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On Certain Classes of Algebras

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Sugiura [4] and Jôichi [2] have studied the classes (A), (A_k), $k\geqslant 2$ and (A_∞) of Lie algebras. In this paper we define these as well as some other classes for general nonassociative algebras and obtain a characterization of alternative algebras over a field of characteristic zero belonging to any one of these classes. Incidentally, we obtain certain results which include striking improvements of earlier ones due to Sugiura (loc. cit.) and Jôichi (loc. cit.).

1. In what follows, A is a finite-dimensional nonassociative algebra over a field F and $L_x(R_x)$ denotes the left (right) multiplication by x in A. The classes mentioned at the outset are defined as follows:

Definition 1.1. A is said to be an (X)-algebra, where (X) is any one of (A), (A'), $(A_k)_{k\geqslant 2}$, $(A'_k)_{k\geqslant 2}$, $(B_k)_{k\geqslant 2}$, $(B'_k)_{k\geqslant 2}$, (A_∞), (A'_∞), (B_∞), (B'_∞), according as the corresponding property (X) given below is satisfied (x, y, z, t in A):

(A) : $x(xy)=0\Rightarrow xy=0$ and $(zt)t=0\Rightarrow zt=0$

(A') : $x(xy)=0=(yx)x\Rightarrow xy=0=yx$

$(A_k)_{k\geqslant 2}$: $L_x^k=0\Rightarrow L_x=0$ and $R_y^k=0\Rightarrow R_y=0$

$(A'_k)_{k\geqslant 2}$: $L_x^k=0=R_x^k\Rightarrow L_x=0=R_x$

$(B_k)_{k\geqslant 2}$: $yL_x^k=0\Rightarrow yL_x=0$ and $zR_t^k=0\Rightarrow zR_t=0$

$(B'_k)_{k\geqslant 2}$: $yL_x^k=0=yR_x^k\Rightarrow yL_x=0=yR_x$

(X_∞) : (X) holds for all $k\geqslant 2$, $(X)=(A_k)$, (A'_k), (B_k), or (B'_k).

Remark 1. The properties (A), (A_k), stated above may be further weakened by considering just the left or right multiplications independently. (This weakening has, of course, no significance for commutative or anticommutative algebras.) The resulting concepts of right (A)-algebra, left (A)-algebra, and (A)-algebra would then be distinct, (e.g.), the algebra A with basis u, v: $u^2=v^2=vu=0$; $uv=u$, is a right (A)-(also right (A_2)-) algebra but not a left (A)-(or left (A_2)-) algebra, i.e., not an (A)-(or (A_2)-) algebra.

The following chart indicates the connections among the properties mentioned in Definition 1.1.

$$
\begin{tikzcd}
 & (B_\infty) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \cdots \arrow[r, Leftrightarrow] & (B_{k+1}) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & (B_k) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \cdots \arrow[r, Leftrightarrow] & (B_2) \arrow[d, Rightarrow] \\
(A) \arrow[ur, Leftrightarrow] \arrow[r, Rightarrow] \arrow[ddr, Rightarrow] & (A_\infty) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \cdots \arrow[r, Rightarrow] & (A_{k+1}) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (A_k) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \cdots \arrow[r, Rightarrow] & (A_2) \arrow[d, Rightarrow] \\
 & (A'_\infty) \arrow[r, Rightarrow] & \cdots \arrow[r, Rightarrow] & (A'_{k+1}) \arrow[r, Rightarrow] & (A'_k) \arrow[r, Rightarrow] & \cdots \arrow[r, Rightarrow] & (A'_2) \\
 & (B'_\infty) \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \cdots \arrow[r, Rightarrow] & (B'_{k+1}) \arrow[r, Rightarrow] \arrow[u, Rightarrow] & (B'_k) \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \cdots \arrow[r, Rightarrow] & (B'_2) = (A'). \arrow[u, Rightarrow]
\end{tikzcd}
$$

Among the properties stated in Definition 1.1, $(A)\Leftrightarrow(A')\Leftrightarrow(B_k)\Leftrightarrow(B'_k)$ and $(A_k)\Leftrightarrow(A'_k)$, when the algebra A is commutative or anticommutative, in particular, when A is a Lie algebra. On the other hand, the properties are distinct for general algebras. For instance, (A) is distinct from (A'), (e.g.), the algebra A with basis u, v: $u^2=v$, $v^2=uv=0$, $vu=-u$ is not an (A)-algebra, but is an (A')-algebra. For an algebra which is a right (A)-algebra, but not an (A')-algebra, we have the example cited in Remark 1.

