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On Kuramochi's Function-theoretic Separative Metrics on Riemann Surfaces

Hiroshi Tanaka

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Introduction

In order to extend Fatou's and Beurling's theorems to arbitrary Riemann surfaces, Z. Kuramochi introduced ([4]; also see [5] and [7]) two notions of function-theoretic separative metrics, i.e., H. B. and H. D. separative metrics.

Since extended Fatou's and Beurling's theorems are stated in terms of compactifications of an open Riemann surface, we shall define separative compactifications rather than separative metrics. In this paper we shall give necessary and sufficient conditions for a compactification to be H. B. or H. D. separative, in terms of the Wiener or the Royden compactification, respectively. Our characterizations are given in a simple form compared with the original definition by Z. Kuramochi and may make it easier to comprehend the meaning of these notions.

In §1, we shall discuss compactifications of a hyperbolic Riemann surface R. §2 (resp. §3) is devoted to the study of harmonic measures (resp. capacitary potentials) which were defined by Z. Kuramochi ([3]). We shall investigate their properties on the Wiener or the Royden boundary of R. In §4 (resp. §5), we shall give our main theorems on H. B. (resp. H. D.) separative compactifications and study the relation between H. B. and H. D. separative compactifications (§5).

As an application, we shall show in §6: 1) for Fatou's theorem, Kuramochi's result ([4], [5], [7]) and Constantinescu and Cornea's result (Satz 14.4 in [2]) are equivalent; 2) for Beurling's theorem, Kuramochi's result ([4], [5], [7]) is independent of a similar result by Constantinescu and Cornea (Satz 18.1 in [2]).

Notation and terminology

Let R be a hyperbolic Riemann surface. For a subset A of R, we denote by ∂A and A^i the (relative) boundary and the interior of A respectively. We shall call a closed subset F of R *regular* if ∂F consists of at most a countable number of analytic arcs clustering nowhere in R.

An *exhaustion* will mean an increasing sequence $\{R_n\}_{n=1}^{\infty}$ of relatively compact domains on R such that $\bigcup_{n=1}^{\infty} R_n = R$ and each ∂R_n consists of a finite number of closed analytic Jordan curves.

A subset A of R is called *polar* if there is a positive superharmonic function s on R such that $s(a) = +\infty$ at every point a of A. A polar set is locally of Lebesgue measure zero. We shall say that a property holds *q.p.* on a set E if it holds on E except for a polar set.

Let u be a harmonic function on R such that $\inf_R u = 0$ and $\sup_R u = 1$. For each α $(0 < \alpha < 1)$, we set $\Omega_\alpha = \{z \in R;\ u(z) \geqq \alpha\}$. If F is a regular closed set in R, then there is a set E of at most a countable number of α in $(0, 1)$ such that $F \cap \Omega_\alpha$ is a regular closed set in R for each α in $(0, 1) - E$.

§1. Compactifications

1.1 *Definition of compactification*

If R^* is a Hausdorff compact space and if there is a homeomorphism of R into R^* such that the image of R is open and dense in R^*, then we may identify the image of R with R and call R^* a compactification of R. $\Delta = R^* - R$ is called an ideal boundary of R.

Let Q be a family of bounded continuous (real valued) functions on R. If a compactification R^* of R satisfies:

1) every $f \in Q$ can be continuously extended over R^*,

2) Q separates points of $\Delta = R^* - R$,

then R^* is called a Q-compactification of R. It is known (cf. [2]) that a Q-compactification always exists and is unique up to a homeomorphism. Thus, it will be denoted by R_Q^* and its ideal boundary by Δ_Q.

We refer to [2] for the definitions and properties of the Martin compactification R_M^*, the Kuramochi compactification R_N^*, the Wiener compactification R_W^*, the Royden compactification R_D^* and harmonic boundaries Γ_W, Γ_D.

For any subset A of R, we shall denote by $\bar{A}^*$ (resp. $\bar{A}^N$, $\bar{A}^W$, $\bar{A}^D$) the closure of A in R^* (resp. R_N^*, R_W^*, R_D^*).

Let R_1^* and R_2^* be two compactifications of R. If there is a continuous mapping of R_1^* onto R_2^* whose restriction on R is the identity mapping, then we shall say that R_2^* is a *quotient space* of R_1^*. It is known ([2]) that, if $Q_1 \subset Q_2$, then $R_{Q_1}^*$ is a quotient space of $R_{Q_2}^*$. Hence R_M^*, R_N^* and R_D^* are quotient spaces of R_W^*. Furthermore, R_N^* is a quotient space of R_D^*.

1.2 *Dirichlet problems*

Let R^* be a compactification of R. Given a function f (extended real valued) on Δ, we consider the following classes:

$$\overline{\mathfrak{S}}_f=\left\{s;\quad \begin{array}{ll}\text{superharmonic, bounded below on } R, & \\ \varliminf_{a\to\xi} s(a)\geqq f(\xi) & \text{for any}\quad \xi\in\Delta\end{array}\right\}\cup\{\infty\},$$

$$\underline{\mathfrak{S}}_f=\{s;\quad -s\in\overline{\mathfrak{S}}_{-f}\},$$

where ∞ means the function which is equal to $+\infty$ everywhere on R. We define $\bar{H}_f(a)=\inf\{s(a);\ s\in\overline{\mathfrak{S}}_f\}$ and $\underline{H}_f(a)=\sup\{s(a);\ s\in\underline{\mathfrak{S}}_f\}$. It is known (Perron-Brelot) that $\bar{H}_f$ (resp. $\underline{H}_f$) is either harmonic, $\equiv+\infty$ or $\equiv-\infty$. If $\bar{H}_f=\underline{H}_f$ and are harmonic, then we say that f is resolutive and $H_f=\bar{H}_f=\underline{H}_f$ is called the Dirichlet solution of f (with respect to R^*).

If any finite continuous function on Δ is resolutive, then R^* is called *resolutive*. It is known that a Q-compactification R_Q^* of R is resolutive if and only if Q consists of bounded continuous Wiener functions (cf. [2]). Hence R_M^*, R_N^*, R_W^* and R_D^* are resolutive. We denote by $\omega=\omega_a$ the harmonic measure on Δ_W at $a\in R$ and note that the support of ω is equal to Γ_W.

1.3 *Some special examples of Q-compactifications*

Example 1. Let R be the unit disk $\{z;\ |z|<1\}$ in the complex plane. Let ω_a be the harmonic measure of the half circle $\{e^{i\theta};\ |\theta|\leqq\pi/2\}$ with respect to R at $a\in R$. We take $\{\omega_a\}$ for Q. Then

a) R_Q^* is a metrizable resolutive compactification,

b) R_Q^* is not a quotient space of R_D^*.

Example 2. Let R be a Riemann surface which belongs to $O_{HD}-O_{HB}$. Then it follows from Folgesatz 11.5 in [2] that Γ_D consists of only one point, say b, and Γ_W contains at least two distinct points, say a_1 and a_2. Then there is a bounded continuous Wiener function f on R such that $\lim_{a\to a_1} f(a)=1$ and $\lim_{a\to a_2} f(a)=0$. Now we set $g=3\max(\min(f, 2/3), 1/3)-1$ and take $\{g\}$ for Q. Then

a) R_Q^* is a metrizable resolutive compactification,

b) R_Q^* is not a quotient space of R_D^*.

§2. Harmonic measure and Wiener boundary

2.1 *Dirichlet problems on an open set*

Let R^* be the one point compactification of R. Let G be a domain on R. Given a function f (extended real valued) on ∂G, we define the function f^* on $\bar{G}^*-G$ which is equal to f on ∂G and 0 on $\bar{G}^*-R$ if G is not relatively compact in R. We may consider $\bar{G}^*$ as a compactification of G and set $\bar{H}_f^G=\bar{H}_{f^*}$ and

$\underline{H}_f^G=\underline{H}_{f^*}$. If $\bar{H}_f^G=\underline{H}_f^G$ and are harmonic, then we say that f is resolutive with respect to G and call $H_f^G=\bar{H}_f^G=\underline{H}_f^G$ the Dirichlet solution of f. If G is an open subset of R, then we decompose G into connected components $\{G_i\}$. Given a function f (extended real valued) on ∂G, we denote by f_i the restriction of f on ∂G_i and define $\bar{H}_f^G=\bar{H}_{f_i}^{G_i}$ and $\underline{H}_f^G=\underline{H}_{f_i}^{G_i}$ on G_i. If each f_i is resolutive with respect to G_i, then we call f resolutive with respect to G and write $H_f^G=\bar{H}_f^G=\underline{H}_f^G$. If $\{f_n\}_{n=1}^{\infty}$ is a monotone sequence of resolutive functions with respect to G and $\{H_{f_n}^G\}$ converges, then the limit function $f=\lim_{n\to\infty} f_n$ is resolutive with respect to G and $H_f^G=\lim_{n\to\infty} H_{f_n}^G$.

