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On Conditionally Upper Continuous Lattices

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1. Introduction

Following the terminology of F. Maeda ([5], p. 87) we say that a lattice L is *conditionally complete* in case: (i) every nonempty subset of L with an upper bound has a supremum and (ii) the dual of (i) holds. A conditionally complete lattice is called *conditionally upper continuous* if $a_\delta \uparrow a$ implies $a_\delta \wedge b \uparrow a \wedge b$ for every b in L. Dually, it is called *conditionally lower continuous* in case $a_\delta \downarrow a$ implies $a_\delta \vee b \downarrow a \vee b$ for all b in L. If L is both conditionally upper and lower continuous, it is called *conditionally continuous.* Finally, as in [5], p. 90, we define a *general continuous geometry* to be a conditionally continuous, relatively complemented modular lattice with 0.

In a lattice L with 0, F. Maeda ([5], Definition 1.1, p. 85) writes $a \triangledown b$ to denote the fact that $a \wedge b=0$ and $(a \vee x) \wedge b = x \wedge b$ for every x in L. In a modular lattice with 0 the relation $\triangledown$ is symmetric. If for each subset S of L, $S^\triangledown = \{x: x \triangledown s$ for each s in $s\}$, $S^\triangledown$ is an ideal of L. An ideal I is called *normal* in case $I=(I^\triangledown)^\triangledown$. In [5], pp. 90-92, Maeda has sketched a proof of the fact that a general continuous geometry can be equipped with a dimension function in much the same way as is done for a continuous geometry. One of the chief differences between the two theories is the fact that the normal ideals of a general continuous geometry have a role analogous to that of the central elements of a continuous geometry. Since, in an arbitrary relatively complemented lattice with 0, the relation $\triangledown$ is symmetric ([4], Corollary 1, p. 3), one can define normal ideals in such a lattice. In [4], Theorem 16, p. 9, we showed that an ideal of a relatively complemented modular lattice L with 0 is normal if and only if it is a central element of $\tilde{L}$, where $\tilde{L}$ denotes the set of ideals J of L such that $J \cap L(0, x)$ is in the completion by cuts of the interval $L(0, x)$ for each x in L. This suggests that it might be possible to start with a general continuous geometry L and equip $\tilde{L}$ with a dimension function whose restriction to L is precisely the dimension function described by F. Maeda in [5]. Our goal in this paper will be to show that this can indeed be done.

In §2 we relate subdirect sum decompositions of a conditionally upper continuous lattice L with 0 to central decompositions of $\tilde{L}$. If L is a conditionally upper continuous, relatively complemented modular lattice with 0, we observe in §3 that $\tilde{L}$ is an upper continuous complemented modular lattice. It follows from this that $\tilde{L}$ is a dimension lattice whose dimension function induces on L a dimension function of the type described in [5].

2. Complete ideals

First we establish some terminology. Given a lattice L with 0 we use the symbol $J(x)$ to denote the principal ideal generated by the element x of L. Following [2] we call an ideal of L *complete* if it is closed under the formation of arbitrary suprema whenever they exist in L. Let $I(L)$ denote the set of all ideals of L and $K(L)$ the set of complete ideals with both sets partially ordered by set inclusion. This clearly makes each of them into a complete lattice with set intersection as the meet operation. In order to avoid confusion we agree to let $I+J$ denote the join operation in $I(L)$ and $I\vee J$ the one in $K(L)$. As is shown in [1], the mapping $x\to J(x)$ embeds L into $K(L)$ in such a way that any existing suprema and infima are preserved. Moreover, if L is conditionally complete, it is easy to show that an ideal J of L is complete if and only if $J\cap J(x)$ is principal for each x in L. It follows that for such a lattice $K(L)$ coincides with the lattice $\tilde{L}$ we discussed in [4], pp. 6–9.

Our goal in this section is to show the relation between direct sum decompositions of L and central decompositions of $K(L)$. If L is a lattice with 0 and if $(S_\alpha\colon \alpha \in A)$ is a family of ideals of L, then L is said to be a *subdirect sum* ([5], Definition 2.3, p. 87) of the ideals $(S_\alpha : \alpha \in A)$, denoted $L=\sum^*(\oplus S_\alpha\colon \alpha \in A)$ in case:

(1) each x in L has a representation of the form $x=\vee_\alpha x_\alpha$ with $x_\alpha \in S_\alpha$ $(\alpha \in A)$.

