J. Sci. HIROSHIMA UNIV. SER. A-I
34 (1970), 73-144

Asymptotic Expansions of the Distributions of Test Statistics in Multivariate Analysis

Yasunori FUJIKOSHI

(Received, April 10, 1970)

Contents

0. Introduction. Non-null distributions of some statistics in multivariate analysis have been expressed by series of zonal polynomials due to James [19], especially in terms of hypergeometric functions of matrix argument (Herz [11], Constantine [6] and James [21]). Such examples have been summarized in James [21], [22]. However the exact distributions of many test statistics are not yet available for the general values of parameters, and the almost all results obtained in some special cases are very complicated. Therefore, the asymptotic approximations for the distributions are very important.

In this paper asymptotic expansions of the distributions of test statistics in multivariate analysis are obtained by inverting asymptotic formulas of the characteristic functions or the moment generating functions expressed in terms of the hypergeometric functions with matrix argument. Based on our formulas, an atempt is also made to compare the powers of (i) likelihood ratio (=LR) criterion, (ii) Hotelling's criterion, and (iii) Pillai's criterion for a multivariate linear hypothesis (Wilks [42], Hotelling [13] and Lawley [24], Pillai [28]). Some statistics have been already treated by our method of asymptotic expansions. They are the generalized variance in Fujikoshi [9] and the LR criteria in multivariate analysis in Sugiura and Fujikoshi [39] and Sugiura [40].

We showed that formulas for weighted sums of zonal polynomials played an important role in our method. To extend the usefulness of our method, additional formulas are required. In part I we derive new formulas for weighted sums of zonal polynomials and the generalized Laguerre polynomials (due to Constantine [7]), and also formulas for Laplace and inverse Laplace transforms of some functions of matrix argument, which yield the asymptotic expressions of the characteristic functions.

Part II deals with the multivariate linear hypothesis. For the Pillai's criterion, certain approximations have been suggested in the null case by Pillai [27], Pillai and Mijares [29], and the limiting distribution has been investigated by Ogawa [26]. In Section 5 we obtain asymptotic expansions of the distributions of the Pillai's criterion both under hypothesis and alternatives up to order N^{-2}, where N denotes the sample size, by using the formulas for weighted sums of zonal polynomials given in Section 2. The tables of the upper 5 and 1% points of the criterion based on our asymptotic expressions are given in Appendix III. Asymptotic expansion of the distribution of the Hotelling's T_0^2 statistic has been investigated by Ito [15], [16] and Siotani [37], [38]. Recently Siotani [38] has extended his result in the non-null case up to order N^{-2}. In Section 6 we give two other methods for obtaining his formula, using the results due to Hsu [14] and Constantine [7], respectively. The numerical comparisons among the powers of three test criteria ((i), (ii), (iii)) are made in Section 7. We note that the observations made by Pillai and Jayachandran [30] in the case of p (=dimension of variates)$=2$ are also valid for $p=3$, when the sample size is moderately large.

In Part III, we investigate asymptotic non-null distributions of the Pillai's criterion and the Hotelling's criterion for testing the hypothesis of independence between two sets of variates under sequence of alternatives converging to the null hypothesis with rate of convergence $N^{-\gamma}(\gamma\geq 0)$. This sequence of alternatives has been considered in testing problems for covariance matrix by Sugiura [40]. By utilizing a close relationship between the tests of the multivariate linear hypothesis and the hypothesis of independence between two sets of variates, we derive asymptotic expressions of the distributions of the Pillai's criterion and the Hotelling's criterion for this problem in the case of $\gamma=1$ in Sections 8 and 9, respectively.

The distributions of the determinant and the trace of a non-ncentral Wishart matrix have been studied by Bagai [4], Hayakawa [10], etc. However, the exact distributions which they have obtained in some special cases are too complicated for numerical computation. In Sections 10 and 11 we investigate asymptotic distributions of the statistics for large n (=degrees of freedom of the non-central Wishart matrix) under the assumption that the non-centrality matrix $\Omega=n^{\delta}\Theta(\delta\geq 0)$ and Θ does not depend on n. In Fujikoshi [9] we obtained an asymptotic expansion of the determinant of the non-central Wishart matrix in the case that Ω is a fixed constant matrix, i. e., in the case of $\delta=0$, by using Lemma 2 in Section 2. In the same way we can derive it in the case of $\delta=\frac{1}{2}$ up to order $n^{-\frac{3}{2}}$ in Section 10. In Section 11, by using the explicit expression of the characteristic function of the non-central Wishart matrix due to Anderson [1] we obtain asymptotic expansions of the distributions of the trace of the non-central Wishart matrix up to order $n^{-\frac{3}{2}}$, when $\delta=0$ and 1, without using the formulas in part I. In Section 12 we obtain asymptotic expansions of the non-null distribution of the modified LR criterion for equality of mean vectors and covariance matrices under the restricted alternatives such as equality of all the covariance matrices.

Part I. Some Useful Formulas

1. Preliminaries. We list some necessary results on zonal polynomials and others which will be used frequently in this paper. The hypergeometric function of matrix argument is defined by Constantine [6] as

$$(1.1)\qquad {}_rF_s(a_1,\cdots,a_r;\ b_1,\cdots,b_s;Z)$$
$$=\sum_{k=0}^{\infty}\sum_{(\kappa)}\{(a_1)_\kappa\cdots(a_r)_\kappa/(b_1)_\kappa\cdots(b_s)_\kappa\}C_\kappa(Z)/k!,$$

where $a_1,\cdots,a_r,b_1,\cdots,b_s$ are real or complex constants, $\kappa=\{k_1,k_2,\cdots,k_p\}$ denotes a partition of the integer k such that $k_1+k_2+\cdots+k_p=k$ and $k_1\geq k_2\geq\cdots\geq k_p\geq 0$. Further

$$(1.2)\qquad (a)_\kappa=\prod_{\alpha=1}^{p}(a-(\alpha-1)/2)(a+1-(\alpha-1)/2)\cdots(a+k_\alpha-1-(\alpha-1)/2),$$

and the symbol $\sum_{(\kappa)}$ denotes summing over all partitions for fixed k. Also, the function $C_\kappa(Z)$ is called a zonal polynomial of the $p\times p$ symmetric matrix Z corresponding to κ and it is a symmetric homogeneous polynomial of degree

k in the p characteristic roots of Z. The detailed discussion may be found in Constantine [6] and James [19], [20], [21]. Tables for zonal polynomials have been given by James [21] up to order 6. For $k=1$, 2, 3 and 4,

$$C_{(1)}(Z)=\mathrm{tr}\ Z, \tag{1.3}$$

$$\begin{pmatrix} C_{(2)}(Z) \\ C_{(1^2)}(Z) \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} (\mathrm{tr}\ Z)^2 \\ \mathrm{tr}\ Z^2 \end{pmatrix},$$

$$\begin{pmatrix} C_{(3)}(Z) \\ C_{(21)}(Z) \\ C_{(1^3)}(Z) \end{pmatrix} = \frac{1}{15}\begin{pmatrix} 1 & 6 & 8 \\ 9 & 9 & -18 \\ 5 & -15 & 10 \end{pmatrix}\begin{pmatrix} (\mathrm{tr}\ Z)^3 \\ (\mathrm{tr}\ Z)\mathrm{tr}\ Z^2 \\ \mathrm{tr}\ Z^3 \end{pmatrix},$$

$$\begin{pmatrix} C_{(4)}(Z) \\ C_{(31)}(Z) \\ C_{(2^2)}(Z) \\ C_{(21^2)}(Z) \\ C_{(1^4)}(z) \end{pmatrix} = \frac{1}{105}\begin{pmatrix} 1 & 12 & 12 & 32 & 48 \\ 20 & 100 & -40 & 80 & -160 \\ 14 & 28 & 98 & -112 & -28 \\ 56 & -56 & -112 & -112 & 224 \\ 14 & -84 & 42 & 112 & -84 \end{pmatrix}\begin{pmatrix} (\mathrm{tr}\ Z)^4 \\ (\mathrm{tr}\ Z)^2\mathrm{tr}\ Z^2 \\ (\mathrm{tr}\ Z^2)^2 \\ (\mathrm{tr}\ Z)\mathrm{tr}\ Z^3 \\ \mathrm{tr}\ Z^4 \end{pmatrix}.$$

The fundamental property of zonal polynomials is the average over the orthogonal group $O(p)$, given by

$$\int_{O(p)} C_\kappa(H'SHT)\,d\mu(H)=C_\kappa(S)C_\kappa(T)/C_\kappa(I), \tag{1.4}$$

where I is the identity matrix of order p and $d\mu(H)$ is the invariant measure on the orthogonal group $O(p)$, so normalized that the measure of the whole group is unity. Special cases of the hypergeometric function of matrix argument are

$$_0F_0(Z)=\sum_{k=0}^{\infty}\sum_{(\kappa)} C_\kappa(Z)/k!=\mathrm{etr}\ Z, \tag{1.5}$$

$$_1F_0(b\,;Z)=\sum_{k=0}^{\infty}\sum_{(\kappa)} (b)_\kappa C_\kappa(Z)/k! = |I-Z|^{-b}, \tag{1.6}$$

where the last formula (1.6) holds when all the absolute values of the characteristic roots of Z are less than one. The following recurrence relations for the hypergeometric function defined in (1.1) are due to Constantine [6]:

$$\{\Gamma_p(a)\}^{-1}\int_{S>0} \{\mathrm{etr}(-S)\}\,|S|^{a-(p+1)/2}{}_rF_s(a_1,\cdots,a_r\,;b_1,\cdots,b_s\,;ST)\,dS \tag{1.7}$$

$$= {}_{r+1}F_s(a_1,\cdots, a_r, a; b_1,\cdots, b_s; T),$$

(1.8) $$\frac{\Gamma_p(b)2^{p(p-1)/2}}{(2\pi i)^{p(p+1)/2}}\int_{\Re(T)=X_0>0}(\mathrm{etr}\ T)|T|^{-b}{}_rF_s(a_1,\cdots, a_r; b_1,\cdots, b_s; T^{-1}S)dT$$

$$= {}_rF_{s+1}(a_1,\cdots, a_r; b_1,\cdots, b_s, b; S),$$

which may be regarded a generalization of Laplace and inverse Laplace transformation. The integral of the last formula (1.8) is taken over all $T(p\times p)=X_0+iY$ for fixed positive definite matrix X_0 and arbitrary real symmetric matrix Y. The function $\Gamma_p(t)$ is defined by

(1.9) $$\Gamma_p(t)=\pi^{p(p-1)/4}\prod_{\alpha=1}^{p}\Gamma(t-(\alpha-1)/2).$$

The following formulas for zonal polynomials are also obtained by Constantine [6].

(1.10) $$\int_{S>0}\{\mathrm{etr}(-ZS)\}|S|^{t-(p+1)/2}C_\kappa(ST)dS=\Gamma_p(t)(t)_\kappa|Z|^{-t}C_\kappa(TZ^{-1}),$$

(1.11) $$\frac{(t)_\kappa\Gamma_p(t)2^{p(p-1)/2}}{(2\pi i)^{p(p+1)/2}}\int_{\Re(Z)=X_0>0}(\mathrm{etr}\ SZ)|Z|^{-t}C_\kappa(Z^{-1})dZ$$

$$=|S|^{t-(p+1)/2}C_\kappa(S),$$

where the first formula (1.10) holds for any symmetric matrix Z whose real part ($=\Re(Z)$) is positive definite and any symmetric matrix T for $\Re(t)>(p-1)/2$. The last formula (1.11) holds for any positive definite matrix S.

The Laguerre polynomial of matrix argument is defined by Constantine [7] as follows:

(1.12) $$L_\kappa^\gamma(\Omega)=\frac{\mathrm{etr}\ \Omega}{\Gamma_p(\gamma+(p+1)/2)}\int_{R>0}\{\mathrm{etr}\ (-R)\}|R|^\gamma C_\kappa(R)$$

$$\cdot {}_0F_1(\gamma+\tfrac{1}{2}(p+1);\ -R\Omega)dR,$$

where $\gamma>-1$. He obtained the following generating function for Laguerre polynomials:

(1.13) $$\sum_{k=0}^{\infty}\sum_{(\kappa)}L_\kappa^\gamma(\Omega)C_\kappa(Z)/\{k!\,C_\kappa(I)\}=|I-Z|^{-\gamma-(p+1)/2}$$

$$\cdot\sum_{k=0}^{\infty}\sum_{(\kappa)}C_\kappa(-\Omega)C_\kappa(Z(I-Z)^{-1})/\{k!\,C_\kappa(I)\}$$

for any symmetric Z such that all the absolute values of the characteristic roots of Z are less than one.

We shall also use the following asymptotic formula for the gamma function (c. f. Anderson [2, p. 204]):

$$(1.14)\quad \log\Gamma(x+h)=\log\sqrt{2\pi}+(x+h-\tfrac{1}{2})\log x-x-\sum_{r=1}^{m}\frac{(-1)^r B_{r+1}(h)}{r(r+1)x^r}+O(|x|^{-m-1}),$$

which holds for large $|x|$ and fixed h with Bernoulli polynomial $B_r(h)$ of degree r. For $r=2$ and 3,

$$(1.15)\quad \begin{aligned} B_2(h)&=h^2-h+(1/6),\\ B_3(h)&=h^3-(3/2)h^2+(1/2)h. \end{aligned}$$

2. Formulas for zonal polynomials. We shall first prove the following lemma which will be useful in deriving some formulas for weighted sums of zonal polynomials.

LEMMA 1. *Let $C_\kappa(Z)$ be a zonal polynomial corresponding to the partition $\kappa=\{k_1, k_2,\cdots, k_p\}$ of k with $k_1+k_2+\cdots+k_p=k$ and $k_1\geq k_2\geq\cdots\geq k_p\geq 0$. Putting*

$$(2.1)\quad \begin{aligned} a_1(\kappa)&=\sum_{\alpha=1}^{p} k_\alpha(k_\alpha-\alpha),\\ a_2(\kappa)&=\sum_{\alpha=1}^{p} k_\alpha(4k_\alpha^2-6k_\alpha\alpha+3\alpha^2), \end{aligned}$$

then the following differential relations hold:

$$(2.2)\quad a_1(\kappa)C_\kappa(Z)=\mathrm{tr}(\Lambda\partial)^2 C_\kappa(\Sigma)\Big|_{\Sigma=\Lambda},$$

$$(2.3)\quad \{3a_1(\kappa)^2-a_2(\kappa)+k\}C_\kappa(Z)=[3\{\mathrm{tr}(\Lambda\partial)^2\}^2+8\,\mathrm{tr}(\Lambda\partial)^3]C_\kappa(\Sigma)\Big|_{\Sigma=\Lambda},$$

where ∂ denote the matrix of differential operators having $\{(1+\delta_{rs})/2\}\dfrac{\partial}{\partial\sigma_{rs}}$ as its (r, s) element for a symmetric matrix $\Sigma=(\sigma_{rs})$ with Kronecker's delta δ_{rs} and $\Lambda=diag(\lambda_1, \lambda_2,\cdots, \lambda_p)$ is a diagonal matrix with p characteristic roots of Z as its non-zero elements.

PROOF. It is sufficient to prove that the formulas (2.2) and (2.3) hold for any positive definite matrix Z. From (1.10) we have the following asymptotic formula for large n with any positive definite matrix Z:

$$(2.4)\quad \begin{aligned} &\left\{\Gamma_p\left(\frac{n}{2}\right)|Z|^{\frac{n}{2}}\right\}^{-1}\int_{S>0}\{\mathrm{etr}(-Z^{-1}S)\}\,|S|^{(n-p-1)/2}C_\kappa\left(\frac{2}{n}S\right)dS\\ &=\left(\frac{2}{n}\right)^k\left(\frac{n}{2}\right)_\kappa C_\kappa(Z)\\ &=\left[1+n^{-1}a_1(\kappa)+(6n^2)^{-1}\{3a_1(\kappa)^2-a_2(\kappa)+k\}+O(n^{-3})\right]C_\kappa(Z). \end{aligned}$$

The left hand side can be expanded asymptotically as follows, by using the method of a matrix of differential operators (see Ito [15], James [23], Siotani [37], Sugiura and Fujikoshi [39], etc. for the method.)

$$(2.5)\qquad \Big[1+n^{-1}\mathrm{tr}(\Lambda\partial)^2+(6n^2)^{-1}\{3(\mathrm{tr}(\Lambda\partial)^2)^2+8\,\mathrm{tr}(\Lambda\partial)^3\} +O(n^{-3})\Big]\, C_\kappa(\Sigma)\Big|_{\Sigma=\Lambda}$$

Comparing the coefficients of each term of orders n^{-1} and n^{-2} in the last equations of (2.4) and (2.5), we see that the formulas (2.2) and (2.3) are true.

The following seven lemmas are fundamental for the asymptotic expansions. Lemma 2 will be used in Sections 10 and 12. Lemma 3 will be used in derivation of the Pillai's criteria for multivariate linear hypothesis and independence.

