1)^{p+(n-p)(q+1)}<S*_1dT,\ \phi>.\end{aligned}$$

Consequently

$$d(S*_1T)=dS*_1w^*T+S*_1dT,$$

which completes the proof of (i).

To prove (ii), we put $\tilde{S}=*S$ and $\tilde{T}=*T$. Since $*i_k(S)=(-1)^{p-1}e_k(\tilde{S})$, it follows from our assumptions that $\tilde{S}*_1\tilde{T}$, $e_k(\tilde{S})*_1\tilde{T}$, $k=1, 2, \cdots, n$, exist. In virtue of (i), $d\tilde{S}*_1\tilde{T}$, $\tilde{S}*_1d\tilde{T}$ exist and

$$d(\tilde{S}*_1\tilde{T})=(d\tilde{S})*_1w^*\tilde{T}+\tilde{S}*_1d\tilde{T}.$$

In view of the relation $\partial=*^{-1}d*w$, a simple caluculation shows that $\partial S*_2T$, $S*_2\partial T$ exist and

$$\partial(S*_2T)=\partial S*_2T+wS*_2\partial T.$$

This is what we wished to show.

REMARK 6. Even if $S*_1T$, $dS*_1T$ and $S*_1dT$ exist, $S*_1T=dS*_1w^*T+S*_1dT$ does not hold in general. Actually, in the case $n=3$, we take $S=dx_1\wedge dx_2$,

$T=g(x)dx_2$, where $g=\int_{-\infty}^{x_3} h(x_1, x_2, t)dt$, $0<h\in \overset{0}{\mathcal{D}}$. Then $S*_1T=0$, $dS*_1w^*T=0$, but $S*_1dT=-\int hdx\neq 0$.

Finally we shall show the following

PROPOSITION 11. (i) *Let r_1, r_2 be non-negative integers $\leqq n$ such that $r_1+r_2=3n-p-q$. Then the convolution $S*_1T$ exists if and only if the following condition is satisfied.*

$$(*)\qquad (S*_1\phi)\wedge(\check{T}*_1\psi)\in \overset{n}{L}_1 \qquad \text{for every} \quad \phi\in\overset{r_1}{\mathcal{D}},\ \psi\in\overset{r_2}{\mathcal{D}},$$

where $\overset{n}{L}_1$ denotes the space of all the summable forms of degree n. Then we have

$$<S*_1T,\ \check{\phi}*_1\psi>=(-1)^{(n-q)(n-r_1)}\int(S*_1\phi)\wedge(\check{T}*_1\psi).$$

(ii) *Let r_1, r_2 be non-negative integers $\leqq n$ such that $r_1+r_2=n-p-q$. Then the convolutions $S*_2T$ exists if and only if the following condition is satisfied.*

$$(S*_2\phi)\wedge(\check{T}*_2\psi)\in \overset{n}{L}_1 \qquad \text{for every} \quad \phi\in\overset{r_1}{\mathcal{D}},\ \psi\in\overset{r_2}{\mathcal{D}}.$$

Then we have

$$<S*_2T,\ \check{\phi}*_2\psi>=(-1)^{qr_1}\int(S*_2\phi)\wedge(T*_2\psi).$$

PROOF. (i) We first note that, for any given r_1, r_2 such that $r_1+r_2=3n-p-q$, the condition $(*)$ is equivalent to the condition

$$(S*_1\phi)\wedge(\check{T}*_1\psi)\in \overset{n}{\mathcal{D}}'_{L^1} \qquad \text{for every} \quad \phi\in\overset{r_1}{\mathcal{D}},\ \psi\in\overset{r_2}{\mathcal{D}},$$

as seen from the procedure given in the proof of Prop. 2 of our paper [5, p. 25].

Assume that S and T are $*_1$-composable. Then $S*_1\phi$ and T becomes $*_1$-composable in view of Prop. 6, and therefore by Prop. 3, $(*)$ will be satisfied as desired.

To show the converse, we put $r_1=n-p+s_1$ and $r_2=n-p+s_2$. Then $s_1+s_2=n$, $p\geqq s_1\geqq 0$, $q\geqq s_2\geqq 0$, and $s_1\geqq n-q$ or $s_2\geqq n-p$. If $s_2>n-p$, then in view of Prop. 3, the condition $(*)$ implies that $S*_1(\check{T}*_1\psi)$ exists, and so, by Prop. 7, $S*_1i_k(\check{T}*_1\psi)$ exists for $k=1, 2, \cdots, n$, and we have for every $\phi'\in\overset{r_1+1}{\mathcal{D}}$

$$(S*_1\phi')\wedge i_k(\check{T}*_1\psi)=(S*_1\phi')\wedge(\check{T}*_1i_k\psi)\in\overset{n}{\mathcal{D}}'_{L^1}.$$

