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On Overrings of a Domain

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1. Introduction

Throughout this paper D will denote an integral domain with $1 \neq 0$ and quotient field K, and by an overring of D will be meant a ring J such that $D \subset J \subset K$. An ideal A of D is called a valuation ideal provided there exists a valuation overring D_V of D such that $AD_V \cap D = A$ ([22; 340], [10]). If Π is a general ring property, then we shall refer to an ideal A of D as a *Π-ideal* provided there exists an overring J of D such that J is a *Π-domain* (i.e. J has the property Π) and $A = AJ \cap D$. It is shown in [10] that if every principal ideal of D is a valuation ideal, then D is a valuation ring. Furthermore, if every proper ideal of D is a Dedekind ideal, then D is a Dedekind domain [2]; and if every proper ideal of D is a Prüfer ideal, then D is a Prüfer domain [7], [10; 238]. In this paper we are mainly concerned with the following question. When does the statement

(a) "there exists a collection $\mathcal{S}$ of Π-ideals of D" imply the statement

(b) "D is a Π-domain" (i.e. D has property Π)? Our main result in this direction is that (a) implies (b) when "Π-domain" = "Krull domain" and $\mathcal{S}$ is the collection of proper principal ideals of D, i.e. if every proper principal ideal is a Krull ideal, then D is a Krull domain. The same result holds in case "Krull domain is replaced by either "integrally closed domain" or "completely integrally closed domain". In addition we show that (a) implies (b) when $\mathcal{S}$ is the collection of proper finitely generated ideals of D and Π is any of the following ring properties: Prüfer, 1-dim. Prüfer, almost Dedekind, or Dedekind.

We remark that (a) does not always imply (b), even in the case that $\mathcal{S}$ is the set of all ideals of D (e.g. if Π is one of *P.I.D.*, Bezout, or *QR*-property-see Section 5).

In general we use the notation and terminology of [21] and [22]. In particular, $\subset$ denotes containment, while $<$ denotes proper containment; and A is a proper ideal of D provided $(0) < A < D$. The theorems considered in this paper are trivial in case D is a field, so we assume throughout that D has at least one proper ideal.

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2. Preliminary results

A domain D is called a Prüfer domain provided D_p (the quotient ring of D with respect to the prime ideal P of D [21; 228]) is a valuation ring for each proper prime ideal P of D (see [1; (b)], [3], [5], [7], [10], [12; 127], [13; 554]); D is an almost Dedekind domain provided D_p is a rank one discrete valuation ring (i.e. a noetherian valuation ring - a valuation ring which is a Dedekind domain) for each proper prime ideal P of D (see [3] and [9]); and D is a Krull domain if D is the intersection of the members of a family $\mathfrak{F}$ of rank one discrete valuation rings D_v such that $D \subset D_v \subset K$ and each non-zero element of D is a non-unit in only finitely many of the $D_v \in \mathfrak{F}$ (see [1; (b)], [12; 104], [19; 8], [22; 82]). An element x of K is said to be almost integral (or quasi-integral) over D provided there exists $d \in D$ such that $d \neq 0$ and $dx^n \in D$ for every positive integer n. If the set of almost integral elements of K over D is equal to D, then D is called completely integrally closed (see [1; (a)] [4], [12; 102], [18]).

LEMMA 2.1. *If S is a multiplicative system in D and A is an ideal of D such that $AJ \cap D = A$ for an overring J of D, then $AD_S = AJ_S \cap D_S$.*

PROOF. It is sufficient to prove that $AJ_S \cap D_S \subset AD_S$. If $x \in AJ_S \cap D_S$, then $x = a/s = d/t$ with s, $t \in S$, $d \in D$, $a \in AJ$. Thus $ta = ds \in AJ \cap D = A$, and $ds(1/st) = x \in AD_S$.

