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# *Duality Theorems in Mathematical Programmings and Their Applications*

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## § 1. Introduction

Extensions of the classical duality theorem in linear programming have been investigated by many authors. We shall be particularly concerned with the results of K. S. Kretschmer [11], K. Isii [9] and M. Ohtsuka [13]. In [11] the program was discussed in paired spaces and the dimension of the classical program was generalized from finite to infinite. A convex program in paired spaces was studied in [9]. The program in [13] is a potential-theoretic generalization of the classical one and is closely related to the theory of capacities.

In this paper, we shall investigate duality theorems and their applications. We reformulate the program in [13] in a form as in [9] and [11] and discuss Ohtsuka's duality theorem and sufficient conditions in it. Then we see that many results in [13] may be interpreted as special cases of those in [11]. We also obtain a new duality theorem in §5 which is a converse of Kretschmer's Theorem 3 and Dieter's strong duality theorem in [4]. Ohtsuka's sufficient conditions are given in a more general form in §6. Those conditions are free from potential theory. We also give some criteria which are different from Kretschmer's. In §§7-11, we indicate how the theory is applied to the potential-theoretic problems treated in [6], [8], [13], [14] and [16].

## § 2. Preliminaries

It is assumed that the reader is familiar with the theory of linear topological spaces as developed in [1] and [2]. The set of real numbers are denoted by $R$ and the set of non-negative real numbers by $R_0$. Let $X$ and $Y$ be linear spaces (over $R$) and $((\ ,\ ))$ be a bilinear functional on $X\times Y$. We say that $X$ and $Y$ are paired under $((\ ,\ ))$ or that $X$ and $Y$ are in duality (relative to $((\ ,\ ))$) if the following two conditions are fulfilled:

(i) For any $x\neq 0$, there exists $y\in Y$ such that $((x, y))\neq 0$.

(ii) For any $y\neq 0$, there exists $x\in X$ such that $((x, y))\neq 0$.

If the linear spaces $X$ and $Y$ are in duality, the weak topology on $X$ is denoted by $w(X, Y)$ and the Mackey topology on $X$ is denoted by $s(X, Y)$. For a cone $C$ in $X$, we set

$$C^{+} = \{y;\ y \in Y \text{ and } ((x,\ y)) \geqq 0 \text{ for all } x \in C\},$$

$$C^{++} = \{x;\ x \in X \text{ and } ((x,\ y)) \geqq 0 \text{ for all } y \in C^{+}\}.$$

The following two lemmas were proved in [2] and [11].

LEMMA 1. *Let $X$ and $Y$ be linear spaces paired under $((\ ,\ ))$ and $C$ be a convex cone in $X$. If $\mathscr{C}_1$ ($\mathscr{C}_2$ resp.) is any topology on $Y$ ($X$ resp.) which is compatible with the duality between $X$ and $Y$, then $C^{+}$ is $\mathscr{C}_1$-closed and $C^{++}$ coincides with the $\mathscr{C}_2$-closure of $C$.*

LEMMA 2. *Let $X$ and $Y$ be linear spaces paired under $((\ ,\ ))_1$, let $Z$ and $W$ be linear spaces paired under $((\ ,\ ))_2$, and suppose that $T$ is a linear transformation from $X$ into $Z$. In order that $T$ be $w(X, Y)-w(Z, W)$ continuous, it is necessary and sufficient that there exists a transformation $T^*$ from $W$ into $Y$ such that*

$$((Tx,\ w))_2 = ((x,\ T^*w))_1$$

*for all $x \in X$ and $w \in W$. If $T^*$ exists, then it is unique and $w(W, Z)-w(Y, X)$ continuous.*

We call $T^*$ the dual transformation of $T$.

## § 3. Convex program

Let $X$ be a linear space and $Z$ and $W$ be linear spaces paired under $((\ ,\ ))_2$. A convex program is a quartet $(\psi, \varphi, C, Q)$; in this quartet, $C$ is a convex set in $X$, $Q$ is a convex cone in $Z$, $\psi$ is a transformation from $C$ into $Z$ which is convex with respect to $Q$, i.e.,

$$t\psi(x_1) + (1-t)\psi(x_2) - \psi(tx_1 + (1-t)x_2) \in Q$$

for any $x_1, x_2 \in C$ and any real number $t \in (0, 1)$, and $\varphi$ is a real-valued convex function defined on $C$. The convex program is said to be *consistent* if there exists $x \in C$ such that $\psi(x) \in Q$. The *value* $N$ of the convex program is defined by

$$N = \inf\{\varphi(x);\ x \in C \quad \text{and} \quad \psi(x) \in Q\}$$

in case the convex program is consistent, and by

$$N = \infty$$

in case the convex program is not consistent.

As a dual quantity, we consider

$$N' = \sup[\inf\{\varphi(x) - ((\psi(x),\ w))_2;\ x \in C\};\ w \in Q^{+}].$$

We have easily

THEOREM 1. *It is always valid that* $N' \leqq N$.

PROOF. We may assume that the convex program is consistent. For any $w \in Q^+$, we observe that

$$N \geqq \inf\{\varphi(x) - ((\psi(x), w))_2;\ x \in C \text{ and } \psi(x) \in Q\}$$
$$\geqq \inf\{\varphi(x) - ((\psi(x), w))_2;\ x \in C\}.$$

By taking the supremum of the last quantity with respect to $w \in Q^+$, we obtain $N \geqq N'$.

Let $Z \times R$ and $W \times R$ be paired under the bilinear functional $((\ ,\ ))$ defined by $(((z, r), (w, s))) = ((z, w))_2 + rs$. Let $U$ be the set in $Z \times R$ defined by

$$U = \{(\psi(x) - z,\ \varphi(x) + r);\ x \in C,\ z \in Q \text{ and } r \in R_0\}.$$

We shall prove

THEOREM 2. *Let the convex program be consistent and have a finite value* $N$. *If* $U$ *is* $w(Z \times R, W \times R)$*-closed, then the equality* $N = N'$ *holds.*

PROOF. By our assumption that the convex program is consistent and that $N$ is finite, it is clear that $(0, N+\varepsilon) \in U$ and $(0, N-\varepsilon) \notin U$ for any number $\varepsilon > 0$. Let $\varepsilon > 0$ be arbitrarily fixed. Since $U$ is a $w(Z \times R, W \times R)$-closed convex set, by a well-known separation theorem, there exist $(w, s) \in W \times R$ and $\alpha \in R$ which satisfy the relation

$$(((0, N-\varepsilon), (w, s))) < \alpha \leqq (((z, r), (w, s)))$$

for all $(z, r) \in U$ ([1], p. 73, Proposition 4; [2], p. 50, Proposition 1). Since $(0, N+\varepsilon) \in U$, $2\varepsilon s > 0$ and hence $s > 0$. Therefore we may assume $s = 1$. Thus we have

$$N - \varepsilon < \alpha \leqq ((z, w))_2 + r$$

for all $(z, r) \in U$. Let us prove $-w \in Q^+$. If we suppose the contrary, then there exists $z_1 \in Q$ such that $((z_1, w))_2 > 0$. For any number $r$ such that $r > N$ and any positive number $t$, $(-tz_1, r)$ belongs to $U$. In fact, there is $x_1 \in C$ such that $\psi(x_1) \in Q$ and $\varphi(x_1) < r$. Then $r_1 = r - \varphi(x_1) > 0$ and

$$(-tz_1, r) = (\psi(x_1) - (\psi(x_1) + tz_1),\ \varphi(x_1) + r_1) \in U.$$

Thus we have

$$-\infty < \alpha \leqq -t((z_1, w))_2 + r.$$

Letting $t \to \infty$, we arrive at a contradiction. Thus $-w \in Q^+$. From the fact that $U$ contains the set $\{(\psi(x), \varphi(x));\ x \in C\}$, it follows that

$$N-\varepsilon<\alpha\leqq((\psi(x),\ w))_2+\varphi(x)$$

for all $x\in C$. Therefore we have

$$N-\varepsilon<\alpha\leqq\inf\{\varphi(x)+((\psi(x),\ w))_2;\ x\in C\}\leqq N'.$$

By the arbitrariness of $\varepsilon$, we conclude $N\leqq N'$. The converse inequality was given in Theorem 1.

THEOREM 3. *Let the convex program have a finite value $N$. If $U$ has a non-empty $s(Z\times R,\ W\times R)$-interior $U^{\circ}$ and if $0$ belongs to the $s(Z,\ W)$-interior $(\psi(C)-Q)^{\circ}$, then the equality $N=N'$ holds. In this case there exists $w\in Q^{+}$ such that*

$$N'=\inf\{\varphi(x)-((\psi(x),\ w))_2;\ x\in C\}.$$

PROOF. Since $(0,\ N)$ is a boundary point of the convex set $U$ which has a non-empty $s(Z\times R,\ W\times R)$-interior, it follows from another separation theorem ([1], p. 71, Proposition 1) that there exists a non-zero $(w,\ s)\in W\times R$ such that

$$Ns\leqq((z,\ w))_2+rs$$

for all $(z,\ r)\in U$. Since $(0,\ N+\varepsilon)\in U$ for $\varepsilon>0$, we have $s\geqq 0$. We show that $s>0$. If $s=0$, then $((z,\ w))_2\geqq 0$ for all $z\in\psi(C)-Q$. By the assumption $0\in(\psi(C)-Q)^{\circ}$, we have $w=0$, which is a contradiction. Therefore we may assume $s=1$. The rest of the proof is carried out by the same argument as in the proof of Theorem 2.