LEMMA 2 (Sugiura and Fujikoshi [39]). *The following identities hold*:

$$(2.6)\qquad \sum_{k=l}^{\infty}\sum_{(\kappa)} C_\kappa(Z)/(k-l)! = (\mathrm{tr}\ Z)^l \mathrm{etr}\ Z,$$

$$(2.7)\qquad \sum_{k=0}^{\infty}\sum_{(\kappa)} C_\kappa(Z)a_1(\kappa)/k! = (\mathrm{tr}\ Z^2)\,\mathrm{etr}\ Z,$$

$$(2.8)\qquad \sum_{k=1}^{\infty}\sum_{(\kappa)} C_\kappa(Z)a_1(\kappa)/(k-1)! = \{2\,\mathrm{tr}\,Z^2+(\mathrm{tr}\ Z)\ \mathrm{tr}\ Z^2\}\,\mathrm{etr}\ Z,$$

$$(2.9)\qquad \sum_{k=0}^{\infty}\sum_{(\kappa)} C_\kappa(Z)a_1(\kappa)^2/k! = \{(\mathrm{tr}\ Z)^2+\mathrm{tr}\ Z^2+4\,\mathrm{tr}\ Z^3+(\mathrm{tr}\ Z^2)^2\}\,\mathrm{etr}\ Z,$$

$$(2.10)\qquad \sum_{k=0}^{\infty}\sum_{(\kappa)} C_\kappa(Z)a_2(\kappa)/k! = \{\mathrm{tr}\ Z+3(\mathrm{tr}\ Z)^2+3\,\mathrm{tr}\ Z^2+4\,\mathrm{tr}\ Z^3\}\mathrm{etr}\ Z,$$

$$(2.11)\qquad \sum_{k=0}^{\infty}\sum_{(\kappa)} C_\kappa(Z)\{a_1(\kappa)^3-a_1(\kappa)(a_2(\kappa)-k+2)+2a_3(\kappa)\}/k!, \\ =\{12\,\mathrm{tr}\ Z^4+8(\mathrm{tr}\ Z^2)\mathrm{tr}\ Z^3+(\mathrm{tr}\ Z^2)^3\}\,\mathrm{etr}\ Z$$

where $a_3(\kappa)$ *is defined by*

$$(2.12)\qquad a_3(\kappa)=\sum_{k=1}^{\infty} k_\alpha\{2k_\alpha^3-4k_\alpha^2\alpha+3k_\alpha\alpha^2-\alpha^3\}.$$

PROOF. For a proof of the formulas (2.6)~(2.10), see Fujikoshi [9], Sugiura and Fujikoshi [39]. In the following we shall prove the formula (2.11). From (1.6) we can write

$$(2.13)\qquad |I-n^{-1}Z|^{-n}=\sum_{k=0}^{\infty}\sum_{(\kappa)}(n)_\kappa C_\kappa(Z)/\{n^k k!\}$$

$$= \sum_{k=0}^{\infty} \sum_{(\kappa)} \Big[1 + (2n)^{-1} a_1(\kappa) + (24n^2)^{-1} \{3a_1(\kappa)^2 - a_2(\kappa) + k\}$$

$$+ (48n^3)^{-1} \{a_1(\kappa)^3 - a_1(\kappa)(a_2(\kappa) - k + 2) + 2a_3(\kappa)\}$$

$$+ O(n^{-4}) \Big] C_\kappa(Z)/k!,$$

which holds for any symmetric matrix Z with large n. The left hand side can be expanded asymptotically as in (2.15) by the well known formula, (2.14).

$$(2.14) \qquad -\log|I - n^{-1}Z| = \sum_{\alpha=1}^{l} \alpha^{-1} \mathrm{tr}(Z/n)^\alpha + O(n^{-l-1}).$$

$$(2.15) \qquad |I - n^{-1}Z|^{-n} = \exp\{-n \log|I - n^{-1}Z|\}$$

$$= (\mathrm{etr}\, Z) \Big[1 + (2n)^{-1} \mathrm{tr}\, Z^2 + (24n^2)^{-1} \{8\mathrm{tr}\, Z^3 + 3(\mathrm{tr}\, Z^2)^2\}$$

$$+ (48n^3)^{-1} \{12\mathrm{tr}\, Z^4 + 8(\mathrm{tr}\, Z^2)\mathrm{tr}\, Z^3 + (\mathrm{tr}\, Z^2)^3\} + O(n^{-3}) \Big].$$

Comparing the coefficients of each term of order n^{-3} in (2.13) and (2.15), we obtain the formula (2.11).

LEMMA 3. *Let Z be any symmetric matrix such that all the absolute values of the characteristic roots are less than one and put $V = Z(I - Z)^{-1}$. Then the following identities hold:*

$$(2.16) \qquad \sum_{k=1}^{\infty} \sum_{(\kappa)} (b)_\kappa C_\kappa(Z)/(k-1)! = b(\mathrm{tr}\, V)|I - Z|^{-b},$$

$$(2.17) \qquad \sum_{k=2}^{\infty} \sum_{(\kappa)} (b)_\kappa C_\kappa(Z)/(k-2)! = b\{b(\mathrm{tr}\, V)^2 + \mathrm{tr}\, V^2\}|I - Z|^{-b},$$

$$(2.18) \qquad \sum_{k=0}^{\infty} \sum_{(\kappa)} (b)_\kappa C_\kappa(Z) a_1(\kappa)/k! = \frac{b}{2} \{(\mathrm{tr}\, V)^2 + (2b+1)\mathrm{tr}\, V^2\}|I - Z|^{-b},$$

$$(2.19) \qquad \sum_{k=1}^{\infty} \sum_{(\kappa)} (b)_\kappa C_\kappa(Z) a_1(\kappa)/(k-1)! = \frac{b}{2} \{2(\mathrm{tr}\, V)^2 + 2(2b+1)\mathrm{tr}\, V^2$$

$$+ b(\mathrm{tr}\, V)^3 + (2b^2 + b + 2)(\mathrm{tr}\, V)\mathrm{tr}\, V^2 + 2(2b+1)\mathrm{tr}\, V^3\}|I - Z|^{-b},$$

$$(2.20) \qquad \sum_{k=0}^{\infty} \sum_{(\kappa)} (b)_\kappa C_\kappa(Z) a_1(\kappa)^2/k! = \frac{b}{4} \{2(2b+1)(\mathrm{tr}\, V)^2 + 2(2b+3)\mathrm{tr}\, V^2$$

$$+ 4(\mathrm{tr}\, V)^3 + 12(2b+1)(\mathrm{tr}\, V)\mathrm{tr}\, V^2 + 8(2b^2 + 3b + 2)\mathrm{tr}\, V^3 + b(\mathrm{tr}\, V)^4$$

$$+ 2(2b^2 + b + 2)(\mathrm{tr}\, V)^2 \mathrm{tr}\, V^2 + (2b+1)(2b^2 + b + 2)(\mathrm{tr}\, V^2)^2$$

$$+ 8(2b+1)(\mathrm{tr}\, V)\mathrm{tr}\, V^3 + 2(8b^2 + 10b + 5)\mathrm{tr}\, V^4\}|I - Z|^{-b},$$

(2.21)
$$\sum_{k=0}^{\infty}\sum_{(\kappa)}(b)_{\kappa}C_{\kappa}(Z)\,a_2(\kappa)/k!=\frac{b}{2}\{2\mathrm{tr}\ V+3(2b+1)(\mathrm{tr}\ V)^2$$
$$+3(2b+3)\,\mathrm{tr}\ V^2+2(\mathrm{tr}\ V)^3+6(2b+1)(\mathrm{tr}\ V)\mathrm{tr}\ V^2$$
$$+4(2b^2+3b+2)\,\mathrm{tr}\ V^3\}\,|I-Z|^{-b}.$$

PROOF. In (1.6) replacing Z by xZ, we have

(2.22)
$$\sum_{k=0}^{\infty}\sum_{(\kappa)}x^k(b)_{\kappa}C_{\kappa}(Z)/k!=|I-xZ|^{-b},$$

which holds for any x such that $|x|\leq 1$. Differentiating (2.22) with respect to x by $\frac{\partial}{\partial x}|U(x)|=|U(x)|\ \mathrm{tr}\ U(x)^{-1}\frac{\partial}{\partial x}U(x)$, we have the formulas (2.16) and (2.17). Multiplying both sides of (2.2) by $(b)_{\kappa}/k!$ and using the formula (1.6), we obtain

(2.23)
$$\sum_{k=0}^{\infty}\sum_{(\kappa)}(b)_{\kappa}C_{\kappa}(Z)a_1(\kappa)/k!=\mathrm{tr}(\Lambda\partial)^2\,|I-\Sigma|^{-b}\Big|_{\Sigma=\Lambda}.$$

After some calculations, we see that the right hand side of (2.23) is equal to the right hand side of (2.18) (see Appendix I). The formula (2.19) follows immediately by replacing Z by xZ in (2.18) and differentiating it with respect to x. Multiplying both sides of (2.2) by $(b)_{\kappa}a_1(\kappa)/k!$ and using the third formula (2.18), we obtain

(2.24)
$$\sum_{k=0}^{\infty}\sum_{(\kappa)}(b)_{\kappa}C_{\kappa}(Z)a_1(\kappa)^2/k!=\mathrm{tr}(\Lambda\partial)^2\Big[\frac{b}{2}\{(\mathrm{tr}\Sigma(I-\Sigma)^{-1})^2$$
$$+(2b+1)\,\mathrm{tr}(\Sigma(I-\Sigma)^{-1})^2\}\,|I-\Sigma|^{-b}\Big]\Big|_{\Sigma=\Lambda},$$

whose right hand side is reduced to the formula (2.20) (see Appendix I). Similarly multiplying both sides of (2.3) by $(b)_{\kappa}/k!$ and using the formula (1.6), we obtain

(2.25)
$$\sum_{k=0}^{\infty}\sum_{(\kappa)}(b)_{\kappa}C_{\kappa}(Z)\,\{3a_1(\kappa)^2-a_2(\kappa)+k\}/k!$$
$$=[3\,\{\mathrm{tr}(\Lambda\partial)^2\}^2+8\,\mathrm{tr}(\Lambda\partial)^3]\,|I-\Sigma|^{-b}\Big|_{\Sigma=\Lambda}$$

Using the formulas (A. 21) and (A. 22) shown in Appendix I, we can write the right hand side of (2.25) as follows:

$$\frac{b}{4}|I-Z|^{-b}\{8(\mathrm{tr}\ V)^3+24(2b+1)(\mathrm{tr}\ V)\,\mathrm{tr}\ V^2+16(2b^2+3b+2)\,tr\ V^3$$
$$+3b(\mathrm{tr}\ V)^4+6(2b^2+b+2)(\mathrm{tr}\ V)^2\mathrm{tr}\ V^2+3(2b+1)(2b^2+b+2)(\mathrm{tr}\ V^2)^2$$

$$+24(2b+1)(\mathrm{tr}\ V)\ \mathrm{tr}\ V^3+6(8b^2+10b+5)\mathrm{tr}\ V^4\}.$$

Combining this result with (2.16) and (2.20), we have the desired formula (2.21).

3. Formulas for Laplace and inverse Laplace transforms of some functions of matrix argument. We will use the following abbreviated notations:

$$\text{(3.1)}\quad \mathrm{L}[f(S,\Omega,\Theta)]=|Z|^{\frac{q}{2}}\left\{\Gamma_p\left(\frac{q}{2}\right)\right\}^{-1}\int_{S>0}\{\mathrm{etr}(-ZS)\}\,|S|^{(q-p-1)/2}f(S,\Omega,\Theta)dS$$

for any symmetric matrix Z whose real part $(=X_0)$ is positive definite, and

$$\text{(3.2)}\quad \mathrm{I}[f(Z^{-1},\Omega,\Theta)]$$

$$=\Gamma_p\left(\frac{q}{2}\right)\frac{2^{p(p-1)/2}}{(2\pi i)^{p(p+1)/2}}\int_{\Re(Z)=X_0>0}(\mathrm{etr}\ Z)|Z|^{-\frac{q}{2}}f(Z^{-1},\Omega,\Theta)\,dZ.$$

In (3.1) and (3.2), Ω and Θ are any symmetric matrices and $q>p-1$. The first integral (3.1) can be regarded as the Laplace transform of $|Z|^{\frac{q}{2}}|S|^{(q-p-1)/2}\cdot f(S,\Omega,\Theta)/\Gamma_p\left(\frac{q}{2}\right)$ and also as the expectation of the statistic $f(S,\Omega,\Theta)$ with respect to the Wishart distribution $W_p(q,\frac{1}{2}Z^{-1})$ on S. The integral (3.2) is related to inverse Laplace transform. The formulas $\mathrm{L}[1]=1$ and $\mathrm{I}[1]=1$ (Herz [11]) will be frequently used in this paper. The formula (1.10) can be written in our notations as

$$\text{(3.3)}\qquad \mathrm{L}[C_\kappa(\Omega S)]=\left(\frac{q}{2}\right)_\kappa C_\kappa(\Omega Z^{-1}).$$

Let us consider the integral $\mathrm{I}[C_\kappa(\Omega Z^{-1})]$. Put $g(\Omega)=\mathrm{I}[C_\kappa(\Omega Z^{-1})]$. From (1.11) we have $g(I)=C_\kappa(I)/\left(\frac{q}{2}\right)_\kappa$. $g(\Omega)$ is clearly a symmetric function of Ω. Hence, by making the transformation $\Omega\to H'\Omega H$ and integrating H over $O(p)$ and using (1.4), we obtain $g(\Omega)=[g(I)/C_\kappa(I)]\,C_\kappa(\Omega)=C_\kappa(\Omega)/\left(\frac{q}{2}\right)_\kappa$. Therefore we can write

$$\text{(3.4)}\qquad \mathrm{I}[C_\kappa(\Omega Z^{-1})]=C_\kappa(\Omega)/\left(\frac{q}{2}\right)_\kappa,$$

which holds for any symmetric matrix Ω.

By inverting the linear relationship (1.3), we have

$$\text{(3.5)}\qquad \mathrm{tr}\ Z=C_{(1)}(Z),$$

$$\begin{pmatrix}(\mathrm{tr}\ Z)^2\\ \mathrm{tr}\ Z^2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}2 & 2\\ 2 & -1\end{pmatrix}\begin{pmatrix}C_{(2)}(Z)\\ C_{(1^2)}(Z)\end{pmatrix},$$

$$\begin{pmatrix}(\mathrm{tr}\ Z)^3\\(\mathrm{tr}\ Z)\mathrm{tr}\ Z^2\\\mathrm{tr}\ Z^3\end{pmatrix}=\frac{1}{12}\begin{pmatrix}12&12&12\\12&2&-6\\12&-3&3\end{pmatrix}\begin{pmatrix}C_{(3)}(Z)\\C_{(21)}(Z)\\C_{(1^3)}(Z)\end{pmatrix},$$

$$\begin{pmatrix}(\mathrm{tr}\ Z)^4\\(\mathrm{tr}\ Z)^2\mathrm{tr}\ Z^2\\(\mathrm{tr}\ Z^2)^2\\(\mathrm{tr}\ Z)\mathrm{tr}\ Z^3\\\mathrm{tr}\ Z^4\end{pmatrix}=\frac{1}{48}\begin{pmatrix}48&48&48&48&48\\48&20&8&-4&-24\\48&-8&28&-8&12\\48&6&-12&-3&12\\48&-8&-2&4&-6\end{pmatrix}\begin{pmatrix}C_{(4)}(Z)\\C_{(31)}(Z)\\C_{(2^2)}(Z)\\C_{(21^2)}(Z)\\C_{(1^4)}(Z)\end{pmatrix}$$

From the formulas (3.3), (3.4) and (3.5) we can easily get the following Lemmas 4 and 5, which will be used for derivation of the distribution of the Hotelling's criterion in Section 6. Lemma 5 will also be used in the case of Pillai's criteria in Sections 5 and 8.

LEMMA 4. *The following identities hold:*

$$\mathrm{L}[\mathrm{tr}\ \Omega S]=\frac{q}{2}\mathrm{tr}\ \Omega Z^{-1}, \tag{3.6}$$

$$\begin{pmatrix}\mathrm{L}[(\mathrm{tr}\ \Omega S)^2]\\\mathrm{L}[\mathrm{tr}(\Omega S)^2]\end{pmatrix}=\frac{q}{4}\begin{pmatrix}q&2\\1&q+1\end{pmatrix}\begin{pmatrix}(\mathrm{tr}\ \Omega Z^{-1})^2\\\mathrm{tr}(\Omega Z^{-1})^2\end{pmatrix}, \tag{3.7}$$

$$\begin{pmatrix}\mathrm{L}[(\mathrm{tr}\ \Omega S)^3]\\\mathrm{L}[(\mathrm{tr}\ \Omega S)\mathrm{tr}(\Omega S)^2]\\\mathrm{L}[\mathrm{tr}(\Omega S)^3]\end{pmatrix}=\frac{q}{8}\begin{pmatrix}q^2&6q&8\\q&q^2+q+4&4(q+1)\\1&3(q+1)&q^2+3q+4\end{pmatrix}\begin{pmatrix}(\mathrm{tr}\ \Omega Z^{-1})^3\\(\mathrm{tr}\ \Omega Z^{-1})\mathrm{tr}(\Omega Z^{-1})^2\\\mathrm{tr}(\Omega Z^{-1})^3\end{pmatrix} \tag{3.8}$$

$$\begin{pmatrix}\mathrm{L}[(\mathrm{tr}\ \Omega S)^4]\\\mathrm{L}[(\mathrm{tr}\ \Omega S)^2\mathrm{tr}(\Omega S)^2]\\\mathrm{L}[\{\mathrm{tr}(\Omega S)^2\}^2]\\\mathrm{L}[(\mathrm{tr}\ \Omega S)\mathrm{tr}(\Omega S)^3]\\\mathrm{L}[\mathrm{tr}(\Omega S)^4]\end{pmatrix}=\frac{q}{16}\begin{pmatrix}q^3&12q^2&12q&32q&48\\q^2&q(q^2+q+10)&2(q^2+q+4)&8(q^2+q+2)&24(q+1)\\q&2(q^2+q+4)&(q+1)(q^2+q+4)&16(q+1)&4(2q^2+5q+5)\\q&3(q^2+q+2)&6(q+1)&q^3+3q^2+16q+12&6(q^2+3q+4)\\1&6(q+1)&2q^2+5q+5&4(q^2+3q+4)&q^3+6q^2+21q+20\end{pmatrix}\begin{pmatrix}(\mathrm{tr}\ \Omega Z^{-1})^4\\(\mathrm{tr}\ \Omega Z^{-1})^2\mathrm{tr}(\Omega Z^{-1})^2\\\{\mathrm{tr}(\Omega Z^{-1})^2\}^2\\(\mathrm{tr}\ \Omega Z^{-1})\mathrm{tr}(\Omega Z^{-1})^3\\\mathrm{tr}(\Omega Z^{-1})^4\end{pmatrix} \tag{3.9}$$