LEMMA 2.2. *Let Π be a ring property such that if J is any domain having property Π and S is a multiplicative system of J, then J_S also has property Π. Then if A is a Π-ideal of D and S is a multiplicative system of D, it follows that AD_S is a Π-ideal of D_S.*

PROOF. An immediate consequence of Lemma 2.1.

COROLLARY 2.3. *Under the conditions of Lemma 2.2 we have the following:*
(a) *every ideal of D is a Π-ideal $\Rightarrow$ every ideal of D_S is a Π-ideal.*
(b) *Every proper principal ideal of D is a Π-ideal $\Rightarrow$ every proper principal ideal of D_S is a Π-ideal.*

PROOF. An immediate consequence of Lemma 2.2.

REMARK 2.4. If we take the ring property Π in Lemma 2.2 to mean that J is a Krull domain, then the conditions of Lemma 2.2 are satisfied and Corollary 2.3 applies (see [1; (b)], [19; 10]). Furthermore we can replace "Krull" by Dedekind, Prüfer, almost Dedekind, noetherian, or integrally closed (see [6; 31], [3; 269], [9], [5], [21]). We note that the property "completely integrally closed" does not satisfy the conditions of Lemma 2.2—the ring E of entire functions is a completely integrally closed Prüfer domain (in fact, Bezout domain) with dimension greater than one ([1-(a); 71], [18; 324]), so that for a non-minimal prime ideal P of E the quotient ring E_P is a valuation ring of rank greater than one and therefore not completely integrally closed [14].

3. Characterizations of domains which are Krull, integrally closed, or completely integrally closed.

The purpose of this section is to establish the following result. Let Π be one of the ring properties "Krull domain", "integrally closed domain", or "completely integrally closed domain"; then if every proper principal ideal of D is a Π-ideal, it follows that D is a Π-domain. Furthermore, a counter-example is given which shows that this result is false if Π is one of the ring properties "Dedekind domain", "almost Dedekind domain", "Prüfer domain", or "Noetherian domain".

LEMMA 3.1. *Let $\mathfrak{F}$ be a family of domains J such that $D \subset J \subset K$ for all $J \in \mathfrak{F}$. If for every x in D there exists $J \in \mathfrak{F}$ such that $xD = xJ \cap D$, then $D = \bigcap_{J \in \mathfrak{F}} J$.*

PROOF. If $\beta \in \bigcap_{J \in \mathfrak{F}} J$, then $\beta = x/y$ with $x, y \in D$. Since $x = \beta y$ implies that $xJ \subset \beta yJ$ for all $J \in \mathfrak{F}$, then $xJ \subset yJ$ for all $J \in \mathfrak{F}$. There exists $J' \in \mathfrak{F}$ such that $yD = yJ' \cap D$. Therefore $xD \subset xJ' \cap D \subset yJ' \cap D = yD$ and $\beta \in D$.

THEOREM 3.2. *If every proper principal ideal of D is an integrally closed ideal (completely integrally closed ideal), then D is integrally closed (completely integrally closed).*

PROOF. This follows directly from Lemma 3.1 since the intersection of a collection of integrally closed (completely integrally closed) domains is integrally closed (completely integrally closed).

COROLLARY 3.3. *If every proper principal ideal of D is a Krull ideal, then D is completely integrally closed.*

PROOF. A Krull domain is completely integrally closed.

COROLLARY 3.4. *If every proper principal ideal of D is a rank one valuation ideal, then D is a rank one valuation ring.*

PROOF. A valuation ring is completely integrally closed if and only if it is rank one [14; 170]; every proper principal ideal of D is a valuation ideal implies that D is a valuation ring [10; 239].

LEMMA 3.5. *If P is a prime ideal of D such that D_P is a Dedekind domain, then the only primary ideals belonging to P are the symbolic powers $P^{(n)}$ of P.*

PROOF. Every proper ideal in D_P is a power of PD_P and is primary for PD_P. Since the primary ideals of D_P which belong to PD_P are in 1–1 correspondence with the primary ideals of D which belong to P, the lemma follows from the definition of symbolic powers [21; 232].