Thus we have only to show the case where $s_2=n-p$, so that $r_1=n$ and

$r_2=2n-p-q$. Since $S*_1\phi=\sum(S_I*\phi_1)dx_I$ for any $\phi=\phi_1 dx \in \overset{n}{\mathcal{D}}$, it is clear that $(*)$ can be written in the form

$$\sum(-1)^{\rho(I,CI)}(S_I*\phi_1)(\check{T}*_1\psi)_{CI} \in \overset{0}{L}_1$$

and in turn

$$\sum(-1)^{\rho(I,CI)} S_I(\check{T}*_1\psi)_{CI} \in \overset{0}{\mathcal{D}}'_{L^1},$$

which implies

$$S\wedge(\check{T}*_1\psi) \in \overset{n}{\mathcal{D}}'_{L^1} \qquad \text{for every} \quad \psi \in \overset{2n-p-q}{\mathcal{D}}.$$

Consequently it follows from Prop. 3 that the condition $(*)$ holds. Moreover we can write

$$\begin{aligned}\int(S*_1\phi)\wedge(\check{T}*_1\psi) &= <(S*_1\phi)*_1 T, \psi> \\ &= (-1)^{(n-q)(n-r_1)}<S*_1 T, \check{\phi}*_1\psi>\end{aligned}$$

as was asserted.

For (ii), the proof will be carried out in the same way as in the case (i), so we omit the proof thereof. This completes the proof.

§3. The convolution maps

Let $\mathcal{E}$ be the space of C^∞-functions defined in R^n, each of which is bounded with its derivatives of every order. We denote by $\dot{\mathcal{E}}$ the closure of $\overset{0}{\mathcal{D}}$ in $\mathcal{E}$. The strong dual of $\dot{\mathcal{E}}$ is the space $\overset{n}{\mathcal{D}}'_{L^1}$. Let $\overset{0}{\mathscr{H}}$ be a *$\dot{\mathcal{E}}$-normal space of distributions* [6, p. 177], that is, a normal space of distributions satisfying the conditions: $\overset{0}{\mathscr{H}}$ is stable under the multiplication by any element of $\dot{\mathcal{E}}$ and linear endomorphism $S\to\alpha S$ of $\overset{0}{\mathscr{H}}$ is uniformly continuous with respect to α when α varies in any bounded subset of $\dot{\mathcal{E}}$. We denote by $\mathscr{H}$ the space of all the currents with coefficients in $\overset{0}{\mathscr{H}}$. A continuous linear map u of $\mathscr{H}$ into $\mathcal{D}'$ is referred to as *a convolution map of the 1st kind* (resp. *of the 2nd kind*), if there exists a current T such that we can write $u(S)=T*_1 S$ (resp. $u(S)=S*_2 T$) for every $S \in \mathscr{H}$. We have shown in [6, p. 178] that a continuous linear map u of $\overset{0}{\mathscr{H}}$ into $\overset{0}{\mathcal{D}}'$ is a convolution map if u is commutative with any translation τ_h on $\overset{0}{\mathcal{D}}$.

We are now ready to prove

THEOREM 1. (i) *A continuous linear map u of $\mathscr{H}$ into $\mathcal{D}'$ is a convolu-*

tion map of the 1*st kind if and only if it is commutative with any translation* τ_h *and with the operators* i_k, $k=1, 2, \cdots, n$, *when* u *is restricted to* $\mathcal{D}$.

(ii) *A continuous linear map* v *of* $\mathscr{H}$ *into* $\mathcal{D}'$ *is a convolution map of* 2*nd kind if and only if it is commutative with any translation* τ_h *and with the operators* e_k, $k=1, 2, \cdots, n$, *when* v *is restricted to* $\mathcal{D}$.

PROOF. The "only if" parts are evident. Now assume that v is commutative with any translation τ_h and with the operator e_k, $k=1, 2, \cdots, n$, when v is restricted to $\mathcal{D}$. v determines the linear maps v_J of $\overset{0}{\mathscr{H}}$ into $\overset{0}{\mathcal{D}}'$ such that

$$v(\overset{0}{S}) = \sum v_J(\overset{0}{S})dx_J, \quad \overset{0}{S} \in \overset{0}{\mathscr{H}},\ v_J(\overset{0}{S}) \in \overset{0}{\mathcal{D}}'.$$

For every J, v_J is continuous and commutative with any translation τ_h when v_J is restricted to $\overset{0}{\mathcal{D}}$. Since $\overset{0}{\mathscr{H}}$ is $\dot{\mathscr{B}}$-normal, it follows from the remark made above that there exists a unique distribution T_J such that $v_J(\overset{0}{S}) = \overset{0}{S} * T_J$. Put $T = \sum_J T_J dx_J$. For any $S = \sum_I S_I dx_I \in \mathscr{H}$ we have

$$v(S) = \sum_I dx_I \wedge v(S) = \sum_{I,J} dx_I \wedge v_J(S_I)dx_J$$

$$= \sum_{I,J} (S_I * T_J)dx_I \wedge dx_J = S *_2 T,$$

which completes the proof of (ii).