LEMMA 5. *The following identities hold*:

$$\frac{q}{2}\mathrm{I}\,[\mathrm{tr}\ \Omega Z^{-1}]=\mathrm{tr}\ \Omega, \tag{3.10}$$

$$d_1\begin{pmatrix}\mathrm{I}\,[(\mathrm{tr}\ \Omega Z^{-1})^2]\\ \mathrm{I}\,[\mathrm{tr}(\Omega Z^{-1})^2]\end{pmatrix}=\begin{pmatrix}q+1 & -2\\ -1 & q\end{pmatrix}\begin{pmatrix}(\mathrm{tr}\ \Omega)^2\\ \mathrm{tr}\Omega^2\end{pmatrix}, \tag{3.11}$$

$$d_2\begin{pmatrix}\mathrm{I}\,[(\mathrm{tr}\ \Omega Z^{-1})^3]\\ \mathrm{I}\,[(\mathrm{tr}\ \Omega Z^{-1})\mathrm{tr}(\Omega Z^{-1})^2]\\ \mathrm{I}\,[\mathrm{tr}(\Omega Z^{-1})^3]\end{pmatrix}=\begin{pmatrix}q^2+3q-2 & -6(q+2) & 16\\ -(q+2) & q^2+2q+4 & -4q\\ 2 & -3q & q^2\end{pmatrix}\begin{pmatrix}(\mathrm{tr}\ \Omega)^3\\ (\mathrm{tr}\ \Omega)\mathrm{tr}\ \Omega^2\\ \mathrm{tr}\ \Omega^3\end{pmatrix}, \tag{3.12}$$

$$d_3\begin{pmatrix}\mathrm{I}\,[(\mathrm{tr}\ \Omega Z^{-1})^4]\\ \mathrm{I}\,[(\mathrm{tr}\ \Omega Z^{-1})^2\mathrm{tr}\,(\Omega Z^{-1})^2]\\ \mathrm{I}\,[\{\mathrm{tr}\,(\Omega Z^{-1})^2\}^2]\\ \mathrm{I}\,[(\mathrm{tr}\Omega Z^{-1})\,\mathrm{tr}\,(\Omega Z^{-1})^3]\\ \mathrm{I}\,[\mathrm{tr}\,(\Omega Z^{-1})^4]\end{pmatrix}=\begin{pmatrix}q^4+7q^3+q^2-35q-6 & -12(q^3+6q^2+3q-6) & 12(q^2+5q+18) & 64q(q+4) & -48(5q+6)\\ -(q^3+6q^2+3q-6) & q^4+6q^3+13q^2+36q+36 & -2q(q^2+5q+18) & -8(q^3+4q^2+5q+6) & 8q(5q+6)\\ q^2+5q+18 & -2q(q^2+5q+18) & q^4+5q^3-6q^2-36q+72 & 16(2q^2+3q-6) & -4q(2q^2+3q-6)\\ 2q(q+4) & -3(q^3+4q^2+5q+6) & 6(2q^2+3q-6) & q^4+4q^3+9q^2+6q+48 & -6q(q^2+q+2)\\ -(5q+6) & 2q(5q+6) & -q(2q^2+3q-6) & -4q(q^2+q+2) & q^2(q^2+q+2)\end{pmatrix}\begin{pmatrix}(\mathrm{tr}\ \Omega)^4\\ (\mathrm{tr}\ \Omega)^2\mathrm{tr}\ \Omega^2\\ (\mathrm{tr}\ \Omega^2)^2\\ (\mathrm{tr}\ \Omega)\ \mathrm{tr}\ \Omega^3\\ \mathrm{tr}\ \Omega^4\end{pmatrix}, \tag{3.13}$$

where $d_i(i=1, 2, 3)$ *are given by*

$$d_1=(q-1)\,q\,(q+2)/4, \tag{3.14}$$

$$d_2=(q-2)(q-1)\,q(q+2)(q+4)/8,$$

$$d_3=(q-3)(q-2)(q-1)q(q+1)(q+2)(q+4)(q+6)/16.$$

For the case of the Hotelling's criterion for independence, we need the following Lemmas 6 and 7, which generalize some results in the Lemmas 4 and 5.

LEMMA 6. *The following identities hold:*

$$\begin{pmatrix} \mathrm{L}[(\mathrm{tr}\ \Omega S)\mathrm{tr}\ \Theta S] \\ \mathrm{L}[\mathrm{tr}\ \Omega S\Theta S] \end{pmatrix} = \frac{q}{4}\begin{pmatrix} q & 2 \\ 1 & q+1 \end{pmatrix}\begin{pmatrix} (\mathrm{tr}\ \Omega Z^{-1})\mathrm{tr}\Theta Z^{-1} \\ \mathrm{tr}\ \Omega Z^{-1}\Theta Z^{-1} \end{pmatrix}, \tag{3.15}$$

$$\begin{pmatrix} \mathrm{L}[(\mathrm{tr}\ \Omega S)^2\mathrm{tr}\ \Theta S] \\ \mathrm{L}[\mathrm{tr}\ (\Omega S)^2\mathrm{tr}\ \Theta S] \\ \mathrm{L}[(\mathrm{tr}\ \Omega S)\mathrm{tr}\ \Omega S\Theta S] \\ \mathrm{L}[\mathrm{tr}(\Omega S)^2\ \Theta S] \end{pmatrix} = \frac{q}{8}\begin{pmatrix} q^2 & 2q & 4q & 8 \\ q & q(q+1) & 4 & 4(q+1) \\ q & 2 & q^2+q+2 & 4(q+1) \\ 1 & q+1 & 2(q+1) & q^2+3q+4 \end{pmatrix} \cdot \begin{pmatrix} (\mathrm{tr}\ \Omega Z^{-1})^2\mathrm{tr}\ \Theta Z^{-1} \\ \mathrm{tr}(\Omega Z^{-1})^2\mathrm{tr}\ \Theta Z^{-1} \\ (\mathrm{tr}\ \Omega Z^{-1})\mathrm{tr}\ \Omega Z^{-1}\Theta Z^{-1} \\ \mathrm{tr}(\Omega Z^{-1})^2\Theta Z^{-1} \end{pmatrix}, \tag{3.16}$$

$$\begin{aligned} \mathrm{L}[\mathrm{tr}(\Omega S)^2\mathrm{tr}(\Theta S)^2] = \frac{q}{16}\{&q(\mathrm{tr}\ \Omega Z^{-1})^2(\mathrm{tr}\ \Theta Z^{-1})^2 \\ &+q(q+1)(\mathrm{tr}\ \Omega Z^{-1})^2\mathrm{tr}(\Theta Z^{-1})^2+q(q+1)\mathrm{tr}(\Omega Z^{-1})^2(\mathrm{tr}\ \Theta Z^{-1})^2 \\ &+8(\mathrm{tr}\ \Omega Z^{-1})(\mathrm{tr}\ \Theta Z^{-1})\mathrm{tr}\ \Omega Z^{-1}\Theta Z^{-1}+q(q+1)^2\mathrm{tr}(\Omega Z^{-1})^2\mathrm{tr}(\Theta Z^{-1})^2 \\ &+4(q+1)(\mathrm{tr}\ \Omega Z^{-1}\Theta Z^{-1})^2+8(q+1)(\mathrm{tr}\ \Omega Z^{-1})\mathrm{tr}\ \Omega Z^{-1}(\Theta Z^{-1})^2 \\ &+8(q+1)(\mathrm{tr}\ \Theta Z^{-1})\mathrm{tr}\ \Theta Z^{-1}(\Omega Z^{-1})^2+8(q+1)^2\mathrm{tr}(\Omega Z^{-1})^2(\Theta Z^{-1})^2 \\ &+4(q+3)\mathrm{tr}(\Omega Z^{-1}\Theta Z^{-1})^2\}. \end{aligned} \tag{3.17}$$

PROOF. Replacing Ω by $\Omega+n^{-1}\Theta$ in (3.7) and (3.8) and comparing the coefficients of each term of order n^{-1} in their asymptotic expansions, we obtain (3.15), and the first and the last formulas of (3.16), as well as the following identity:

$$\begin{aligned} \mathrm{L}[\mathrm{tr}(\Omega S)^2\mathrm{tr}\ \Theta S]+2\mathrm{L}[(\mathrm{tr}\ \Omega S)\mathrm{tr}\ \Omega S\Theta S] = \frac{q}{8}\{&3q(\mathrm{tr}\ \Omega Z^{-1})^2\mathrm{tr}\ \Theta Z^{-1} \\ &+(q^2+q+4)\mathrm{tr}(\Omega Z^{-1})^2\mathrm{tr}\ \Theta Z^{-1}+2(q^2+q+4)(\mathrm{tr}\ \Omega Z^{-1})\mathrm{tr}\ \Omega Z^{-1}\Theta Z^{-1} \end{aligned} \tag{3.18}$$

$$+12(q+1)\operatorname{tr}(\Omega Z^{-1})^2\Theta Z^{-1}\}.$$

From the second identity in (3.7), we note that (3.19) holds for large n such that the real part of $Z-\frac{1}{n}\Theta$ is positive definite.

$$(3.19)\qquad |Z|^{\frac{q}{2}}\left\{\Gamma_p\left(\frac{q}{2}\right)\right\}^{-1}\int_{S>0}\left[\operatorname{etr}\left\{-\left(Z-\frac{1}{n}\Theta\right)S\right\}\right]|S|^{(q-p-1)/2}\operatorname{tr}(\Omega S)^2 dS$$

$$=\frac{q}{4}\left[\left\{\operatorname{tr}\,\Omega\left(Z-\frac{1}{n}\Theta\right)^{-1}\right\}^2+(q+1)\operatorname{tr}\left\{\Omega\left(Z-\frac{1}{n}\Theta\right)^{-1}\right\}^2\right]|Z|^{\frac{q}{2}}\left|Z-\frac{1}{n}\Theta\right|^{-\frac{q}{2}}.$$

Equating the coefficients of each term of order n^{-1} on both sides of the above identity, we obtain the formula for $\mathrm{L}[\operatorname{tr}(\Omega S)^2\operatorname{tr}\Theta S]$. The formula for $\mathrm{L}[(\operatorname{tr}\Omega S)\operatorname{tr}\Omega S\Theta S]$ is an immediate consequence of this result and (3.18). Now we will prove the last formula (3.17). Let us consider the following identity for large n:

$$(3.20)\qquad |Z|^{\frac{q}{2}}\left\{\Gamma_p\left(\frac{q}{2}\right)\right\}^{-1}\int_{S>0}[\operatorname{etr}\{-(Z+\Omega)S\}]|S|^{(q-p-1)/2}\operatorname{tr}(\Theta S)^2$$

$$\cdot\left[\left\{\Gamma_p\left(\frac{n}{2}\right)\right\}^{-1}\int_{R>0}\{\operatorname{etr}(-R)\}|R|^{(n-p-1)/2}\left\{\operatorname{etr}\left(\frac{2}{n}R\Omega^{\frac{1}{2}}S\Omega^{\frac{1}{2}}\right)\right\}dR\right]dS$$

$$=\left\{\Gamma_p\left(\frac{n}{2}\right)\right\}^{-1}\int_{R>0}\{\operatorname{etr}(-R)\}|R|^{(n-p-1)/2}\left[|Z|^{\frac{q}{2}}\left\{\Gamma_p\left(\frac{q}{2}\right)\right\}^{-1}\right.$$

$$\left.\int_{S>0}[\operatorname{etr}\{-(Z+\Omega-\Omega^{\frac{1}{2}}\frac{2}{n}R\Omega^{\frac{1}{2}})\}S]|S|^{(q-p-1)/2}\operatorname{tr}(\Theta S)^2\,dS\right]dR.$$

Applying the last formula of (3.7) to the second integral in the right hand side of the above identity, and expanding it in a Taylor series with respect to $\frac{2}{n}R$ about I as in (2.5), we can write it as follows:

$$\frac{q}{4}\{(\operatorname{tr}\Theta Z^{-1})^2+(q+1)\operatorname{tr}(\Theta Z^{-1})^2\}+\frac{1}{n}\frac{q}{4}|Z|^{\frac{q}{2}}\operatorname{tr}\partial^2|Z+\Omega-\Omega^{\frac{1}{2}}\Sigma\Omega^{\frac{1}{2}}|^{-\frac{q}{2}}$$

$$\cdot\left[\{\operatorname{tr}\Theta(Z+\Omega-\Omega^{\frac{1}{2}}\Sigma\Omega^{\frac{1}{2}})^{-1}\}^2+(q+1)\operatorname{tr}\{\Theta(Z+\Omega-\Omega^{\frac{1}{2}}\Sigma\Omega^{\frac{1}{2}})^{-1}\}^2\right]\Bigg|_{\Sigma=I}$$

$$+O(n^{-2}).$$

The calculation of the operation ∂ in the above expression is given in Appendix II, which shows that the coefficient of order n^{-1} is equal to the right hand side of (3.17). Noting that the reductions

$$(3.21)\qquad \{\operatorname{etr}(-\Omega S)\}\left\{\Gamma_p\left(\frac{n}{2}\right)\right\}^{-1}\int_{R>0}\{\operatorname{etr}(-R)\}|R|^{(n-p-1)/2}$$

$$\cdot\left\{\operatorname{etr}\left(\frac{n}{2}R\Omega^{\frac{1}{2}}S\Omega^{\frac{1}{2}}\right)\right\}dR=\{\operatorname{etr}(-\Omega S)\}\left|I-\frac{n}{2}\Omega^{\frac{1}{2}}S\Omega^{\frac{1}{2}}\right|^{-\frac{n}{2}}$$

$$=1+n^{-1}\operatorname{tr}(\Omega S)^2+O(n^{-2}),$$

hold for large n, the left hand side of (3.20) can be expanded as follows:

$$\mathrm{L}[\operatorname{tr}(\Theta S)^2]+n^{-1}\mathrm{L}[\operatorname{tr}(\Omega S)^2\operatorname{tr}(\Theta S)^2]+O(n^{-2}).$$

Therefore we see that the formula (3.17) is true.

LEMMA 7. *The following identities hold:*

$$(3.22)\qquad d_1\begin{pmatrix}\mathrm{I}[(\operatorname{tr}\Omega Z^{-1})\operatorname{tr}\Theta Z^{-1}]\\ \mathrm{I}[\operatorname{tr}\Omega Z^{-1}\Theta Z^{-1}]\end{pmatrix}=\begin{pmatrix}q+1 & -2\\ -1 & q\end{pmatrix}\begin{pmatrix}(\operatorname{tr}\Omega)\operatorname{tr}\Theta\\ \operatorname{tr}\Omega\Theta\end{pmatrix},$$

$$(3.23)\qquad d_2\begin{pmatrix}\mathrm{I}[(\operatorname{tr}\Omega Z^{-1})^2\operatorname{tr}\Theta Z^{-1}]\\ \mathrm{I}[\operatorname{tr}(\Omega Z^{-1})^2\operatorname{tr}\Theta Z^{-1}]\\ \mathrm{I}[(\operatorname{tr}\Omega Z^{-1})\operatorname{tr}\Omega Z^{-1}\Theta Z^{-1}]\\ \mathrm{I}[\operatorname{tr}(\Omega Z^{-1})^2\Theta Z^{-1}]\end{pmatrix}=\begin{pmatrix}q^2+3q-2 & -2(q+2) & -4(q+2) & 16\\ -(q+2) & q^2+2q-4 & 8 & -4q\\ -(q+2) & 4 & q(q+2) & -4q\\ 2 & -q & -2q & q^2\end{pmatrix}\begin{pmatrix}(\operatorname{tr}\Omega)^2\operatorname{tr}\Theta\\ \operatorname{tr}\Omega^2(\operatorname{tr}\Theta)\\ (\operatorname{tr}\Omega)\operatorname{tr}\Omega\Theta\\ \operatorname{tr}\Omega^2\Theta\end{pmatrix},$$

where d_1 and d_2 are given by (3.14).

PROOF. The formula (3.22) is proved from (3.11) by the same method as in the proof of the formula (3.15). Also from (3.12) we have the first and the last formulas of (3.23) and the following identity:

$$(3.24)\qquad d_2\mathrm{I}[\operatorname{tr}(\Omega Z^{-1})^2\operatorname{tr}\Theta Z^{-1}]+2d_2\mathrm{I}[(\operatorname{tr}\Omega Z^{-1})\operatorname{tr}\Omega Z^{-1}\Theta Z^{-1}]$$

$$=-3(q+2)(\operatorname{tr}\Omega)^2\operatorname{tr}\Theta+(q^2+2q+4)\operatorname{tr}\Omega^2(\operatorname{tr}\Theta)$$

$$+2(q^2+2q+4)(\operatorname{tr}\Omega)\operatorname{tr}\Theta-12q\operatorname{tr}\Omega^2\Theta.$$

To prove the second and the third identities in (3.23), we use the following asymptotic formula for large n:

$$(3.25)\qquad \Gamma_p\left(\frac{q}{2}\right)\frac{2^{p(p-1)/2}}{(2\pi i)^{p(p+1)/2}}\int_{\Re(Z)=X_0>0}(\operatorname{etr}Z)\,|Z|^{-\frac{q}{2}}$$

$$\left| \mathrm{I}+\frac{2}{nq}\Theta Z^{-1}\right|^{-\frac{q}{2}} \mathrm{tr}\,(\Omega Z^{-1})^2\, dZ = \mathrm{I}\,[\mathrm{tr}\,(\Omega Z^{-1})^2] - n^{-1}\mathrm{I}[\mathrm{tr}(\Omega Z^{-1})^2 \cdot \mathrm{tr}\;\Theta Z^{-1}] + O(n^{-2}) \quad \text{(using (2.14))}.$$

Making the transformation $Z \to T = Z + \frac{2}{nq}\Theta$, we can expand the left hand side of (3.25) as follows:

$$\mathrm{I}[\mathrm{tr}\,(\Omega Z^{-1})^2] - 2(nq)^{-1}\{(\mathrm{tr}\;\Theta)\mathrm{I}[\mathrm{tr}\,(\Omega Z^{-1})^2] - 2\mathrm{I}[\mathrm{tr}\,(\Omega Z^{-1})^2\Theta Z^{-1}]\} + O(n^{-2}). \tag{3.26}$$

From (3.25) and the result obtained by applying the formulas for $\mathrm{I}[\mathrm{tr}(\Omega Z^{-1})^2]$ in (3.11) and $\mathrm{I}[\mathrm{tr}\,(\Omega Z^{-1})^2\Theta Z^{-1}]$ in (3.23) to the coefficient of order n^{-1} in (3.26), we obtain the formula for $\mathrm{I}[\mathrm{tr}\,(\Omega Z^{-1})^2\,\mathrm{tr}\;\Theta Z^{-1}]$. The formula for $\mathrm{I}[(\mathrm{tr}\;\Omega Z^{-1})\mathrm{tr}\;\Omega Z^{-1}\Theta Z^{-1}]$ is immediately obtained from this result and (3.24).