LEMMA 3.6. *If xD is a proper principal ideal of D such that xD is a Krull ideal, then xD has an irredundant representation as a finite intersection of strong primary ideals (i.e. primary ideals which contain a power of their radical). Furthermore, if P is an associated prime ideal of xD, then $\bigcap_n P^n=(0)$.*

PROOF. There is a Krull domain J such that $D\subset J\subset K$ and $xD=xJ\cap D$. Furthermore, since xJ is a non-zero principal ideal in the Krull domain J, we have $xJ=\bar{Q}_1\cap\cdots\cap\bar{Q}_n$ where the $\bar{Q}_i$ are symbolic powers of the finite number of minimal primes of J which contain xJ (see [12; 119] and [20; Coro. 2.14]). Since $\bar{Q}_i$ is a symbolic power of $P_i=\sqrt{\bar{Q}_i}$ in J, then there is an integer n_i such that $\bar{Q}_i\supset\bar{P}_i^{n_i}$ for $i=1,\cdots,n$. It is clear that $Q_i=\bar{Q}_i\cap D$ is primary for $P_i=\bar{P}_i\cap D$ and $Q_i\supset P_i^{n_i}$ for $i=1,\cdots,n$. Furthermore $xD=Q_1\cap\cdots\cap Q_n$. Since $J_{\bar{P}_i}$ is a Dedekind domain, $\bigcap_n\bar{P}_i^n D_{\bar{P}_i}=(0)$ and therefore $\bigcap_n P_i^n=(0)$.

LEMMA 3.7. *If $M\neq(0)$ is a maximal ideal in a completely integrally closed domain D and $M^{-1}=\{x\in K\mid xM\subset D\}>D$, then M is invertible.*

PROOF. Since MM^{-1} is an ideal in D and $M\subset MM^{-1}\subset D$, then $MM^{-1}=M$ or $MM^{-1}=D$. If $MM^{-1}=M$, then $M(M^{-1})^n=M\subset D$ for every positive integer n. If $0\neq d\in M$ and $x\in M^{-1}$, then $dx^n\in D$ and x is almost integral over D. Since D is completely integrally closed, then $x\in D$ and $M^{-1}=D$—a contradiction. Hence $MM^{-1}=D$ and M is invertible.

LEMMA 3.8. *If every proper principal ideal of D is a Krull ideal and P is a (non-zero) minimal prime ideal of D, then D_P is a rank one discrete valuation ring (i.e. a Dedekind domain with exactly one proper prime ideal).*

PROOF. It follows from Corollary 2.3 and Remark 2.4 that every proper principal ideal of D_P is a Krull ideal, and Corollary 3.3 implies that D_P is completely integrally closed. Since PD_P is the only proper prime ideal in D_P, it is sufficient to show that PD_P is invertible ([6] or [17]). Therefore, in view of Lemma 3.7, it suffices to show that $(PD_P)^{-1}>D_P$. Let $0\neq x\in PD_P$. By Lemma 3.6 there exists an irredundant strong primary representation for xD_P, and since every proper ideal in D_P is primary for PD_P, it follows that $xD_P\supset P^nD_P$ for some positive integer n. If $n=1$ then PD_P is invertible, so we may assume that $n\geq 2$ and minimal, that is $xD_P\supset(PD_P)^n$ and $xD_P\not\supset(PD_P)^{n-1}$. Choose $y\in(PD_P)^{n-1}$ such that $y\notin xD_P$. Then $y/x\notin D_P$, and it follows easily that $y/x\in(PD_P)^{-1}$.