2.2 *Reduced functions*

Let F be a closed set in R and let s be a non-negative superharmonic function on R. We consider the following function

$$s_F=\inf\{v;\quad \text{superharmonic} \geqq 0 \text{ on } R,\ v\geqq s \text{ q.p. on } F\}.$$

Then s_F is superharmonic on R and $0\leqq s_F\leqq s$.

The following properties are known (cf. [2]):

(A1) $s_F=H_s^{R-F}$ on $R-F$ and $s_F=s$ on F except for irregular boundary points of $R-F$.

(A2) If $F_1\subset F_2$ and $s_1\leqq s_2$, then $(s_1)_{F_1}\leqq (s_2)_{F_2}$.

(A3) If $F_1\subset F_2$, then $s_{F_1}=(s_{F_1})_{F_2}=(s_{F_2})_{F_1}$.

(A4) $(a_1s_1+a_2s_2)_F=a_1(s_1)_F+a_2(s_2)_F$ $(a_1,\ a_2\geqq 0)$.

(A5) $s_{F_1\cup F_2}+s_{F_1\cap F_2}\leqq s_{F_1}+s_{F_2}$.

2.3 PROPOSITION 1. *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of closed sets in R such that $F_n\supset F_{n+1}$ $(n=1, 2, \dots)$ and $\bigcap_{n=1}^{\infty} F_n=\emptyset$. Set $u=\lim_{n\to\infty} 1_{F_n}$ and $\Omega_\alpha=\{z\in R;\ u(z)\geqq\alpha\}$ $(0<\alpha<1)$. Then*

(a) *if F is a closed set in R such that $F\supset F_{n_0}$ for some n_0, then $u_F=u$ on R,*

(b) *if u is positive, then $\sup_{F_n} u=1$ for each n,*

(c) $\lim_{n\to\infty} 1_{F_n-\Omega_\alpha^i}=0$ *for each α,*

(d) $u_{\Omega_\alpha}=u$ *on R for each α.*

PROOF. (a) Since $F\supset F_n$ for $n\geqq n_0$, it follows from (A3) and (A1) that

$$1_{F_n}=H_{1_{F_n}}^{R-F} \qquad \text{on} \quad R-F \qquad (n\geqq n_0).$$

Since 1_{F_n} decreases to u as $n\to\infty$, we have $u=H_u^{R-F}$ on $R-F$. Since the set of irregular boundary points of $R-F$ is polar (cf. Satz 4.7 in [2]), we have $u_F=u$ q.p. on R by (A1). Hence $u_F=u$ on R.

(b) Set $K_n=\sup_{F_n} u$ for each n. Then K_n decreases with n. Since u is positive, K_n is positive. By (a), we have

$$u=u_{F_m}\leqq K_m 1_{F_m}\leqq K_n 1_{F_m} \qquad \text{on} \quad R$$

for $m\geqq n$. By letting $m\to\infty$, we have $u\leqq K_n u$ on R. $u>0$ implies $K_n=1$.

(c) We may assume that u is positive. Set $v_\alpha=\lim_{n\to\infty} 1_{F_n-\Omega_\alpha^i}$. Suppose v_α is positive for some α. Then, by (b),

$$1=\sup_{F_n-\Omega_\alpha^i} v_\alpha\leqq \sup_{F_n-\Omega_\alpha^i} u\leqq\alpha<1.$$

This is a contradiction. Thus $v_\alpha=0$ for each α.

(d) We may assume that u is positive. We note that $\lim_{n\to\infty} u_{F_n-\Omega_\alpha^i}\leqq \lim_{n\to\infty} 1_{F_n-\Omega_\alpha^i}=0$ by (c). Since $u=u_{F_n\cup\Omega_\alpha}\leqq u_{\Omega_\alpha}+u_{F_n-\Omega_\alpha^i}\to u_{\Omega_\alpha}$ as $n\to\infty$ for each α, we have $u\leqq u_{\Omega_\alpha}\leqq u$ on R for each α.

Remark. This proposition is a generalization of a result given by Z. Kuramochi ([3]).

2.4 *Harmonic measure on Wiener boundary*

Let R^* be a compactification of R. Let u be a non-negative superharmonic function on R. Given a closed subset A of Δ, we consider the following class:

$$\mathcal{S}_{A,R^*}^u=\left\{v\,;\ \begin{array}{l}\text{superharmonic}\geqq 0 \text{ on } R,\ v\geqq u \text{ on } U\cap R\\ \text{for some neighborhood } U \text{ of } A \text{ in } R^*\end{array}\right\}.$$

Then the function

$$u_A=\inf\{v\,;\ v\in\mathcal{S}_{A,R^*}^u\}$$

is harmonic on R and $0\leqq u_A\leqq u$.

The following lemma is due to M. Brelot [1]:

Lemma 1. *A metrizable compactification R^* of R is resolutive if and only if $(1_A)_B=0$ for any mutually disjoint compact subsets A and B of $\Delta=R^*-R$.*

Lemma 2. *If A is a closed subset of Δ_W, then $1_A=\omega(A)$.*

Proof. It is easy to see that $1_A=\bar{H}_{\phi_A}$, where ϕ_A is the characteristic function of A. On the other hand, it follows from Hilfssatz 8.3 in [2] that $\bar{H}_{\phi_A}=\omega(A)$. Hence $1_A=\omega(A)$.

Lemma 3. *For any closed subset A of Δ_W, there exists a sequence $\{F_n\}_{n=1}^\infty$ of regular closed sets in R which has the following properties:*

a) *each* $\bar{F}_n^W$ *is a neighborhood of* A *in* R_W^*,

b) $F_n \supset F_{n+1}$ $(n=1, 2, \dots)$ *and* $\bigcap_{n=1}^{\infty} F_n = \emptyset$,

c) 1_{F_n} *decreases to* $\omega(A)$ *as* $n \to \infty$.

PROOF. Let a be a fixed point in R. Then there exists a sequence $\{s_n\}_{n=1}^{\infty}$ in $\mathcal{S}_{A,R_W^*}^1$ such that $s_n(a) \to 1_A(a)$ as $n \to \infty$. Each s_n dominates 1 on $U_n \cap R$ for some neighborhood U_n of A in R_W^*. We can choose a sequence $\{F_n\}_{n=1}^{\infty}$ of regular closed sets in R which satisfies a), b) and $F_n \subset U_n \cap R$ for each n. Set $u = \lim_{n\to\infty} 1_{F_n}$. Then u is harmonic and $u \geqq 1_A$ on R by the definition of 1_A. Since $s_n \geqq 1_{F_n}$ on R for each n, we see that

$$1_A(a) = \lim_{n\to\infty} s_n(a) \geqq \lim_{n\to\infty} 1_{F_n}(a) = u(a) \geqq 1_A(a).$$

Hence $u = 1_A$ on R. Therefore c) is valid by Lemma 2.

COROLLARY 1. *If* $\{F_n\}_{n=1}^{\infty}$ *is a sequence of regular closed sets in* R *such that* $F_n \supset F_{n+1}$ $(n=1, 2, \dots)$ *and* $\bigcap_{n=1}^{\infty} F_n = \emptyset$, *then* 1_{F_n} *decreases to* $\omega(\bigcap_{n=1}^{\infty} \bar{F}_n^W)$ *as* $n \to \infty$.

COROLLARY 2. *If* F *is a regular closed set in* R, *then*

$$h_{1_F}{}^{1)} = \omega(\bar{F}^W \cap \Delta_W) = \lim_{n\to\infty} 1_{F-R_n},$$

where $\{R_n\}_{n=1}^{\infty}$ *is an exhaustion of* R.

PROOF. It follows from the above corollary that the second equality is valid. By Hilfssatz 8.7 in [2], we obtain that $h_{1_F} \leqq \bar{H}_f$, where f is the characteristic function of $\bar{F}^W \cap \Delta_W$. Hence $h_{1_F} \leqq \omega(\bar{F}^W \cap \Delta_W)$ by Hilfssatz 8.3 in [2]. Since $(1_F)_{R-R_n} \geqq 1_{F-R_n}$, by letting $n \to \infty$, we have

$$h_{1_F} \geqq \lim_{n\to\infty} 1_{F-R_n} = \omega(\bar{F}^W \cap \Delta_W).$$

This completes the proof.

REMARK. The functions in Corollary 2 are denoted by $w(F \cap B, z)$ in [3], [4], [5] and by $w(B(F), z)$ in [7].

LEMMA 4. *If* F *is a regular closed set in* R, *then* 1_F *can be continuously extended over* R_W^*. *Furthermore,* $1_F = 1$ *on* $\bar{F}^W$ *and* $1_F = 0$ *on* $\Gamma_W - \bar{F}^W$.

PROOF. Since 1_F is a bounded continuous Wiener function on R, it can be continuously extended over R_W^*, so that $1_F = 1$ on $\bar{F}^W$. Since $1_F = h_{1_F} + p$ (p: a continuous Green potential), it follows from Corollary 2 to Lemma 3 and Folgesatz 9.2 in [2] that $1_F = h_{1_F} = \omega(\bar{F}^W \cap \Delta_W) = 0$ on $\Gamma_W - \bar{F}^W$.