To prove the sufficiency in (i), we consider the map $u'(S) = *^{-1}u(*S)$ for every $S \in \mathscr{H}$. Putting $e'_k S = S \wedge dx_k = S *_2 \delta dx_k$, $k=1, 2, \cdots, n$, it is easy to see that

$$e'_k u'(S) = u'(e'_k S) \qquad \text{for every} \quad S \in \mathscr{H}.$$

In a similar way as in the proof of (ii), we can infer that there exists a unique current U such that $u'(S) = U *_2 S$. Putting $T = *U$, we have

$$u(S) = *u'(*^{-1}S) = *\big(U *_2 (*^{-1}S)\big)$$

$$= *\big((*^{-1}T) *_2 (*^{-1}S) = T *_1 S.$$

Thus the proof is complete.

As an immediate consequence of Theorem 1 we have

COROLLARY. *A continuous linear map* $u: \mathcal{D}' \to \mathcal{D}'$ *is commutative with any translation* τ_h *and with the operators* i_k (*resp. the operators* e_k), $k=1, 2, \cdots, n$, *when* u *is restricted to* $\mathcal{D}$, *if and only if there exists a unique current* T *with compact support such that* $u(S) = T *_1 S$ (*resp.* $u(S) = S *_2 T$).

§4. The exchange formula for Fourier transformation

Let $\overset{0}{S}$, $\overset{0}{T}$ be tempered distributions. If $\overset{0}{S}$, $\overset{0}{T}$ are $\mathscr{S}'$-composable, the multiplicative product $\mathcal{F}(\overset{0}{S})\mathcal{F}(\overset{0}{T})$ is defined and $\mathcal{F}(\overset{0}{S}*\overset{0}{T})=\mathcal{F}(\overset{0}{S})\mathcal{F}(\overset{0}{T})$, where $\mathcal{F}$ stands for the Fourier transform for distributions [1, 7]. In the following we shall extend this formula to the currents. To begin with, we shall define the exterior product and $\mathscr{S}'$-convolutions.

By *a restricted δ-sequence* we shall understand every sequence of non-negative functions $\rho_k \in \overset{0}{\mathcal{D}}$ with the following properties:

(i) Supp ρ_k converges to $\{0\}$ as $k\to\infty$;

(ii) $\int\rho_k(x)dx$ converges to 1 as $k\to\infty$;

(iii) $\int|x|^{|p|}|D^p\rho_k(x)|\,dx\leqq K_p$, a constant independent of k.

We note that a sequence $\{\rho_k\}$ satisfying the conditions (i) and (ii) is called a δ-sequence. Let S, $T\in\mathcal{D}'$.

If the sequence of the exterior product $\{S\wedge(T*_2\rho_k)\}$, $k=1, 2, \cdots$, converges to the current in $\mathcal{D}'$ as $k\to\infty$, then the limit is called *the exterior product* which will be denoted by $S\wedge T$. We can show that $S\wedge T$ exists if and only if $\lim_{k\to\infty}(S*_2\rho_k)\wedge(T*_2\tilde{\rho}_k)$ or $\lim_{k\to\infty}(S*_2\rho_k)\wedge T$ exists in $\mathcal{D}'$ for arbitrary restricted δ-sequences $\{\rho_k\}$ and $\{\tilde{\rho}_k\}$ and that in either case the limit equals $S\wedge T$. Indeed, this will follow from the same reasoning as in the proof of Prop. 5 in [7, p. 95].

Let us denote by $\mathscr{S}$ the space of rapidly decreasing C^∞-forms and by $\mathscr{S}'$ its dual, that is, $\mathscr{S}'$ is the space of all currents whose coefficients are tempered distributions.