4. Formulas for Laguerre polynomials. In this section we derive the formulas similar to Lemma 2 for Laguerre polynomials, which are used only for the second derivation of an asymptotic expansion of the non-null distribution of the Hotelling's criterion for multivariate linear hypothesis.

Putting $\gamma=(q-p-1)/2$ and $Z=xI_p$ in (1.13), we have

$$\sum_{k=0}^{\infty}\sum_{(\kappa)} x^k L_\kappa^{(q-p-1)/2}(\Omega)/k! = (1-x)^{-pq/2}\,\mathrm{etr}\left(-\frac{x}{1-x}\Omega\right), \tag{4.1}$$

for $|x|<1$. Moreover, we prove the following lemma:

LEMMA 8. *Let x be any number such that $|x|<1$, and put $\gamma=(q-p-1)/2$ with $\gamma>-1$. Then the following identities hold:*

$$\sum_{k=1}^{\infty}\sum_{(\kappa)} x^k L_\kappa^\gamma(\Omega)/(k-1)! = (1-x)^{-pq/2}\frac{x}{1-x}\left\{\frac{pq}{2}-\frac{\mathrm{tr}\;\Omega}{1-x}\right\}\mathrm{etr}\left(-\frac{x}{1-x}\Omega\right), \tag{4.2}$$

$$\sum_{k=2}^{\infty}\sum_{(\kappa)} x^k L_\kappa^\gamma(\Omega)/(k-2)! = (1-x)^{-pq/2}\left(\frac{x}{1-x}\right)^2\left\{\frac{pq}{2}\left(\frac{pq}{2}+1\right) - (pq+2)\frac{\mathrm{tr}\;\Omega}{1-x}+\left(\frac{\mathrm{tr}\;\Omega}{1-x}\right)^2\right\}\mathrm{etr}\left(-\frac{x}{1-x}\Omega\right), \tag{4.3}$$

$$\sum_{k=0}^{\infty}\sum_{(\kappa)} x^k L_\kappa^\gamma(\Omega)a_1(\kappa)/k! = (1-x)^{-pq/2}\left(\frac{x}{1-x}\right)^2\left\{\frac{1}{4}pq(p+q+1) - (p+q+1)\frac{\mathrm{tr}\;\Omega}{1-x}+\frac{\mathrm{tr}\;\Omega^2}{(1-x)^2}\right\}\mathrm{etr}\left(-\frac{x}{1-x}\Omega\right), \tag{4.4}$$

$$\sum_{k=1}^{\infty}\sum_{(\kappa)} x^k L_\kappa^\gamma(\Omega)a_1(\kappa)/(k-1)! = (1-x)^{-pq/2}\left(\frac{x}{1-x}\right)^2\left[\frac{1}{4}pq(p+q+1)\{2\right. \tag{4.5}$$

$$+\left(\frac{pq}{2}+2\right)\frac{x}{1-x}\Big\}-(p+q+1)\Big\{2+3\left(\frac{pq}{4}+1\right)\frac{x}{1-x}\Big\}\frac{\mathrm{tr}\ \Omega}{1-x}$$

$$+(p+q+1)\frac{x}{1-x}\left(\frac{\mathrm{tr}\ \Omega}{1-x}\right)^2+\Big\{2+\left(\frac{pq}{2}+4\right)\frac{x}{1-x}\Big\}\frac{\mathrm{tr}\ \Omega^2}{(1-x)^2}$$

$$-\frac{x}{1-x}\frac{(\mathrm{tr}\ \Omega)\,\mathrm{tr}\ \Omega^2}{(1-x)^3}\Big]\mathrm{etr}\left(-\frac{x}{1-x}\Omega\right),$$

(4.6) $$\sum_{k=0}^{\infty}\sum_{(\kappa)}x^kL_\kappa^\gamma(\Omega)\,a_1(\kappa)^2/k!=(1-x)^{-pq/2}\left(\frac{x}{1-x}\right)^2\Big[\frac{pq}{16}\Big\{4((q+1)p+q+3)$$

$$+8(p^2+3(q+1)p+q^2+3q+4)\frac{x}{1-x}+(qp^3+2(q^2+q+4)p^2$$

$$+(q+1)(q^2+q+20)p+4(2q^2+5q+5))\left(\frac{x}{1-x}\right)^2\Big\}-\Big\{(q+1)p+q+3$$

$$+3(p^2+3(q+1)p+q^2+3q+4)\frac{x}{1-x}+\frac{1}{2}(qp^3+2(q^2+q+4)p^2$$

$$+(q+1)(q^2+q+20)p+4(2q^2+5q+5))\left(\frac{x}{1-x}\right)^2\Big\}\frac{\mathrm{tr}\ \Omega}{1-x}$$

$$+\Big\{1+\frac{6x}{1-x}+(p^2+2(q+1)p+q^2+2q+7)\left(\frac{x}{1-x}\right)^2\Big\}\left(\frac{\mathrm{tr}\ \Omega}{1-x}\right)^2$$

$$+\Big\{1+6(p+q+2)\frac{x}{1-x}+\frac{1}{2}(qp^2+(q^2+q+20)p$$

$$+4(5q+8))\left(\frac{x}{1-x}\right)^2\Big\}\frac{\mathrm{tr}\ \Omega^2}{(1-x)^2}-2(p+q+1)\left(\frac{x}{1-x}\right)^2\frac{(\mathrm{tr}\ \Omega)\mathrm{tr}\ \Omega^2}{(1-x)^3}$$

$$-\frac{4x}{1-x}\left(1+\frac{2x}{1-x}\right)\frac{\mathrm{tr}\ \Omega^3}{(1-x)^3}+\left(\frac{x}{1-x}\right)^2\frac{(\mathrm{tr}\ \Omega^2)^2}{(1-x)^4}\Big]\mathrm{etr}\left(-\frac{x}{1-x}\Omega\right),$$

(4.7) $$\sum_{k=0}^{\infty}\sum_{(\kappa)}x^kL_\kappa^\gamma(\Omega)\,a_2(\kappa)/k!=(1-x)^{-pq/2}\frac{x}{1-x}\Big[\frac{pq}{4}\Big\{2+3((q+1)p+q+3)$$

$$\cdot\frac{x}{1-x}+2(p^2+3(q+1)p+q^2+3q+4)\left(\frac{x}{1-x}\right)^2\Big\}-\{1+3((q+1)p$$

$$+q+3)\frac{x}{1-x}+3(p^2+3(q+1)p+q^2+3q+4)\left(\frac{x}{1-x}\right)^2\Big\}\frac{\mathrm{tr}\ \Omega}{1-x}$$

$$+\frac{3x}{1-x}\Big\{\left(1+\frac{2x}{1-x}\right)\left(\frac{\mathrm{tr}\ \Omega}{1-x}\right)^2+\left(1+2(p+q+2)\frac{x}{1-x}\right)\frac{\mathrm{tr}\ \Omega^2}{(1-x)^2}\Big\}$$

$$-4\left(\frac{x}{1-x}\right)^2\frac{\mathrm{tr}\ \Omega^3}{(1-x)^3}\Big]\mathrm{etr}\left(-\frac{x}{1-x}\Omega\right).$$

PROOF. Differentiation of (4.1) with respect to x yields (4.2) and (4.3). Similarly the formula (4.5) follows from (4.4). In the following, we prove the formulas (4.4), (4.6) and (4.7). It is sufficient to show that these formulas are true for any positive semidefinite matrix Ω. Noting that $L_\kappa^\gamma(\Omega)$ is expressed as follows by (1.8) and (1.12)

$$L_\kappa^\gamma(\Omega)=(\mathrm{etr}\ \Omega)\mathrm{IL}_\Omega[C_\kappa(R)] \tag{4.8}$$

with an abbreviated notation

$$\mathrm{IL}_\Omega[\{\quad\}]=\frac{2^{p(p-1)/2}}{(2\pi i)^{p(p+1)/2}}\int_{\Re(Z)=X_0>0}(\mathrm{etr}\ Z)|Z|^{-\frac{q}{2}} \cdot\left\{\int_{S>0}\{\mathrm{etr}-(I+\Omega^{\frac{1}{2}}Z^{-1}\Omega^{\frac{1}{2}})R\}|R|^{(q-p-1)/2}\{\quad\}dR\right\}dZ, \tag{4.9}$$

and using Lemma 2, we obtain the following identities:

$$\sum_{k=0}^{\infty}\sum_{(\kappa)}x^k L_\kappa^\gamma(\Omega)a_1(\kappa)/k!=(\mathrm{etr}\ \Omega)x^2\mathrm{IL}_\Omega[(\mathrm{tr}\ R^2)\mathrm{etr}\ xR], \tag{4.10}$$

$$\sum_{k=0}^{\infty}\sum_{(\kappa)}x^k L_\kappa^\gamma(\Omega)a_1(\kappa)^2/k!=(\mathrm{etr}\ \Omega)x^2\mathrm{IL}_\Omega[\{(\mathrm{tr}\ R)^2+\mathrm{tr}\ R^2 +4x\,\mathrm{tr}\ R^3+x^2(\mathrm{tr}\ R^2)^2\}\mathrm{etr}\ xR], \tag{4.11}$$

$$\sum_{k=0}^{\infty}\sum_{(\kappa)}x^k L_\kappa^\gamma(\Omega)a_2(\kappa)/k!=(\mathrm{etr}\ \Omega)x\mathrm{IL}_\Omega[\{\mathrm{tr}\ R+3x(\mathrm{tr}\ R)^2 +3x\ \mathrm{tr}\ R^2+4x^2\ \mathrm{tr}\ R^3\}\ \mathrm{etr}\ xR]. \tag{4.12}$$

Using Lemmas 4 and 5, we simplify each of the right hand sides of above expressions. For example, by Lemma 4 the right hand side of (4.10) can be written as follows:

$$\frac{q}{4}x^2(\mathrm{etr}\ \Omega)\mathrm{I}[|(1-x)I+\Omega Z^{-1}|^{-\frac{q}{2}}\{(\mathrm{tr}((1-x)I+\Omega Z^{-1})^{-1})^2 +(q+1)\mathrm{tr}((1-x)I+\Omega Z^{-1})^{-2}\}] \tag{4.10}$$

with $\mathrm{I}[\{\quad\}]$ defined by (3.2). Making the transformation $Z\to Z+(1-x)^{-1}\Omega$, we can write it as follows:

$$\frac{q}{4}(1-x)^{-pq/2}\left(\frac{x}{1-x}\right)^2\left\{\mathrm{etr}\left(-\frac{x}{1-x}\Omega\right)\right\}\mathrm{I}\left[p(p+q+1) -2(p+q+1)\frac{\mathrm{tr}\ \Omega Z^{-1}}{1-x}+\left(\frac{\mathrm{tr}\ \Omega Z^{-1}}{1-x}\right)^2+(q+1)\frac{\mathrm{tr}(\Omega Z^{-1})^2}{(1-x)^2}\right]. \tag{4.13}$$

Applying Lemma 5 to the above expression, we obtain the right hand side of (4.4). Similarly the right hand sides of (4.11) and (4.12) also imply (4.6) and (4.7), respectively.

PART II. MULTIVARIATE LINEAR HYPOTHESIS

5. Asymptotic expansions of the distributions of the Pillai's criterion

5.1. *The moment generating functions of the criterion.* The multivariate linear hypothesis model has been discussed by many authors (e.g., Anderson [2], Das Gupta, Anderson and Mudholkar [8], Roy [35], Seber [36], etc.). The following canonical form is well known: Let each column vector of $p\times N$ matrix $X'=(X_1'(p\times q),\ X_2'(p\times(N-s)),\ X_3'(p\times(s-q)))$ with $q\leq s$ be distributed independently according to a p-variate normal distribution with the common covariance matrix Σ and expectations given by

$$\mathrm{E}[X_1]=M(q\times p),\ \mathrm{E}[X_2]=0((N-s)\times p),\ \mathrm{E}[X_3]=\Gamma((s-q)\times p).$$

Then multivariate linear hypothesis is defined by testing the hypothesis

$$(5.1)\qquad H: M=0 \qquad \text{against alternative} \qquad K: M\neq 0,$$

where Γ is a matrix of nuisance parameters. The Pillai's criterion (Pillai [28]) for this problem is based on the statistic

$$(5.2)\qquad V=m\ \mathrm{tr}\ S_h(S_h+S_e)^{-1},$$

where $m=N-s+q=n+q$, $S_e=X_2'X_2$ and $S_h=X_1'X_1$ are the matrices of sums of squares and products due to error and due to the hypothesis, respectively. The matrix S_e has the Wishart distribution $W_p(n,\Sigma)$. The matrix S_h has the non-central Wishart distribution $W_p(q,\Sigma,\Omega)$, where the matrix of non-centrality parameters is given by $\Omega=\frac{1}{2}\Sigma^{-\frac{1}{2}}M'M\Sigma^{-\frac{1}{2}}$.

We can easily see that in the case of $q\geq p$ under the hypothesis H the matrix $B=(S_h+S_e)^{-\frac{1}{2}}S_h(S_h+S_e)^{-\frac{1}{2}}$ has the following multivariate beta distribution

$$(5.3)\qquad \Gamma_p\left(\frac{n+q}{2}\right)\left\{\Gamma_p\left(\frac{q}{2}\right)\Gamma_p\left(\frac{n}{2}\right)\right\}^{-1}|B|^{(q-p-1)/2}|I-B|^{(n-p-1)/2}\,dB.$$

Therefore, by James [21] the moment generating fonction of V under the hypothesis H with $q\geq p$ is expressed as follows:

$$(5.4)\qquad M_H(t)={}_1F_1\left(\frac{q}{2};\frac{m}{2};mtI_p\right).$$

By using a well known fact that the density function of the characteristic roots of $S_hS_e^{-1}$, in the degenerate case of $q<p$, is obtained from its density function in the case of $p\leq q$ by making the substitutions (c.f., Anderson [2, p. 318], Roy [36, p. 46])

(5.5) $$q \to p, \qquad n \to n+q-p, \qquad p \to q.$$

We can write the $M_H(t)$ in the degenerate case as ${}_1F_1\left(\frac{p}{2}; \frac{m}{2}; mtI_q\right)$. It is easily seen that this expression is equal to the expression (5.4) by the formula $C_\kappa(I_q) = d(\kappa)\left(\frac{q}{2}\right)_\kappa$ of Constantine [6], which vanishes when the number of parts in a partition of k is larger than q. Therefore the formula (5.4) also holds for $q < p$.

Moreover, Pillai [31] has obtained the following moment generating function of V under alternative K with $p \leq q$, which covers the formula (5.4) as a special case:

(5.6) $$M_K(t) = \{\mathrm{etr}(-\Omega)\} \mathrm{I}\left[\left\{\Gamma_p\left(\frac{m}{2}\right)\right\}^{-1} \int_{S>0} \{\mathrm{etr}(-S)\} |S|^{(m-p-1)/2} \cdot {}_1F_1\left(\frac{q}{2}; \frac{m}{2}; mtI + \Omega^{\frac{1}{2}} S \Omega^{\frac{1}{2}} T^{-1}\right) dS\right]$$

where I[ ] is defined by (3.2) with respect to T. We note that if $q < p$, V has the moment generating function obtained from (5.6) by making the substitutions (5.5) and putting $\Omega = \frac{1}{2} M \Sigma^{-1} M'$.

5.2. *Approximate null distribution.* First we derive an asymptotic expansion of the null distribution of the Pillai's criterion V, which is also derived as a special case of the general result in the Section 5.3. The moment generating function $M_H(t)$ can be written, by the definition (1.1), as follows:

(5.7) $$M_H(t) = \sum_{k=0}^{\infty} \sum_{(\kappa)} \left(\frac{q}{2}\right)_\kappa \left(\frac{m}{2}\right)^k C_\kappa(2tI) \Big/ \left\{\left(\frac{m}{2}\right)_\kappa k!\right\}.$$

From (2.4) we can write $M_H(t)$ as

(5.8) $$\sum_{k=0}^{\infty} \sum_{(\kappa)} \left(\frac{q}{2}\right)_\kappa \left[1 - \frac{1}{m} a_1(\kappa) + \frac{1}{6m^2}\{3a_1(\kappa)^2 + a_2(\kappa) - k\} + O(m^{-3})\right] C_\kappa(2tI)/k!.$$

By Lemma 3, we simplify the above expression, and obtain

(5.9) $$M_H(t) = (1-2t)^{-pq/2}\left[1 - \frac{pq}{4m}(p+q+1)\{1 - 2(1-2t)^{-1} + (1-2t)^{-2}\} + \frac{pq}{96m^2}\left\{\sum_{\alpha=0}^{4} (-1)^\alpha h_\alpha (1-2t)^{-\alpha}\right\} + O(m^{-3})\right],$$

which holds for $|t|<\frac{1}{2}$, where the coefficients $h_\alpha(\alpha=0,1,\cdots,4)$ are given by (5.10).

$$(5.10)\quad \begin{aligned} h_0&=3qp^3+2(3q^2+3q-4)p^2+3(q+1)(q^2+q-4)p-4(2q^2+3q-1),\\ h_1&=12pq(p+q+1)^2,\\ h_2&=6\{3qp^3+2(3q^2+3q+4)p^2+(q+1)(3q^2+3q+16)p+8(q+1)^2\},\\ h_3&=4\{3qp^3+2(3q^2+3q+8)p^2+3(q+1)(q^2+q+12)p+4(4q^2+9q+7)\},\\ h_4&=3\{qp^3+2(q^2+q+4)p^2+(q+1)(q^2+q+20)p+4(2q^2+5q+5)\}.\end{aligned}$$

Since $(1-2t)^{-\frac{f}{2}}$ is the moment generating function of the χ^2 distribution with f degrees of freedom, χ^2_f, we obtain the following theorem:

THEOREM 5.1. *The null distribution of the Pillai's criterion* (5.2) *for multivariate linear hypothesis can be approximated asymptotically up to order* m^{-2} *by the following distribution*:

$$(5.11)\quad P_H(V<z)=P(\chi^2_f<z)-\frac{pq}{4m}(p+q+1)\{P(\chi^2_f<z)$$
$$-2P(\chi^2_{f+2}<z)+P(\chi^2_{f+4}<z)\}+\frac{pq}{96m^2}\left\{\sum_{\alpha=0}^{4}(-1)^\alpha h_\alpha P(\chi^2_{f+2\alpha}<z)\right\}+O(m^{-3}),$$

where $m=N-s+q$, $f=pq$ *and the coefficients* $h_\alpha(\alpha=0,1,\cdots,4)$ *are given by* (5.10).