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1) See p. 55 in [2] for this definition.

LEMMA 5. *If F is a regular closed set in R, then*

$$\omega(\overline{F-\Omega_{\alpha}^{i}}^{W}\cap\Delta_{W})=0$$

for each α in $(0, 1)$, *where* $\Omega_{\alpha}=\{z\in R;\ \omega_{z}(\bar{F}^{W}\cap\Delta_{W})\geqq\alpha\}$ $(0<\alpha<1)$.

PROOF. Let $\{R_n\}_{n=1}^{\infty}$ be an exhaustion of R such that each $F-R_n$ is a regular closed set in R. We assume that $F-\Omega_{\alpha}^{i}$ is a regular closed set in R for each α in $(0, 1)-E$, where E is a set of at most a countable number of α in $(0, 1)$. By Corollary 2 to Lemma 3 and (c) in Proposition 1, we obtain that

$$\omega(\overline{F-\Omega_{\alpha}^{i}}^{W}\cap\Delta_{W})=0$$

for each α in $(0, 1)-E$. Since $\overline{F-\Omega_{\alpha_1}^{i}}^{W}\subset\overline{F-\Omega_{\alpha_2}^{i}}^{W}$ if $\alpha_1<\alpha_2$ and $(0, 1)-E$ is dense in $(0, 1)$, $\omega(\overline{F-\Omega_{\alpha}^{i}}^{W}\cap\Delta_{W})=0$ for every $\alpha\in(0, 1)$.

2.5 Let K_0 be a closed disk in R and let $R_0=R-K_0$. Given a closed subset A of Δ_W, we consider the following class:

$$\mathcal{S}_A=\left\{v;\ \begin{array}{l}\text{superharmonic}\geqq 0 \text{ on } R_0,\ v\geqq 1 \text{ on } U\cap R_0\\ \text{for some neighborhood } U \text{ of } A \text{ in } R_W^*\end{array}\right\}.$$

Then the function

$$\omega^{K_0}(A)=\inf\{v;\ v\in\mathcal{S}_A\}$$

is harmonic on R_0 and $0\leqq\omega^{K_0}(A)\leqq 1$.

We shall prove

LEMMA 6. *If A is a closed subset of Δ_W, then*

$$\omega^{K_0}(A)=\omega(A)-H_{\omega(A)}^{R_0}\qquad\text{on}\quad R_0.$$

PROOF. By a discussion similar to the proof of Lemma 3, we can choose a sequence $\{F_n\}_{n=1}^{\infty}$ of regular closed sets in R which has the following properties:

a) each $\bar{F}_n^W$ is a neighborhood of A in R_W^*,

b) $F_n\supset F_{n+1}$ $(n=1, 2, \dots)$ and $\bigcap_{n=1}^{\infty}F_n=\emptyset$,

c) $1_{F_n}\searrow\omega(A)$ and $H_{f_n}^{R_0-F_n}\searrow\omega^{K_0}(A)$ as $n\to\infty$,

where $f_n=0$ on ∂K_0 and 1 on ∂F_n. Let $\{R_n\}_{n=1}^{\infty}$ be an exhaustion of R. We may assume that $F_n\cap R_n=\emptyset$ for each n. Let g_n be the continuous function on $\partial F_n\cup\partial K_0$ such that $g_n=0$ on ∂F_n and $=1_{F_n}$ on ∂K_0. Obviously,

$$H_{f_n}^{R_0-F_n}=1_{F_n}-H_{g_n}^{R_0-F_n}\qquad\text{on}\quad R_0-F_n.$$

By an elementary discussion, we can show that $H_{g_n}^{R_0-F_n}\to H_{\omega(A)}^{R_0}$ on R_0 as $n\to\infty$. Hence, by letting $n\to\infty$ in the above equality, we complete the proof.

COROLLARY. *$\omega(A)=0$ if and only if $\omega^{K_0}(A)=0$.*

§3. Capacity on Royden boundary

Let K_0 be a closed disk in R and let $R_0=R-K_0$.

3.1 *Dirichlet principle*

Let f be a Dirichlet function[2] on R and let F be a non-polar closed set in R. Then there exists a uniquely determined Dirichlet function f^F which minimizes the Dirichlet norm $\|g\|$ among Dirichlet functions g such that $g=f$ q.p. on F and which is equal to f on F and is harmonic on $R-F$.

The following properties are known ([2]):

(B1) $\|f^F\|\leqq\|f\|$, and $(f^F, g-f^F)=0$ for any Dirichlet function g such that $g=f$ q.p. on F.
(B2) $(a_1f_1+a_2f_2)^F=a_1(f_1)^F+a_2(f_2)^F$ (a_1, a_2: real).
(B3) If $f\equiv$ constant, then $f^F=f$.
(B4) If $f\geqq 0$, then $f^F\geqq 0$.
(B5) If $F_1\subset F_2$, then $f^{F_1}=(f^{F_1})^{F_2}=(f^{F_2})^{F_1}$.
(B6) If G is a component of $R-F$, then $f^F=f^{\partial G}$ on G.
(B7) If $f\geqq 0$, then $f^F\geqq H_f^{R-F}$ on $R-F$.

The following property is an immediate consequence of (B4):

(B8) If $f\geqq 0$ on F, then $f^F\geqq 0$.

LEMMA 7. *If f is a bounded continuous Dirichlet function on R, then $f^F(a)\to f(b)$ as a in $R-F$ tends to every regular boundary point b of $R-F$.*

PROOF. Suppose $|f|\leqq M<\infty$. Then we have

$$H_{f+M}^{R-F}-M\leqq f^F\leqq H_{f-M}^{R-F}+M \qquad \text{on} \quad R-F$$

by (B2), (B3) and (B7). Hence we have the lemma.

3.2. A continuous function on an open set G in R will be called *piecewise smooth* if it is continuously differentiable in an open subset $G'\subset G$ such that $G-G'$ locally consists of a finite number of points and open analytic arcs. Let F be a regular closed set in R and let ϕ be a given continuous function on ∂F.

We denote by $\mathcal{D}_{R-F}(\phi)$ the family of piecewise smooth functions f on $R-F$ with boundary values ϕ on ∂F and with finite Dirichlet norm $\|f\|_{R-F}$. Any function in $\mathcal{D}_{R-F}(\phi)$ is a Dirichlet function on $R-F$.

The following formulation of Dirichlet principle is due to M. Ohtsuka

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2) This is called eine Dirichletsche Function in [2].

([8]):

Let F be a regular closed set in R and let ϕ be a given continuous function on ∂F. If $\mathcal{D}_{R-F}(\phi)\neq\emptyset$, then there exists a uniquely determined function ϕ_F in $\mathcal{D}_{R-F}(\phi)$ such that

a) *ϕ_F is harmonic on $R-F$,*

b) $\|\phi_F\|_{R-F}\leqq\|g\|_{R-F}$ *for any g in $\mathcal{D}_{R-F}(\phi)$.*

Furthermore, if $\{R_n\}_{n=1}^{\infty}$ is an exhaustion of R, then there is a uniquely determined harmonic function h_n on R_n-F with boundary values $h_n=\phi$ on the closure of $\partial F\cap R_n$ and $\partial h_n/\partial\nu=0$ on the rest of the boundary. h_n tends to ϕ_F locally uniformly on $R-F$ and in Dirichlet norm as $n\to\infty$.

We shall prove

LEMMA 8. *Let F be a regular closed set in R. If f is a bounded continuous Dirichlet function on R, then $\phi_F=f^F$ on $R-F$, where ϕ is the restriction of f on ∂F.*

PROOF. By Lemma 7, we see that $f^F\in\mathcal{D}_{R-F}(\phi)$. Hence $\|\phi_F\|_{R-F}\leqq\|f^F\|_{R-F}$. Conversely, let $g=\phi_F$ on $R-F$ and $=f$ on F. Then we can show that g is a Dirichlet function on R. Hence, by (B1), we have $(\phi_F-f^F, f^F)_{R-F}=0$. Thus

$$\|f^F-\phi_F\|_{R-F}^2=\|\phi_F\|_{R-F}^2-\|f^F\|_{R-F}^2\leqq 0.$$

Since both f^F and ϕ_F are harmonic on $R-F$ and take the same boundary values ϕ on ∂F, it holds that $\phi_F=f^F$ on $R-F$.

3.3 *Full-superharmonic functions*

Let s be a non-negative (K_0-) full-superharmonic function[3)] on R_0 and let F be a closed set in R. We consider the following function

$$s_{\tilde{F}}=\inf\left\{v;\ \begin{array}{l}\text{full-superharmonic}\geqq 0\text{ on }R_0,\\ v\geqq s\text{ q.p. on }F\cap R_0\end{array}\right\}.$$

Then the function $s_{\tilde{F}}$ is full-superharmonic on R_0 and $0\leqq s_{\tilde{F}}\leqq s$.

The following properties are known ([2]):

(C1) $s_{\tilde{F}}=s$ on F except for irregular boundary points of R_0-F and $s_{\tilde{F}}$ is harmonic on R_0-F.