In the case of alternative algebras, we can supplement, as follows, the implications in the above chart.

PROPOSITION 1.2. *For an alternative algebra, any property in each one of the following families is equivalent to any other in the same family.*
(i) $(A_k)_{k\geqslant 2}$, (A_∞) (ii) $(A'_k)_{k\geqslant 2}$, (A'_∞) (iii) $(B'_k)_{k\geqslant 2}$, (B'_∞).

PROOF. We only give the proof in the case when A is a (B'_k)-algebra, leaving other similar proofs. After the implications in the above chart, we need only show that an alternative (B'_2)-algebra A is also a (B'_k)-algebra for all k. Suppose $yL_x^k=0=yR_x^k$ for some x, y in A, and let t be an integer such that 2^t is greater than k. Then $yL_x^{2^t}=0=yR_x^{2^t}$; by biassociativity property (see [3] for this and related concepts) of the alternative algebra, $yL_{x^{2^t}}=0=yR_{x^{2^t}}$. Thus $y(L_{x^{2^{t-1}}})^2=0=y(R_{x^{2^{t-1}}})^2$. Hence, by (B'_2)-property, $yL_{x^{2^{t-1}}}=0=yR_{x^{2^{t-1}}}$. We can repeat this argument to show that $yL_{x^2}=0=yR_{x^2}$ and ultimately $yL_x=0=yR_x$, or that A is a (B'_k)-algebra.

We note that any subalgebra of an (A)-, (A')- or (B'_k)-algebra is evidently of the same type. We also have the

PROPOSITION 1.3 (cf. [2, Proposition 2]). *Let B be an ideal of an algebra A. Then*

(i) *if A is an (A_{k+1}) (respectively (A'_{k+1}))-algebra, then B is an (A_k) (respectively (A'_k))-algebra;*

(ii) *for B contained in the annihilator ideal $I\equiv\{x\in A\mid xy=0=yx$ for all y in $A\}$, if A is an (A_{k+1}) (respectively (A'_{k+1}), (B'_{k+1}))-algebra, then A/B is an (A_k) (respectively (A'_k), (B'_k))-algebra. On the other hand, if A is an (A)-algebra, A/B is also an (A)-algebra.*

The proof of Proposition 1.3 is similar to that of the result referred to against it and is omitted.

PROPOSITION 1.4 (cf. [2, Corollary to Proposition 4]). *Let A be an algebra such that $A=A_1\oplus A_2\oplus\cdots\oplus A_r$ for ideals A_i of A. Then A is an (A)-(respec-*

tively (A')-, (A_k)-, (A'_k)-, (B'_k)-) algebra if and only if each $A_i (i=1, 2, \cdots, r)$ is respectively so.

PROOF. We give only the proof for the case when A is an (A_k)-algebra, leaving similar proofs for the rest. Let x be in A_i such that $R_x^k=0$ on A_i. Since R_x is already 0 on $A_j (j \neq i)$, $R_x^k=0$ on A, so that by (A_k)-property of A, $R_x=0$. Similarly for the left multiplications. Thus A_i is an (A_k)-algebra. Conversely, suppose A_i are (A_k)-algebras. Let for an $x=x_1+x_2+\cdots+x_r$ in A, $R_x^k=0$. Thus, for $y=y_1+y_2+\cdots+y_r$ in A, $(y_1+y_2+\cdots+y_r)R_x^k=0$, or $y_1R_{x_1}^k+y_2R_{x_2}^k+\cdots+y_rR_{x_r}^k=0$; by (A_k)-property in A_i, $R_{x_i}=0$ $(i=1, 2, \cdots, r)$, i.e., $yR_x=\sum y_iR_{x_i}=0$ and $R_x=0$, since y_i are arbitrary in A_i. Hence A is an (A_k)-algebra.

2. In this section we generalize certain results of Jôichi [2] and Sugiura [4]. The generalization, in fact, comprises of extending their results to more general algebras, and in some cases simultaneously removing the restriction that the base field is of characteristic zero.