Isii [9] proved

PROPOSITION 1. *Any one of the following conditions assures that $U$ has a non-empty $s(Z\times R,\ W\times R)$-interior and $0\in(\psi(C)-Q)^{\circ}$:*

(A) *$Q^{\circ}\neq\phi$ and $0\in(\psi(C)-Q)^{\circ}$, where $\phi$ is the empty set,*

(B) *There exist an $s(Z,\ W)$-neighborhood $V$ of $0$ in $Z$ and a constant $k$ such that $h(z)>k$ in $V$, where $h(z)$ is defined by*

$$h(z)=\inf\{\varphi(x);\ x\in C \text{ and } \psi(x)-z\in Q\}.$$

Note that condition (B) is closely related to Rockafellar's stability condition in [15]. We remark here that Rockafellar's method is also useful in our case and that a duality theorem of his type is valid.

In the rest of this section, we assume that $C$ is a convex cone $P$, that $\varphi$ is a convex function which is positively homogeneous (of order 1) and that $\psi(x)=Ax-z_0$, where $A$ is a linear transformation from $X$ into $Z$ and $z_0$ is an element of $Z$.

LEMMA 3. *If $N'>-\infty$, then we have*

$$N'=\sup\{((z_0,\ w))_2;\ w\in Q^{+} \text{ and } \varphi(x)\geqq((Ax,\ w))_2 \text{ for all } x\in P\}.$$

PROOF. For a fixed $w\in Q^{+}$, we have

$$\inf\{\varphi(x)-((\psi(x), w))_2;\ x\in P\}$$
$$=\inf\{\varphi(x)-((Ax, w))_2;\ x\in P\}+((z_0, w))_2.$$

If $\varphi(x)-((Ax, w))_2<0$ for some $x\in P$, then

$$\varphi(tx)-((A(tx), w))_2=t[\varphi(x)-((Ax, w))_2]$$

for any positive number $t$. Since $P$ is a convex cone, we have $\inf\{\varphi(x)-((Ax, w))_2;\ x\in P\}=-\infty$. Such $w$ can be neglected in the calculation of $N'$. Hence we can restrict $w\in Q^+$ to those satisfying $\varphi(x)-((Ax, w))_2\geqq 0$ for all $x\in P$.

Since $A(P)$ and $Q$ are convex cones, we see that

$$(A(P)-Q)^\circ\subset A(P)-Q^\circ$$

if the $s(Z, W)$-interior $Q^\circ$ is non-empty ([11], Lemma 1). Thus condition (A) implies

$$Q^\circ\neq\phi\quad\text{and}\quad z_0\in A(P)-Q^\circ.$$

Making use of Theorem 3, Proposition 1 and Lemma 3, we have

THEOREM 4. *Assume that $N$ is finite and that there exists $x\in P$ such that $Ax-z_0\in Q^\circ$. Then we have*

$$\inf\{\varphi(x);\ x\in P\text{ and }Ax-z_0\in Q\}$$
$$=\max\{((z_0, w))_2;\ w\in Q^+\text{ and }((Ax, w))_2\leqq\varphi(x)\text{ for all }x\in P\}.$$

We shall prove

THEOREM 5. *If $N'$ is finite and $z_0\in Q^\circ$, then there exists $x\in P$ such that $Ax-z_0\in Q^\circ$.*

PROOF. Obviously $Q^\circ$ and $A(P)-z_0$ are non-empty convex sets. We show $(A(P)-z_0)\cap Q^\circ\neq\phi$. If we suppose the contrary, then we see by a separation theorem that there exist non-zero $w_1\in W$ and $\alpha\in R$ such that

$$\sup\{((z, w_1))_2;\ z\in A(P)-z_0\}\leqq\alpha\leqq\inf\{((z, w_1))_2;\ z\in Q^\circ\}.$$

Making use of the fact that the $w(Z, W)$-closure of $Q$ is equal to the $s(Z, W)$-closure of $Q^\circ$ ([1], p. 50, Corollaire 2 and [2], p. 67, Proposition 4), we have $((z, w_1))_2\geqq\alpha$ for all $z\in Q$. Since $Q$ is a cone, $w_1\in Q^+$ and hence we may take $\alpha=0$. From the relation $((Ax, w_1))_2\leqq((z_0, w_1))_2$ for all $x\in P$ and the fact that $P$ is a cone, it follows that $((Ax, w_1))_2\leqq 0$ for all $x\in P$. Let $w$ be an element of $Q^+$ such that $\varphi(x)\geqq((Ax, w))_2$ for all $x\in P$. Then $w+tw_1\in Q^+$ and $\varphi(x)\geqq((Ax, w+tw_1))_2$ for any $t\in R_0$ and for all $x\in P$. On the other hand, since $z_0\in Q^\circ$, $w_1\in Q^+$ and $w_1\neq 0$, we see that $((z_0, w_1))_2>0$ and

$$N' \geqq ((z_0, w+tw_1))_2 = ((z_0, w))_2 + t((z_0, w_1))_2$$

by Lemma 3. Letting $t \to \infty$, we have $N' = \infty$, which contradicts our assumption. Thus $(A(P) - z_0) \cap Q^\circ \neq \phi$.

## § 4. Linear programs in paired spaces

Let $X$ and $Y$ be linear spaces paired under $((\ ,\ ))_1$ and $Z$ and $W$ be linear spaces paired under $((\ ,\ ))_2$. A linear program for these paired spaces is a quintuple $(A, P, Q, y_0, z_0)$. In this quintuple, $A$ is a linear transformation from $X$ into $Z$ which is $w(X, Y) - w(Z, W)$ continuous, $P$ is a convex cone in $X$ which is $w(X, Y)$-closed, $Q$ is a convex cone in $Z$ which is $w(Z, W)$-closed, $y_0$ is an element of $Y$, and $z_0$ is an element of $Z$. In the rest of this paper, a program will always be a linear program unless otherwise stated. The program is said to be *consistent* if there exists $x \in P$ such that $Ax - z_0 \in Q$. Such an $x$ is called *feasible*. The *value* $M$ of the program is defined by

$$M = \inf \{((x, y_0))_1 ;\ x \in P \text{ and } Ax - z_0 \in Q\}$$

in case the program is consistent, and by

$$M = \infty$$

in case the program is not consistent. The program is said to be *convergent* if it is consistent, has a finite value and there is a feasible $x$ such that $((x, y_0))_1 = M$.

The *dual program* is the program $(A^*, Q^+, -P^+, -z_0, y_0)$ for $W$ and $Z$ paired under ${}_2((\ ,\ ))$ and for $Y$ and $X$ paired under ${}_1((\ ,\ ))$. The bilinear functionals ${}_2((\ ,\ ))$ and ${}_1((\ ,\ ))$ are defined by ${}_2((w, z)) = ((z, w))_2$ for all $w \in W$ and $z \in Z$ and ${}_1((y, x)) = ((x, y))_1$ for all $y \in Y$ and $x \in X$. The dual transformation $A^*$ is determined by Lemma 2. It is easily seen that the dual program is well-defined. The value of the dual program is denoted by $M'$. The dual program is consistent if and only if there exists $w \in Q^+$ such that $y_0 - A^*w \in P^+$. In this case, we have

$$M' = -\sup\{((z_0, w))_2 ;\ w \in Q^+ \text{ and } y_0 - A^*w \in P^+\}.$$

We have easily

THEOREM 6.[1] (a) *It is always valid that* $-M' \leqq M$.
(b) *The dual of the dual program is equal to the program itself.*

PROOF. (a) By our convention, it suffices to show the inequality in the case where both the program and the dual program are consistent. Let $x$ and $w$ fulfill the following conditions: $x \in P$, $Ax - z_0 \in Q$; $w \in Q^+$, $y_0 - A^*w \in P^+$.

1) [11], Theorem 1, p. 225.

Then we have

$$((x,\ y_0))_1 \geqq ((x,\ A^*w))_1 = ((Ax,\ w))_2 \geqq ((z_0,\ w))_2.$$

The proof of (b) is easily seen.

REMARK 1. By taking $\psi(x) = Ax - z_0$ and $\varphi(x) = ((x,\ y_0))_1$, we see that the linear program $(A, P, Q, y_0, z_0)$ is the convex program $(\psi, \varphi, P, Q)$ in §3 and $M = N$. Noting that $((x,\ y_0))_1 \geqq ((Ax,\ w))_2 = ((x,\ A^*w))_1$ for all $x \in P$ is equivalent to $y_0 - A^*w \in P^+$, we have by Lemma 3 that $-M' = N'$.

## § 5. Duality theorems

Let $X \times R$ and $Y \times R$ be paired under the bilinear functional $((\ ,\ ))$ defined by $(((x, r), (y, s))) = ((x,\ y))_1 + rs$. Let $G$ be the set in $Y \times R$ defined by

$$G = \{(A^*w + y,\ r - ((z_0,\ w))_2);\ y \in P^+,\ w \in Q^+ \text{ and } r \in R_0\}.$$

Kretschmer proved

THEOREM 7.[2] *Let the program $(A, P, Q, y_0, z_0)$ be consistent and have a finite value $M$. If $G$ is $w(Y \times R,\ X \times R)$-closed, then the dual program $(A^*, Q^+, -P^+, -z_0, y_0)$ is convergent and has $-M$ as its value.*

We shall apply Theorem 2 in §3 to the present case. Let $Z \times R$ and $W \times R$ be paired under the bilinear functional $((\ ,\ ))$ defined by $(((z, r), (w, s))) = ((z,\ w))_2 + rs$. Let $H$ be the set in $Z \times R$ defined by

$$H = \{(Ax - z,\ r + ((x,\ y_0))_1);\ x \in P,\ z \in Q \text{ and } r \in R_0\}.$$

By Remark 1, the set $U$ in §3 can be written as follows:

$$\begin{aligned} U &= \{(Ax - z - z_0,\ ((x,\ y_0))_1 + r);\ x \in P,\ z \in Q \text{ and } r \in R_0\} \\ &= H - (z_0, 0). \end{aligned}$$

On account of Theorem 2 and Remark 1, we have

THEOREM 8. *Let the program $(A, P, Q, y_0, z_0)$ be consistent and have a finite value $M$. If $H$ is $w(Z \times R,\ W \times R)$-closed, then the dual program $(A^*, Q^+, -P^+, -z_0, y_0)$ is consistent and has $-M$ as its value.*

This theorem seems to be new in the theory of linear programs.