(C2) If $F_1\subset F_2$ and $s_1\leqq s_2$, then $(s_1)_{\tilde{F}_1}\leqq(s_2)_{\tilde{F}_2}$.

(C3) If $F_1\subset F_2$, then $s_{\tilde{F}_1}=(s_{\tilde{F}_1})_{\tilde{F}_2}=(s_{\tilde{F}_2})_{\tilde{F}_1}$.

(C4) $(a_1s_1+a_2s_2)_{\tilde{F}}=a_1(s_1)_{\tilde{F}}+a_2(s_2)_{\tilde{F}}$ $(a_1, a_2\geqq 0)$.

3) This is called $\overline{\text{s}}$uperharmonic by Z. Kuramochi ([3]) and "positive vollsuperharmonisch" in [2].

(C5) $s_{\widetilde{F_1\cup F_2}}+s_{\widetilde{F_1\cap F_2}}\leqq s_{\tilde{F}_1}+s_{\tilde{F}_2}$.

(C6) If s is a Dirichlet function on R, $s=0$ on K_0 and s is non-negative full-superharmonic on R_0, then

$$s_{\tilde{F}}=s^{F\cup K_0} \qquad \text{on} \quad R_0-F.$$

3.4 PROPOSITION 2. *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of regular closed subsets of R_0 such that $F_n\supset F_{n+1}(n=1, 2, \dots)$ and $\bigcap_{n=1}^{\infty}F_n=\emptyset$. Then $1_{\tilde{F}_n}$ converges locally uniformly on R_0 and in Dirichlet norm as $n\to\infty$. Furthermore, setting $u=\lim_{n\to\infty}1_{\tilde{F}_n}$ and $\Omega_\alpha=\{z\in R_0;\ u(z)\geqq\alpha\}$ $(0<\alpha<1)$, we have*

(α) *if F is a regular closed subset of R_0 such that $F\supset F_{n_0}$ for some n_0, then $u_{\tilde{F}}=u$ on R_0,*

(β) *if u is positive, then $\sup_{F_n} u=1$ for each n,*

(γ) $\lim_{n\to\infty} 1_{\widetilde{F_n-\Omega_\alpha^i}}=0$ *for each α,*

(δ) $u_{\tilde{\Omega}_\alpha}=u$ *on R_0 for each α.*

PROOF. Let D be an open disk in R such that $D\supset K_0$ and $F_1\cap(D\cup\partial D)=\emptyset$. Let v be the harmonic function on $D-K_0$ with boundary values 0 on ∂K_0 and 1 on ∂D. We extend v over R by 0 on K_0 and by 1 on $R-D$ and denote by f the extended function. Then f is a continuous Dirichlet function on R and is full-superharmonic on R_0. By (C6), we see that

$$f^{K_0\cup F_n}=1_{\tilde{F}_n}.$$

It follows from (B1) that $\{1_{\tilde{F}_n}\}_{n=1}^{\infty}$ is a Cauchy sequence in Dirichlet norm. Hence $1_{\tilde{F}_n}$ tends to u locally uniformly on R_0 and in Dirichlet norm as $n\to\infty$. Note that u vanishes on ∂K_0, so that

$$u^{K_0\cup F}=u_{\tilde{F}}.$$

(α) By (B5) and (B1), we have

$$\begin{aligned}\|u_{\tilde{F}}-u\|&=\lim_{n\to\infty}\|u_{\tilde{F}}-1_{\tilde{F}_n}\|=\lim_{n\to\infty}\|u^{K_0\cup F}-f^{K_0\cup F_n}\|\\&=\lim_{n\to\infty}\|(u-f^{K_0\cup F_n})^{K_0\cup F}\|\\&\leqq\lim_{n\to\infty}\|u-f^{K_0\cup F_n}\|=\lim_{n\to\infty}\|u-1_{\tilde{F}_n}\|=0.\end{aligned}$$

Since every boundary point of R_0-F is regular, we have $u_{\tilde{F}}=u$ on R_0 by (C1).

(β) can be proved by a discussion similar to the proof of (b) in Proposition 1.

(γ) We may assume that u is positive. Set $v_\alpha=\lim_{n\to\infty}1_{\widetilde{F_n-\Omega_\alpha^i}}$. Then $F_n-\Omega_\alpha^i$ is a regular closed subset of R_0 except for a set E of at most a countable number of α in $(0, 1)$. Suppose v_α is positive for some α in $(0, 1)-E$. Then, by (β),

$$1=\sup_{F_n-\Omega_\alpha^i} v_\alpha \leqq \sup_{F_n-\Omega_\alpha^i} u \leqq \alpha<1.$$

This is a contradiction. Since $v_{\alpha_1} \leqq v_{\alpha_2}$ if $\alpha_1<\alpha_2$ and $(0, 1)-E$ is dense in $(0, 1)$, we have $v_\alpha=0$ on R for each α.

(δ) can be proved by a discussion similar to the proof of (d) in Proposition 1.

3.5 Let G be an open set in R with piecewise analytic boundary. Let F be a closed subset of G such that $\overline{R-G}^D \cap \bar{F}^D=\emptyset$. Then there is a bounded continuous Dirichlet function on R such that $f=0$ on $R-G$ and 1 on F. Since $f^{(R-G)\cup F}$ does not depend on the choice of f, we shall denote it by 1_G^F.[4] If F is a regular closed set, then 1_G^F is continuous on G. Let $\{F_n\}_{n=1}^\infty$ be a sequence of regular closed subsets of G such that $F_n \supset F_{n+1}$ $(n=1, 2, \cdots)$ and $\bigcap_{n=1}^\infty F_n=\emptyset$. Suppose $\overline{R-G}^D \cap \bar{F}_1^D=\emptyset$. Then $1_G^{F_n}$ tends to a function, say u, on R locally uniformly and in Dirichlet norm as $n \to \infty$ and u is harmonic on G. Let $\Omega_\alpha=\{z \in G;\ u(z) \geqq \alpha\}$ $(0<\alpha<1)$.

The following lemma is known ([7]):

Lemma 9. *If $u \not\equiv 0$, then*

$$\int_{\partial\Omega_\alpha} \frac{\partial u}{\partial \nu}\, ds=\|u\|_G^2 \qquad \text{for almost all} \quad \alpha,\ 0<\alpha<1.$$

By Lemma 3 in [7] and Lemma 8, we have

Lemma 10. *If f is a bounded continuous Dirichlet function on R, then*

$$\int_{\partial\Omega_\alpha} f^{R-G} \frac{\partial u}{\partial \nu}\, ds$$

is a constant for almost all α, $0<\alpha<1$.

We shall prove

Proposition 3. *Let G, $\{F_n\}_{n=1}^\infty$ and u be as above. Suppose $u \not\equiv 0$. Let F be a regular closed subset of G such that $\overline{R-F^i}^D \cap \bar{F}_1^D=\emptyset$. Then $\overline{(R-G)\cup F_n}^D \cap \overline{\Omega_\alpha-F^i}^D=\emptyset$ for every α $(0<\alpha<1)$ and n, and $1_{G-F_n}^{\Omega_\alpha-F^i}$ tends to 0 locally uniformly on $G-F_n$ and in Dirichlet norm as $\alpha \to 1$ for each n. Furthermore, $1_G^{\Omega_\alpha-F^i}$ tends to 0 locally uniformly on G and in Dirichlet norm as $\alpha \to 1$.*

Proof. By the assumption on F, there is a bounded continuous Dirichlet

4) This function is denoted by $\omega(\partial F, z, G-F)$ in [3], [7].

function f on R such that $f=0$ on F_1 and 1 on $R-F^i$. Then $g=\min(1_G^{\Omega_\alpha}, f)$ is a bounded continuous Dirichlet function on R. Since $g=0$ on $(R-G)\cup F_n$ and 1 on $\Omega_\alpha - F^i$, $\overline{(R-G)\cup F_n{}^D}\cap\overline{\Omega_\alpha - F^{iD}}=\emptyset$ for every α $(0<\alpha<1)$ and n.