For an ideal B of an algebra A, we define $B^{(1)}=B$, $B^{(2)}=B^{(1)}B^{(1)}$, $\cdots$, and inductively $B^{(n+1)}=B^{(n)}B^{(n)}$, where, for subspaces C_1, C_2 of A, C_1C_2 denotes the subspace of A generated by all elements of the form x_1x_2 for x_i in C_i $(i=1, 2)$. B is said to be *solvable* in case there exists an integer n such that $B^{(n)}=0$. The set I of all absolute divisors of zero in A, viz., $\{x \in A \mid xy=yx=0 \text{ for all } y \text{ in } A\}$, is an ideal of A and is called the *annihilator ideal* of A. A is said to be a *zero algebra* if $A^{(2)}$ is the zero ideal.

PROPOSITION 2.1 (cf. [4, Proposition 4]). *Any solvable (A')-algebra A is a zero algebra.*

PROOF. Suppose $A^{(1)}, A^{(2)}, \cdots, A^{(k)}, \cdots$ is the derived series of A. Let n be the integer such that $A^{(n)} \neq 0$, $A^{(n+1)}=0$. If $n=1$, there is nothing to prove. Hence, let n be greater than 1. Then we have $A^{(n)}(A^{(n-1)}A^{(n-1)})=0=(A^{(n-1)}A^{(n-1)})A^{(n)}$. Since $A^{(n)}$ is contained in $A^{(n-1)}$, $A^{(n)}(A^{(n)}A^{(n-1)})=0=(A^{(n-1)}A^{(n)})A^{(n)}$. Consequently, for x in $A^{(n)}$, y in $A^{(n-1)}$, $x(xy)=0=(yx)x$. But, by (A')-property, $xy=0=yx$, i.e., $A^{(n)}A^{(n-1)}=0=A^{(n-1)}A^{(n)}$, or $A^{(n-1)}(A^{(n-1)}A^{(n-1)})=0=(A^{(n-1)}A^{(n-1)})A^{(n-1)}$. Again, for z, t in $A^{(n-1)}$, $t(tz)=0=(zt)t$. Then, by (A')-property, $tz=0=zt$. In other words, $A^{(n)}=A^{(n-1)}A^{(n-1)}=0$, a contradiction. We should therefore have $n=1$, in which case A is a zero algebra.

COROLLARY 2.2. *Any solvable ideal B of an (A')-algebra A is contained in the annihilator ideal I of A.*

PROOF. B, which is a subalgebra of an (A')-algebra, is itself an (A')-algebra. By Proposition 2.1, B is a zero algebra. Consequently, B being an ideal of A, for x in B and y in A, $x(xy)=0=(yx)x$. Then, by (A')-property $xy=0=yx$. In other words, B is contained in I.

COROLLARY 2.3. *The solvable radical (maximal solvable ideal) of an (A')-algebra A is the annihilator ideal of A.*

For the proof of Corollary 2.3, it suffices to observe that the annihilator ideal of an algebra is a solvable ideal.

We remark that Sugiura's arguments in the proof of his result referred to against Proposition 2.1, are valid only for Lie algebras, and use the restriction of zero characteristic on the base field heavily. We observe that a weakening of the hypothesis of Proposition 2.1 may not be possible. More explicitly, a solvable right (A)-algebra need not be a zero algebra, as the example of a right (A)-algebra (which is also a non-(A')-algebra) given in Remark 1, shows.

We recall now that an algebra A is said to be *nilpotent* if there exists a fixed integer n such that all products of n elements of A are zero irrespective of how they are associated [3]. For such algebras we have the following

PROPOSITION 2.4 (cf. [2, Theorem 1(a)]). *If A is a nilpotent algebra over a field F, then the properties (A_2), $\cdots$, (A_k), $\cdots$, (A_∞), (A), (A'_2), $\cdots$, (A'_k), $\cdots$, (A'_∞), (A'), (B'_2), $\cdots$, (B'_k), $\cdots$, (B'_∞) are all equivalent to the property that A is a zero algebra.*

PROOF. If A is a zero algebra, it is evidently an (A)-algebra. After this, in view of the relations indicated by the chart in Section 1, it remains to show that a nilpotent (A'_2)-algebra is a zero algebra. Let then A be a nilpotent (A'_2)-algebra, and $A^1=A$, $A^2=AA$, $\cdots$, $A^n=\{A^{n-1}A,\ AA^{n-1}\}$, $\cdots$. A being nilpotent, there exists an integer n such that $A^n\neq 0$, $A^{n+1}=0$. Then $A^nA=0=AA^n$; in particular, $A(A^{n-1}A)=0=(AA^{n-1})A$, i.e., for x in A^{n-1}, $AL_x^2=0=AR_x^2$. By (A'_2)-property, $L_x=0=R_x$, or $A^{n-1}A=0=AA^{n-1}$. In other words $A^n=0$, a contradiction to our assumption. As in the proof of Proposition 2.1, this contradiction leads to the conclusion that A is a zero algebra.