By means of Theorem 6 (b), we have the dual statements of the above theorems:

THEOREM 7*. *Let the dual program $(A^*, Q^+, -P^+, -z_0, y_0)$ be consistent*

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2) [11], Theorem 3, p. 226. Cf. [4], p. 110.

*and have a finite value $M'$. If the set $H$ is $w(Z\times R,\ W\times R)$-closed, then the program $(A, P, Q, y_0, z_0)$ is convergent and has $-M'$ as its value.*

THEOREM 8*. *Let the dual program $(A^*, Q^+, -P^+, -z_0, y_0)$ be consistent and have a finite value $M'$. If $G$ is $w(Y\times R,\ X\times R)$-closed, then the program $(A, P, Q, y_0, z_0)$ is consistent and has $-M'$ as its value.*

Combining Theorem 7 with Theorem 8*, we have

THEOREM 9. *Let the dual program $(A^*, Q^+, -P^+, -z_0, y_0)$ be consistent and have a finite value $M'$. If $G$ is $w(Y\times R,\ X\times R)$-closed, then the dual program is convergent.*

Similarly we have

THEOREM 9*. *Let the program $(A, P, Q, y_0, z_0)$ be consistent and have a finite value $M$. If $H$ is $w(Z\times R,\ W\times R)$-closed, then the program is convergent.*

REMARK 2. The condition that $G$ is $w(Y\times R,\ X\times R)$-closed does not necessarily imply that $H$ is $w(Z\times R,\ W\times R)$-closed. This will be shown by Example 1 in §7.

## § 6. Sufficient conditions

When one intends to make use of duality theorems in the preceding section, one may pose the following problems:

When is the set $G$ $w(Y\times R,\ X\times R)$-closed?

When is the set $H$ $w(Z\times R,\ W\times R)$-closed?

In order to study these problems, we define condition (K) and condition (K*) as follows:

(K) *$Q$ has a non-empty $s(Z, W)$-interior $Q^\circ$ and there exists $x\in P$ such that $Ax-z_0\in Q^\circ$.*

(K*) *$P^+$ has a non-empty $s(Y, X)$-interior $(P^+)^\circ$ and there exists $w\in Q^+$ such that $y_0-A^*w\in(P^+)^\circ$.*

Kretschmer gave the following useful criteria.

THEOREM 10.[3] *If condition* (K) *is fulfilled, then $G$ is $w(Y\times R,\ X\times R)$-closed.*

THEOREM 10*. *If condition* (K*) *is fulfilled, then $H$ is $w(Z\times R,\ W\times R)$-closed.*

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3) [11], Lemma 5 and Theorem 3. It should be observed that the proof of Lemma 5 in [11] is not complete in case $((x_0, y))_1=0$ for all $x_0\in P\cap T^{-1}(Q^\circ)$ (p. 223, *l*. 11 from below). In this case by taking $x'_\alpha=((Tx_0, w))_2x_\alpha+x_0$ and $z'_\alpha=((Tx_0, w))_2z_\alpha$ with $x_0\in P\cap T^{-1}(Q^\circ)$ and $w\in Q^+$, $w\neq 0$, we can complete the proof.

For simplicity, we denote by $\Gamma$ and $\Gamma^*$ the totalities of feasible elements of the program and the dual program respectively:

$$\Gamma = \{x \in P;\ Ax - z_0 \in Q\},\ \Gamma^* = \{w \in Q^+;\ y_0 - A^*w \in P^+\}.$$

We have

PROPOSITION 2. *Any one of the following conditions* (C. 1) *and* (C. 2) *implies condition* (K):

(C. 1) $-z_0 \in Q^\circ$,

(C. 2) *there is* $x_1 \in P$ *such that* $Ax_1 \in Q^\circ$.

PROOF. Condition (C. 1) obviously implies condition (K). Assume condition (C. 2). Then there exists a number $t>0$ such that $Ax_1 - tz_0 \in Q^\circ$. Taking $x_2 = x_1/t$, we see $x_2 \in P$ and $Ax_2 - z_0 = (Ax_1 - tz_0)/t \in Q^\circ$ ([1], p. 51, Corollaire 2).

COROLLARY. *If* $z_0 \in Q^\circ$ *and* $\Gamma \neq \phi$, *then condition* (K) *is fulfilled.*

PROOF. Since $\Gamma \neq \phi$, there exists $x_1 \in P$ such that $Ax_1 - z_0 \in Q$. We have

$$A(x_1/2) = (Ax_1 - z_0)/2 + z_0/2 \in Q^\circ$$

([1], p. 51, Proposition 15). Thus (C. 2) is fulfilled.

By Theorem 5, we have

PROPOSITION 3. *The following condition* (C. 3) *implies condition* (K):

(C. 3) $\quad \Gamma^* \neq \phi,\ -\infty < M' < \infty$ *and* $z_0 \in Q^\circ$.

REMARK 3. In general, the condition $z_0 \in Q^\circ$ is not enough to ensure condition (K). This will be shown in Example 2 in §7.

For a locally convex Hausdorff topological linear space $(E, \mathscr{C})$, we denote the strong dual of $(E, \mathscr{C})$ by $E^*$. It is clear that $E$ and $E^*$ are paired under the bilinear functional $((\ ,\ ))_1$ defined by $((e, e^*))_1 = e^*(e)$ and that $E^*$ and $E$ are paired under the bilinear functional ${}_1((\ ,\ ))$ defined by ${}_1((e^*, e)) = e^*(e)$.

Observe that $s(E, E^*) = \mathscr{C}$ if $(E, \mathscr{C})$ is a disk space[4] ([2], p. 70, Proposition 5) or metrizable ([2], p. 71, Proposition 6).

We shall prove

PROPOSITION 4. *Assume that* $Z$ *is a disk space and* $W = Z^*$. *If* $\Gamma^*$ *is a non-empty* $w(W, Z)$*-compact set and* $Q^\circ \neq \phi$, *then condition* (K) *is valid.*

PROOF. If condition (K) is not valid, then we see by the same argument as in the proof of Theorem 5 that there exists $w_1 \in Q^+$ such that $w_1 \neq 0$ and $((x, A^*w_1))_1 = ((Ax, w_1))_2 \leqq 0$ for all $x \in P$. Hence $-A^*w_1 \in P^+$. Taking $w \in \Gamma^*$,

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we have $w+tw_1\in\Gamma^*$ for all $t\in R_0$. Since $Q^\circ\neq\phi$, this contradicts our assumption that $\Gamma^*$ is $w(W, Z)$-compact.

We have some criteria of another type. In case $Q^+=\{0\}$, the equality $G=P^+\times R_0$ holds and hence $G$ is $w(Y\times R, X\times R)$-closed. In the rest of this section, we always assume that $Q^+\neq\{0\}$.

PROPOSITION 5. *Let $Z$ be a normed space, let $W=Z^*$, and assume that* $\{w; w\in Q^+$ and $\|w\|=1\}$[5)] *is $w(W, Z)$-compact. If $\Gamma^*$ is non-empty and $w(W, Z)$-compact, then $G$ is $w(Y\times R, X\times R)$-closed.*

PROOF. Let $\{(y_\alpha, r_\alpha); \alpha\in D\}$ be a net in $G$ which $w(Y\times R, X\times R)$-converges to $(y, r)\in Y\times R$. We prove $(y, r)\in G$. There exists $w_\alpha\in Q^+$ such that $y_\alpha-A^*w_\alpha\in P^+$ and $((z_0, w_\alpha))_2\geqq -r_\alpha$. Then there exists a subnet $\{w_\alpha; \alpha\in D_0\}$ such that $\{\|w_\alpha\|; \alpha\in D_0\}$ is bounded. In fact, if we suppose the contrary, then there exists a subnet $\{w_\alpha; \alpha\in D'\}$ such that $\|w_\alpha\|\to\infty$ along $D'$. We set $w'_\alpha=w_\alpha/\|w_\alpha\|$ and choose a $w(W, Z)$-convergent subnet of $\{w'_\alpha; \alpha\in D'\}$. We shall denote it again by $\{w'_\alpha; \alpha\in D'\}$ and let $w'_0$ be the limit. Then for any $x\in P$, we have

$$((x, A^*w'_0))_1=\lim_{D'}((x, A^*w'_\alpha))_1=\lim_{D'}((x, A^*w_\alpha))_1/\|w_\alpha\|$$
$$\leqq\lim_{D'}((x, y_\alpha))_1/\|w_\alpha\|=0,$$

and hence $-A^*w'_0\in P^+$. Take $w\in\Gamma^*$. Then we see easily that $w+tw'_0\in\Gamma^*$ for all $t\in R_0$. Since $\|w'_0\|=1$ and $w'_0\in Q^+$, there is $z\in Z$ such that $((z, w'_0))_2>0$. Therefore we have

$$((z, w+tw'_0))_2=((z, w))_2+t((z, w'_0))_2\to\infty \quad \text{as} \quad t\to\infty,$$

which means that $\Gamma^*$ is not $w(W, Z)$-bounded. This is a contradiction ([2], p. 65, Théorème 1). We choose a subnet $\{w_\alpha; \alpha\in D'\}$ of $\{w_\alpha; \alpha\in D_0\}$ which $w(W, Z)$-converges to $w\in Q^+$. Then it follows that

$$((z_0, w))_2=\lim_{D'}((z_0, w_\alpha))_2\geqq\lim_{D'}(-r_\alpha)=-r$$

and

$$((x, y-A^*w))_1=\lim_{D'}((x, y_\alpha))_1-\lim_{D'}((Ax, w_\alpha))_2$$
$$=\lim_{D'}((x, y_\alpha-A^*w_\alpha))_1\geqq 0$$

for all $x\in P$. Consequently $y-A^*w\in P^+$ and hence $(y, r)\in G$.