Now fix an integer $n>0$ arbitrarily. By (B1), we see that

$$\|1_G^{\Omega_\alpha - F^i}\|_G \leqq \|1_{G-F_n}^{\Omega_\alpha - F^i}\|_{G-F_n} \leqq \|g\|_G < \infty$$

and $1_{G-F_n}^{\Omega_\alpha - F^i}$ (resp. $1_G^{\Omega_\alpha - F^i}$) tends to a harmonic function, say v (resp. v_0), locally uniformly on $G-F_n$ (resp. G) and in Dirichlet norm as $\alpha\to 1$, so that $\|v_0\|_G \leqq \|v\|_{G-F_n}$. Suppose $v\not\equiv 0$. We set $\delta_\alpha=\{z\in G-F_n;\ v(z)\geqq\alpha\}$ $(0<\alpha<1)$. Then, by Lemma 9, we obtain that

$$\int_{\partial\delta_\alpha}\frac{\partial v}{\partial\nu}\,ds=\|v\|_{G-F_n}^2$$

for almost all α, $0<\alpha<1$. We can show that $u^{(R-G)\cup F_n}=u$ on $G-F_n$ by a discussion similar to the proof of (α) in Proposition 2. Hence, by Lemma 10,

$$\int_{\partial\delta_\alpha} u\frac{\partial v}{\partial\nu}\,ds$$

is a constant, say m, for almost all α, $0<\alpha<1$. Since $0\leqq u<1$ on G, we have

$$0<m=\int_{\partial\delta_\alpha} u\frac{\partial v}{\partial\nu}\,ds<\int_{\partial\delta_\alpha}\frac{\partial v}{\partial\nu}\,ds=\|v\|_{G-F_n}^2.$$

Let β_0 $(0<\beta_0<1)$ be a real number such that $m<\beta_0\|v\|_{G-F_n}^2$. On the other hand, since $u^{(R-G)\cup F_n}=u$ on $G-F_n$, $u^{(R-G)\cup F_n\cup(\Omega_\alpha - F^i)}=u$ on $G-F_n$. Since $g\leqq u/\alpha$ on $(R-G)\cup F_n\cup(\Omega_\alpha-F^i)$, we have $1_{G-F_n}^{\Omega_\alpha-F^i}=g^{(R-G)\cup F_n\cup(\Omega_\alpha - F^i)}\leqq u/\alpha$ on $G-F_n$ by (B8). Hence $v\leqq u$ on $G-F_n$. Thus

$$\beta\|v\|_{G-F_n}^2=\int_{\partial\delta_\beta} v\frac{\partial v}{\partial\nu}\,ds\leqq\int_{\partial\delta_\beta} u\frac{\partial v}{\partial\nu}\,ds=m$$
$$<\beta_0\|v\|_{G-F_n}^2$$

for almost all β, $0<\beta<1$. This is a contradiction. Therefore, $v=0$ and hence $v_0=0$.

COROLLARY. *Let $\{F_n\}_{n=1}^\infty$ be a sequence of regular closed subsets of R_0 such that $F_n\supset F_{n+1}$ $(n=1, 2, \dots)$ and $\bigcap_{n=1}^\infty F_n=\emptyset$. Set $\Omega_\alpha=\{z\in R_0;\ \lim_{n\to\infty} 1_{F_n}(z)\geqq\alpha\}$ $(0<\alpha<1)$. If $\overline{R-F_n^{iD}}\cap\bar F_{n+1}^D=\emptyset$ for each n, then $1_{\widetilde{\Omega_\alpha - F_n^i}}$ tends to 0 locally uniformly on R_0 and in Dirichlet norm as $\alpha\to 1$ for each n.*

This is proved easily by the aid of the identity $1_{R_0}^{\Omega_\alpha - F_n^i}=1_{\widetilde{\Omega_\alpha - F_n^i}}$ on R_0 for each n.

REMARK. The above Proposition and Corollary are essentially due to Z. Kuramochi [4] and [6].

3.6 *Capacity on Royden boundary*

Given a closed subset A of Δ_D, we consider the following class:

$$\tilde{\mathfrak{S}}_A = \left\{ v;\ \begin{array}{l} \text{full-superharmonic} \geqq 0 \text{ on } R_0,\ v \geqq 1 \text{ on } U \cap R_0 \\ \text{for some neighborhood } U \text{ of } A \text{ in } R_D^* \end{array} \right\}.$$

Then the function

$$\tilde{\omega}(A) = \tilde{\omega}_a(A) = \inf \{v(a);\ v \in \tilde{\mathfrak{S}}_A\} \qquad (a \in R_0)$$

is full-superharmonic, harmonic on R_0 and $0 \leqq \tilde{\omega}(A) \leqq 1$. The following lemma will show that $\|\tilde{\omega}(A)\| < \infty$.

LEMMA 11. *For any closed subset A of Δ_D, there is a sequence $\{F_n\}_{n=1}^{\infty}$ of regular closed subsets of R_0 which has the following properties:*

a) *each $\bar{F}_n^D$ is a neighborhood of A in R_D^*,*

b) *$F_n \supset F_{n+1} (n=1, 2, \dots)$ and $\bigcap_{n=1}^{\infty} F_n = \emptyset$,*

c) *$1_{\tilde{F}_n} \searrow \tilde{\omega}(A)$ and $\|1_{\tilde{F}_n} - \tilde{\omega}(A)\| \to 0$ as $n \to \infty$.*

PROOF. By a discussion similar to the proof of Lemma 3, we can choose a sequence $\{F_n\}_{n=1}^{\infty}$ of regular closed subsets of R_0 which satisfies a), b) and $1_{\tilde{F}_n} \searrow \tilde{\omega}(A)$ as $n \to \infty$. By Proposition 2, we see that $1_{\tilde{F}_n}$ tends to $\tilde{\omega}(A)$ as $n \to \infty$ in Dirichlet norm.

COROLLARY 1. *If $\{F_n\}_{n=1}^{\infty}$ is a sequence of regular closed subsets of R_0 such that $F_n \supset F_{n+1}$ $(n=1, 2, \dots)$ and $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then $1_{\tilde{F}_n}$ decreases to $\tilde{\omega}(\bigcap_{n=1}^{\infty} \bar{F}_n^D)$,*

$$\|1_{\tilde{F}_n} - \tilde{\omega}(\bigcap_{n=1}^{\infty} \bar{F}_n^D)\| \to 0 \qquad as \quad n \to \infty$$

and $\|1_{\tilde{F}_n}\|$ decreases with n.

COROLLARY 2. *If F is a regular closed set in R, then*

$$\tilde{\omega}(\bar{F}^D \cap \Delta_D) = \lim_{n \to \infty} 1_{\widetilde{F-R_n}},$$

where $\{R_n\}_{n=1}^{\infty}$ is an exhaustion of R.

REMARK. The functions in Corollary 2 are denoted by $\omega(F \cap B, z)$ in [3], [4], [5] and by $\omega(B(F), z)$ in [7].

For any closed subset A of Δ_D, we define

$$C(A) = \frac{1}{2\pi} \int_{\partial K_0} \frac{\partial \tilde{\omega}(A)}{\partial \nu} ds.$$

We call $C(A)$ the capacity of A (with respect to K_0). Let $\{F_n\}_{n=1}^{\infty}$ be a sequence which has the properties in Lemma 1. By Lemma 9 and Green's formula, we have

$$\frac{1}{2\pi}\int_{\partial K_0}\frac{\partial 1_{\tilde{F}_n}}{\partial \nu}ds=\frac{1}{2\pi}\|1_{\tilde{F}_n}\|^2.$$

Since $\partial 1_{\tilde{F}_n}/\partial\nu$ tends to $\partial\tilde{\omega}(A)/\partial\nu$ uniformly on ∂K_0 and $\|1_{\tilde{F}_n}\|^2$ tends to $\|\tilde{\omega}(A)\|^2$ as $n\to\infty$ by c) in Lemma 11, we see that $C(A)=(2\pi)^{-1}\|\tilde{\omega}(A)\|^2$.

We see that both $A\to\tilde{\omega}_a(A)$ and $A\to C(A)$ are Choquet's capacities.

LEMMA 12. *Let F be a regular closed set in R and let $\Omega_\alpha=\{z\in R_0;\ \tilde{\omega}_z(\bar{F}^D\cap\Delta_D)\geqq\alpha\}$ $(0<\alpha<1)$. Then*

$$C(\overline{F-\Omega_\alpha^i}^D\cap\Delta_D)=0$$

for each α.

PROOF. By the aid of Corollary 2 to Lemma 11 and (γ) in Proposition 2, we can prove the lemma in a way similar to the proof of Lemma 5.

LEMMA 13. *Let $\{F_n\}_{n=1}^\infty$ be a sequence of regular closed subsets of R_0 such that $F_n\supset F_{n+1}$ $(n=1, 2, \dots)$ and $\bigcap_{n=1}^\infty F_n=\emptyset$ and let $\Omega_\alpha=\{z\in R_0;\ \tilde{\omega}_z(\bigcap_{n=1}^\infty\bar{F}_n^D)\geqq\alpha\}$ $(0<\alpha<1)$. If $\overline{R-F_n^i}^D\cap\bar{F}_{n+1}^D=\emptyset$ for each n, then $C(\overline{\Omega_\alpha-F_n^i}^D\cap\Delta_D)$ tends to 0 as $\alpha\to 1$ for each n.*

PROOF. Each $\Omega_\alpha-F_n^i$ is a regular closed subset of R_0 for $\alpha\in(0, 1)-E$, where E is a set of at most a countable number of α in $(0, 1)$. By the aid of Corollary 2 to Lemma 11, we have

$$\begin{aligned}C(\overline{\Omega_\alpha-F_n^i}^D\cap\Delta_D)&=\frac{1}{2\pi}\|\tilde{\omega}(\overline{\Omega_\alpha-F_n^i}^D\cap\Delta_D)\|^2\\&\leqq\frac{1}{2\pi}\|1_{\widetilde{\Omega_\alpha-F_n^i}}\|^2\end{aligned}$$

for each $\alpha\in(0, 1)-E$ and each n. It follows from the Corollary to Proposition 3 that

$$C(\overline{\Omega_\alpha-F_n^i}^D\cap\Delta_D)\to 0$$

as $\alpha\to 1$ $(\alpha\in(0, 1)-E)$ for each n. Since $C(\overline{\Omega_{\alpha_1}-F_n^i}^D\cap\Delta_D)\geqq C(\overline{\Omega_{\alpha_2}-F_n^i}^D\cap\Delta_D)$ if $\alpha_1<\alpha_2$ and $(0, 1)-E$ is dense in $(0, 1)$, $C(\overline{\Omega_\alpha-F_n^i}^D\cap\Delta_D)\to 0$ as $\alpha\to 1$ for each n.