5.3. *Approximate non-null distribution.* In this section we derive the asymptotic expansion of the Pillai's criterion V given by (5.2) under the alternative K, by expanding the moment generating function (5.6). From (2.4) and Lemma 3 in Section 2 we have

$$(5.12)\quad {}_1F_1\left(\frac{q}{2};\frac{m}{2};mtI+\Omega^{\frac{1}{2}}T^{-1}\Omega^{\frac{1}{2}}S\right)=\left|(1-2t)I-\Omega^{\frac{1}{2}}\frac{2}{m}S\Omega^{\frac{1}{2}}T^{-1}\right|^{-\frac{q}{2}}\left\{1\right.$$
$$\left.-\frac{q}{4m}U_1\left(\frac{2}{m}S,T\right)+\frac{q}{96m^2}U_2\left(\frac{2}{m}S,T\right)+O(m^{-3})f\left(\frac{2}{m}S,T\right)\right\},$$

which holds for sufficiently small $|t|$ and large m, where $U_1\left(\frac{2}{m}S,T\right)$ and $U_2\left(\frac{2}{m}S,T\right)$ are given by (5.13).

$$(5.13)\quad U_1\left(\frac{2}{m}S,T\right)=(\mathrm{tr}\ W)^2+(q+1)\mathrm{tr}\ W^2,$$

$$U_2\Big(\frac{2}{m}S,\ T\Big)=24(q+1)(\mathrm{tr}\ W)^2+24(q+3)\,\mathrm{tr}\ W^2+32(\mathrm{tr}\ W)^3$$

$$+96(q+1)(\mathrm{tr}\ W)\,\mathrm{tr}\ W^2+32(q^2+3q+4)\,\mathrm{tr}\ W^3+3q(\mathrm{tr}\ W)^4$$

$$+6(q^2+q+4)(\mathrm{tr}\ W)^2\,\mathrm{tr}\ W^2+3(q+1)(q^2+q+4)(\mathrm{tr}\ W^2)^2$$

$$+48(q+1)(\mathrm{tr}\ W)\,\mathrm{tr}\ W^3+12(2q^2+5q+5)\,\mathrm{tr}\ W^4,$$

with $W=\Big(2tI+\Omega^{\frac{1}{2}}\frac{2}{m}S\Omega^{\frac{1}{2}}T^{-1}\Big)\Big\{(1-2t)I-\Omega^{\frac{1}{2}}\frac{2}{m}S\Omega^{\frac{1}{2}}T^{-1}\Big\}^{-1}$, and $f\Big(\frac{2}{m}S,\ T\Big)$ is a remainder term. By using the above formula and the same method as in the expansion of the left hand side of (2.4), the following asymptotic identity is obtained:

$$\text{(5.14)}\qquad \Big\{\Gamma_p\Big(\frac{m}{2}\Big)\Big\}^{-1}\int_{S>0}\{\mathrm{etr}(-S)\}\,|S|^{(m-p-1)/2}{}_1F_1\Big(\frac{q}{2};\frac{m}{2};mtI+\Omega^{\frac{1}{2}}S\Omega^{\frac{1}{2}}T^{-1}\Big)dS$$

$$=(1-2t)^{-pq/2}|T|^{\frac{q}{2}}|T-\phi\Omega|^{-\frac{q}{2}}\Big[1-\frac{1}{m}\Big\{\frac{1}{4}qU_1(I,\ T)-|T-\phi\Omega|^{\frac{q}{2}}$$

$$\cdot\mathrm{tr}\ \partial^2|T-\phi\Omega^{\frac{1}{2}}\Sigma\Omega^{\frac{1}{2}}|^{-\frac{q}{2}}\Big|_{\Sigma=I}\Big\}+\frac{1}{m^2}\Big\{\frac{1}{96}qU_2(I,\ T)-\frac{1}{4}q|T-\phi\Omega|^{\frac{q}{2}}$$

$$\cdot\mathrm{tr}\ \partial^2|T-\phi\Omega^{\frac{1}{2}}\Sigma\Omega^{\frac{1}{2}}|^{-\frac{q}{2}}U_1(\Sigma,\ T)\Big|_{\Sigma=I}+|T-\phi\Omega|^{\frac{q}{2}}\Big(\frac{1}{2}(\mathrm{tr}\ \partial^2)^2+\frac{4}{3}\mathrm{tr}\ \partial^3\Big)$$

$$T-\phi\Omega^{\frac{1}{2}}\Sigma\Omega^{\frac{1}{2}}|^{-\frac{q}{2}}\Big|_{\Sigma=I}\Big\}+O(m^{-3})\Big],$$

where $\phi=(1-2t)^{-1}$ is used for abbreviation. Now we have to carry out the operations ∂ appeared in the right hand side of the above expression, which is given in Appendix II. Inserting the formulas (A, 37), (A, 38), (A, 39) and (A, 47) in Appendix II to the right hand side of (5.14) and using the transformation $T\to Z=T-(1-2t)^{-1}\Omega$ we can express $M_K(t)$ as follows:

$$\text{(5.15)}\qquad (1-2t)^{-pq/2}\Big\{\mathrm{etr}\Big(\frac{2t}{1-2t}\Omega\Big)\Big\}\,\mathrm{I}\Big[1-\frac{q}{4m}\Big\{p(p+q+1)(\phi-1)^2$$

$$+2(p+q+1)\phi^2(\phi-1)\mathrm{tr}\ \Omega Z^{-1}+\phi^2(\phi^2-1)((\mathrm{tr}\ \Omega Z^{-1})^2+(q+1)\mathrm{tr}(\Omega Z^{-1})^2)\Big\}$$

$$+\frac{q}{96m^2}\Big\{\gamma_0+\gamma_1\,\mathrm{tr}\ \Omega Z^{-1}+\gamma_2(\mathrm{tr}\ \Omega Z^{-1})^2+\gamma_3\,\mathrm{tr}(\Omega Z^{-1})^2+\gamma_4(\mathrm{tr}\ \Omega Z^{-1})^3$$

$$+\gamma_5(\mathrm{tr}\ \Omega Z^{-1})\,\mathrm{tr}(\Omega Z^{-1})^2+\gamma_6\,\mathrm{tr}(\Omega Z^{-1})^3+3\phi^4(\phi^2-1)^2\{q(\mathrm{tr}\ \Omega Z^{-1})^4$$

$$+2(q^2+q+4)(\mathrm{tr}\ \Omega Z^{-1})^2\,\mathrm{tr}(\Omega Z^{-1})^2+(q+1)(q^2+q+4)(\mathrm{tr}(\Omega Z^{-1})^2)^2$$

$$+16(q+1)(\mathrm{tr}\Omega Z^{-1})\mathrm{tr}(\Omega Z^{-1})^3+4(2q^2+5q+5)\mathrm{tr}(\Omega Z^{-1})^4\}\Big\}+O(m^{-3})\Big],$$

where the notation I[ ] is defined by (3.2) and the coefficients $\gamma_\alpha(\alpha=0, 1,\cdots,6)$ are given, with h_α in (5.10), by (5.16).

(5.16) $\gamma_0=p\sum_{\alpha=0}^{4}(-1)^\alpha\phi^\alpha h_\alpha$,

$$\gamma_1=\phi^2\{-h_1+2h_2\phi-3h_3\phi^2+4h_4\phi^3\},$$

$$\begin{aligned}\gamma_2=6\phi^2\Big[&-pq(p+q+1)+2\{qp^2+(q^2+q+4)p+4(q+1)\}\phi+2\{qp^2\\&+2(q^2+q-1)p+(q+1)(q^2+q-4)\}\phi^2-2\{3qp^2+(5q^2+5q+12)p\\&+2(q+1)(q^2+q+8)\}\phi^3+\{3qp^2+5(q^2+q+4)p\\&+2(q+1)(q^2+q+16)\}\phi^4\Big],\end{aligned}$$

$$\begin{aligned}\gamma_3=6\phi^2\Big[&-pq(q+1)(p+q+1)+2(q+1)\{qp^2+(q^2+q+4)p+4(q+1)\}\phi\\&+4\{p^2+(q+1)p-(q+3)\}\phi^2-2\{(q^2+q+4)p^2+(q+1)(q^2+q+20)p\\&+4(4q^2+9q+7)\}\phi^3+\{(q^2+q+4)p^2+(q+1)(q^2+q+28)p\\&+12(2q^2+5q+5)\phi^4\Big],\end{aligned}$$

$$\begin{aligned}\gamma_4=4\phi^3\Big\{&4+3q(p+q+1)\phi-3(qp+q^2+q+4)\phi^2\\&-(3qp+3q^2+3q+4)\phi^3+3(qp+q^2+q+4)\phi^4\Big\},\end{aligned}$$

$$\begin{aligned}\gamma_5=12\phi^3\Big[&4(q+1)+(q^2+q+4)(p+q+1)\phi-\{(q^2+q+4)p+(q+1)\\&\cdot(q^2+q+16)\}\phi^2-\{(q^2+q+4)p+(q+1)(q^2+q+8)\}\phi^3\\&+\{(q^2+q+4)p+(q+1)(q^2+q+16)\}\phi^4\Big],\end{aligned}$$

$$\begin{aligned}\gamma_6=16\phi^3\Big[&q^2+3q+4+3(q+1)(p+q+1)\phi-3\{(q+1)p+2q^2+5q+5\}\phi^2\\&-\{3(q+1)p+4q^2+9q+7\}\phi^3+3\{(q+1)p+2q^2+5q+5\}\phi^4\Big].\end{aligned}$$

By Lemma 5 in Section 3, we finally obtain the following asymptotic formula for the moment generating function of V:

(5.17) $$M_K(t)=(1-2t)^{-pq/2}\left\{\operatorname{etr}\left(\frac{2t}{1-2t}\Omega\right)\right\}\left[1-\frac{1}{4m}\left\{pq(p+q+1)(1-2(1-2t)^{-1})\right.\right.$$
$$+(pq(p+q+1)-4(p+q+1)\operatorname{tr}\Omega-4\operatorname{tr}\Omega^2)(1-2t)^{-2}$$
$$\left.+4(p+q+1)(1-2t)^{-3}\operatorname{tr}\Omega+4(1-2t)^{-4}\operatorname{tr}\Omega^2\right\}$$
$$\left.+\frac{1}{96m^2}\left\{pq(h_0-h_1(1-2t)^{-1})+\sum_{\alpha=2}^{8}A_\alpha(\Omega)(1-2t)^{-\alpha}\right\}+O(m^{-3})\right],$$

with the coefficients $A_\alpha(\Omega)$ $(\alpha=2,3,\cdots,8)$ given by (5.18).

(5.18) $$A_2(\Omega)=pqh_2-2h_1\operatorname{tr}\Omega-24pq(p+q+1)\operatorname{tr}\Omega^2,$$
$$A_3(\Omega)=-pqh_3+4h_2\operatorname{tr}\Omega+48\{qp^2+(q^2+q+4)p+4(q+1)\}\operatorname{tr}\Omega^2$$
$$+128\operatorname{tr}\Omega^3,$$
$$A_4(\Omega)=pqh_4-6h_3\operatorname{tr}\Omega+48\{p^2+2(q+1)p+q^2+2q-1\}(\operatorname{tr}\Omega)^2$$
$$-96(p+q+2)\operatorname{tr}\Omega^2+96(p+q+1)(\operatorname{tr}\Omega)\operatorname{tr}\Omega^2+48(\operatorname{tr}\Omega^2)^2,$$
$$A_5(\Omega)=8h_4\operatorname{tr}\Omega-96\{p^2+2(q+1)p+q^2+2q+3\}(\operatorname{tr}\Omega)^2-48\{qp^2$$
$$+(q^2+q+12)p+4(3q+4)\}\operatorname{tr}\Omega^2-96(p+q+1)(\operatorname{tr}\Omega)\operatorname{tr}\Omega^2$$
$$-384\operatorname{tr}\Omega^3,$$
$$A_6(\Omega)=48\{p^2+2(q+1)p+q^2+2q+7\}(\operatorname{tr}\Omega)^2+24\{qp^2+(q^2+q+20)p$$
$$+4(5q+8)\}\operatorname{tr}\Omega^2-96(p+q+1)(\operatorname{tr}\Omega)\operatorname{tr}\Omega^2-128\operatorname{tr}\Omega^3$$
$$-96(\operatorname{tr}\Omega^2)^2,$$
$$A_7(\Omega)=96(p+q+1)(\operatorname{tr}\Omega)\operatorname{tr}\Omega^2+384\operatorname{tr}\Omega^3,$$
$$A_8(\Omega)=48(\operatorname{tr}\Omega^2)^2$$

Noting that the expression (5.17) is a symmetric function with respect to p and q, we can easily see that the asymptotic expansion of $M_K(t)$ in the generate case is also given by (5.17). By inverting this moment generating function using the fact that $(1-2t)^{-f/2}\exp\{2t\delta^2/(1-2t)\}$ is the moment generating function of the non-central χ^2 distribution with f degrees of freedom and non centrality parameter δ^2, we obtain the following theorem:

THEOREM 5.2. *The non-null distribution of the Pillai's criterion* (5.2) *for multivariate linear hypothesis can be approximated asymptotically up to order* m^{-2} *by*

(5.19) $$P_K(V<z)=P(\chi_f^2(\delta^2)<z)-\frac{1}{4m}\left[pq(p+q+1)P(\chi_f^2(\delta^2)<z)\right.$$

$$-2pq(p+q+1)P(\chi^2_{f+2}(\delta^2)<z)+\{pq(p+q+1)-4(p+q+1)\operatorname{tr}\Omega-4\operatorname{tr}\Omega^2\}$$

$$P(\chi^2_{f+4}(\delta^2)<z)+4(p+q+1)\operatorname{tr}\Omega\cdot P(\chi^2_{f+6}(\delta^2)<z)+4\operatorname{tr}\Omega^2\cdot P(\chi^2_{f+8}(\delta^2)<z)\Big]$$

$$+\frac{1}{96m^2}\Big\{pqh_0P(\chi^2_f(\delta^2)<z)-pqh_1P(\chi^2_{f+2}(\delta^2)<z)+\sum_{\alpha=2}^{8}A_\alpha(\Omega)P(\chi^2_{f+2\alpha}(\delta^2)<z)\Big\}$$

$$+O(m^{-3}),$$

where $m=N-s+q$, $f=pq$, $\delta^2=\operatorname{tr}\Omega=\frac{1}{2}\operatorname{tr}\Sigma^{-1}M'M$ *and the coefficients* $h_\alpha(\alpha=0,1,\cdots,4)$ *and* $A_\alpha(\Omega)$ $(\alpha=2,3,\cdots,8)$ *are given by* (5.10) *and* (5.18), *respectively. The non-central* χ^2*-variate with* f *degrees of freedom and non-centrality parameter* δ^2 *is denoted by* $\chi^2_f(\delta^2)$.

If we specialize Ω to the null matrix, we can obtain the asymptotic expansion (5.11) of the null distribution of the Pillai's criterion (5.2).