We define condition (C. 4) as follows:

5) For $w\in W=Z^*$, $\|w\|$ is defined by $\sup\{|((z, w))_2|; z\in Z$ and $\|z\|=1\}$.

(C. 4) $\qquad \Gamma^* \neq \phi,\ -\infty < M' < \infty,\ P \neq \{0\}\quad$ *and* $\quad -A^*w \in (P^+)^\circ$

*for all* $w \in Q^+$, $w \neq 0$.

We shall prove

PROPOSITION 6. *Assume that* $Y$ *and* $Z$ *are normed spaces, that* $X = Y^*$ *and* $W = Z^*$ *and that* $\{x;\ x \in P$ *and* $\|x\| = 1\}$ *and* $\{w;\ w \in Q^+$ *and* $\|w\| = 1\}$ *are* $w(X, Y)$*-compact and* $w(W, Z)$*-compact respectively. If we further assume condition* (C. 4), *then* $G$ *is* $w(Y \times R, X \times R)$*-closed.*

REMARK 4. Condition (C. 4) does not necessarily imply condition (K). This will be shown by Example 3 in §7.

We prove the following lemma under the same assumptions as in Proposition 6.

LEMMA 4. *Let* $\bar{w} \in Q^+$ *satisfy* $y - A^*\bar{w} \in P^+$. *Then we can find* $\bar{w}' \in Q^+$ *such that* $\bar{w} - \bar{w}' \in Q^+$, $\|\bar{w}'\| \leqq -\beta(y)/\delta(A)$, $((z_0, \bar{w}'))_2 \geqq ((z_0, \bar{w}))_2$ *and* $y - A^*\bar{w}' \in P^+$, *where* $\delta(A)$ *and* $\beta(y)$ *are given by*

$$\delta(A) = \inf_{\substack{x \in P \\ \|x\|=1}} \inf_{\substack{w \in Q^+ \\ \|w\|=1}} |((Ax, w))_2|, \quad \beta(y) = \min(0, \inf_{\substack{x \in P \\ \|x\|=1}} ((x, y))_1).$$

PROOF. Since $((Ax, w))_2$ is continuous on $\{x \in P;\ \|x\| = 1\} \times \{w \in Q^+;\ \|w\| = 1\}$ which is the product of a $w(X, Y)$-compact set and a $w(W, Z)$-compact set, it is clear by condition (C. 4) that $\delta(A) > 0$. We may assume $\bar{w} \neq 0$. Suppose that there is $w_1 \in Q^+$ such that $((z_0, w_1))_2 > 0$. Taking $w \in \Gamma^*$, we see that $w + tw_1 \in \Gamma^*$ for all $t \in R_0$ and hence

$$-M' \geqq ((z_0, w + tw_1))_2 = ((z_0, w))_2 + t((z_0, w_1))_2.$$

Letting $t \to \infty$, we have $M' = -\infty$, which contradicts our assumption. Therefore $((z_0, w))_2 \leqq 0$ for all $w \in Q^+$ and hence $-z_0 \in Q^{++} = Q$ by Lemma 1. If $y \in P^+$, then $\bar{w}' = 0$ satisfies the conditions. If $y \notin P^+$, then we set

$$\gamma = \sup_{\substack{x \in P \\ \|x\|=1}} ((x, A^*\bar{w}))_1 \quad \text{and} \quad \beta = \inf_{\substack{x \in P \\ \|x\|=1}} ((x, y))_1.$$

It is easily seen that $\beta(y) = \beta \leqq 0$ and

$$-\gamma = \inf_{\substack{x \in P \\ \|x\|=1}} ((x, -A^*\bar{w}))_1 \geqq \delta(A)\|\bar{w}\|.$$

If $\gamma \geqq \beta$, then $\beta \leqq -\delta(A)\|\bar{w}\|$ and $\bar{w}$ itself satisfies the conditions. If $\gamma < \beta$, we consider $\bar{w}' = \bar{w}\beta/\gamma$. We see that $\bar{w} - \bar{w}' \in Q^+$ and $\|\bar{w}'\| \leqq -\beta/\delta(A)$. It is not difficult to verify that $y - A^*\bar{w}' \in P^+$. Since $-z_0 \in Q$, we have $((z_0, \bar{w}))_2 \leqq ((z_0, \bar{w}'))_2$. Thus $\bar{w}'$ satisfies all the requirements.

PROOF OF PROPOSITION 6. Since $G$ is convex, it suffices to show that $G$ is

$s(Y\times R, X\times R)$-closed ([2], p. 67, Proposition 4). Let $\{(y_n, r_n)\}$ be a sequence in $G$ which $s(Y\times R, X\times R)$-converges to $(y, r)\in Y\times R$. Then there exists $\bar{w}_n\in Q^+$ such that $y_n-A^*\bar{w}_n\in P^+$ and $((z_0, \bar{w}_n))_2\geqq -r_n$. For every $\bar{w}_n$, we take $\bar{w}_n'$ by Lemma 4. Since $s(Y, X)$-topology is the topology induced by the norm on $Y$, $\{\beta(y_n)\}$ are bounded. Consequently $\{\|\bar{w}_n'\|\}$ are bounded. Choose a $w(W, Z)$-convergent subsequence, denote it by $\{\bar{w}_n'\}$ again and let $\bar{w}'$ be the limit. Then we have

$$((z_0, \bar{w}'))_2=\lim_{n\to\infty}((z_0, \bar{w}_n'))_2\geqq\lim_{n\to\infty}((z_0, \bar{w}_n))_2$$

$$\geqq\lim_{n\to\infty}(-r_n)=-r,$$

$$((x, A^*\bar{w}'))_1=((Ax, \bar{w}'))_2=\lim_{n\to\infty}((Ax, \bar{w}_n'))_2$$

$$=\lim_{n\to\infty}((x, A^*\bar{w}_n'))_1\leqq\lim_{n\to\infty}((x, y_n))_1=((x, y))_1$$

for all $x\in P$ and hence $y-A^*\bar{w}'\in P^+$. This means $(y, r)\in G$.

In case $Q=\{0\}$, we can not apply some of the above criteria. In this case, we have

PROPOSITION 7. *Let $Y$ and $Z$ be normed spaces, $X=Y^*$ and $W=Z^*$. If $z_0\in A(P)^\circ$, then $G$ is $w(Y\times R, X\times R)$-closed.*

PROOF. It suffices to show that $G$ is $s(Y\times R, X\times R)$-closed. Let $\{(y_n, r_n)\}$ be a sequence in $G$ which $s(Y\times R, X\times R)$-converges to $(y, r)\in Y\times R$. Then there exists $w_n\in Q^+$ such that $y_n-A^*w_n\in P^+$ and $((z_0, w_n))_2\geqq -r_n$. If we prove that $\{w_n\}$ is relatively $w(W, Z)$-compact, then we see $(y, r)\in G$ by the same argument as in the proof of Proposition 6. Let $x_0$ be an element of $P$ such that $Ax_0=z_0$. Then we have

$$-r_n\leqq((z_0, w_n))_2=((Ax_0, w_n))_2=((x_0, A^*w_n))_1\leqq((x_0, y_n))_1$$

and hence $\{((z_0, w_n))_2\}$ are bounded. Since $z_0\in A(P)^\circ$, it is easily seen that $\{((z, w_n))_2\}$ are bounded for every $z\in Z$.

The dual statements of the results in this section are also valid.

## § 7. Potential-theoretic linear program

In this section, we shall study how the preceding theory is applied to the potential-theoretic duality problem treated in [13]. Let $E$ and $F$ be compact Hausdorff spaces, $M(E)$ be the totality of Radon measures of any sign on $E$, $M^+(E)$ be the totality of non-negative Radon measures on $E$, $C(E)$ be the totality of finite real-valued continuous functions on $E$ and $C^+(E)$ be the subset of $C(E)$ which consists of non-negative functions. We use this notation in the rest of this paper.

We set

$$X=M(E),\quad Y=C(E),\quad Z=C(F)\quad \text{and}\quad W=M(F).$$

It is easily seen that $X$ and $Y$ are paired under the bilinear functional $((\ ,\ ))_1$ defined by

$$((\sigma,\ g))_1=\int g d\sigma \qquad \text{for all } \sigma\in X \text{ and } g\in Y,$$

and $Z$ and $W$ are paired under the bilinear functional $((\ ,\ ))_2$ defined by

$$((f,\ \tau))_2=\int f d\tau \qquad \text{for all } f\in Z \text{ and } \tau\in W.$$

Let $\Phi(u,\ v)$ be a continuous kernel, i.e., a finite real-valued continuous function on $E\times F$ and let $A$ be a linear transformation from $X$ into $Z$ given by

$$A\sigma=\Phi(\sigma,\ \cdot)=\int \Phi(u,\ \cdot)d\sigma(u).$$

The linear transformation $A$ is $w(X,\ Y)-w(Z,\ W)$ continuous and $A^*$ is given by

$$A^*\tau=\Phi(\cdot,\ \tau)=\int \Phi(\cdot,\ v)d\tau(v).$$

Let $g_0$ be an element of $C(E)$ and $f_0$ be an element of $C(F)$.
Set

$$P=M^+(E),\quad Q=C^+(F),\quad y_0=g_0\quad \text{and}\quad z_0=f_0.$$

It is clear that $P$ is a $w(X,\ Y)$-closed convex cone in $X$ and that $Q$ is a $w(Z,\ W)$-closed convex cone in $Z$. In this way, the program $(A,\ P,\ Q,\ g_0,\ f_0)$ is well-defined.