Let S, $T\in\mathscr{S}'$ be homogeneous currents of degree p and q respectively. S, T are called to be *$*_1$-$\mathscr{S}'$-composable* if

$$S(x)\wedge T(y)\wedge\phi(x+y)\in(\overset{2n}{\mathcal{D}}'_{L^1})_{x,y} \qquad \text{for every} \quad \phi\in\overset{2n-p-q}{\mathscr{S}}.$$

If this is the case, the closed graph theorem implies that the map $\phi\to S(x)\wedge T(y)\wedge\phi(x+y)$ of $\overset{2n-p-q}{\mathscr{S}}$ into $\overset{2n}{\mathcal{D}}'_{L^1}$ is continuous. Then *the $\mathscr{S}'$-convolution of the 1st kind* $S*_1T\in\mathscr{S}'$ is defined as follows:

$$<S*_1T,\ \phi>=(-1)^{(n-p)q}\iint S(x)\wedge T(y)\wedge\phi(x+y).$$

Similarly we can define the *$\mathscr{S}'$-convolution of the 2nd kind* $S*_2T\in\mathscr{S}'$ as follows:

$$S*_2T=*^{-1}((*S)*_1(*T)),$$

when $*S$ and $*T$ are $*_1$-$\mathscr{S}'$-composable. Replacing $\mathscr{D}$ and $\mathscr{D}'$ by $\mathscr{S}$ and $\mathscr{S}'$ respectively we can show that the discussions given in the section 1 are also valid in this situation.

Let $S=\sum_I S_I dx_I \in \mathscr{S}'$. *The Fourier transform of the* 1*st kind* $\mathfrak{F}_1(S)$ is defined by

$$\mathfrak{F}_1(S)=\sum_I \mathfrak{F}(S_I)*d\xi_I \in \mathscr{S}',$$

where ξ denotes a generic point of Ξ^n, the dual of R^n. Actually this is the Fourier transform defined by R. Scarfiello [4]. Further we shall define *the Fourier transform of the* 2*nd kind* $\mathfrak{F}_2(S)$ as follows:

$$\mathfrak{F}_2(S)=\mathfrak{F}_1(*^{-1}S)=\sum_I \mathfrak{F}(S_I)d\xi_I.$$

Let $K(x, y)$ be any kernel distribution belonging to $(\overset{0}{\mathscr{S}'})_{x,y}$. Then K is called to be $\mathscr{S}'$-*composable* if

$$(*)_{\mathscr{S}'} \qquad K(x, y)\phi(x+y) \in (\overset{2n}{\mathscr{D}'_{L^1}})_{x,y} \qquad \text{for every} \quad \phi \in \overset{2n}{\mathscr{S}},$$

and the $\mathscr{S}'$-convolution $\overset{*}{K} \in \overset{0}{\mathscr{S}'}$ of K is defined by

$$<\overset{*}{K}, \phi> = \iint K(x, y)\phi(x+y).$$

In our previous work [2, p. 549], we have discussed the various conditions equivalent to $(*)_{\mathscr{S}'}$.

LEMMA 1. *Let K be an $\mathscr{S}'$-composable kernel distribution. Then we have for every δ-sequence $\{\rho_k\}$*

$$\mathfrak{F}(\overset{*}{K})=\lim_{k\to\infty} <\mathfrak{F}K(\xi, \xi-\eta), \qquad \rho_k(\eta)d\eta>_\eta,$$

where $\lim_{k\to\infty}$ means the distributional limit.

PROOF. Putting $\hat{K}=\mathfrak{F}K$, we have for any $\phi d\xi \in \overset{n}{\mathscr{S}}$

$$\begin{aligned} &<<\hat{K}(\xi, \xi-\eta), \rho_k(\eta)d\eta>_\eta, \phi(\xi)d\xi>_\xi \\ &=<\hat{K}(\xi, \xi-\eta), \rho_k(\eta)\phi(\xi)d\xi\wedge d\eta>_{\xi,\eta} \\ &=<\hat{K}(\xi, \eta), \rho_k(\xi-\eta)\phi(\xi)d\xi\wedge d\eta>_{\xi,\eta}. \end{aligned}$$

By Parseval's formula, it follows that

$$\begin{aligned} &<\hat{K}(\xi, \eta), \rho_k(\xi-\eta)\phi(\xi)d\xi\wedge d\eta>_{\xi,\eta} \\ &=<K(x, y), \hat{\rho}_k(-y)\hat{\phi}(x+y)dx\wedge dy>_{x,y}. \end{aligned}$$

Since $\hat{\rho}_k(\cdot - y)$ tends to 1 in $(\mathscr{B}_c)_{x,y}$ as $k \to \infty$,

$$\lim_{k\to\infty} < K(x, y), \hat{\rho}_k(-y)\hat{\phi}(x+y)dx \wedge dy >_{x,y}$$

$$= < \overset{*}{K}, \hat{\phi}(x)dx >_x = < \hat{\overset{*}{K}}, \phi(\xi)d\xi >_\xi .$$

Consequently, $\mathcal{J}\overset{*}{K} = \lim_{k\to\infty} < \hat{K}(\xi, \xi-\eta), \rho_k(\eta)d\eta >_\eta$, which completes the proof.