5.4. *Numerical accuracy of the approximations.* When $p=2$, Pillai and Jayachandran [30] have given the exact 5 and 1% points of tr $S_h(S_h+S_e)^{-1}$ and its powers under certain alternatives for some values of q and n. Hence it is possible to put our results (5.11) and (5.19) to the test of a numerical comparison. From (5.11) the approximate $100\alpha\%$ point of the Pillai's criterion V are obtained by solving the equation

$$Q(z)=\alpha, \tag{5.20}$$

where $Q(z)$ is given by

$$Q(z)=P(\chi^2_f>z)-pq(4m)^{-1}(p+q+1)\{P(\chi^2_f>z)-2P(\chi^2_{f+2}>z)+P(\chi^2_{f+4}>z)\}+pq(96m^2)^{-1}\{\sum_{\alpha=0}^{4}(-1)^\alpha h_\alpha P(\chi^2_{f+2\alpha}>z)\}, \tag{5.21}$$

with the notations defined in Theorem 5.1. To solve the equation (5.20), we use the Newton's iterative method. It may be remarked that the $100\alpha\%$ point of the Pillai's criterion V can also be expressed in terms of the $100\alpha\%$ point u of the χ^2 distribution with $f=pq$ degrees of freedom, by applying the general inverse expansion formula of Hill and Davis [12] to the asymptotic null distribution of V given by (5.11), giving

$$u-\frac{p+q+1}{2m(f+2)}u(u-f-2)+\frac{u}{48m^2(f+2)(f+4)(f+6)}\Big\{h_4u^3+(h_4-h_3)(f+6)u^2+(h_1-h_0)(f+4)(f+6)u-h_0(f+2)(f+4)(f+6)\Big\}$$
$$-\frac{(p+q+1)^2}{16m^2(f+2)^2}u(u-f-2)\{u^2-2(f+4)u+(f+2)^2\}+O(n^{-3}). \tag{5.22}$$

TABLE 1. *Comparison of approximations to the upper* 5% *points of* tr $S_h(S_h+S_e)^{-1}$ *for* $p=2$

n	q	Neglecting terms of order $O(n^{-1})$	$O(n^{-2})$	$O(n^{-3})$	Exact
13	3	0.78698	0.71712	0.69996	0.69762
	7	1.18424	1.06873	1.04051	1.03905
	13	1.49558	1.35214	1.31479	1.30525
33	3	0.34977	0.33435	0.33262	0.33257
	7	0.59212	0.55968	0.55580	0.55598
	13	0.84533	0.79386	0.78622	0.78628
63	3	0.19078	0.18602	0.18573	0.18573
	7	0.33835	0.32729	0.32659	0.32662
	13	0.51165	0.49181	0.49020	0.49030

TABLE 2. *Comparison of approximations to the upper* 1% *points of* tr $S_h(S_h+S_e)^{-1}$ *for* $p=2$

n	q	Neglecting terms of order $O(n^{-1})$	$O(n^{-2})$	$O(n^{-3})$	Exact
13	3	1.05074	0.85894	0.83389	0.85427
	7	1.45707	1.18805	1.15056	1.18472
	13	1.75545	1.44859	1.39781	1.42313
33	3	0.46700	0.42472	0.42424	0.42557
	7	0.72853	0.65227	0.65238	0.65685
	13	0.99221	0.87953	0.87606	0.88550
63	3	0.25473	0.24194	0.24206	0.24219
	7	0.41630	0.39102	0.39191	0.39243
	13	0.60055	0.55810	0.55966	0.56119

TABLE 3. *Comparison of approximations to the powers of* tr $S_h(S_h+S_e)^{-1}$ *for* $p=2$ *and* $\alpha=0.05$

ω_1	ω_2	n	q	Neglecting terms of order $O(n^{-1})$	$O(n^{-2})$	$O(n^{-3})$	Exact
0.125	.125	33	3	0.08634	0.06942	0.06785	0.06788
			7	0.09018	0.06308	0.06001	0.06007
		83	3	0.07670	0.06963	0.06939	0.06939
			7	0.07349	0.06178	0.06131	0.06131
0	1	33	3	0.17137	0.13228	0.12973	0.12992
			7	0.14709	0.09709	0.09343	0.09389
		83	3	0.15620	0.13964	0.13924	0.13926
			7	0.12367	0.10169	0.10119	0.10123
2	2	63	3	0.55171	0.50766	0.50469	0.505
			7	0.41336	0.34238	0.33716	0.337
0	5	63	3	0.66328	0.60115	0.59401	0.594
			7	0.51352	0.41007	0.39904	0.398

Tables 1 and 2 give the upper 5 and 1% points of tr $S_h(S_h+S_e)^{-1}$ for $q=3$, 7, 13 and $n=13$, 33, 63. Table 3 gives approximate powers of the Pillai's criterion V for various pairs of values of the characteristic roots (ω_1, ω_2) of Ω, based on our 5% points. As is shown by all three tables, the agreement between the results derived from our formulas and the exact values in Pillai and Jayachandran [30] (extracted from their tables 7a, 8, 10, by noting that their notations n, m, ω_i mean by our notations $2n+3$, $2q+3$, $\frac{1}{2}\omega_i$, respectively) is excellent and is still excellent in the case when terms of $O(m^{-2})$ are neglected. Tables 1 and 2 also shows that the approximations to the upper 5% points are better than those of the upper 1% points. It is worthwhile to note that our asymptotic formulas (5.11) and (5.19) hold for any p, q, n such that $p\leq n$ and without any assumption on the rank of noncentrality matrix Ω. By using (5.21), the upper 5 and 1% points were computed for values $p=2(1)7$, $q=2(1)12$, and $n=25$, 30, 40, 60, 80, 100, 130, 160, 200, 250, 350, 500. These results are presented in Appendix III.

6. New derivation of an asymptotic expansion of the non-null distribution of the Hotelling's T_0^2 statistic

6.1. *The characteristic function and the Laplace transform of the statistic.* The Hotelling's criterion for testing the multivariate linear hypothesis given in (5.1) is based on the statistic $T_0^2=n$ tr $S_hS_e^{-1}$, which is called the Hotelling's T_0^2 statistic. The exact distribution of this statistic has been studied by various authors, e.g. Hsu [14], Hotelling [13], Constantine [7], Pillai and Jayachandran [30]. However the exact distribution of T_0^2 is available only for some particular values of p and q or under the condition $T_0^2<n$. On the other hand, an asymptotic expansion of the distribution has been given for general values of the parameters p, q and the non-centrality matrix Ω. An asymptotic expansion of the null distribution of T_0^2 was given by Ito [15] up to order n^{-2}. The non-null distribution was given by Siotani [37] and later by Ito [16] up to order n^{-1}. Recently Siotani [38] obtained the non-null distribution up to order n^{-2}. In Section 6.2 we give two other methods of obtaining the non-null distribution up to order n^{-2} by using hypergeometric function and Laguerre polynomial of matrix argument. First we shall express the characteristic function and the Laplace transform in a form convenient for our method.

THEOREM 6.1. *Under the alternative K with $p\leq q$, the characteristic function $C(t)$ of the T_0^2 statistic can be expressed as follows:*

$$C(t)=\left\{\Gamma_p\left(\frac{n+q}{2}\right)\Big/\left(\frac{n}{2}\right)^{pq/2}\Gamma_p\left(\frac{n}{2}\right)\right\}\left\{\Gamma_p\left(\frac{q}{2}\right)\right\}^{-1}\int_{S>0}\left|I+\frac{2}{n}S\right|^{-(n+q)/2} \tag{6.1}$$

$$\cdot\{\mathrm{etr}(2itS)\}\ |S|^{(q-p-1)/2}{}_0F_1\Big(\frac{q}{2};\ 2it\Omega S\Big)dS.$$

PROOF. From Hsu [14] and Ito [16] we can write $C(t)$ as follows:

$$(6.2)\qquad C(t)=\Big\{\Gamma_p\Big(\frac{n+q}{2}\Big)/(n\pi)^{pq/2}\Gamma_p\Big(\frac{n}{2}\Big)\Big\}\int_{X\in\mathfrak{X}}\Big|I+\frac{1}{n}X'X\Big|^{-(n+q)/2}$$

$$\cdot\{\mathrm{etr}\,(itX'X+\sqrt{2it}\tilde{D}X)\}\,dX,$$

where $\mathfrak{X}=\{X(q\times p)=(x_{ij})\,|\,-\infty<x_{ij}<\infty,\ i=1,2,\cdots,q,\ j=1,2,\cdots,p\}$, $\tilde{D}(p\times q)=(D^{\frac{1}{2}},\ 0(p\times(q-p))$ and $\frac{1}{2}D=\mathrm{diag}\,(\omega_1,\omega_2,\cdots,\omega_p)$ is a diagonal matrix with p characteristic roots ω_i of Ω as its non-zero elements. We decompose X as follows:

$$(6.3)\qquad X=AS^{\frac{1}{2}},$$

where $A=X(X'X)^{-\frac{1}{2}}$ is a $q\times p$ matrix satisfying $A'A=I_p$ and $S=X'X$ is a positive definite matrix. The Jacobian of this transformation is (Herz [11], James [18])

$$(6.4)\qquad dX=\pi^{pq/2}\Big\{\Gamma_p\Big(\frac{q}{2}\Big)\Big\}^{-1}\ |S|^{(q-p-1)/2}d\nu(A)\,dS,$$

where $d\nu(A)$ is a normalized invariant measure on Stiefel manifold with volume unity. Inserting (6.3) and (6.4) to (6.2), and using the formula

$$\int_{A'A=I_p}\mathrm{etr}(\sqrt{2it}AS^{\frac{1}{2}}\tilde{D})\,d\nu(A)={}_0F_1\Big(\frac{q}{2};\ itDS\Big)$$

(see Constantine [6], p. 1277–8), we obtain the expression (6.1).

COROLLARY 6.1. *The characteristic function of T_0^2 in the degenerating case, i.e. $q<p$, is given by* (6.5),

$$(6.5)\quad C(t)=\Big\{\Gamma_p\Big(\frac{n+q}{2}\Big)/\Big(\frac{n}{2}\Big)^{pq/2}\Gamma_p\Big(\frac{n}{2}\Big)\Big\}\Big\{\Gamma_q\Big(\frac{p}{2}\Big)\Big\}^{-1}\int_{S(q\times q)>0}\Big|I+\frac{2}{n}S\Big|^{-(n+q)/2}$$

$$\cdot\{\mathrm{etr}\,(2itS)\}\,|S|^{(p-q-1)/2}{}_0F_1\Big(\frac{p}{2};2it\Omega S\Big)\,dS,$$

where Ω is defined by $\frac{1}{2}M\Sigma^{-1}M'$.

The corollary is obtained by considering the transformation $X'=AS^{\frac{1}{2}}$, where

$A=X'(XX')^{-\frac{1}{2}}$ and $S=XX'$, instead of (6.3), with the obvious changes of the matrix $\tilde{D}$ and the formula $d\nu(A)$.

Constantine [7] obtained the non-null density function of T_0^2 for $T_0^2<n$ by a series of Laguerre polynomials of matrix argument. A slight reduction from his expression yields the following theorem:

THEOREM 6.2. *Under the alternative K with $p\leq q$, the Laplace transform $g(t)$ of a density function of T_0^2 can be expressed asymptotically as* (6.6),

$$(6.6)\quad g(t)=\left\{\Gamma_p\left(\frac{n+q}{2}\right)\Big/\left(\frac{n}{2}\right)^{pq/2}\Gamma_p\left(\frac{n}{2}\right)\right\}\{\mathrm{etr}(-\Omega)\}(2t)^{-pq/2}$$

$$\cdot\sum_{k=0}^{\infty}\sum_{(\kappa)}\left(-\frac{1}{2t}\right)^k L_\kappa^{(q-p-1)/2}(\Omega)\left[1+\frac{1}{n}(a_1(\kappa)+qk)\right.$$

$$\left.+\frac{1}{6n^2}\{3a_1(\kappa)^2-a_2(\kappa)+k+6q(k-1)a_1(\kappa)+3q^2k(k-1)\}+O(n^{-3})\right]\Big/k!,$$

for $|t|>\frac{1}{2}$, *where $a_1(\kappa)$ and $a_2(\kappa)$ are given by* (2.1).

PROOF. Substituting nt for t in the expression (36) in Constantine [7], we have

$$(6.7)\qquad g(t)=\mathrm{E}\left[e^{-tT_0^2}\right]$$

$$=\left\{\Gamma_p\left(\frac{n+q}{2}\right)\Big/\left(\frac{n}{2}\right)^{pq/2}\Gamma_p\left(\frac{n}{2}\right)\right\}\{\mathrm{etr}(-\Omega)\}(2t)^{-pq/2}\left\{\Gamma_p\left(\frac{n+q}{2}\right)\right\}^{-1}$$

$$\cdot\int_{S>0}\{\mathrm{etr}(-S)\}\,|S|^{(n+q-p-1)/2}\sum_{k=0}^{\infty}\sum_{(\kappa)}\left(-\frac{1}{2t}\right)^k L_\kappa^{(q-p-1)/2}(\Omega)$$

$$\cdot(n+q)^k n^{-k}C_\kappa\left(\frac{2}{n+q}S\right)\Big/\{C_\kappa(I)k!\}\,dS.$$

Expanding the above integrand in a Taylor series with respect to $\frac{2}{n+q}S$ about I as in (2.5), we can get the following

$$(6.8)\quad g(t)=\left\{\Gamma_p\left(\frac{n+q}{2}\right)\Big/\left(\frac{n}{2}\right)^{pq/2}\Gamma_p\left(\frac{n}{2}\right)\right\}\{\mathrm{etr}(-\Omega)\}(2t)^{-pq/2}$$

$$\cdot\sum_{k=0}^{\infty}\sum_{(\kappa)}\left(-\frac{1}{2t}\right)^k L_\kappa^{(q-p-1)/2}(\Omega)\left\{1+\frac{1}{n}qk+\frac{1}{2n^2}q^2k(k-1)\right.$$

$$\left.+O(n^{-3})\right\}\left[1+\frac{1}{n+q}\mathrm{tr}\,\partial^2+\frac{1}{6(n+q)^2}\{3(\mathrm{tr}\,\partial^2)^2+8\mathrm{tr}\,\partial^3\}+O(n^{-3})\right]$$

$$\cdot C_\kappa(\Sigma)\Big|_{\Sigma=I}\Big/\{C_\kappa(I)k!\}.$$

Now, the result follows from the Lemma 1 in part I.

COROLLARY 6.2. *The Laplace transform of a density function of T_0^2 in the degenerate case, i.e. $q<p$, is obtained from* (6.6) *by making the substitutions*

$$L_\kappa^{(q-p-1)/2}(\Omega)\rightarrow L_\kappa^{(p-q-1)/2}\left(\frac{1}{2}M\Sigma^{-1}M'\right),\ \kappa=\{k_1, k_2,\cdots, k_p\}\rightarrow\{k_1, k_2,\cdots, k_q\}.$$

The corollary follows from the fact that for the degenerate case the substitutions (5.5) and $\Omega\rightarrow\frac{1}{2}M\Sigma^{-1}M'$ for each parameter except two terms $\left(\frac{n}{2}\right)^{pq/2}$ and n^{-k} should be made in (6.7). We also note that $\Gamma_p\left(\frac{n+q}{2}\right)/\Gamma_p\left(\frac{n}{2}\right)$ is invariant for this transformation.

6.2. *Approximate non-null distribution.* In this section we give two alternative methods of obtaining the asymptotic formula of the non-null distribution of T_0^2 up to order n^{-2}. At first we expand the characteristic function $C(t)$ of T_0^2 expressed in (6.1) with respect to n. Let

$$C(t)=\left\{\Gamma_p\left(\frac{n+q}{2}\right)/\left(\frac{n}{2}\right)^{pq/2}\Gamma_p\left(\frac{n}{2}\right)\right\}C_1(t). \tag{6.9}$$

Then the first factor can be expanded easily by (1.14) as follows:

$$\Gamma_p\left(\frac{n+q}{2}\right)/\left\{\left(\frac{n}{2}\right)^{pq/2}\Gamma_p\left(\frac{n}{2}\right)\right\}=1+\frac{pq}{4n}(q-p-1)+\frac{pq}{96n^2}\{3qp^3 -2(3q^2-3q+4)p^2+3(q-1)(q^2-q+4)p-4(2q^2-3q-1)\}+O(n^{-3}). \tag{6.10}$$

The remainder factor $C_1(t)$ in (6.9) can be written, by using (1.8) and (2.14), in the following form for large n:

$$\mathcal{A}_\Omega[1+n^{-1}(\mathrm{tr}\ S^2-q\mathrm{tr}\ S)+(6n^2)^{-1}\{3q^2(\mathrm{tr}\ S)^2+6q\mathrm{tr}\ S^2-8\mathrm{tr}\ S^3 -6q(\mathrm{tr}\ S)\mathrm{tr}\ S^2+3(\mathrm{tr}\ S^2)^2\}+O(n^{-3})], \tag{6.11}$$

where $\mathcal{A}_\Omega[\{\ \ \}]$ is an abbreviation for

$$2^{p(p-1)/2}(2\pi i)^{-p(p+1)/2}\int_{\Re(T)=X_0>0}(\mathrm{etr}\ T)|T|^{-\frac{q}{2}} \cdot\left[\int_{S>0}\{\mathrm{etr}-((1-2it)I-2it\Omega^{\frac{1}{2}}T^{-1}\Omega^{\frac{1}{2}})S\}|S|^{(q-p-1)/2}\{\ \ \}dS\right]dT. \tag{6.12}$$

We must carry out each integral $\mathcal{A}_\Omega[(\mathrm{tr}\ S)^2]$, $\mathcal{A}_\Omega[\mathrm{tr}\ S^2]$, etc. in (6.11). For example, let us consider $\mathcal{A}_\Omega[(\mathrm{tr}\ S)^2]$. By using Lemma 4, we can write

$$\mathcal{A}_\Omega[(\mathrm{tr}\ S)^2]=\Gamma_p\left(\frac{q}{2}\right)\frac{2^{p(p-1)/2}}{(2\pi i)^{p(p+1)/2}}\int_{\Re(T)=X_0>0}(\mathrm{etr}\ T)|T|^{-\frac{q}{2}}|(1-2it)I \tag{6.13}$$

$$-2it\Omega^{\frac{1}{2}}T^{-1}\Omega^{\frac{1}{2}}|^{-\frac{q}{2}}\frac{q}{4}\Big[q\{\mathrm{tr}((1-2it)I-2it\Omega^{\frac{1}{2}}T^{-1}\Omega^{\frac{1}{2}})^{-1}\}^2$$

$$+2\mathrm{tr}\{(1-2it)I-2it\Omega^{\frac{1}{2}}T^{-1}\Omega^{\frac{1}{2}}\}^{-2}\Big]dT.$$