Following [13], we introduce two families of measures:

$$\mathscr{M}=\{\mu;\ \mu\in M^+(F) \text{ and } \Phi(u,\ \mu)\leqq g_0(u) \text{ on } E\},$$

$$\mathscr{M}'=\{\nu;\ \nu\in M^+(E) \text{ and } \Phi(\nu,\ v)\geqq f_0(v) \quad \text{on } F\}.$$

Then it is valid that $\mathscr{M}=\Gamma^*$ and $\mathscr{M}'=\Gamma$. The program $(A,\ P,\ Q,\ g_0,\ f_0)$ is consistent if and only if $\mathscr{M}'\neq\phi$. The value of the program is equal to

$$M=M_0'=\inf\left\{\int g_0 d\nu;\ \nu\in\mathscr{M}'\right\}^{6)}.$$

The dual program $(A^*,\ Q^+,\ -P^+,\ -f_0,\ g_0)$ is consistent if and only if $\mathscr{M}\neq\phi$. The value of the dual program is equal to

6) As to notation, note that $g_0$ ($f_0$ resp.) and $M_0'$ ($M_0$ resp.) in this section play the roles of $g$ ($f$ resp.) and $M'$ ($M$ resp.) in [13].

$$-M' = M_0 = \sup \left\{ \int f_0 d\mu;\ \mu \in \mathscr{M} \right\}.$$

The sets $G$ and $H$ in §5 are written as follows:

$$G = \left\{ \left( \Phi(\cdot,\ \mu) + g,\ r - \int f_0 d\mu \right);\ \mu \in M^+(F),\ g \in C^+(E) \text{ and } r \in R_0 \right\},$$

$$H = \left\{ \left( \Phi(\nu,\ \cdot) - f,\ r + \int g_0 d\nu \right);\ \nu \in M^+(E),\ f \in C^+(F) \text{ and } r \in R_0 \right\}.$$

By Theorem 9, there exists $\mu \in \mathscr{M}$ such that $M_0 = \int f_0 d\mu$ provided that $\mathscr{M} \neq \phi$, $-\infty < M_0 < \infty$ and that $G$ is $w(Y \times R,\ X \times R)$-closed. This is a generalization of Theorem 2 in [13].

From Theorems 7* and 8*, we derive an extension of Theorem 3 in [13].

THEOREM 11. *Assume that $\mathscr{M} \neq \phi$ and $-\infty < M_0 < \infty$. If we further assume either that $G$ is $w(Y \times R,\ X \times R)$-closed or that $H$ is $w(Z \times R,\ W \times R)$-closed, then we have $\mathscr{M}' \neq \phi$ and $M_0 = M_0'$.*

Next we are concerned with sufficient conditions given in §6. The topology $s(Z,\ W)$ is the topology induced by the norm on $Z$ defined by $\|f\| = \sup \{ |f(v)|;\ v \in F \}$. Conditions (K) and (K*) may be stated as follows:

(K) There is $\nu \in M^+(E)$ such that $\Phi(\nu,\ \cdot) - f_0 > 0$.

(K*) There is $\mu \in M^+(F)$ such that $g_0 - \Phi(\cdot,\ \mu) > 0$.

In order to complement the remarks in §5 and §6, we give some examples.

EXAMPLE 1.[7] Let $F = \{1\}$ and $E = \{N,\ \omega\}$ be the Alexandroff one point compactification of the discrete space $N$ of all natural numbers. Let $f_0(1) = 1$, $g_0(n) = 1/n^2$, $g_0(\omega) = 0$ and define $\Phi$ by $\Phi(n,\ 1) = 1/n$, $\Phi(\omega,\ 1) = 0$. Then we have $M_0 = M_0' = 0$. We see by Theorem 10 and Proposition 3 that $G$ is $w(Y \times R,\ X \times R)$-closed. Since the program $(A,\ P,\ Q,\ g_0,\ f_0)$ is not convergent, we see by Theorem 9* that $H$ is not $w(Z \times R,\ W \times R)$-closed.

EXAMPLE 2. Let $E$ and $F$ be the compact interval $[0,\ 1]$ in the real line, $\Phi = 0$, $f_0 = 1$ and $g_0 = 1$. Then we see that $\Phi(\nu,\ \cdot) - f_0 = -1 \notin Q^\circ$ for any $\nu \in P$.

EXAMPLE 3. Let $E$ and $F$ be the same as in Example 2, $\Phi = -1$, $f_0 = 0$ and $g_0 = 1$. Then $\mathscr{M}' = \{0\}$, $\mathscr{M} = M^+(F)$ and $\Phi(\nu,\ \cdot) - f_0 = -\nu(E) \notin Q^\circ$ for all $\nu \in M^+(E) = P$.

As in [13], we consider the following conditions relative to $\Phi$, $f_0$ and $g_0$:
(i) $f_0 > 0$ *on* $F$,

7) [17], Example 3.

(ii) *there is* $u_0 \in E$ *such that* $\Phi(u_0, \cdot) > 0$ *on* $F$,

(iii) $f_0 < 0$ *on* $F$,

(iv) $\Phi < 0$ *on* $E \times F$,

(v) $g_0 < 0$ *on* $E$,

(vi) *there is* $v_0 \in F$ *such that* $\Phi(\cdot, v_0) < 0$ *on* $E$,

(vii) $g_0 > 0$ *on* $E$,

(viii) $\Phi > 0$ *on* $E \times F$.

Clearly any one of conditions (ii), (iii) and (viii) implies condition (K), and any one of conditions (iv), (vi) and (vii) implies condition (K*) (Proposition 2). In case $\mathscr{M} \neq \phi$ and $-\infty < M_0 < \infty$, condition (C. 3) is equivalent to condition (i) and hence condition (K) follows from condition (i) by Proposition 3. In case $\mathscr{M} \neq \phi$ and $-\infty < M_0 < \infty$, condition (C. 4) is led by condition (iv) and hence condition (iv) implies that $G$ is $w(Y \times R, X \times R)$-closed (Proposition 6). Condition (K*) is derived from $\mathscr{M} \neq \phi$ and condition (v). This is the dual result of the corollary of Proposition 2.

Thus we have

PROPOSITION 8. *Assume that* $\mathscr{M} \neq \phi$ *and* $-\infty < M_0 < \infty$. *If one of conditions* (i)–(viii) *is satisfied, then* $\mathscr{M}' \neq \phi$ *and* $M_0 = M_0'$.

PROPOSITION 8*. *Assume that* $\mathscr{M}' \neq \phi$ *and* $-\infty < M_0' < \infty$. *If we assume one of conditions* (i)–(viii), *then we have* $\mathscr{M} \neq \phi$ *and* $M_0 = M_0'$.

REMARK 5. In the case where $E = F$, the following condition (ix) also implies condition (K):

(ix) $\Phi \geqq 0$ *on* $E \times F$ *and* $\Phi(u, u) > 0$ *for every* $u \in E$.

This is an immediate consequence of Kishi's existence theorem in [10] and Proposition 2 (cf. §11).

## § 8. Lower semicontinuous kernel

In this section, we extend Proposition 8* in a form similar to Ohtsuka's duality theorem in [13].

Let $\Phi$ be a lower semicontinuous kernel on $E \times F$, i.e., a lower semicontinuous function on $E \times F$ which takes values in $(-\infty, \infty]$. Let $g_0$ be a bounded Borel function on $E$ and $f_0$ an upper semicontinuous function on $F$ which does not take the value $+\infty$. We define $\mathscr{M}$, $\mathscr{M}'$, $M_0$ and $M_0'$ in the same way as in §7. It is easily verified that $M_0 \leqq M_0'$.

We shall prove

THEOREM 12. *Assume that* $\mathscr{M}' \neq \phi$ *and* $-\infty < M_0' < \infty$. *If there exists*

*$\nu_0 \in M^+(E)$ such that $\Phi(\nu_0, v) - f_0(v) > 0$ on $F$, then $\mathscr{M} \neq \phi$ and $M_0' = M_0 = \int f_0 d\mu$ for some $\mu \in \mathscr{M}$.*

PROOF. First we consider the case where $\Phi$ and $f_0$ are continuous. Let $X$, $Z$, $W$, $P$, $Q$, $z_0$, $A$ and $((\ ,\ ))_2$ be the same as in §7. Take for $Y$ the class $B(E)$ of all bounded Borel functions on $E$ and take $g_0$ for $y_0$. Then $X$ and $Y$ are linear spaces paired under the bilinear functional $((\ ,\ ))_1$ defined by

$$((\sigma, g))_1 = \int g d\sigma \qquad \text{for all } \sigma \in X \text{ and } g \in Y.$$

In this way, the program $(A, P, Q, y_0, z_0)$ is well-defined. Since condition (K) is fulfilled by our assumption, our assertion follows from Theorems 7 and 10.