COROLLARY 2.5. *A nilpotent ideal B of an (A'_3)-algebra A is contained in the annihilator ideal I of A.*

PROOF. B, which is an ideal of an (A'_3)-algebra, is itself an (A'_2)-algebra, by Proposition 1.3(i). Hence, by Proposition 2.4, B is a zero algebra, i.e., for x in B, y in A, $x(xy)=0=(yx)x$; $L_x^2=0=R_x^2$. Since an (A'_3)-algebra is also (A'_2), this means that $L_x=0=R_x$. In other words, B is contained in the annihilator ideal of A.

COROLLARY 2.6. *The nilradical (maximal nilpotent ideal) of an (A'_3)-algebra A, whenever it exists, is precisely the annihilator ideal I of A.*

Corollary 2.7. *The radical of a Jordan* (A_3)*-algebra* A (*the maximal solvable, or nilpotent ideal of* A (see [3])) *is precisely the annihilator ideal* I *of* A.

The essential part of Corollary 2.7 holds for another class of algebras, namely, the algebras in which the derived series of an ideal consists of ideals; we have

Proposition 2.8. *Let* B *be a solvable ideal of an* (A_3')*-algebra* A *such that the derived series of* B *consists of ideals of* A. *Then,* B *is contained in the annihilator ideal* I *of* A.

Proof. Let B be a solvable ideal such that $B^{(k)}$ are ideals of A. Let $B^{(n)}\neq 0$, $B^{(n+1)}=0$. Then $B^{(n)}B^{(n)}=0$, so that $B^{(n)}(B^{(n)}A)=0=(AB^{(n)})B^{(n)}$, since $B^{(n)}$ is an ideal of A. For x in $B^{(n)}$, this means that $L_x^2=0=R_x^2$, and by (A_3')-property, $L_x=0=R_x$, i.e., $B^{(n)}A=0=AB^{(n)}$. Hence $(B^{(n-1)}B^{(n-1)})A=0=A(B^{(n-1)}B^{(n-1)})$; $B^{(n-1)}(B^{(n-1)}B^{(n-1)})=0=(B^{(n-1)}B^{(n-1)})B^{(n-1)}$. Since $B^{(n-1)}$ is an ideal of A, this means that $B^{(n-1)}(B^{(n-1)}(B^{(n-1)}A))=0=((AB^{(n-1)})B^{(n-1)})B^{(n-1)}$. Hence, for x in $B^{(n-1)}$, $L_x^3=0=R_x^3$, and by (A_3')-property, $L_x=0=R_x$, i.e., $B^{(n-1)}A=0=AB^{(n-1)}$. In particular, $B^{(n)}=B^{(n-1)}B^{(n-1)}=0$, a contradiction to our assumption. Thus $B^{(2)}=0$. Now, for x in B, $AR_x^2=(AR_x)R_x\subseteq BR_x=0$. Similarly $AL_x^2=0$. Since an (A_3')-algebra is also an (A_2')-algebra, this means that $AB=BA=0$; thus B is contained in the annihilator ideal I of A.

Corollary 2.9 (cf. [2, Theorem 1 (b)]). *A solvable ideal of a Lie* (A_3)*-algebra is contained in the center of the Lie algebra.*

Corollary 2.10. *The radical of a Lie* (A_3)*-algebra is precisely its center.*

Corollary 2.11. *The radical of an alternative* (*associative*) (A_3')*-algebra* A *is precisely its annihilator ideal.*

Corollary 2.11 can be derived either from Proposition 2.8, using the fact that the derived series consists of ideals, or from Corollary 2.6, noting that the radical of an alternative algebra is also its maximal nilpotent ideal (see [3]). On the other hand, Corollary 2.7 relating to Jordan algebras (whose derived series in general does not consist of ideals) and falling in the class of these two results, rests on the equivalence of solvability and nilpotency for a Jordan algebra over a field of characteristic $\neq 2$ (see [3]).