Considering the transformation $T\rightarrow Z=T-2it(1-2it)^{-1}\Omega$, we can simplify $\mathcal{A}_\Omega[(\mathrm{tr}\ S)^2]$ as follows:

$$\frac{q}{4}(1-2it)^{-pq/2-2}\left\{\mathrm{etr}\left(\frac{2it}{1-2it}\Omega\right)\right\}\mathrm{I}\Big[p(pq+2)+\frac{4it(pq+2)}{1-2it}\mathrm{tr}\ \Omega Z^{-1}$$

$$+\left(\frac{2it}{1-2it}\right)^2\{q(\mathrm{tr}\ \Omega Z^{-1})^2+2\mathrm{tr}(\Omega Z^{-1})^2\}\Big],$$

with the notation I[ ] in (3.2). Applying Lemma 5 to the above expression, we obtain

$$(6.14)\qquad \mathcal{A}_\Omega[(\mathrm{tr}\ S)^2]=(1-2it)^{-pq/2-2}\left\{\frac{1}{4}pq(pq+2)+\frac{2it(pq+2)}{1-2it}\mathrm{tr}\ \Omega\right.$$

$$\left.+\left(\frac{2it}{1-2it}\right)^2(\mathrm{tr}\ \Omega)^2\right\}\mathrm{etr}\left(\frac{2it}{1-2it}\Omega\right).$$

The similar computation gives us (6.15).

$$(6.15)\qquad \mathcal{A}_\Omega[\mathrm{tr}S]=(1-2it)^{-pq/2-1}\left\{\frac{1}{2}pq+\frac{2it}{1-2it}\mathrm{tr}\ \Omega\right\}\mathrm{etr}\left(\frac{2it}{1-2it}\Omega\right),$$

$$\mathcal{A}_\Omega[\mathrm{tr}\ S^2]=(1-2it)^{-pq/2-2}\left\{\frac{1}{4}pq(p+q+1)+\frac{2it(p+q+1)}{1-2it}\mathrm{tr}\ \Omega\right.$$

$$\left.+\left(\frac{2it}{1-2it}\right)^2\mathrm{tr}\ \Omega^2\right\}\mathrm{etr}\left(\frac{2it}{1-2it}\Omega\right),$$

$$\mathcal{A}_\Omega[(\mathrm{tr}\ S)\mathrm{tr}\ S^2]=(1-2it)^{-pq/2-3}\Big[\{qp^2+(q^2+q+4)p+4(q+1)\}\Big\{\frac{1}{8}pq$$

$$+\frac{3it}{2(1-2it)}\mathrm{tr}\ \Omega\Big\}+\frac{1}{2}\left(\frac{2it}{1-2it}\right)^2\{2(p+q+1)(\mathrm{tr}\ \Omega)^2+(pq+8)\mathrm{tr}\ \Omega^2\}$$

$$+\left(\frac{2it}{1-2it}\right)^3(\mathrm{tr}\Omega)\mathrm{tr}\ \Omega^2\Big]\mathrm{etr}\left(\frac{2it}{1-2it}\Omega\right),$$

$$\mathcal{A}_\Omega[\mathrm{tr}\ S^3]=(1-2it)^{-pq/2-3}\Big[\{p^2+3(q+1)p+q^2+3q+4\}\Big\{\frac{1}{8}pq$$

$$+\frac{3it}{2(1-2it)}\mathrm{tr}\ \Omega\Big\}+\frac{3}{2}\left(\frac{2it}{1-2it}\right)^2\{(\mathrm{tr}\Omega)^2+(p+q+2)\mathrm{tr}\ \Omega^2\}$$

$$+\left(\frac{2it}{1-2it}\right)^3 \mathrm{tr}\ \Omega^3\Big]\mathrm{etr}\left(\frac{2it}{1-2it}\right)\Omega\ ,$$

$$\mathcal{A}_\Omega[(\mathrm{tr}\ S^2)^2]=(1-2it)^{-pq/2-4}\Big[\{qp^3+2(q^2+q+4)p^2+(q+1)(q^2+q+20)p$$

$$+4(2q^2+5q+5)\}\left\{\frac{1}{16}pq+\frac{it}{1-2it}\mathrm{tr}\ \Omega\right\}+\frac{1}{2}\left(\frac{2it}{1-2it}\right)^2\{2(p^2$$

$$+2(q+1)p+q^2+2q+7)(\mathrm{tr}\ \Omega)^2+(qp^2+(q^2+q+20)p$$

$$+4(5q+8))\mathrm{tr}\ \Omega^2\}+2\left(\frac{2it}{1-2it}\right)^3\{(p+q+1)(\mathrm{tr}\ \Omega)\mathrm{tr}\ \Omega^2$$

$$+4\mathrm{tr}\ \Omega^3\}+\left(\frac{2it}{1-2it}\right)^4(\mathrm{tr}\ \Omega^2)^2\Big]\ \mathrm{etr}\left(\frac{2it}{1-2it}\Omega\right).$$

Therefore, we obtain (6.16).

(6.16) $$C(t)=(1-2it)^{-pq/2}\left\{\mathrm{etr}\left(\frac{2it}{1-2it}\Omega\right)\right\}\Big[1+\frac{1}{4n}\{pq(q-p-1)$$

$$-2q(pq-2\mathrm{tr}\ \Omega)(1-2it)^{-1}+(pq(p+q+1)-4(p+2q+1)\mathrm{tr}\ \Omega$$

$$+4\mathrm{tr}\ \Omega^2)(1-2it)^{-2}+4((p+q+1)\ \mathrm{tr}\ \Omega-2\mathrm{tr}\ \Omega^2)(1-2it)^{-3}$$

$$+4\mathrm{tr}\ \Omega^2(1-2it)^{-4}\}+\frac{1}{96n^2}\{\sum_{\alpha=0}^{8}B_\alpha(\Omega)(1-2it)^{-\alpha}\}+O(n^{-3})\Big],$$

where $B_\alpha(\Omega)$ ($\alpha=0, 1,\cdots, 8$) are given by (6.17).

(6.17) $$B_0(\Omega)=pql_0,$$

$$B_1(\Omega)=-l_1(pq-2\mathrm{tr}\ \Omega),$$

$$B_2(\Omega)=pql_2-2(l_1+2l_2)\mathrm{tr}\ \Omega+48q^2(\mathrm{tr}\ \Omega)^2$$

$$-24q\{p^2-(q-1)p-4\}\mathrm{tr}\ \Omega^2,$$

$$B_3(\Omega)=-pql_3+2(2l_2+3l_3)\mathrm{tr}\ \Omega-96(qp+2q^2+q+2)(\mathrm{tr}\ \Omega)^2$$

$$+48\{qp^2-(2q^2-q+4)p-8(2q+1)\}\mathrm{tr}\ \Omega^2+96q(\mathrm{tr}\ \Omega)\mathrm{tr}\ \Omega^2$$

$$+128\mathrm{tr}\ \Omega^3,$$

$$B_4(\Omega)=pql_4-2(3l_3+4l_4)\mathrm{tr}\ \Omega+48\{p^2+2(3q+1)p$$

$$+3(2q^2+2q+5)\}(\mathrm{tr}\ \Omega)^2+48\{3(q^2+6)p+4(9q+8)\}\mathrm{tr}\ \Omega^2$$

$$-96(p+4q+1)(\mathrm{tr}\ \Omega)\mathrm{tr}\ \Omega^2-768\mathrm{tr}\ \Omega^3+48(\mathrm{tr}\ \Omega^2)^2,$$

$$B_5(\Omega)=8l_4(\mathrm{tr}\ \Omega)-96\{p^2+(3q+2)p+2q^2+3q+9\}(\mathrm{tr}\ \Omega)^2-48\{qp^2$$

$$+(2q^2+q+24)p+8(4q+5)\}\mathrm{tr}\,\Omega^2+288(p+2q+1)(\mathrm{tr}\,\Omega)\mathrm{tr}\,\Omega^2$$
$$+1536\,\mathrm{tr}\,\Omega^3-192(\mathrm{tr}\,\Omega^2)^2,$$

$$B_6(\Omega)=48\{p^2+2(q+1)p+q^2+2q+7\}(\mathrm{tr}\,\Omega)^2+24\{qp^2+(q^2+q+20)p$$
$$+4(5q+8)\}\mathrm{tr}\,\Omega^2-96(3p+4q+3)(\mathrm{tr}\,\Omega)\,\mathrm{tr}\,\Omega^2-1280\,\mathrm{tr}\,\Omega^3$$
$$+288(\mathrm{tr}\,\Omega^2)^2,$$

$$B_7(\Omega)=96\{(p+q+1)(\mathrm{tr}\,\Omega)\mathrm{tr}\,\Omega^2+4\mathrm{tr}\,\Omega^3-2(\mathrm{tr}\,\Omega^2)^2\},$$

$$B_8(\Omega)=48(\mathrm{tr}\,\Omega^2)^2,$$

and $l_\alpha(\alpha=0, 1,\cdots, 4)$ are given by (6.18).

(6.18) $$l_0=3qp^3-2(3q^2-3q+4)p^2+3(q-1)(q^2-q+4)p-4(2q^2-3q-1),$$
$$l_1=-12pq^2(p-q+1),$$
$$l_2=-6q\{p^3+2p^2-3(q^2+1)p-4(2q+1)\},$$
$$l_3=4\{(3q^2+4)p^2+3(q^3+q^2+8q+4)p+8(2q^2+3q+2)\},$$
$$l_4=3\{qp^3+2(q^2+q+4)p^2+(q+1)(q^2+q+20)p+4(2q^2+5q+5)\}.$$

From Corollary 6.1 and the fact that the formulas (6.14) and (6.15) are symmetric with respect to p and q, we see that this asymptotic formula holds also for $q<p$. Inverting this characteristic function, we have the following asymptotic formula with the same notations in (5.19):

(6. 19) $$P_K(T_0^2<z)=P(\chi_f^2(\delta^2)<z)+\frac{1}{4n}\{pq(q-p-1)P(\chi_f^2(\delta^2)<z)$$
$$-2q(pq-2\mathrm{tr}\,\Omega)P(\chi_{f+2}^2(\delta^2)<z)+(pq(p+q+1)-4(p+2q+1)\mathrm{tr}\,\Omega$$
$$+4\mathrm{tr}\,\Omega^2)P(\chi_{f+4}^2(\delta^2)<z)+4((p+q+1)\mathrm{tr}\,\Omega-2\mathrm{tr}\,\Omega^2)P(\chi_{f+6}^2(\delta^2)<z)$$
$$+4\mathrm{tr}\,\Omega^2\cdot P(\chi_{f+8}^2(\delta^2)<z)\}+\frac{1}{96n^2}\left\{\sum_{\alpha=0}^{8}B_\alpha(\Omega)P(\chi_{f+2\alpha}^2(\delta^2)<z)\right\}+O(n^{-3}),$$

which agrees with the result of Siotani [39], after minor changes of notation and some calculations. By putting $\Omega=0$ and $\delta=0$ in (6.19), we have the following asymptotic expansion of the null distribution of T_0^2:

(6.20) $$P_H(T_0^2<z)=P(\chi_f^2<z)+\frac{pq}{4n}\{(q-p-1)P(\chi_f^2<z)-2qP(\chi_{f+2}^2<z)$$
$$+(p+q+1)P(\chi_{f+4}^2<z)\}+\frac{pq}{96n^2}\left\{\sum_{\alpha=0}^{4}(-1)^\alpha l_\alpha P(\chi_{f+2\alpha}^2<z)\right\}+O(n^{-3}),$$

where $l_\alpha(\alpha=0, 1,\cdots, 4)$ are given by (6.18).

We now show another derivation of the non-null distribution of T_0^2 by simplifing the Laplace transform of the non-null density function of T_0^2 given in (6.6). The expansion of the first factor in (6.6) is given by (6.10), and each term of the infinite series can be simplified by using the formulas for weighted sums of Laguerre polynomials given by Lemma 8, which gives exactly the same formula as (6.16) with it replaced by $-t$. This result also holds for $q<p$ from Corollary 6.2 and the fact that each formula in Lemma 8 is symmetric with respect to p and q. Hence (6.19) can be also drived by our second method.

7. Numerical results of the powers. For testing the multivariate linear hypothesis H defined in Section 5.1, various criteria have been suggested (see e.g. Anderson [2], Seber [36]). In this paper we are especially concerned with three criteria, namely, (i) LR criterion due to Wilks [42], (ii) Hotelling's T_0^2 criterion due to Lawley [24] and Hotelling [13], and (iii) Pillai's criterion due to Pillai [28]. For a given level of significance $\alpha(0<\alpha<1)$, their rejection regions are given by

$$\text{Criterion (i)} \quad : \quad W=|S_e|/|S_e+S_h|<w$$

$$\text{Criterion (ii)} \quad : \quad T_0^2=n\,\mathrm{tr}\; S_hS_e^{-1}>t$$

$$\text{Criterion (iii)} : \quad V=m\mathrm{tr}\; S_h(S_h+S_e)^{-1}>v.$$

Here constants w, t and v are defined by the equations

$$P_H(W<w)=P_H(T>t)=P_H(V>v)=\alpha.$$

Except for particular values of p and q, the exact distributions of these test statistics are not available in closed forms, even for the null case. However asymptotic expansion of the distributions of these test statistic with respect to n are available for any p, q, n and Ω, such that $p\leq n$ as we have seen for V and T_0^2 in Sections 5 and 6. Asymptotic expansion of the null distribution of W was obtained by Rao [33] as follows:

$$(7.1) \qquad P_H(-n'\log W<z)=P(\chi_f^2<z)+\frac{\beta_1}{n'^2}\{P(\chi_{f+4}^2<z)-P(\chi_f^2<z)\}$$
$$+\frac{1}{n'^4}\Big[\beta_2\{P(\chi_{f+8}^2<z)-P(\chi_f^2<z)\}-\beta_1^2\{P(\chi_{f+4}^2<z)-P(\chi_f^2<z)\}\Big]+O(n'^{-6}),$$

where $f=pq$, $n'=n+(q-p-1)/2$, and β_1, β_2 are defined by

$$(7.2) \qquad \beta_1=pq(p^2+q^2-5)/48,$$
$$\beta_2=\beta_1^2/2+pq\{3p^4+3q^4-50p^2-50q^2+10p^2q^2+159\}/1920.$$

This and similar approximations were also given by Box [5]. Recently Sugiura and Fujikoshi [39] have obtained the non-null distribution of W up to order n^{-2}, based on Lemma 2, as follows:

$$(7.3)\qquad P_K(-n'\log W<z)=P(\chi_f^2(\delta^2)<z)+\frac{1}{2n'}\Big[(p+q+1)\mathrm{tr}\ \Omega\cdot P(\chi_{f+2}^2(\delta^2)<z)$$
$$-\{(p+q+1)\mathrm{tr}\ \Omega-2\mathrm{tr}\ \Omega^2\}P(\chi_{f+4}^2(\delta^2)<z)-2\mathrm{tr}\ \Omega^2\cdot P(\chi_{f+6}^2(\delta^2)<z)\Big]$$
$$+\frac{1}{n'^2}\Big[\beta_1\{P(\chi_{f+4}^2(\delta^2)<z)-P(\chi_f^2(\delta^2)<z)\}$$
$$+\sum_{\alpha=2}^{6}G_\alpha(\Omega)P(\chi_{f+2\alpha}^2(\delta^2)<z)\Big]+O(n'^{-3}),$$

where $\delta^2=\mathrm{tr}\ \Omega$ and the coefficients $G_\alpha(\Omega)(\alpha=2,3,\cdots,6)$ are given by (7.4).

$$(7.4)\qquad G_2(\Omega)=\frac{1}{8}(p+q+1)^2\{(\mathrm{tr}\ \Omega)^2-2\mathrm{tr}\ \Omega\}+\frac{1}{2}(p+q+1)\mathrm{tr}\Omega^2,$$

$$G_3(\Omega)=\frac{1}{4}(p+q+1)^2\mathrm{tr}\ \Omega-\Big\{1+\frac{1}{4}(p+q+1)^2\Big\}(\mathrm{tr}\ \Omega)^2$$
$$-\{1+2(p+q+1)\}\mathrm{tr}\ \Omega^2+\frac{4}{3}\mathrm{tr}\ \Omega^3+\frac{1}{2}(p+q+1)(\mathrm{tr}\ \Omega)\mathrm{tr}\ \Omega^2,$$

$$G_4(\Omega)=\Big\{1+\frac{1}{8}(p+q+1)^2\Big\}(\mathrm{tr}\ \Omega)^2+\Big\{1+\frac{3}{2}(p+q+1)\Big\}\mathrm{tr}\ \Omega^2-4\mathrm{tr}\ \Omega^3$$
$$-(p+q+1)(\mathrm{tr}\ \Omega)\mathrm{tr}\ \Omega^2+\frac{1}{2}(\mathrm{tr}\ \Omega^2)^2,$$

$$G_5(\Omega)=\frac{8}{3}\mathrm{tr}\ \Omega^3+\frac{1}{2}(p+q+1)(\mathrm{tr}\ \Omega)\mathrm{tr}\ \Omega^2-(\mathrm{tr}\ \Omega^2)^2,$$

$$G_6(\Omega)=\frac{1}{2}(\mathrm{tr}\ \Omega^2)^2.$$

These formulas, together with (5.11), (5.19), (6.19) and (6.20), give the numerical results shown in Table 4, when $p=3$, $q=3, 5, 7$, $n=85, 170$ and $\alpha=0.05$ for specified values of the characteristic roots of $\Omega=\{\omega_1, \omega_2, \omega_3\}(\omega_1\leq\omega_2\leq\omega_3)$. Numerical comparisons of the powers of tests of (i), (ii) and (iii) have been made in some special cases by several authors. For example, Ito [17] has made power comparison of tests of (i) and (ii) using the formula (6.19) up to order n^{-1} when rank $\Omega=1$. For the tests of (i), (ii) and (iii) Mikhail [25] has given such comparison by employing an approximate method when $p=2$. For test of (i) Posten and Bargmann [32] have computed the power by using the formula (5.19) when p or $q=1$ and rank $\Omega=1$, and Roy [34] has also computed the power by using an approximate method when rank $\Omega=1$. Recently Pillai and Jayachandran [30] has made a thorough investigation of power comparison by the exact powers expressed in terms of zonal polynomials when $p=2$. However their method is available only for $p=2$.