We can show

LEMMA 14. *If F_1, F_2 are regular closed sets in R, then*

$$\omega^{K_0}(\bar{F}_1^W\cap\bar{F}_2^W\cap\Delta_W)\leqq\tilde{\omega}(\bar{F}_1^D\cap\bar{F}_2^D\cap\Delta_D)\qquad\text{on}\quad R_0$$

and

$$C(\bar{F}_1^D \cap \bar{F}_2^D \cap \Delta_D) \leqq \tilde{C}(\bar{F}_1^N \cap \bar{F}_2^N \cap \Delta_N),$$

where $\tilde{C}$ is the Kuramochi capacity[5] *on Δ_N (with respect to K_0).*

Corollary. *$C(\Gamma_D)$ is positive for any hyperbolic Riemann surface.*

Proof. By Satz 8.6 in [2], we see that $\omega^{K_0}(\Gamma_W) \leqq \tilde{\omega}(\Gamma_D)$ on R_0. By the aid of Lemma 6, we can show that $\omega^{K_0}(\Gamma_W) > 0$. Therefore $C(\Gamma_D) > 0$.

Remark. Let π be the continuous mapping of R_D^* onto R_N^* whose restriction on R is the identity mapping. If X is a closed subset of Δ_N, then $C(\pi^{-1}(X)) = \tilde{C}(X)$.

§4. H.B. separative compactification

In this section we shall denote by $\bar{A}$ the closure of any subset A of R in R_W^*.

4.1 *Definition of H.B. separative compactification*

Definition. A compactification R^* of R is said to be *H.B. separative* if the following Condition B is satisfied:

Condition B: If F_1, F_2 are regular closed sets in R such that $\bar{F}_1^* \cap \bar{F}_2^* = \emptyset$ in R^*, then

$$\omega\Big(\bigcap_{0<\alpha<1} \overline{\delta_\alpha \cap F_2} \cap \Delta_W\Big) = 0,$$

where $\delta_\alpha = \{z \in R;\, 1_{F_1}(z) \geqq \alpha\}$ $(0<\alpha<1)$.

By virtue of the remark in 2.4, we see that the above definition is equivalent to that introduced by Z. Kuramochi in [7]; also see [4] and [5].

4.2 Proposition 4. *If F_1, F_2 are regular closed sets in R, then*

$$\omega\Big(\bigcap_{0<\alpha<1} \overline{\Omega_\alpha \cap F_2} \cap \Delta_W\Big) = \omega\Big(\bigcap_{0<\alpha<1} \overline{\delta_\alpha \cap F_2} \cap \Delta_W\Big)$$

$$= \omega(\bar{F}_1 \cap \bar{F}_2 \cap \Delta_W),$$

where $\Omega_\alpha = \{z \in R;\, \omega_z(\bar{F}_1 \cap \Delta_W) \geqq \alpha\}$ and $\delta_\alpha = \{z \in R;\, 1_{F_1}(z) \geqq \alpha\}$ $(0<\alpha<1)$.

Proof. By Lemma 4, we see that

$$\bar{F}_1 \cap \Gamma_W = \bigcap_{0<\alpha<1} \bar{\delta}_\alpha \cap \Gamma_W.$$

Since $\omega_z(\bar{F}_1 \cap \Delta_W) \leqq 1_{F_1}(z)$ on R by Corollary 1 to Lemma 3, $\Omega_\alpha \subset \delta_\alpha$ for each α.

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5) See [2] and [3] for this definition.

Hence we have

$$\omega(\bigcap_{0<\alpha<1}\overline{\Omega_\alpha\cap F_2}\cap\Delta_W)\leqq\omega(\bigcap_{0<\alpha<1}\overline{\delta_\alpha\cap F_2}\cap\Delta_W)$$
$$\leqq\omega(\bar{F}_1\cap\bar{F}_2\cap\Delta_W).$$

Since $\bar{F}_1\cap\bar{F}_2\subset(\overline{F_1\cap\Omega_\alpha}\cap\bar{F}_2)\cup\overline{F_1-\Omega_\alpha^i}$ and $\omega(\overline{F_1-\Omega_\alpha^i}\cap\Delta_W)=0$ by Lemma 5, we see that

$$\omega(\bar{F}_1\cap\bar{F}_2\cap\Delta_W)\leqq\omega(\bigcap_{0<\alpha<1}\overline{\Omega_\alpha\cap F_1}\cap\bar{F}_2\cap\Delta_W).$$

Next we shall prove $\bigcap_{0<\alpha<1}\overline{\Omega_\alpha\cap F_1}\cap\bar{F}_2\subset\bigcap_{0<\alpha<1}\overline{\Omega_\alpha\cap F_2}$. We may suppose that $\omega(\bar{F}_1\cap\Delta_W)$ is not a constant. Then $\Omega_\alpha\neq\emptyset$ and $\Omega_\alpha\neq R$ for each α by Lemma 3 and (b) in Proposition 1. Let a be an arbitrary point of $\bigcap_{0<\alpha<1}\overline{\Omega_\alpha\cap F_1}\cap\bar{F}_2$. Suppose there is an α_0 such that $a\notin\overline{\Omega_{\alpha_0}\cap F_2}$. Then there is a neighborhood U of a in R_W^* such that $U\cap\overline{\Omega_{\alpha_0}\cap F_2}=\emptyset$. Since $\{z\in R_W^*;\ \omega_z(\bar{F}_1\cap\Delta_W)\geqq\alpha_0\}$ is a neighborhood of a in R_W^* and its restriction on R is Ω_{α_0}, we may assume that $U\cap R\subset\Omega_{\alpha_0}$. Hence $U\cap F_2=\emptyset$. This shows that a does not belong to $\bar{F}_2$. This is a contradiction. Thus $\bigcap_{0<\alpha<1}\overline{\Omega_\alpha\cap F_1}\cap\bar{F}_2\subset\bigcap_{0<\alpha<1}\overline{\Omega_\alpha\cap F_2}$. Therefore we have

$$\omega(\bar{F}_1\cap\bar{F}_2\cap\Delta_W)\leqq\omega(\bigcap_{0<\alpha<1}\overline{\Omega_\alpha\cap F_2}\cap\Delta_W).$$

This completes the proof.

COROLLARY 1. *Condition B is equivalent to that $\omega(\bar{F}_1\cap\bar{F}_2)=0$ for any regular closed sets F_1, F_2 in R such that $\bar{F}_1^*\cap\bar{F}_2^*=\emptyset$.*

COROLLARY 2. *Let R_1^* and R_2^* be two compactifications of R. Suppose R_1^* is a quotient space of R_2^*. If R_2^* is H.B. separative, then so is R_1^*.*

4.3 *Main theorem on H.B. separative compactifications*

THEOREM 1. *A compactification R^* of R is H.B. separative if and only if it is resolutive.*

PROOF. Let F_1, F_2 be regular closed sets in R. If R^* is resolutive, then $\bar{F}_1^*\cap\bar{F}_2^*=\emptyset$ implies $\bar{F}_1\cap\bar{F}_2=\emptyset$. Hence R^* is H.B. separative by Corollary 1 to Proposition 4.