Secondly we consider the case where $f_0$ is continuous but $\Phi$ may not be continuous. Let $D$ be the directed set of continuous functions not greater than $\Phi$. We use the notation $\mathscr{M}_\Psi$, $\mathscr{M}_\Psi'$, $M_\Psi = M_{0\Psi}$ and $M_\Psi' = M_{0\Psi}'$ when $\Psi \in D$ is taken as a kernel. If $\Psi, \Psi' \in D$ and $\Psi \leqq \Psi'$, then $\mathscr{M} \subset \mathscr{M}_{\Psi'} \subset \mathscr{M}_\Psi$ and $\mathscr{M}_\Psi' \subset \mathscr{M}_{\Psi'}' \subset \mathscr{M}'$. Hence $\{M_\Psi\}$ and $\{M_\Psi'\}$ are decreasing along $D$ and $\lim_D M_\Psi' \geqq M_0'$. We can show that there exists $\Psi_0 \in D$ such that $\Psi(\nu_0, v) - f_0(v) > 0$ on $F$ for all $\Psi \in D$, $\Psi \geqq \Psi_0$. In fact, writing $V_\Psi = \{v;\ v \in F \text{ and } \Psi(\nu_0, v) - f_0(v) > 0\}$ for $\Psi \in D$, we see that $V_\Psi$ is open, that $V_\Psi \subset V_{\Psi'}$ if $\Psi \leqq \Psi'$ and that $F = \cup\{V_\Psi;\ \Psi \in D\}$. Since $F$ is compact, there is a finite subset $D_0$ of $D$ such that $F = \cup\{V_\Psi;\ \Psi \in D_0\}$. It suffices to take the upper envelope of $D_0$ for $\Psi_0$. Thus we see that $\mathscr{M}_\Psi' \neq \phi$, $-\infty < M_\Psi' < \infty$ and condition (K) is satisfied for $\Psi \in D$, $\Psi \geqq \Psi_0$. Then it follows from the first step that $\mathscr{M}_\Psi \neq \phi$ and $M_\Psi' = M_\Psi = \int f_0 d\mu_\Psi$ for some $\mu_\Psi \in \mathscr{M}_\Psi$. We show that $\{\mu_\Psi(F);\ \Psi \in D, \Psi \geqq \Psi_0\}$ is bounded. Suppose that $\mu_\Psi(F) \to \infty$ along a subdirected set $D'$ of $D$. We set $\lambda_\Psi = \mu_\Psi / \mu_\Psi(F)$ and choose a vaguely convergent[8] subnet of $\{\lambda_\Psi;\ \Psi \in D'\}$. We denote it again by $\{\lambda_\Psi;\ \Psi \in D'\}$ and let $\lambda_0$ be the limit. We have

$$\Psi'(u, \lambda_\Psi) \leqq \Psi(u, \lambda_\Psi) \leqq g_0(u)/\mu_\Psi(F)$$

for all $\Psi, \Psi' \in D'$, $\Psi' \leqq \Psi$. Hence

$$\Phi(u, \lambda_0) = \sup_{\Psi' \in D'} \Psi'(u, \lambda_0) \leqq \lim_{\Psi \in D'} g_0(u)/\mu_\Psi(F) = 0$$

and

$$\int f_0 d\lambda_0 = \lim_{D'} \left( \int f_0 d\mu_\Psi \right) \Big/ \mu_\Psi(F) = \lim_{D'} M_\Psi / \mu_\Psi(F) = 0.$$

We have

8) The topology $w(M(F), C(F))$ is the vague topology on $M(F)$. Cf. [3].

$$0<\int[\Phi(\nu_0, v)-f_0(v)]d\lambda_0(v)=\int\Phi(\nu_0, v)d\lambda_0(v)$$

$$=\int\Phi(u, \lambda_0)d\nu_0(u)\leqq 0,$$

which is a contradiction. Now we choose a subnet $\{\mu_\Psi;\ \Psi\in D'\}$ of $\{\mu_\Psi;\ \Psi\in D,\ \Psi\geqq\Psi_0\}$ which converges vaguely to $\mu_0\in M^+(F)$. We observe that $\mu_0\in\mathscr{M}$ and

$$M_0'\leqq\lim_{D'} M_\Psi'=\lim_{D'}\int f_0 d\mu_\Psi=\int f_0 d\mu_0\leqq M_0,$$

and hence $M_0'=M_0=\int f_0 d\mu_0$.

Finally we consider the general case where $f_0$ may not be continuous. We consider the directed set $H$ of continuous functions $h$ not smaller than $f_0$ and use the notation $\mathscr{M}_h$, $\mathscr{M}_h'$, $M_h=M_{0h}$ and $M_h'=M_{0h}'$ when $h\in H$ is taken as $f_0$. $\{M_h\}$ and $\{M_h'\}$ are decreasing along $H$ and the inequalities $M_h\geqq M_0$ and $M_h'\geqq M_0'$ hold. By the same argument as in the second step, we see that there is $h_0\in H$ such that $\Phi(\nu_0, v)-h(v)>0$ on $F$ for all $h\in H$, $h\leqq h_0$. Hence for $h\in H$, $h\leqq h_0$, we infer that $\mathscr{M}_h\neq\phi$ and $M_h'=M_h=\int h d\mu_h$ for some $\mu_h\in\mathscr{M}$. We can prove as in the second step that $\{\mu_h(F);\ h\in H,\ h\leqq h_0\}$ is bounded. Choose a subnet $\{\mu_h;\ h\in H'\}$ of $\{\mu_h;\ h\in H,\ h\leqq h_0\}$ which converges vaguely to $\mu_0\in M^+(F)$. Then it follows that $\mu_0\in\mathscr{M}$ and

$$M_0'\leqq\lim_{H'}\int h d\mu_h\leqq\lim_{H'}\int h' d\mu_h=\int h' d\mu_0$$

for every $h'\in H'$, and hence

$$M_0'\leqq\int f_0 d\mu_0\leqq M_0.$$

This completes the proof.

Making use of Theorems 8* and 10 and Proposition 3 in the first step of the above proof, we can prove

PROPOSITION 9. *Assume that* $\mathscr{M}\neq\phi$ *and* $-\infty<M_0<\infty$. *If we assume that* $f_0>0$ *on* $F$, *then* $\mathscr{M}'\neq\phi$ *and* $M_0'=M_0=\int f_0 d\mu$ *for some* $\mu\in\mathscr{M}$.

This is an extension of Theorem 4 in [13].

## § 9. Duality theorems of the Minkowski type

Hustad [8] obtained duality theorems of the Minkowski type and showed some applications of them. We here give a simple proof for his theorems.

Let $\mu_0$ be an element of $M^+(E)$, $g$ be an element of $C(E)$ and $P$ be a convex cone in $C(E)$. Hustad gave

THEOREM 13. *Suppose that one of the following conditions is true*:
(1) $g>0$ *on* $E$,
(2) $P$ *contains a strictly negative function.*
*Then the following equality holds*:

$$\sup\{\mu_0(f); f\in P, f\leqq g\} = \min\{\mu(g); \mu\in M^+(E), \mu\geqq\mu_0 \text{ on } P\}.$$

PROOF. Let us take

$$X=Z=C(E), \quad W=M(E), \quad Q=C^+(E) \quad \text{and} \quad z_0=-g.$$

Then $Z$ and $W$ are linear spaces paired under the bilinear functional $((\ ,\ ))_2$ defined by $((f, \mu))_2=\mu(f)$ for all $f\in Z$ and $\mu\in W$. Let $A$ be a linear transformation from $X$ into $W$ defined by $Af=-f$, $\psi(f)=Af-z_0$ and $\varphi(f)=-\mu_0(f)$. In this way, the convex program $(\psi, \varphi, P, Q)$ is well-defined. It is easily seen that $N=\inf\{\varphi(f); f\in P, Af-z_0\in Q\}$ and $N'=\sup\{((z_0, w))_2; w\in Q^+, ((Af, w))_2\leqq\varphi(f) \text{ on } P\}$ are finite. By condition (1), we have $Af-z_0=g\in Q^\circ$ for $f=0$. By condition (2), there exists $f_0\in P$ such that $f_0<0$. Since $E$ is compact, there is a positive number $t$ satisfying $t(\max\{f_0(u); u\in E\})<\min\{g(u); u\in E\}$. We have $tf_0\in P$ and $A(tf_0)-z_0=-tf_0+g\in Q^\circ$. Since $Q^+=M^+(E)$, our assertion follows from Theorem 4.

## § 10. Application to the theory of capacities

Recently Ohtsuka [14] showed that Kretschmer's duality theorem is applied to the theory of capacities in the potential theory. We follow these lines and apply our duality theorem to a problem similar to the one in [14].

Let $E$ and $F$ be compact Hausdorff spaces, $B(E)$ the metric space of bounded Borel functions on $E$ given the distance $\sup|f-g|$ for $f, g\in B(E)$, and $B^+(E)$ the subset of $B(E)$ which consists of non-negative functions. The strong dual of $B(E)$ is denoted by $B(E)^*$. Note that $B(E)^*$ contains the set $M(E)$. Let $\Phi(u, v)$ be a lower semicontinuous kernel on $E\times F$ and $m$ be a non-negative Radon measure on $E$.