(2) $\alpha\neq\beta$ implies $S_\alpha\subseteq S_\beta^\nabla$.

If $L=\sum^*(\oplus S_\alpha\colon \alpha \in A)$ and if for any family $\{x_\alpha\colon x_\alpha \in S_\alpha\}$, $\vee_\alpha x_\alpha$ exists in L, we call L the *direct sum* of the ideals $(S_\alpha\colon \alpha \in A)$ and write $L=\sum(\oplus S_\alpha\colon \alpha \in A)$. In case $A=\{1, 2, \cdots, n\}$ we will use the notation $L=S_1\oplus S_2\oplus\cdots\oplus S_n$ to denote a direct sum decomposition.

THEOREM 1. ([4], Theorem 1, p. 1). *Let L be a lattice with* 0. *Then $L=S_1\oplus S_2\oplus\cdots\oplus S_n$ if and only if $\{S_i\colon i=1, 2, \cdots, n\}$ is a family of pairwise disjoint central elements of $I(L)$ whose supremum in $I(L)$ is L.*

For the remainder of this paper L will denote a conditionally upper continuous lattice with 0.

LEMMA 2. *Let $\{S_\alpha\colon \alpha \in A\}$ be a family of ideals of L. Then $L=\sum^*(\oplus S_\alpha\colon \alpha \in A)$ if and only if each x in L has a unique representation of the form $x=\vee_\alpha x_\alpha$ with $x_\alpha \in S_\alpha$ $(\alpha \in A)$.*

PROOF: Suppose first that each x in L has a unique representation of the indicated form. If $x \leq y = \vee_\alpha y_\alpha$ then $y = x \vee y = (\vee_\alpha x_\alpha) \vee (\vee_\alpha y_\alpha) = \vee_\alpha (x_\alpha \vee y_\alpha) = \vee_\alpha y_\alpha$, and by uniqueness of the representation for y, $y_\alpha = y_\alpha \vee x_\alpha \geq x_\alpha$ for each index α. If on the other hand $x_\alpha \leq y_\alpha$ for every α in A then x is clearly a subelement of y. It follows from this that if $x = \vee_\alpha x_\alpha$ and $y = \vee_\alpha y_\alpha$, then $x \vee y = \vee_\alpha (x_\alpha \vee y_\alpha)$ and $x \wedge y = \vee_\alpha (x_\alpha \wedge y_\alpha)$. Using this fact it is now easy to show that if $x \in S_\alpha$, $y \in S_\beta$ with $\alpha \neq \beta$ then $x \nabla y$. Hence $L = \sum^*(\oplus S_\alpha \colon \alpha \in A)$. The converse implication can be found in [5], Lemma 2.2, p. 88.

Lemma 3. *The center of $I(L)$ coincides with the center of $K(L)$.*

Proof: Let S be central in $I(L)$ with T as its complement. By Theorem 1, $L = S \oplus T$ and by [5], Theorem 1.1, p. 86, the mapping $(x, y) \rightarrow x \vee y$ is an isomorphism of $S \times T$ onto L. It follows from this that S and T are complete ideals of L. By Theorem 1, S is central in $I(L)$, so for each ideal I of L we have

$$I = (I \cap S) + (I \cap T) = (I + S) \cap (I + T). \tag{1}$$

By [1], every polynomial identity valid in $I(L)$ is also valid in $K(L)$. It follows that for every complete ideal I,

$$I = (I \cap S) \vee (I \cap T) = (I \vee S) \cap (I \vee T). \tag{2}$$

By [3], Theorem 7.2, p. 299, S is central in $K(L)$.