Conversely, suppose R^* is H.B. separative. First we assume that R^* is metrizable. Let A, B be any mutually disjoint compact subsets of Δ. Then there are two regular closed sets F_1, F_2 in R such that $\bar{F}_1^*$ is a neighborhood of A, $\bar{F}_2^*$ is a neighborhood of B and $\bar{F}_1^*\cap\bar{F}_2^*=\emptyset$. We set $u=\omega(\bar{F}_1\cap\Delta_W)$, $\Omega_\alpha=\{z\in R;\ u(z)\geqq\alpha\}$ $(0<\alpha<1)$ and $\delta_\alpha=R-\Omega_\alpha^i$. Then there is a set E of at most a countable number of α in $(0, 1)$ such that each $F_2 \cap \Omega_\alpha$ is a regular closed set in R for $\alpha \in (0, 1)-E$. We obtain that $1_A \leqq u$ and

$$(1_A)_B \leqq (1_A)_{F_2-R_n} \leqq u_{F_2-R_n} \leqq 1_{\Omega_\alpha \cap F_2-R_n} + u_{\delta_\alpha},$$

where $\{R_n\}_{n=1}^{\infty}$ is an exhaustion of R. By letting $n \to \infty$, we have

$$(1_A)_B \leqq \omega(\overline{\Omega_\alpha \cap F_2} \cap \Delta_W) + u_{\delta_\alpha}$$

on R for each $\alpha \in (0, 1)-E$ by Corollary 2 to Lemma 3. Let $0<\alpha<\beta<1$ $(\alpha \in (0, 1)-E)$. Then $u_{\Omega_\alpha}=u$ on R by (d) in Proposition 1 and $u_{\delta_\alpha} \leqq 1_{\delta_\alpha}$. Hence $u_{\delta_\alpha} \leqq \min(1_{\delta_\alpha}, 1_{\Omega_\beta})$. Since $\bar{\delta}_\alpha \cap \bar{\Omega}_\beta = \emptyset$, u_{δ_α} vanishes on Γ_W by Lemma 4. It follows from the minimum principle (cf. Satz 8.4 in [2]) that

$$(1_A)_B \leqq \omega(\overline{\Omega_\alpha \cap F_2} \cap \Delta_W)$$

on R for each $\alpha \in (0, 1)-E$. Since $\omega(\overline{\Omega_{\alpha_1} \cap F_2} \cap \Delta_W) \geqq \omega(\overline{\Omega_{\alpha_2} \cap F_2} \cap \Delta_W)$ if $\alpha_1 < \alpha_2$ and $(0, 1)-E$ is dense in $(0, 1)$, we obtain that

$$(1_A)_B \leqq \omega(\overline{\Omega_\alpha \cap F_2} \cap \Delta_W)$$

on R for each α in $(0, 1)$. Thus

$$(1_A)_B \leqq \omega(\bigcap_{0<\alpha<1} \overline{\Omega_\alpha \cap F_2} \cap \Delta_W) = 0$$

by Proposition 4 and Condition B. Therefore R^* is resolutive by Lemma 1.

Next we consider the case where R^* is not necessarily metrizable. We can find a family Q of bounded continuous functions on R such that $R^* = R_Q^*$. Let f_0 be any function in Q and set $Q_0 = \{f_0\}$. It follows from Corollary 2 to Proposition 4 that $R_{Q_0}^*$ is H.B. separative. Since $R_{Q_0}^*$ is metrizable, by the above discussion, we see that $R_{Q_0}^*$ is resolutive. Hence, it follows from Hilfssatz 8.2 in [2] that f_0 is a Wiener function on R. Thus any function in Q is a Wiener function. Therefore $R^* = R_Q^*$ is resolutive (cf. Satz 9.3 in [2]).

COROLLARY 1 (Z. Kuramochi [4], [7]). *The Martin and Kuramochi compactifications are H.B. separative.*

COROLLARY 2. *A compactification R^* of R is resolutive if and only if $\omega(\bar{F}_1 \cap \bar{F}_2)=0$ for any regular closed sets F_1, F_2 in R such that $\bar{F}_1^* \cap \bar{F}_2^* = \emptyset$ in R^*.*

§5. H.D. separative compactification

In this section we shall denote by $\bar{A}$ the closure of any subset A of R in R_D^*. Let K_0 be a closed disk in R and let $R_0 = R - K_0$.

5.1. *Definition of H.D. separative compactification*

Definition. A compactification R^* of R is said to be *H.D. separative* if the following Condition D is satisfied:

Condition D: If F_1, F_2 are regular closed sets in R such that $\bar{F}_1^* \cap \bar{F}_2^* = \emptyset$ in R^*, then there is an increasing sequence $\{V_n\}_{n=1}^{\infty}$ such that

a) each V_n is a relatively open subset of F_2 such that $F_2 - V_n$ is a regular closed set in R,

b) $C(\overline{F_2 - V_n} \cap \Delta_D) \to 0$ as $n \to \infty$,

c) $\bar{F}_1 \cap \overline{V_n \cup \partial V_n} = \emptyset$ for each n.

REMARK. (i) The property b) does not depend on the choice of K_0.

(ii) The property c) is equivalent to the fact that there is a bounded continuous Dirichlet function f_n on R such that $f_n = 0$ on F_1 and 1 on $V_n \cup \partial V_n$.

By virtue of the first remark in 3.6, we see that the above definition is equivalent to that defined by *Z.* Kuramochi in [4] and [5] (cf. [7], §3; in particular footnote 4) and Lemma 5).

5.2 PROPOSITION 5. *If F_1, F_2 are regular closed sets in R, then*

$$C(\bigcap_{0<\alpha<1} \overline{\Omega_\alpha \cap F_2}) = C(\bar{F}_1 \cap \bar{F}_2 \cap \Delta_D),$$

where $\Omega_\alpha = \{z \in R_0;\ \tilde{\omega}_z(\bar{F}_1 \cap \Delta_D) \geqq \alpha\}$ $(0<\alpha<1)$.

PROOF. By a discussion similar to the proof of Proposition 4, we can prove that

$$C(\bar{F}_1 \cap \bar{F}_2 \cap \Delta_D) \leqq C(\bigcap_{0<\alpha<1} \overline{\Omega_\alpha \cap F_2})$$

by using Lemma 12. Now we shall prove the converse inequality.

For $A = \bar{F}_1 \cap \bar{F}_2 \cap \Delta_D$ (resp. $A = \bar{F}_1 \cap \Delta_D$), there is a sequence $\{\Omega_n\}_{n=1}^{\infty}$ (resp. $\{\delta_n\}_{n=1}^{\infty}$) of regular closed subsets of R_0 which satisfies a), b) and c) in Lemma 11. Since R_D^* is a normal space, we may assume that $\overline{R - \delta_n^i} \cap \bar{\delta}_{n+1} = \emptyset$ and $\bar{\delta}_n \cap \bar{F}_2 \cap \Delta_D \subset \bar{\Omega}_n \cap \Delta_D$ for each n. Since $\overline{\Omega_\alpha \cap F_2} \subset \overline{\delta_n \cap F_2} \cup \overline{\Omega_\alpha - \delta_n^i}$, we have

$$C(\overline{\Omega_\alpha \cap F_2} \cap \Delta_D) \leqq C(\overline{\delta_n \cap F_2} \cap \Delta_D) + C(\overline{\Omega_\alpha - \delta_n^i} \cap \Delta_D).$$

It follows from Lemma 13 and the definition of capacity that

$$C(\bigcap_{0<\alpha<1} \overline{\Omega_\alpha \cap F_2}) \leqq C(\overline{\delta_n \cap F_2} \cap \Delta_D) \leqq C(\bar{\delta}_n \cap \bar{F}_2 \cap \Delta_D)$$

$$\leqq C(\bar{\Omega}_n \cap \Delta_D) \leqq \frac{1}{2\pi} \|1_{\tilde{\Omega}_n}\|^2.$$

By letting $n \to \infty$, we have $C(\bigcap_{0<\alpha<1} \overline{\Omega_\alpha \cap F_2}) \leqq C(\bar{F}_1 \cap \bar{F}_2 \cap \Delta_D)$ and thus we complete the proof.

5.3 *Main theorems on H.D. separative compactifications*

THEOREM 2. *A compactification R^* of R is H.D. separative if and only if $C(\bar{F}_1 \cap \bar{F}_2)=0$ for any regular closed sets F_1, F_2 in R such that $\bar{F}_1^* \cap \bar{F}_2^* = \emptyset$ in R^*.*

PROOF. Let F_1, F_2 be regular closed sets in R such that $\bar{F}_1^* \cap \bar{F}_2^* = \emptyset$ in R^*. Suppose R^* is H.D. separative. Let $\{V_n\}_{n=1}^{\infty}$ be a sequence which satisfies a), b) and c) in Condition D. Since $\bar{F}_1 \cap \bar{F}_2 \subset \overline{F_2 - V_n}$, we see that

$$C(\bar{F}_1 \cap \bar{F}_2) \leqq C(\overline{F_2 - V_n} \cap \varDelta_D) \to 0$$

as $n \to \infty$.

Conversely, suppose $C(\bar{F}_1 \cap \bar{F}_2)=0$. We set $A=\bar{F}_1 \cap \bar{F}_2$. If $A=\emptyset$, then we can take F_2 for V_n $(n=1, 2, \dots)$ and see that R^* is H.D. separative. Hence we may assume that $A \neq \emptyset$. Let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence for A which satisfies a), b) and c) in Lemma 11. Then $V_n=F_2-\delta_n$ is a relatively open subset of F_2 and increases with n. We may assume that each V_n is a regular closed set in R. It is easy to see that

$$\bar{F}_1 \cap \overline{V_n \cup \partial V_n} = \emptyset \qquad \text{for each} \quad n.$$

Since $F_2 - V_n \subset \delta_n$ for each n, we see that

$$C(\overline{F_2 - V_n} \cap \varDelta_D) \leqq C(\bar{\delta}_n \cap \varDelta_D) \leqq \frac{1}{2\pi} \|1_{\tilde{\delta}_n}\|^2$$

$$\to \frac{1}{2\pi} \|\tilde{\omega}(A)\|^2 = C(\bar{F}_1 \cap \bar{F}_2) = 0$$

as $n \to \infty$. Therefore R^* is H.D. separative.