It is evident from the proof of Proposition 2.8, that the replacement of the hypothesis of property (A_3') in that proposition by that of property (A_2') does not seem to be possible for general algebras (see [2, Section 3]). However, in the case of alternative algebras, Proposition 1.2 and Corollary 2.11 establish the validity of Proposition 2.8 for an (A_2')-algebra. We thus have

Lemma 2.12. *The radical* (*maximal nilpotent ideal*) *of an alternative* (A_2')-

algebra is precisely its annihilator ideal.

Before concluding this section and passing on to the application of Lemma 2.12 in the proof of the main theorem of this paper in Section 3, we consider a power-associative (A)- or (A')- or (B'_k)-algebra A with radical R (maximal nilideal of A). For x in R, $x^n=0$ for some n. By repeated application of (A')-property we deduce that $x^2=0$ for every x in R. Let x, y be in R. Then $(x+y)^2=0$ implies $xy+yx=0$. Hence, when A is a commutative power-associative (A')-(here, $(B'_k)=(A)=(A')$) algebra over a field F of characteristic $\neq 2$, $xy=0$ for x, y in R. Since x in R, y in A imply xy in R, this means $x(xy)=0$ and by (A)-property, $xy=0$. Hence, *the radical R is the annihilator ideal* $I\equiv\{x \in A \mid xy=0$ for all y in $A\}$, since I is a nilideal. This result, which is of the same type as Corollaries 2.7, 2.11 or Lemma 2.12, can be reformulated as follows: *A commutative power-associative (A)-algebra without absolute divisors of zero is semi-simple.* However, there are semisimple commutative power-associative algebras which are not (A)-algebras, e.g., the simple Jordan algebra, got by introducing the Jordan multiplication $x\circ y=xy+yx/2$ in a full matrix algebra of order $n>1$ over a field F. (See also Section 3.)

3. We first obtain (Theorem 3.1) a characterization of semisimple alternative (A'_2)-algebras, which eventually leads to the main characterization theorem (Theorem 3.2). Any such algebra A is a direct sum of ideals A_i $(i=1, 2, \cdots, r)$, which are simple as algebras [3, Theorem 3.12]. By Proposition 1.4, A_i are (A'_2)-algebras. Further, A_i, which is a simple alternative algebra, is either an associative algebra, or an alternative non-associative division algebra, over its center, or the vector-matrix algebra of Zorn (i.e., the split Cayley algebra with divisors of zero) [3, pp. 52–57]. First, when A_i is a simple associative algebra, it is a full matrix algebra of order n_i over an associative division algebra D_i [1, p. 39]. If $n_i>1$, and if e is the identity of D_i, the matrix E_{12} with e at the (1,2)-th entry and 0 elsewhere, is such that $E_{12}^2=0$, $E_{12}\neq 0$, i.e., $L_{E_{12}^2}=0=R_{E_{12}^2}$, but $L_{E_{12}}$, $R_{E_{12}}\neq 0$ (L, R denote the left, right multiplications in A_i; since A_i contains an identity, $E_{12}\neq 0$ implies that $L_{E_{12}}$, $R_{E_{12}}\neq 0$). Hence A_i is not an (A'_2)-algebra, a contradiction. Consequently, $n_i=1$, i.e., A_i is a division algebra. Since a division algebra is evidently an (A'_2)-algebra, A_i is a division algebra over the base field F, in this case. Secondly, alternative non-associative division algebras are Cayley-Dickson (division) algebras over their centers (see [3, Theorem 3.17]), and being division algebras they are also (A'_2)-algebras. Lastly, the vector-matrix algebra is an algebra which is not even (A'_2). This assertion follows from the fact that the existence of a nonzero element x in an alternative algebra with identity, such that $x^2=0$, implies that the algebra cannot be an (A'_2)-algebra. From these observations we now have

THEOREM 3.1. *An alternative algebra is a semisimple* (A_2')*-algebra iff it is the direct sum of ideals which are either associative division algebras or Cayley-Dickson division algebras over their centers.*

Theorem 3.1 is essentially, the assertion that the only alternative semisimple (A_2')-algebras are the ones which are direct sums of ideals, which are alternative division algebras.