Suppose conversely that S is central in $K(L)$, and let T be its complement. In view of Theorem 1, if we can show that $L = S \oplus T$, it will follow that S is central in $I(L)$. Given x in L. If we let $J(a) = J(x) \cap S$ and $J(b) = J(x) \cap T$ we have

$$\begin{aligned} J(x) &= J(x) \cap (S \vee T) = (J(x) \cap S) \vee (J(x) \cap T) \\ &= J(a) \vee J(b) = J(a \vee b). \end{aligned}$$

This shows that $x = a \vee b$ with a in S and b in T. Uniqueness of the representation follows from the fact that if $x = c \vee d$ with $c \in S$ and $d \in T$ then

$$S \cap J(x) = S \cap (J(c) \vee J(\mathrm{d})) = (S \cap J(c)) \vee (S \cap J(d)) = J(c).$$

Similarly, $T \cap J(x) = J(d)$. By Lemma 2, $L = S \oplus T$.

Theorem 4. *Let $\{S_\alpha \colon \alpha \in A\}$ be a family of ideals of L. A necessary and sufficient condition that $L = \sum^*(\oplus S_\alpha \colon \alpha \in A)$ is that $\{S_\alpha \colon \alpha \in A\}$ be a family of pairwise desjoint central elements of $K(L)$ whose supremum in $K(L)$ is L.*

Proof: Let $L = \sum^*(\oplus S_\alpha \colon \alpha \in A)$. By Lemma 2 each x in L can be represented uniquely in the form $x = \vee_\alpha x_\alpha$ with $x_\alpha \in S_\alpha$ $(\alpha \in A)$. Fix an index β and let $T = \{x \in L \colon x = \vee_\alpha x_\alpha \text{ with } x_\alpha \in S_\alpha,\ x_\beta = 0\}$. Then each x in L can be represented in the form $x=x_1\vee x_2$ with $x_1\in S_\beta$ and $x_2\in T$. Suppose also $x=y_1\vee y_2$ with $y_1\in S_\beta$ and $y_2\in T$. Then write $y_2=\vee_\alpha y_\alpha$ with $y_\alpha\in S_\alpha$ and $y_\beta=0$ and observe that $x=\vee_\alpha x_\alpha=y_1\vee(\vee_\alpha y_\alpha)$. By uniqueness of the representation for x, it follows that $x_1=y_1$ and $x_\alpha=y_\alpha$ for $\alpha\neq\beta$, so $x_2=y_2$. By Lemma 2, $L=S_\beta\oplus T$, so by Theorem 1, each S_β is a central element of $K(L)$. Clearly $\alpha\neq\beta$ implies $S_\alpha\cap S_\beta=(0)$ and the fact that each x can be represented as a join of elements from $\cup_\alpha S_\alpha$ shows that $\vee_\alpha S_\alpha=L$ in $K(L)$.

Suppose conversely that the ideals S_α ($\alpha\in A$) are pairwise disjoint central elements of $K(L)$ whose join in $K(L)$ is L. Since by [1], $K(L)$ is upper continuous, we may apply [6], Hilfssatz 3.6, p. 29, to conclude that for each x in L,

$$J(x)=J(x)\cap(\vee_\alpha S_\alpha)=\vee_\alpha(S_\alpha\cap J(x))=\vee_\alpha J(x_\alpha)$$

where $J(x_\alpha)=J(x)\cap S_\alpha$. It follows that $x=\vee_\alpha x_\alpha$ in L. If $x\in S_\alpha$, $y\in S_\beta$ with $\alpha\neq\beta$, the fact that S_α and S_β are disjoint central elements of $K(L)$ will now imply that $x\bigtriangledown y$. Therefore $L=\sum^*(\oplus S_\alpha\colon \alpha\in A)$.

3. The modular case

If L is a conditionally upper continuous relatively complemented modular lattice with 0, then by [1], $K(L)$ is an upper continuous modular lattice and by [8], Satz 1.4, p. 5, it is also complemented. Note now that by [4], Theorem 16, p. 9, the center of $K(L)$ is precisely the set of normal ideals of L. By [7], Theorem 9.1, p. 395, $K(L)$ can be equipped with a dimension function. The restriction of this function to L (via the embedding $x\to J(x)$) now provides L with a dimension function of the type described in [5].

One can obtain a concrete example of this situation by thinking of L as being the lattice of finite dimensional subspaces of an infinite dimensional vector space V. It is easily seen that $K(L)$ is isomorphic to the lattice of all subspaces of V. Notice that L is a general continuous geometry but that $K(L)$ is *not* lower continuous. This shows that one cannot expect $K(L)$ to be a continuous geometry, even if L is a general continuous geometry.

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