THEOREM 3. *If a compactification R^* of R is a quotient space of R_D^*, then it is H.D. separative. The converse is not true.*

PROOF. Theorem 2 shows that if R^* is a quotient space of R_D^*, then it is H.D. separative. Now we shall prove that the converse is not true. Let R be a unit disk $\{z;\ |z|<1\}$ in the complex plane. We take R_Q^* which is defined in Example 1 in 1.3. Then R_Q^* is not a quotient space of R_D^*. We shall prove that R_Q^* is H.D. separative. We take $\{z \in R;\ |z| \leqq 1/2\}$ for K_0. We may identify the Kuramochi compactification of R with the closed disk $\{z;\ |z| \leqq 1\}$ (see p. 167 in [2]). We denote by $\tilde{C}$ the Kuramochi capacity on $\{z;\ |z|=1\}$ with respect to K_0. Let F_1, F_2 be regular closed sets in R such that $\bar{F}_1^* \cap \bar{F}_2^* = \emptyset$ in R_Q^*. Then we can show that

$$\bar{F}_1^N \cap \bar{F}_2^N \subset \{e^{i\pi/2},\ e^{-i\pi/2}\}.$$

Hence

$$C(\bar{F}_1\cap\bar{F}_2)\leqq\tilde{C}(\bar{F}_1^N\cap\bar{F}_2^N)\leqq\tilde{C}(\{e^{i\pi/2},\ e^{-i\pi/2}\})=0$$

by Lemma 14. Therefore R_Q^* is H.D. separative by Theorem 2.

COROLLARY (Z. Kuramochi [4], [7]). *The Kuramochi compactification of R is H.D. separative.*

THEOREM 4. *If a compactification R^* of R is H.D. separative, then it is resolutive. The converse is not true.*

PROOF. Let F_1, F_2 be regular closed sets in R such that $\bar{F}_1^*\cap\bar{F}_2^*=\emptyset$ in R^*. Then

$$\omega^{K_0}(\bar{F}_1^W\cap\bar{F}_2^W)\leqq\tilde{\omega}(\bar{F}_1\cap\bar{F}_2)=0$$

by Lemma 14 and Theorem 2. It follows from the Corollary to Lemma 6 that $\omega(\bar{F}_1^W\cap\bar{F}_2^W)=0$. Hence R^* is resolutive by Corollary 1 to Proposition 4. Now we shall prove that the converse is not true. Let R be a Riemann surface which belongs to $O_{HD}-O_{HB}$. We take R_Q^* which is defined in Example 2 in 1.3. Then R_Q^* is resolutive. Let $F_1=\{z\in R;\ g(z)\geqq 2/3\}$ and $F_2=\{z\in R;\ g(z)\leqq 1/3\}$. Then $\bar{F}_1^*\cap\bar{F}_2^*=\emptyset$ in R_Q^*. We may assume that both F_1 and F_2 are regular closed sets in R. Since $\bar{F}_k^W(k=1,2)$ is a neighborhood of a_k $(k=1,2)$ in R_W^*, it follows from Satz 8.6 in [2] that $b\in\bar{F}_1\cap\bar{F}_2$. Hence we have

$$0<C(\{b\})\leqq C(\bar{F}_1\cap\bar{F}_2)$$

by the Corollary to Lemma 14. Therefore R_Q^* is not H.D. separative by Theorem 2.

§6. Remarks on Fatou's and Beurling's theorems

Let ϕ be an analytic mapping of an open Riemann surface R into another open Riemann surface R'. Suppose R' is hyperbolic. Let Δ_1 (resp. $\tilde{\Delta}_1$) be the set of all minimal points of the Martin (resp. Kuramochi) boundary of R. For each $b\in\Delta_1$ (resp. $\tilde{b}\in\tilde{\Delta}_1$), we denote by $\mathfrak{G}_b$ (resp. $\mathfrak{G}_{\tilde{b}}$) the system of fine neighborhoods of b (resp. $\tilde{b}$).[6] Let R'^* be a metrizable compactification of R' and consider the following sets:

$$\mathfrak{F}(\phi)=\{b\in\Delta_1;\ \bigcap_{G\in\mathfrak{G}_b}\overline{\phi(G)}^*\text{ is one point}\}$$

and

$$\tilde{\mathfrak{F}}(\phi)=\{\tilde{b}\in\tilde{\Delta}_1;\ \bigcap_{G\in\mathfrak{G}_{\tilde{b}}}\overline{\phi(G)}^*\text{ is one point}\},$$

where $\overline{\phi(G)}^*$ means the closure of $\phi(G)$ in R'^*. It is known (cf. [2], [4], [6],

6) See p. 145 and p. 221 in [2]; also see § 2 in [7].

[7]) that both $\mathfrak{F}(\phi)$ and $\widetilde{\mathfrak{F}}(\phi)$ are Borel sets.

6.1 *Fatou's theorem*

We shall denote by χ the harmonic measure on $R_M^* - R$.

THEOREM F1 (Z. Kuramochi [4], [6], [7]). *If R'^* is H.B. separative, then*

$$\chi(\Delta_1 - \mathfrak{F}(\phi)) = 0.$$

THEOREM F2 (C. Constantinescu and A. Cornea; Satz 14.4 in [2]). *If R'^* is resolutive, then*

$$\chi(\Delta_1 - \mathfrak{F}(\phi)) = 0.$$

The following theorem is an immediate consequence of Theorem 1:

THEOREM 5. *Theorems* F1 *and* F2 *are equivalent.*

6.2 *Beurling's theorem*

We shall denote by $\tilde{C}$ the Kuramochi capacity on $R_N^* - R$ with respect to a fixed closed disk K_0.

THEOREM B1 (Z. Kuramochi [4], [6], [7]). *If R'^* is H.D. separative and ϕ is an almost finitely sheeted mapping,*[7] *then*

$$\tilde{C}(\tilde{\Delta}_1 - \widetilde{\mathfrak{F}}(\phi)) = 0.$$

THEOREM B2 (C. Constantinescu and A. Cornea; Satz 18.1 in [2]). *If R'^* is a quotient space of $R_D'^*$ and ϕ is a Dirichlet mapping,*[8] *then*

$$\tilde{C}(\tilde{\Delta}_1 - \widetilde{\mathfrak{F}}(\phi)) = 0.$$

We shall prove

THEOREM 6. *Theorems* B1 *and* B2 *are independent.*

PROOF. First we take $R = R' = \{z;\ |z| < 1\}$, $w = \phi(z) = z$ and $R'^* = R_Q'^*$ in Example 1 in 1.3. By the proof of Theorem 3, we see that the conditions of Theorem B1 are satisfied by this example. However, the assumptions in Theorem B2 are not satisfied.

Next we set

$$R = \left\{z;\ z = x + iy,\ -1 < x < 1,\ 0 < y < \frac{\pi}{\sqrt{1+x}} + \frac{\pi}{\sqrt{1-x}}\right\},$$

$$R' = \{w;\ e^{-1} < |w| < e\},\ w = \phi(z) = e^z \text{ and } R'^* = R_N'^*.$$

7) See [4], [6] and [7] for this definition.

8) This is called eine Dirichletsche Abbildung in [2].

It follows from Folgesatz 10.3 in [2] that ϕ is a Dirichlet mapping. On the other hand, we can show that ϕ is not a finitely sheeted mapping. Hence the conditions of Theorem B1 are satisfied but the conditions of Theorem B2 are not. Thus we complete the proof.

References

[1] M. Brelot: *Le problème de Dirichlet axiomatique et frontière de Martin*, J, Math. Pures Appl., (9) **35** (1956), 297-335.

[2] C. Constantinescu and A. Cornea: *Ideale Ränder Riemannscher Flächen*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[3] Z. Kuramochi: *Potentials on Riemann surfaces*, J. Fac. Sci. Hokkaido Univ. Ser. I, **16** (1962), 5-79.

[4] Z. Kuramochi: *On the behaviour of analytic functions on the ideal boundary, I-IV*, Proc. Japan Acad., **38** (1962), 150-155, 188-193, 194-198, 199-203.

[5] Z. Kuramochi: *Correction to the paper "On the behaviour of analytic functions"*, Proc. Japan Acad., **39** (1963), 27-32.

[6] Z. Kuramochi: *Correspondence of boundaries of Riemann surfaces*, J. Fac. Sci. Hokkaido Univ. Ser. I, **17** (1963), 96-122.

[7] Z. Kuramochi: *On Beurling's and Fatou's theorems*, Lecture Notes in Mathematics, Vol. **58**, Kuramochi boundaries of Riemann surfaces, Springer-Verlag, 1968.

[8] M. Ohtsuka: *Dirichlet principle on Riemann surfaces*, J. Analyse Math., **19** (1967), 295-311.

Department of Mathematics,
Faculty of Science,
Hiroshima University