LEMMA 3.9. *If every proper principal ideal of D is a Krull ideal and if $P\neq(0)$ is a maximal ideal in D such that P is an associated prime ideal of a* *non-zero principal ideal yD of D (see Lemma 3.6), then P is invertible.*

PROOF. Let $yD=Q_1\cap\cdots\cap Q_n$ be an irredundant strong primary representation such that $P_i=\sqrt{Q_i}$ for each i and $P=P_1$. If $yD: P\neq yD$ and $x\in (yD: P)\setminus yD$, then $x/y\notin D$ and $x/y\in P^{-1}$ so that P is invertible by Corollary 3.3 and Lemma 3.7. Suppose that $yD: P=yD$, and therefore $yD: P^k=yD$ for all k. There exists an integer t such that $P^t\subset Q_1$ and hence $Q_1: P^t=D$. It follows that $yD=yD: P^t=(\bigcap_1^n Q_i): P^t=\bigcap_1^n (Q_i: P^t)=\bigcap_{i\neq 1}(Q_i: P^t)\supset \bigcap_{i\neq 1} Q_i\supset yD$ and $yD=\bigcap_{i\neq 1} Q_i$, a contradiction.

LEMMA 3.10. *If every proper principal ideal in D is a Krull ideal, then the associated prime ideals of any proper principal ideal yD of D (see Lemma 3.6) are minimal prime ideals of D (and consequently a proper principal ideal of D has no imbedded components). Hence yD is contained in only finitely many minimal prime ideals of D.*

PROOF. Let P be an associated prime ideal of a proper principal ideal yD of D. Then the maximal ideal PD_P of D_P is an associated prime ideal of yD_P and P is a minimal prime of D if and only if PD_P is a minimal prime ideal of D_P. Since every proper principal ideal of D_P is a Krull ideal, by Corollary 2.3 and Remark 2.4, we may assume that P is a maximal ideal of D. Under this assumption, Lemma 3.9 implies that P is invertible. Suppose that P_1 is a prime ideal of D such that $P_1<P$. Then P_1P^{-1} is an ideal of D and $P(P_1P^{-1})=P_1$ implies that $P_1P^{-1}=P_1$ and $P_1=P_1P$. Hence $P_1=P_1P^n\subset P^n$ for all n. Since $\bigcap_n P^n=(0)$ by Lemma 3.6, then $P_1=(0)$ and P is a minimal prime ideal of D.

THEOREM 3.11. *If every proper principal ideal in D is a Krull ideal, then D is a Krull domain.*

PROOF. If $\mathfrak{M}$ denotes the set of minimal prime ideals of D, then D_P is a rank one discrete valuation ring for each $P\in\mathfrak{M}$ by Lemma 3.7. For each non-zero $d\in D$, Lemma 3.10 implies that the principal ideal dD is contained in at most a finite number of minimal prime ideals of D, and therefore d is a unit in D_P for all except possibly a finite number of ideals $P\in\mathfrak{M}$.

To complete the proof it suffices to prove that $\bigcap_{P\in\mathfrak{M}} D_P=D^*\subset D$. If $0\neq \alpha\in D^*$, then $\alpha=x/y$ with $x, y\in D$. Denote by X, Y the collections of associated prime ideals of xD, yD respectively (Lemma 3.6) and note that $Y\subset X\subset\mathfrak{M}$ (Lemma 3.10). For each $P\in\mathfrak{M}$ there exists an integer $\underset{P}{v}(x)\geq 0$ such that $xD_P=(PD_P)^{v_P(x)}$, since D_P is a Dedekind domain. It follows from Lemmas 3.5, 3.6, and 3.8 that $xD=\bigcap_{P\in X} P^{(v_P(x))}$. Similarly $yD=\bigcap_{P\in Y} P^{(v_P(y))}$. Since $\alpha\in D^*$, we have $xD_P\subset yD_P$ for each $P\in X$, and therefore $v_P(x)\geq v_P(y)$ for each $P\in X$.

Consequently $\bigcap_{P\in X} P^{(v_P(x))} \subset \bigcap_{P\in Y} P^{(v_P(y))}$, which implies that $\alpha \in D$.