We assume the following two potential-theoretic conditions:

$$\text{(PT. 1)} \quad \check{U}^m(v)=\int\Phi(u, v)dm(u)\in C(F),$$

$$\text{(PT. 2)} \quad \int\Phi(u, v)f(u)dm(u)\in C(F) \qquad \textit{for every} \quad f\in B(E).$$

We shall prove

THEOREM 14. *Let $g$ be an element of $C(F)$ such that $g>0$. If the value* $\sup\left\{\int g d\nu;\ \nu\in M^+(F),\ \int_e U^\nu dm \leqq m(e) \text{ for every Borel set } e\subset E\right\}$ *is finite, then we have*

$$\max\left\{\int g d\nu;\ \nu\in M^+(F),\ \int_e U^\nu dm \leqq m(e) \text{ for every Borel set } e\subset E\right\}$$

$$=\inf\left\{\int f dm;\ f\in B^+(E),\ \int \Phi(u,\ v)f(u)dm(u)\geqq g(u) \text{ on } F\right\},$$

*where*

$$U^\nu(u)=\int \Phi(u,\ v)d\nu(v).$$

PROOF. We set

$$X=B(E),\quad Y=B(E)^*,\quad Z=B(F),\quad W=B(F)^*,\quad P=B^+(E),\quad Q=B^+(F).$$

Then $X$ and $Y$ ($Z$ and $W$ resp.) are linear spaces paired under the natural bilinear functional $((\ ,\ ))_1$ ($((\ ,\ ))_2$ resp.) mentioned in §6. Define $Af$ for $f\in X$ by

$$Af(v)=\int \Phi(u,\ v)f(u)dm(u).$$

Take $m$ for $y_0\in Y$ and $g$ for $z_0\in Z$. Then the quintuple $(A,\ P,\ Q,\ y_0,\ z_0)$ is a program by our assumption. In case $\sigma\in M(F)$, $((h,\ \sigma))_2$ signifies $\int h d\sigma$ for every $h\in B(F)$. For every $w\in W=B(F)^*$, there exists a unique Radon measure $\sigma$ such that $((h,\ w))_2$ is equal to $\int h d\sigma$ for all $h\in C(F)$. If a measure belongs to $Q^+$, then it is non-negative, i.e., $M(F)\cap Q^+=M^+(F)$. We have by condition (PT. 2)

$$((f,\ A^*w))_1=((Af,\ w))_2=\int Af d\sigma$$

$$=\int\int \Phi(u,\ v)f(u)dm(u)d\sigma(v)=\int f U^\sigma dm$$

for all $f\in B(E)$. Since $z_0=g\in C(F)\cap Q^\circ$ and the value

$-M'=\sup\{((z_0,\ w))_2;\ w\in Q^+,\ ((f,\ y_0-A^*w))_1\geqq 0 \text{ for all } f\in P\}$ is finite by our assumption, our assertion follows from Theorems 8*, 9 and 10 and Proposition 3.

We consider the case where $E=F$ is a compact set $K$ in $R^3$ and $\Phi$ is the Newtonian kernel, i.e., $\Phi(u,\ v)=1/|u-v|$. Let $m$ be the restriction of Lebesgue measure in $R^3$ onto $K$. Then conditions (PT. 1) and (PT. 2) are fulfilled (cf. [14]). We have

COROLLARY. *If the value* $\sup\{\nu(K);\ \nu\in M^+(K),\ U^\nu\leqq 1$ a.e. on $K\}$[9] *is finite, then*

$$\max\{\nu(K);\ \nu\in M^+(K),\ U^\nu\leqq 1 \text{ a.e. on } K\}$$

$$=\inf\{\mu(K);\ \mu\in M^+(K),\ \mu \text{ is absolutely continuous, } U^\mu\geqq 1 \text{ on } K\}.$$

Next we shall consider another application of our results to the theory of capacities. Let $1\leqq p\leqq\infty$ and $1/p+1/q=1$. Assume that there exists $m\in M^+(E)$ whose support is equal to $E$. We use usual norms $\|\ \|_p$ and $\|\ \|_q$ on $L^p(E, m)$ and $L^q(E, m)$.

Fuglede [6] proved the following theorem by making use of a generalized minimax theorem.

THEOREM 15. *Assume that $\Phi$ is a non-negative lower semicontinuous kernel on $E\times F$ which satisfies the condition that for every $v\in F$ there exists $u\in E$ such that $\Phi(u, v)>0$. Then*

$$\max\{\mu(E);\ \mu\in M^+(F),\ \|\Phi(\cdot,\mu)\|_q\leqq 1\}$$

$$=\inf\{\|f\|_p;\ f\in L_p^+(E, m),\ \Phi(fm,\cdot)\geqq 1 \text{ on } F\}.$$

We remark here that this theorem can be proved by a generalized duality theorem. We only consider the case where $\Phi$ is a continuous kernel. For the general case, we may repeat the same argument as in §8. We set

$$X=L^p(E, m),\quad Z=C(F),\quad W=M(F),\quad P=L_p^+(E, m),\quad Q=C^+(F),$$

$$\varphi(f)=\|f\|_p \quad\text{and}\quad \psi(f)=Af-z_0,$$

where $Af=\Phi(fm,\cdot)$ and $z_0$ is an element of $C(F)$ such that $z_0(v)=1$ for every $v\in F$. $Z$ and $W$ are linear spaces paired under the bilinear functional $((\ ,\ ))_2$ defined in §7. Thus the convex program $(\psi, \varphi, P, Q)$ in §3 is well-defined. Since $N'=\sup\{\mu(E);\ \mu\in M^+(F)$ and $\|\Phi(\cdot,\mu)\|_q\leqq 1\}$ is finite and $z_0\in Q^\circ$, we have the desired equality by Theorems 4 and 5.

Note that the hypothesis on the kernel assures that $N'$ is finite.

## § 11. Existence theorems in potential theory

Let $E$ and $F$ be compact Hausdorff spaces, $G(u, u')$, $\Phi(u, v)$, $f(u)$ and $g(v)$ be real-valued continuous functions on $E\times E$, $E\times F$, $E$ and $F$ respectively. A measure will always be a non-negative Radon measure on $E$ or $F$. For a measure $\mu$, we denote the support of $\mu$ by $S\mu$. In this section, we shall be concerned with the following two problems:

9) a.e. = almost everywhere with respect to $m$.

PROBLEM I. Do there exist measures $\mu$ on $E$ and $\nu$ on $F$ satisfying

(I. 1) $G(u, \mu)+\Phi(u, \nu)\geqq f(u)$ on $E$,

(I. 2) $G(u, \mu)+\Phi(u, \nu)\leqq f(u)$ on $S\mu$,

(I. 3) $\Phi(\mu, v)\leqq g(v)$ on $F$,

(I. 4) $\Phi(\mu, v)= g(v)$ on $S\nu$ ?

Let $E$ consist of a finite number of mutually disjoint compact sets $\{E_k\}$, $k=1, \cdots, n$, $h$ be a real-valued continuous function on $E$ and $\{t_k\}$ be numbers. We define $h_k$ by $h_k=h$ on $E_k$ and $h_k=0$ on the complement of $E_k$ in $E$.

PROBLEM II. Do there exist a measure $\mu$ on $E$ and numbers $\{\gamma_k\}$ satisfying

(II. 1) $G(u, \mu)+ \sum_{k=1}^{n}\gamma_k h_k(u)\geqq f(u)$ on $E$,

(II. 2) $G(u, \mu)+ \sum_{k=1}^{n}\gamma_k h_k(u)\leqq f(u)$ on $S\mu$,

(II. 3) $\int h_k d\mu = t_k$ $(k=1, \cdots, n)$ ?

These problems are closely related to the conditional Gauss variational problem treated in [12], [14] and [16]. In the case where $G$ is symmetric, i.e., $G(u, u')=G(u', u)$ for all $u, u'\in E$, Problems I and II (in a more general form as in Remark 8 below) were studied by Ohtsuka [14] by means of the Gauss variational method. However this method can not be applied to our problems, since $G$ is not symmetric. We use Glicksberg-Fan's fixed point theorem and generalized duality theorems obtained in §5 and §6 in the present paper.

First we study Problem I. Let $\mathscr{M}$ be the set of measures $\lambda$ on $E$ satisfying

$$\Phi(\lambda, v)\leqq g(v) \qquad \text{on } F.$$

We have

THEOREM 16. *Assume that $g>0$ on $F$ and that $\mathscr{M}$ is vaguely compact. Then Problem I is solvable.*

PROOF. It is easily seen that $\mathscr{M}$ is a non-empty vaguely compact convex set. For a measure $\mu$ on $E$, we set

$$f_\mu(u)=f(u)-G(u, \mu).$$

We define a point-to-set mapping $\varphi: \mu\rightarrow\varphi(\mu)$ on $\mathscr{M}$ by

$$\varphi(\mu)=\left\{\lambda\in\mathscr{M};\ \int f_\mu d\lambda=M_\mu\right\},\quad \text{where}\quad M_\mu=\sup\left\{\int f_\mu d\tau;\ \tau\in\mathscr{M}\right\}.$$

Since $G(u, u')$ is continuous, $\varphi(\mu)$ is non-empty and convex, and the mapping $\varphi$ is closed in the following sense: If nets $\{\mu_\alpha; \alpha\in D\}$ and $\{\lambda_\alpha; \alpha\in D\}$ ($D$ is a directed set) converge vaguely to $\mu$ and $\lambda$ respectively and $\lambda_\alpha\in\varphi(\mu_\alpha)$ for any $\alpha\in D$, then $\lambda\in\varphi(\mu)$. Consequently by Glicksberg-Fan's fixed point theorem ([5], [7]) there exists a measure $\mu_0\in\mathscr{M}$ such that $\mu_0\in\varphi(\mu_0)$.

Since $g>0$ on $F$ and $M_{\mu_0}$ is finite, there exists a measure $\nu_0$ on $F$ such that $\Phi(u, \nu_0)\geqq f_{\mu_0}(u)$ on $E$ and $M_{\mu_0}=\int g d\nu_0=\min\left\{\int g d\nu;\ \Phi(u,\nu)\geqq f_{\mu_0}(u) \text{ on } E\right\}$. In fact, we see that $f_{\mu_0}$ and $g$ play the roles of $f_0$ and $g_0$ in the programs in §7. Since the condition that $g>0$ on $F$ implies condition (K*), our assertion follows from Theorems 7* and 10*. By the relation

$$M_{\mu_0}=\int f_{\mu_0}d\mu_0\leqq\int\Phi(u,\nu_0)d\mu_0(u)=\int\Phi(\mu_0, v)d\nu_0(v)\leqq\int g d\nu_0,$$

we see easily that $\mu_0$ and $\nu_0$ satisfy (I. 1)-(I. 4).

REMARK 6. Let $\mu$ and $\nu$ be a solution of Problem I. If $g>0$ on $F$ and there exists $u_0\in E$ such that $f(u_0)>0$, then $\mu\neq 0$ by (I. 1) and (I. 4).