Now, let A be an alternative (A_2')-algebra over a field of characteristic zero. Then, by Lemma 2.12, the radical R of A is precisely the annihilator ideal I of A. We can appeal to the Wedderburn factor theorem for alternative algebras (see [3, Theorem 3.18]) to get A as the direct sum of I and a semisimple subalgebra B. I being the annihilator ideal of A, B is actually an ideal of A, so that A is the direct sum of ideals I and B. Hence, by Proposition 1.4, B will be an (A_2')-algebra, so that we can appeal to Theorem 3.1 to obtain

THEOREM 3.2. *An alternative algebra over a field F of characteristic zero is an* (A_2')*-algebra iff it is a direct sum of a zero ideal, and ideals which are (alternative) division algebras over F.*

Since an alternative algebra which is a direct sum of division algebras and a zero ideal is a priori an (A)-algebra, we can replace (A_2')- in the statement of Theorem 3.2 by (A_k)-, (A_∞)-, (A_k')-, (A_∞')-, (B_k')-, (B_∞')-, (A')-, or (A)-, in view of the chart of Section 1. We have incidentally proved

THEOREM 3.3. *For an alternative algebra over a field of characteristic zero, the properties* $(A_2), \cdots, (A_k), \cdots, (A_\infty), (A_2'), \cdots, (A_k'), \cdots, (A_\infty'), (B_2'), \cdots, (B_k'), \cdots, (B_\infty'), (A)$ *and* (A') *are all equivalent.*

REMARKS. (i) In view of Theorem 3.1, *all the properties defined in Section 1 are equivalent for semisimple alternative algebras over a field of arbitrary characteristic.*

However, it is interesting to note that the equivalence of properties (A_k), (A_k') for a semisimple associative algebra A can be established directly as follows (without appealing to the classification of simple associative algebras as in the proof of Theorem 3.1): For this, it suffices to show, in view of Proposition 1.2, and the chart in Section 1, that for a semisimple associative algebra, (A_2')-property implies (A_2)-property. To this end, we note that in an associative algebra A, $I_1 \equiv \{x \mid xy=0 \text{ for all } y \text{ in } A\}$ is a characteristic right ideal of A and hence is a characteristic ideal of A; it is also solvable. Consequently, when A is semisimple, I_1 is the zero ideal. Let $L_x^2=0$ for an x in A. Since $I_1=0$ and $L_x^2=L_{x^2}$, $x^2=0$. Therefore $L_x^2=L_{x^2}=0=R_{x^2}=R_x^2$. By (A_2')-property, $L_x=0=R_x$. Similarly when $R_y^2=0$ for a y in A, $R_y=0=L_y$. In other words, A is an (A_2)-algebra.

(ii) When the algebra in the hypothesis of Theorem 3.2 is associative, the division algebras in the conclusion of the same theorem will be associative ones.

We conclude with an analogue of Theorem 3.2 for a Jordan (A_k)-algebra over a field of characteristic zero. In this case, the radical is, by Corollary 2.7, the annihilator ideal. Further, the Wedderburn factor theorem for a Jordan algebra over a field of zero characteristic (see [3, pp. 106–107]) enables us to deduce that a Jordan (A_k)-algebra A (for $k \geqslant 3$) is the direct sum of its annihilator ideal I, and a semisimple ideal $B \equiv A^2$, such that B is an (A_k)-algebra. Since B is a semisimple algebra, its annihilator ideal is the zero ideal, i.e., $\{x \in B \mid L'_x \equiv R'_x = 0\}$ (L', R' are the left $\equiv$ right multiplications in B) $=0$. Since B is an (A_k)-algebra, this means that there exists no element x in B such that $L_x^k = 0$ (L, the left multiplication in A). We have thus proved

PROPOSITION 3.4 (cf. [2, Theorem 3]). *A Jordan algebra A over a field of characteristic zero is an (A_k)-algebra for $k \geqslant 3$, iff it is either a zero algebra or is a direct sum of its annihilator ideal and the semisimple ideal A^2 such that there exists no non-zero x in A^2 with $L_x^k = 0$.*

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References

[1] A. A. Albert, *Structure of algebras*, Amer. Math. Soc. Colloq. Publns., V. **24** (1939).
[2] A. Jôichi, *On certain properties of Lie algebras*, J. Sci. Hiroshima Univ. Ser. A-I, **31** (1967), 25-33.
[3] R. D. Schafer, *An introduction to nonassociative algebras*, Academic Press, New York (1967).
[4] M. Sugiura, *On a certain property of Lie algebras*, Sci. Pap. Coll. Gen. Edu. Univ. Tokyo, **5** (1955), 1-12.

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