COROLLARY 3.12. *If every proper principal ideal of D is a Noetherian integrally closed ideal, then D is a Krull domain.*

PROOF. This is an immediate consequence of Theorem 3.11 since a Noetherian integrally closed domain is a Krull domain.

If F is a non-zero fractional ideal of D, let $F^{-1}=\{\alpha \in K \mid \alpha F \subset D\}$. An ideal $A \neq (0)$ of D is called a v-ideal provided $A=(A^{-1})^{-1}$ (see [12; 118], [20], and the material on divisorial ideals in [1] and [19]).

PROPOSITION 3.13. *Every v-ideal of a Krull domain D is a Dedekind ideal (in fact, a P.I.D.-ideal).*

PROOF. The set of v-ideals of D is exactly the set of finite intersections of symbolic powers of minimal prime ideals of D ([12; 119] and [20; Coro. 2.14]). Let $A=\bigcap_{i=1}^{k} P_i^{(n_i)}$ be a v-ideal of D, where $P_i^{(n_i)}$ is a symbolic power of the minimal prime ideal P_i of D. Now D_{P_i} is a Dedekind domain for $i=1, \cdots, k$ and $D_S=\bigcap_{i=1}^{k} D_{P_i}$ where $S=D\setminus \bigcup_{i=1}^{k} P_i$. It follows from [16; 38] and [3; 276] that D_S is a Dedekind domain with a finite number of prime ideals (and therefore a principal ideal domain). Or, alternately, D_S is a quotient ring of a Krull ring, and therefore a Krull ring [19; 10]; D_S is one dimensional with a finite number of prime ideals by [21; 225], and hence D_S is a Dedekind domain [22; 84]. Finally, $AD_S \cap D=A$ by [21; 225].

EXAMPLE 3.14. Let $J=F[x_1, x_2, \cdots, x_n, \cdots]$ be the polynomial ring in infinitely many indeterminates over a field F. Then J is a unique (element) factorization domain, and therefore J is a Krull domain [19]. It is clear that an invertible ideal is a v-ideal—in particular, every non-zero principal ideal is a v-ideal. A domain is Prüfer if and only if every finitely generated non-zero ideal is invertible [5]; consequently J is not a Prüfer domain since (x_1, x_2) is not an invertible ideal in J ($J>(x_1, x_2)>(x_1)$ and a proper invertible ideal can not properly contain an invertible prime ideal). It is clear that J is not Noetherian. Therefore, in the hypothesis of Theorem 3.11 we can not replace "Krull ideal" by any of the following: "Dedekind ideal", "almost Dedekind ideal", "Prüfer ideal", "Noetherian ideal", "*P.I.D.* ideal".

4. Characterizations of Prüfer domains, almost Dedekind domains, and Dedekind domains.

In this section we show that the statement "every proper finitely generated ideal of D is a Π-ideal implies that D is a Π-domain" holds in case the ring property Π is one of the following; Prüfer domain, 1-dim. Prüfer domain, almost Dedekind domain, or Dedekind domain.

If A is an ideal of D, then $x \in K$ is said to be integral over A provided there exist $a_1, \cdots, a_n$ such that $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ and $a_i \in A^i$ for $i = 1, \cdots, n$; the set $A^* = \{x \in K \mid x \textit{ is integral over } A\}$ is called the integral closure of A in K (see [16] and [22; 350]). The set $A' = \cap AD_v$, where D_v ranges over all of the valuation over rings of D, is called the completion of A and A is said to be complete if $A = A'$; furthermore $A' = A^*$ [22; 350]. In addition, D is a Prüfer domain if and only if $A = A'$ for every ideal A of D [10]. In particular, if D is a Prüfer domain, then $A = A^*$ for every finitely generated ideal A of D; the converse is also true and is established below.