In the case where $F=\{v\}$, $g(v)=1$ and $f(u)=\Phi(u, v)=1$ for every $u\in E$, the assumptions in the above theorem are satisfied. If $G(u, u')\geqq 0$ and $G(u, u)>0$ for every $u$, $u'\in E$, then we see easily by Theorem 16 and Remark 6 that there exists a measure $\mu$ on $E$ such that

$$G(u,\mu)\geqq 1 \qquad \text{on } E,$$

$$G(u,\mu)\leqq 1 \qquad \text{on } S\mu.$$

We shall call this result Kishi's existence theorem; see [10].

REMARK 7. We shall show by an example that Problem I has not always a solution if we omit the condition $g>0$ on $F$. Let $E$ and $F$ be the interval $[0, 1]$ in the real line, $G=1$, $f(u)=u+1$, $\Phi(u, v)=uv$ and $g(v)=v^2$. If there exist measures $\mu$ and $\nu$ which satisfy (I. 1)-(I. 4), then $\mu=a\varepsilon_0$ and $\nu=b\varepsilon_0$ with $a\geqq 0$ and $b\geqq 0$ by (I. 3) and (I. 4), where $\varepsilon_0$ is the unit point measure at $x=0$. This contradicts (I. 1) and (I. 2).

We can weaken the compactness condition for $\mathscr{M}$ as follows:

PROPOSITION 10. *Assume that $G>0$, $f>0$, $g>0$ and $\Phi\geqq 0$. Then Problem I is solvable.*

Next we shall study Problem II.
We have

THEOREM 17. *Assume that $h>0$ on $E$ and $t_k>0$ for every $k$. Then Prob-*

*lem II is solvable.*

PROOF. Let $\mathscr{F}$ be the set of measures $\mu$ on $E$ satisfying $\int h_k d\mu = t_k$ for every $k$. Then $\mathscr{F}$ is vaguely compact and convex. By the same argument as in the proof of Theorem 16, we see that there exists a fixed point $\mu_0 \in \mathscr{F}$ of the point-to-set mapping $\varphi$: $\mu \to \varphi(\mu)$ on $\mathscr{F}$ defined by

$$\varphi(\mu) = \left\{\lambda \in \mathscr{F};\ \int f_\mu d\lambda = M_\mu\right\}$$

where $M_\mu = \max\left\{\int f_\mu d\nu;\ \nu \in \mathscr{F}\right\}$ and $f_\mu(u) = f(u) - G(u, \mu)$.

We shall show that there exist numbers $\{r_k\}$ such that

$$\sum_{k=1}^n r_k h_k(u) \geqq f(u) \qquad \text{on } E \text{ and } \int f_{\mu_0} d\mu_0 = \sum_{k=1}^n r_k \int h_k d\mu_0.$$

Set $X = M(E)$, $Y = C(E)$, $Z = W = R^n$, $P = M^+(E)$, $Q = \{0\}$, $y_0 = -f_{\mu_0}$ and $z_0 = (-t_1, \cdots, -t_n)$. $X$ and $Y$ are paired under the natural bilinear functional $((\ ,\ ))_1$ (cf. §7). $Z$ and $W$ are paired under the bilinear functional $((\ ,\ ))_2$ defined by $((z, w))_2 = zw$, where $zw$ means the inner product of $z \in R^n$ and $w \in R^n$. Let $A$ be the linear transformation from $X$ into $Z$ defined by

$$A\mu = \left(-\int h_1 d\mu, \cdots, -\int h_n d\mu\right).$$

Thus the program $(A, P, Q, y_0, z_0)$ is well-defined. Since the value of the program is equal to $M_{\mu_0} = \int f_{\mu_0} d\mu_0$ and $z_0 \in A(P)^\circ$, our assertion follows from Theorem 7 and Proposition 7. It is easily seen that $\mu_0$ and $\{r_k\}$ satisfy (II. 1)–(II. 3).

COROLLARY. *There exist a measure $\mu$ on $E$ and a real number $r$ satisfying* (1) $G(u, \mu) + r \geqq f(u)$ *on* $E$, (2) $G(u, \mu) + r \leqq f(u)$ *on* $S\mu$ *and* (3) $\mu(E) = 1$.

REMARK 8. It is easily seen that Problem II is a special case of the following problem which was discussed in [14] in the case where $G$ is symmetric:

PROBLEM II′. Do there exist a non-negative Radon measure $\mu$ on $E$ and a signed Radon measure $\nu$ on $F$ satisfying

(II′. 1) $G(u, \mu) + \Phi(u, \nu) \geqq f(u)$ on $E$,

(II′. 2) $G(u, \mu) + \Phi(u, \nu) \leqq f(u)$ on $S\mu$,

(II′. 3) $\Phi(\mu, v) = g(v)$ on $F$ ?

This problem seems however to be beyond the application of the duality theorems in this paper. In fact, if we study Problem II′ by the same method as in the proof of Theorems 16 and 17, then we need a new duality theorem which is valid in the case where the constraints are given by equalities. Though we can transform equality constraints into inequality constraints as in the classical case, we can not apply most of the criteria in §6 to this case. Needless to say, equality constraints correspond to the condition $Q=\{0\}$ or $P^+=\{0\}$ in the programs. The answer to Problem II′ in [14] is not therefore yet complete, a remark with which Professor Ohtsuka has been kind enough to say that he agrees.

## § 12. Appendix

As remarked in the introduction, our main theorem is a converse of Kretschmer's duality theorem. In this section, we give a converse of Dieter's strong duality theorem in [4].

Let $X$ be a real locally convex linear space and $X^*$ be the strong dual of $X$. Let $C$ and $D$ be convex sets in $X$, $f(x)$ a convex function on $C$ and $g(x)$ a concave function on $D$. We recall some definitions in [4].

$$[f,\ C]=\{(r,\ x);\ x\in C,\ r\geqq f(x)\}\subset R\times X,$$

$$f^*(x^*)=\sup\{x^*(x)-f(x);\ x\in C\},$$

$$C^*=\{x^*\in X^*;\ f^*(x^*)<\infty\},$$

$$[g,\ D]=\{(r,\ x);\ x\in D,\ r\leqq g(x)\}\subset R\times X,$$

$$g^*(x^*)=\inf\{x^*(x)-g(x);\ x\in D\},$$

$$D^*=\{x^*\in X^*;\ g^*(x^*)>-\infty\}.$$

It is well-known that $f^*(x^*)$ is a convex function on $C^*$ and $g^*(x^*)$ is a concave function on $D^*$.

We set

$$V=\sup\{g(x)-f(x);\ x\in C\cap D\} \qquad \text{if} \quad C\cap D\neq\phi,$$

$$V=-\infty \qquad \text{if} \quad C\cap D=\phi,$$

$$V^*=\inf\{f^*(x^*)-g^*(x^*);\ x^*\in C^*\cap D^*\} \qquad \text{if} \quad C^*\cap D^*\neq\phi,$$

$$V^*=\infty \qquad \text{if} \quad C^*\cap D^*=\phi,$$

$$S=[f^*,\ C^*]+[-g^*,\ -D^*].$$

Dieter proved

THEOREM 18 (strong duality theorem). *Let $[f,\ C]$ and $[g,\ D]$ be closed. If $V$ is finite and if the set $S$ is weak* closed, then $V=V^*=\min\{f^*(x^*)-g^*(x^*);\ x^*\in C^*\cap D^*\}$.*

We shall prove

THEOREM 19. *Let $[f, C]$ and $[g, D]$ be closed. If $V^*$ is finite and if the set $S$ is weak* closed, then the equality $V=V^*$ holds.*

PROOF. Since $V^*$ is finite, we see that $(V^*+\varepsilon, 0)\in S$ and $(V^*-\varepsilon, 0)\notin S$ for every $\varepsilon>0$. Let $\varepsilon$ be an arbitrarily fixed positive number. Since $S$ is weak* closed and convex, there exist $(s_1, x_1)\in R\times X$ and $\alpha_1\in R$, by a well-known separation theorem, which satisfy

$$(V^*-\varepsilon)s_1<\alpha_1\leqq rs_1+x^*(x)$$

for all $(r, x^*)\in S$. From the fact $(V^*+\varepsilon, 0)\in S$, we derive $s_1>0$. Writing $\alpha=\alpha_1/s_1$ and $x=-x_1/s_1$, we have

$$V^*-\varepsilon<\alpha\leqq r-x^*(x)$$

for all $(r, x^*)\in S$. By the relation $\{(f^*(x_1^*)-g^*(x_2^*),\ x_1^*-x_2^*);\ x_1^*\in C^*,\ x_2^*\in D^*\}\subset S$, we have

$$\alpha+x_1^*(x)-f^*(x_1^*)\leqq x_2^*(x)-g^*(x_2^*)$$

for all $x_1^*\in C^*$ and $x_2^*\in D^*$. Making use of the fact that

$$f(x)=\sup\{x_1^*(x)-f^*(x_1^*);\ x_1^*\in C^*\}$$

and

$$g(x)=\inf\{x_2^*(x)-g^*(x_2^*);\ x_2^*\in D^*\}$$

([4], p. 99, Hilfssatz 7), we have $\alpha+f(x)\leqq g(x)$, which implies $x\in C\cap D$. Consequently $V^*-\varepsilon<\alpha\leqq g(x)-f(x)\leqq V$. By the arbitrariness of $\varepsilon$, we have $V^*\leqq V$. The converse inequality is always valid ([4], p. 102).

COROLLARY. *Let $[f, C]$ and $[g, D]$ be closed. If $V^*$ is finite and if the set $S$ is weak* closed, then there exists $x^*\in C^*\cap D^*$ such that $V^*=f^*(x^*)-g^*(x^*)$.*

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