THEOREM 2. *Let* $S \in \overset{p}{\mathscr{S}'}$ *and* $T \in \overset{q}{\mathscr{S}'}$.

(i) *If* S, T *are* $*_1$-$\mathscr{S}'$-*composable, then* $\mathcal{J}_1(S) \wedge \mathcal{J}_1(T)$ *is defined and* $\mathcal{J}_1(S *_1 T) = \mathcal{J}_1(S) \wedge \mathcal{J}_1(T)$.

(ii) *If* $S *_2 T$ *exists,* $\mathcal{J}_2(S) \wedge \mathcal{J}_2(T)$ *is defined and* $\mathcal{J}_2(S *_2 T) = \mathcal{J}_2(S) \wedge \mathcal{J}_2(T)$.

PROOF. Let $S = \sum S_I dx_I \in \overset{p}{\mathscr{S}'}$ and $T = \sum T_J dx_J \in \overset{q}{\mathscr{S}'}$ be $*_1$-$\mathscr{S}'$-composable. Then we have

$$\mathcal{J}_1(S *_1 T) = \sum_K \mathcal{J}\left\{ \int \sum_{I,J} \epsilon_{I,J,K} S_I(y) T_J(x-y) dy \right\} * d\xi_K .$$

If we put $K(x, y) = \sum_{I,J} \epsilon_{I,J,K} S_I(x) T_J(y) \in \overset{0}{(\mathscr{S}')}_{x,y}$ and apply Lemma 1, then for every δ-sequence $\{\rho_k\}$

$$\mathcal{J}\overset{*}{K} = \lim_{k\to\infty} < \hat{K}(\xi, \xi-\eta), \rho_k(\eta)d\eta >_\eta$$

$$= \lim_{k\to\infty} \sum_{I,J} \epsilon_{I,J,K} \hat{S}_I(\xi)(\hat{T}_J * \rho_k)(\xi).$$

Therefore we can conclude that $\mathcal{J}_1(S) \wedge \mathcal{J}_1(T)$ is defined and

$$\mathcal{J}_1(S) \wedge \mathcal{J}_1(T) = \lim_{k\to\infty} \sum_{I,J} \hat{S}_I(\xi)(\hat{T}_J * \rho_k)(\xi) \divideontimes d\xi_I \wedge * d\xi_J$$

$$= \lim_{k\to\infty} \sum_K \sum_{I,J} \epsilon_{I,J,K} \hat{S}_I(\xi)(\hat{T}_J * \rho_k)(\xi) \divideontimes d\xi_K$$

$$= \mathcal{J}_1(S *_1 T)$$

as was asserted.

(ii) follows from (i) because of the following relations

$$\mathcal{J}_2(S *_2 T) = \mathcal{J}_1\big((\divideontimes^{-1} S) *_1 (\divideontimes^{-1} T)\big)$$

$$= \mathcal{J}_1(\divideontimes^{-1} S) \wedge \mathcal{J}_1(\divideontimes^{-1} T)$$

$$= \mathcal{J}_2(S) \wedge \mathcal{J}_2(T).$$

This establishes the theorem.

References

[1] Y. Hirata and H. Ogata, *On the exchange formula for distributions*, this Journal, **22** (1958), 147-152.
[2] ——— and R. Shiraishi, *On partial summability and convolutions in the theory of vector valued distributions*, this Journal, **24** (1960), 535-562.
[3] F. Norguet, *Problémes sur les formes différentielles et les courants*, Ann. Inst. Fourier, **11** (1961), 1-88.
[4] R. Scarfiello, *Sur le changement de variables dans les distributions et leurs transformées de Fourier*, Nuovo Cimento, **12** (1954), 471-482.
[5] R. Shiraishi, *On the definition of convolutions for distributions*, this Journal, **23** (1959), 19-32.
[6] ———, *On θ-convolutions of vector valued distributions*, this Journal, **27** (1963), 173-212.
[7] ——— and M. Itano, *On the multiplicative products of distributions*, this Journal, **28** (1964), 223-235.
[8] ———, *On the value of distributions at a point and the multiplicative products*, this Journal, **31** (1967), 89-104.
[9] L. Schwartz, *Théorie des distributions à valeurs vectorielles*, Chap. I, Ann. Inst. Fourier, **7** (1957), 1-141.
[10] ———, *Théorie des distributions*, Paris, Hermann (1966).

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