THEOREM 4.1. *The domain D a is Prüfer domain if and only if $A = A^*$ for every finitely generated ideal A of D.*

PROOF. The domain D is Prüfer if and only if $AB = AC$ implies $B = C$, when B and C are ideals of D and $A \neq (0)$ is a finitely generated ideal of D (see [5], [10]). Suppose $H = H^*$ for every finitely generated ideal H of D, and let B, C, $A = (x_1, \cdots, x_n) \neq (0)$ be ideals of D such that $AB = AC$. If $b \in B$, there exist $c_{ij} \in C$ such that $bx_i = \sum_{j=1}^{n} c_{ij} x_j$ for $i = 1, \cdots, n$. Let C_1 be the ideal in D generated by the c_{ij}. Since C_1 is finitely generated, then $C_1 = C_1^* \subset C$. Let $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$, and solve the system of equations $\sum_{j=1}^{n} (c_{ij} - b\delta_{ij}) x_j = 0$ $(i = 1, \cdots, n)$ by Cramer's rule. Since not all of the x_j are 0, it follows that the determinant $|c_{ij} - b\delta_{ij}| = 0$; expanding the determinant, we have elements $c_1, \cdots, c_n$ such that $b^n + c_1 b^{n-1} + \cdots + c_n = 0$ with $c_i \in C_l^i$. Therefore b is integral over C_1, and $b \in C_1^* = C_1 \subset C$. Hence $B \subset C$, and similarly $C \subset B$, so that $B = C$.

THEOREM 4.2. *If every finitely generated ideal in D is a Prüfer ideal, then D is a Prüfer domain.*

PROOF. Since a Prüfer domain is integrally closed, it follows from Theorem 3.2 that D is integrally closed. Moreover, since a Prüfer ideal is an intersection of valuation ideals [10], every finitely generated ideal of D is an intersection of valuation ideals. An intersection of valuation ideals is a complete ideal [22; 353], and therefore every finitely generated ideal of D is complete. Hence $A = A^*$ for every finitely generated ideal A of D, and D is a Prüfer domain by Theorem 4.1.

We say that a domain D has property ρ provided the following holds: if P is a proper prime ideal of D and $a, b \in D$ with $b \neq 0$, then there exist c, $d \in D$ such that $a/b = c/d$ and either $c \notin P$ or $d \notin P$ (this is an abstraction of a very useful property in classical algebraic number theory, where fractions can not in general be "reduced to lowest terms" but can be "reduced to lowest terms with respect to a given prime ideal").

LEMMA 4.3. *A domain D is Prüfer if and only if D has property ρ.*

PROOF. It is easy to show that D has property ρ if and only if for every non-zero $\alpha \in K$, that either α or α^{-1} belongs to D_P for every proper prime ideal P of D.

LEMMA 4.4. *If J is an overring of a Prüfer domain D and P is a prime ideal of J, then $J_P = D_{P \cap D}$.*

PROOF. It is clear that $J_P \supset D_{P \cap D}$. Let $\alpha = \alpha_1/\alpha_2$ with $\alpha_1, \alpha_2 \in J$ and $\alpha_2 \notin P$. Since D has property ρ, it follows directly that $\alpha_1 = x_1/y_1$ and $\alpha_2 = x_2/y_2$ with $x_1, x_2, y_1, y_2 \in D$, $y_1 \notin P$, $x_2 \notin P$ and the reverse inclusion is established.

COROLLARY 4.5. *An overring of a Prüfer domain is a Prufer domain.*

THEOREM 4.6. *If every proper finitely generated ideal of D is a 1-dim. Prüfer ideal, then D is a 1-dim. Prüfer domain.*

PROOF. Theorem 4.2 implies that D is a Prüfer domain. If P is a proper prime ideal of D, every proper finitely generated ideal of the valuation ring D_P is a 1-*dim.* Prüfer ideal by lemma 2.2. Lemma 3.1 implies that D_P is an intersection of 1-*dim.* Prufer domains; therefore the valuation ring D_P is completely integrally closed and must be rank one. Hence D is a 1-*dim.* Prufer domain.

COROLLARY 4.7. *If every proper finitely generated ideal of D is an almost Dedekind ideal, then D is an almost Dedekind domain.*

PROOF. We have from Theorem 4.6 that D is a 1-*dim.* Prüfer domain, so that D_P is a rank one valuation ring for each proper prime ideal P of D. Lemma 2.2 implies that every proper finitely generated ideal of D_P is an almost Dedekind ideal, and since D_P is a maximal subring of the quotient field of D it follows that D_P is a rank one discrete valuation ring.

COROLLARY 4.8. *If every proper finitely generated ideal of D is a Dedekind ideal, then D is a Dedekind domain.*

PROOF. Since a Dedekind domain is a Krull domain, it follows from Theorem 3.11 that D is a Krull domain; D is 1-*dim.* by Theorem 4.6, and therefore D is a Dedekind domain [22; 84].

REMARK 4.9. It follows immediately from proposition 3.13 and Corollary 4.8 that in any Krull domain with dimension $\neq 1$ there is a finitely generated ideal which is not a v-ideal.

5. Some miscellaneous results.

PROPOSITION 5.1. *If $\mathfrak{F}$ is a finite family of Noetherian overrings J of D such that for each proper ideal A of D there exists $J \in \mathfrak{F}$ such that $AJ \cap D = A$, then D is Noetherian.*

PROOF. If $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$ is a chain of ideals in D, then $A_1J \subset A_2J \subset \cdots \subset A_nJ \subset \cdots$ for each $J \in \mathfrak{F}$. There exists an integer k such that $A_nJ = A_{n+1}J$ for all $n \geq k$ and all $J \in \mathfrak{F}$, and there exists a domain $J' \in \mathfrak{F}$ such that $A_kJ' \cap D = A_k$. It follows that $A_n = A_{n+1}$ for all $n \geq k$, and D is Noetherian.

By an ascending ring property we understand a ring property Π such that if D has property Π and J is an overring of D, then J has property Π. We consider the following question: if Π is an ascending ring property and every proper ideal of D is a Π-ideal, does it follow that D is a Π-domain? We give three ascending ring properties for which the answer to this question is negative.

PROPOSITION 5.2. *If J is an overring of a Bezout domain D (i.e. every finitely generated ideal of D is principal), then J is a Bezout domain.*

PROOF. It is sufficient to prove that an ideal (x, y) of J, generated by two elements, is principal. Since $x, y \in K$ then $x = a/c$ and $y = b/c$ with $a, b, c \in D$. Hence $c(x, y)J = (a, b)J = (a, b)D \cdot J = rD \cdot J = rJ$ for some $r \in D$, and $(x, y)J = (r/c)J$.

COROLLARY 5.3. *Every overring of a principal ideal domain (P.I.D.) is a P.I.D.*

PROOF. A domain D is a *P.I.D.* if and only if D is a Bezout domain and a Dedekind domain (both of which are ascending ring properties).

EXAMPLE 5.4. Denote by J the domain of algebraic integers in the field $R(\sqrt{-6})$, where R is the field of rational numbers. Then J is a Dedekind domain which is not a Bezout domain [14; 43, 49], and every proper ideal of J is invertible and therefore a v-ideal. It follows from Proposition 3.13 that every proper ideal of J is a *P.I.D.* ideal.

A domain D has the QR-property provided every overring of D is a quotient ring of D with respect to some multiplicative system in D. It is shown in [11; 98] that the QR-property is an ascending ring property; furthermore, an example is given [11; 102] of a Dedekind domain D^* which does not have the QR-property. Proposition 3.13 shows that every proper ideal in a Dedekind domain is a P. I. D.-ideal, and hence a QR-ideal since a *P.I.D.* is a QR-domain [11; 99]. Consequently, every proper ideal of D^* is a QR-ideal and D^* is not a QR-domain.

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