TABLE 4. *Approximate powers of the W test, the T_0^2 test and the V test for $p=3$ and $\alpha=0.05$*

ω_1	ω_2	ω_3	q	$n=85$ W	$n=85$ T_0^2	$n=85$ V	$n=170$ W	$n=170$ T_0^2	$n=170$ V
0	0	0.9	3	0.1116	0.1119	0.1111	0.1140	0.1142	0.1138
			5	0.0942	0.0946	0.0938	0.0964	0.0966	0.0962
			7	0.0855	0.0858	0.0851	0.0874	0.0876	0.0872
0	0.3	0.6	3	0.1120	0.1119	0.1122	0.1143	0.1142	0.1144
			5	0.0946	0.0946	0.0946	0.0966	0.0965	0.0966
			7	0.0857	0.0858	0.0857	0.0876	0.0875	0.0875
0	0.45	0.45	3	0.1121	0.1119	0.1123	0.1143	0.1142	0.1144
			5	0.0946	0.0945	0.0947	0.0966	0.0965	0.0966
			7	0.0858	0.0858	0.0858	0.0876	0.0875	0.0876
0.15	0.3	0.45	3	0.1122	0.1119	0.1125	0.1144	0.1142	0.1146
			5	0.0947	0.0945	0.0949	0.0966	0.0965	0.0967
			7	0.0858	0.0858	0.0859	0.0876	0.0875	0.0877
0.15	0.375	0.375	3	0.1122	0.1119	0.1126	0.1144	0.1142	0.1146
			5	0.0947	0.0945	0.0949	0.0967	0.0965	0.0967
			7	0.0858	0.0858	0.0859	0.0876	0.0875	0.0877
0.15	0.15	0.6	3	0.1121	0.1119	0.1123	0.1143	0.1142	0.1144
			5	0.0946	0.0945	0.0947	0.0966	0.0965	0.0966
			7	0.0858	0.0858	0.0858	0.0876	0.0875	0.0876
0.3	0.3	0.3	3	0.1123	0.1119	0.1127	0.1144	0.1142	0.1146
			5	0.0948	0.0945	0.0950	0.0967	0.0965	0.0968
			7	0.0859	0.0858	0.0860	0.0876	0.0875	0.0877
0	0	4.5	3	0.473	0.478	0.467	0.491	0.494	0.488
			5	0.370	0.376	0.361	0.390	0.393	0.385
			7	0.308	0.315	0.298	0.328	0.332	0.323
0	1.5	3	3	0.481	0.479	0.483	0.496	0.495	0.496
			5	0.378	0.377	0.378	0.394	0.394	0.394
			7	0.316	0.316	0.314	0.332	0.332	0.331
0	2.25	2.25	3	0.482	0.480	0.485	0.496	0.495	0.497
			5	0.379	0.377	0.380	0.395	0.394	0.395
			7	0.317	0.316	0.316	0.332	0.332	0.332
0.75	1.5	2.25	3	0.485	0.480	0.489	0.497	0.495	0.499
			5	0.381	0.378	0.384	0.396	0.394	0.397
			7	0.319	0.316	0.320	0.334	0.332	0.334
0.75	1.875	1.875	3	0.485	0.480	0.489	0.498	0.495	0.500
			5	0.382	0.378	0.385	0.396	0.394	0.397
			7	0.319	0.316	0.321	0.334	0.332	0.335
0.75	0.75	3	3	0.482	0.480	0.485	0.496	0.495	0.497
			5	0.379	0.377	0.380	0.395	0.394	0.395
			7	0.317	0.316	0.316	0.332	0.332	0.332
1.5	1.5	1.5	3	0.486	0.480	0.491	0.498	0.495	0.500
			5	0.382	0.378	0.386	0.396	0.394	0.398
			7	0.320	0.316	0.322	0.334	0.332	0.335

It is well known that all three tests are good for testing the hypothesis (5.1). This fact can also be seen in Table 4. Moreover, Table 4 shows that from the power point of view, for moderately large values of n, these tests differ in their powers if we consider different classes of alternatives. For example, we can see that $T_0^2 > W > V$ when ω_3 is far apart from ω_1 and ω_2, and $V > W > T_0^2$ when ω_1, ω_2 and ω_3 are close, as having been pointed out in the case of $p=2$ by Pillai and Jayachandran [30].

PART III. TESTS OF INDEPENDENCE

8. Asymptotic non-null distributions of the Pillai's criterion under local alternatives

8.1. *Test criteria for independence.* Let $(p+q)\times 1$ vectors $\begin{pmatrix} x_1(p\times 1) \\ y_1(q\times 1)\end{pmatrix}$, $\begin{pmatrix} x_2 \\ y_2\end{pmatrix}$, $\dots$, $\begin{pmatrix} x_N \\ y_N\end{pmatrix}$ be a random sample from a multivariate normal distribution with mean vector μ and covariance matrix Σ. Put

$$S=\sum_{\alpha=1}^{N}\begin{pmatrix} x_\alpha-\bar{x} \\ y_\alpha-\bar{y}\end{pmatrix}\begin{pmatrix} x_\alpha-\bar{x} \\ y_\alpha-\bar{y}\end{pmatrix}', \qquad \bar{x}=\frac{1}{N}\sum_{\alpha=1}^{N} x_\alpha, \qquad \bar{y}=\frac{1}{N}\sum_{\alpha=1}^{N} y_\alpha$$

and let us partition Σ and S into p and q rows and columns as

$$\Sigma=\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}, \qquad S=\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22}\end{pmatrix}.$$

Without loss of generality we may assume $p\leq q$. To test the hypothesis of independence between two sets of variates, H: $\Sigma_{12}=0(p\times q)$ against all alternativse K: $\Sigma_{12}\neq 0$, the following three test criteria can be considered (c.f. Pillai [28], Pillai and Jayachandran [30]):

(i) LR criterion:

$$W=|S|/(|S_{11}||S_{22}|)=|I-R|,$$

(ii) Hotelling's T_0^2 criterion:

$$T_0^2=m\mathrm{tr}\ S_{12}S_{22}^{-1}S_{21}(S_{11}-S_{12}S_{22}^{-1}S_{21})^{-1}=m\mathrm{tr}\ R(I-R)^{-1},$$

(iii) Pillai's criterion:

$$V=n\mathrm{tr}\ S_{12}S_{22}^{-1}S_{21}S_{11}^{-1}=n\mathrm{tr}\ R,$$

where $n=N-1$, $m=n-q$ and $R=\mathrm{diag}(r_1^2, r_2^2,\dots, r_p^2)$ with p characteristic roots of $S_{12}S_{22}^{-1}S_{21}S_{11}^{-1}$ as its non-zero elements. The $r_j(j=1, 2,\cdots, p)$ are called the sample canonical correlations.

Sugiura and Fujikoshi [39] have obtained asymptotic expansion of the distribution of the LR criterion W under a fixed alternative hypothesis up to order N^{-1}, by using the characteristic function expressed in terms of hypergeometric function with matrix argument. The limiting non-null distribution degenerates at the null hypothesis, so that the asymptotic formula does not give good approximation when the alternative hypothesis is near the null hypothesis as having been pointed out by Sugiura [40]. He derived asymptotic non-null distributions of the LR criteria for covariance matrix under sequences of alternatives converging to the null hypothesis at the rate of convergence $N^{-\gamma}$ for arbitrary positive number γ and also derived the asymptotic expansion of the distribution of W for this problem in the case of $\gamma=1$. The main purpose of part III is to give asymptotic expressions of the non-null distributions of the Pillai's criterion and Hotelling's T_0^2 criterion for this problem under the local alternative in the above sense.

8.2. *The moment generating function of the Pillai's criterion.* The moment generating function of the Pillai's criterion for the hypothesis of independence between two sets of variates has been given under alternative K by Pillai [31] as follows:

$$(8.1) \qquad M(t)=|I-P|^{\frac{n}{2}}\mathrm{I}\Big[\Big\{\Gamma_p\Big(\frac{n}{2}\Big)\Big\}^{-1}\int_{S_2>0}\{\mathrm{etr}(-S_2)\}\,|S_2|^{(n-p-1)/2}$$

$$\cdot\Big\{\Big\{\Gamma_p\Big(\frac{n}{2}\Big)\Big\}^{-1}\int_{S_1>0}\{\mathrm{etr}(-S_1)\}\,|S_1|^{(n-p-1)/2}{}_1F_1\Big(\frac{q}{2};\frac{n}{2};ntI$$

$$+(P^{\frac{1}{2}}S_2P^{\frac{1}{2}})^{\frac{1}{2}}S_1(P^{\frac{1}{2}}S_2P^{\frac{1}{2}})^{\frac{1}{2}}T^{-1}\Big)dS_1\Big\}dS_2\Big],$$

where $\mathrm{I}[\ \ ]$ is defined by (3.2) with respect to T and $P=\mathrm{diag}\ (\rho_1^2, \rho_2^2,\cdots, \rho_p^2)$ with population canonical correlation ρ_j. This expression is in a form covenient for our asymptotic expansion.

8.3. *Asymptotic distribution when* $\gamma=1$. Since the assumption $\Sigma_{12}=0$ is equivalent to $P=0$, the alternative K: $\Sigma_{12}\neq 0$ can be expressed as K: $P\neq 0$. In this section we derive asymptotic expansion of the distribution V under the sequence of alternatives K_γ: $P=2n^{-\gamma}\Theta$ with $\gamma=1$, by expanding the moment generating function. Using the same method as in the derivation of the expansion (5.14), we can rewrite the part in the brackets $[\ \ ]$ in (8.1) as (8.2).

(8.2)
$$(1-2t)^{-pq/2}|T|^{\frac{q}{2}}|T-\phi\Theta|^{-\frac{q}{2}}\Big[1-\frac{1}{n}\Big\{\frac{q}{4}U_1(I,\ T)-\sum_{i=1}^{2}|T-\phi\Theta|^{\frac{q}{2}}$$
$$\cdot\mathrm{tr}\ \partial_i^2|T-\phi\Theta^{\frac{1}{2}}\Sigma_i\Theta^{\frac{1}{2}}|^{-\frac{q}{2}}\Big|_{\Sigma_i=I}\Big\}+\frac{1}{n^2}\Big\{\frac{q}{96}U_2(I,\ T)-\sum_{i=1}^{2}\frac{q}{4}|T-\phi\Theta|^{\frac{q}{2}}$$
$$\cdot\mathrm{tr}\ \partial_i^2|T-\phi\Theta^{\frac{1}{2}}\Sigma_i\Theta^{\frac{1}{2}}|^{-\frac{q}{2}}U_1(\Sigma_i,\ T)\Big|_{\Sigma_i=I}+\sum_{i=1}^{2}|T-\phi\Theta|^{\frac{q}{2}}\Big(\frac{1}{2}(\mathrm{tr}\ \partial_i^2)^2$$
$$+\frac{4}{3}\mathrm{tr}\ \partial_i^3\Big)|T-\phi\Theta^{\frac{1}{2}}\Sigma_i\Theta^{\frac{1}{2}}|^{-\frac{q}{2}}\Big|_{\Sigma_i=I}+|T-\phi\Theta|^{\frac{q}{2}}(\mathrm{tr}\ \partial_2^2)\mathrm{tr}\ \partial_1^2$$
$$\cdot|T-\phi(\Theta^{\frac{1}{2}}\Sigma_2\Theta^{\frac{1}{2}})^{\frac{1}{2}}\Sigma_1(\Theta^{\frac{1}{2}}\Sigma_2\Theta^{\frac{1}{2}})^{\frac{1}{2}}|^{-\frac{q}{2}}\Big|_{\Sigma_1=I,\ \Sigma_2=I}\Big\}+O(n^{-3})\Big],$$

which holds for sufficiently small $|t|$ and large n, and $\phi=(1-2t)^{-1}$, $\partial_i(i=1,2)$ denote the $p\times p$ matrices of differential operators having $\{(1+\delta_{rs})/2\}\dfrac{\partial}{\partial\sigma_{rs}^{(i)}}$ as its (r,s) element for symmetric matrices $\Sigma=(\sigma_{rs}^{(i)})$ and $U_i(i=1,2)$ are given by (5.13) with $\Omega=\Theta$. The above operations ∂_i, except for the operation (tr ∂_2^2) tr ∂_1^2, are carried out in Appendix II. We obtain the following formula from (A.37) and (A.47):

(8.3)
$$(\mathrm{tr}\ \partial_2^2)\mathrm{tr}\ \partial_1^2|T-\phi(\Theta^{\frac{1}{2}}\Sigma_2\Theta^{\frac{1}{2}})^{\frac{1}{2}}\Sigma_1(\Theta^{\frac{1}{2}}\Sigma_2\Theta^{\frac{1}{2}})^{\frac{1}{2}}|^{-\frac{q}{2}}\Big|_{\Sigma_1=I,\ \Sigma_2=I}$$
$$=\frac{q}{16}\phi^2|T-\phi\Theta|^{-\frac{q}{2}}\Big[4(q+1)(\mathrm{tr}_1)^2+4(q+3)\mathrm{tr}_2+8\phi\{(\mathrm{tr}_1)^3$$
$$+3(q+1)\mathrm{tr}_1\,\mathrm{tr}_2+(q^2+3q+4)\mathrm{tr}_3\}+\phi^2\{q(\mathrm{tr}_1)^4+2(q^2+q+4)(\mathrm{tr}_1)^2\mathrm{tr}_2$$
$$+(q+1)(q^2+q+4)(\mathrm{tr}_2)^2+16(q+1)\mathrm{tr}_1\,\mathrm{tr}_3+4(2q^2+5q+5)\mathrm{tr}_4\}\Big],$$

where tr_j is an abbreviation for tr $\{\Theta(T-\phi\Theta)^{-1}\}^j$. By using the formulas (A.37), (A.38), (A.39) and (A.47) in Appendix II, and considering the transformation $T\to Z=T-\phi\Theta$, we can write $M(t)$ as follows:

(8.4)
$$\Big|I-\frac{2}{n}\Theta\Big|^{\frac{n}{2}}(1-2t)^{-pq/2}\Big\{\mathrm{etr}\Big(\frac{1}{1-2t}\Theta\Big)\Big\}\ \mathrm{I}\Big[1-\frac{q}{4n}\{(\phi-1)^2p(p+q+1)$$
$$+2\phi^2(\phi-1)(p+q+1)\mathrm{tr}\ \Theta Z^{-1}+\phi^2(\phi^2-2)((\mathrm{tr}\,\Theta Z^{-1})^2+(q+1)\mathrm{tr}(\Theta Z^{-1})^2)\}$$
$$+\frac{q}{96n^2}\{\gamma_0+\gamma_1\mathrm{tr}\ \Theta Z^{-1}+\tilde{\gamma}_2(\mathrm{tr}\ \Theta Z^{-1})^2+\tilde{\gamma}_3\mathrm{tr}\ (\Theta Z^{-1})^2+\tilde{\gamma}_4(\mathrm{tr}\ \Theta Z^{-1})^3$$
$$+\tilde{\gamma}_5(\mathrm{tr}\ \Theta Z^{-1})\mathrm{tr}(\Theta Z^{-1})^2+\tilde{\gamma}_6\mathrm{tr}(\Theta Z^{-1})^3+3\phi^4(\phi^2-2)^2(q(\mathrm{tr}\ \Theta Z^{-1})^4$$
$$+2(q^2+q+4)(\mathrm{tr}\ \Theta Z^{-1})^2\mathrm{tr}(\Theta Z^{-1})^2+(q+1)(q^2+q+4)(\mathrm{tr}(\Theta Z^{-1})^2)^2$$

$$+16(q+1)(\mathrm{tr}\,\Theta Z^{-1})\mathrm{tr}(\Theta Z^{-1})^3+4(2q^2+5q+5)\mathrm{tr}(\Theta Z^{-1})^4)\}+O(n^{-3})\Big],$$

where γ_0 and γ_1 are given by (5.16) and $\tilde{\gamma}_\alpha(\alpha=2, 3,\cdots, 6)$ are given by (8.5).

$$(8.5)\quad \tilde{\gamma}_2=6\phi^2\Big[-2\{qp^2+q(q+1)p-2(q+1)\}+4\{qp^2+(q^2+q+4)p+4(q+1)\}\phi$$
$$+\{qp^2+3(q^2+q-4)p+2(q+1)(q^2+q-10)\}\phi^2-2\{3qp^2+(5q^2+5q+12)p$$
$$+2(q+1)(q^2+q+8)\}\phi^3+\{3qp^2+5(q^2+q+4)p+2(q+1)(q^2+q+16)\}\phi^4\Big],$$

$$\tilde{\gamma}_3=6\phi^2\Big[-2\{q(q+1)p^2+q(q+1)^2p-2(q+3)\}+4(q+1)\{qp^2+(q^2+q+4)p$$
$$+4(q+1)\}\phi-\{(q^2+q-4)p^2+(q+1)(q^2+q+4)p+8(q^2+3q+4)\}\phi^2$$
$$-2\{(q^2+q+4)p^2+(q+1)(q^2+q+20)p+4(4q^2+9q+7)\}\phi^3$$
$$+\{(q^2+q+4)p^2+(q+1)(q^2+q+28)p+12(2q^2+5q+5)\}\phi^4\Big],$$

$$\tilde{\gamma}_4=4\phi^3\{20+6q(p+q+1)\phi-6(qp+q^2+q+4)\phi^2-(3qp+3q^2+3q+4)\phi^3$$
$$+3(qp+q^2+q+4)\phi^4\Big\},$$

$$\tilde{\gamma}_5=12\phi^3\Big[20(q+1)+2(p+q+1)(q^2+q+4)\phi-2\{(q^2+q+4)p$$
$$+(q+1)(q^2+q+16)\}\phi^2-\{(q^2+q+4)p+(q+1)(q^2+q+8)\}\phi^3$$
$$+\{(q^2+q+4)p+(q+1)(q^2+q+16)\}\phi^4\Big],$$

$$\tilde{\gamma}_6=16\phi^3\Big[5(q^2+3q+4)+6(q+1)(p+q+1)\phi-6\{(q+1)p+2q^2+5q+5\}\phi^2$$
$$-\{3(q+1)p+4q^2+9q+7\}\phi^3+3\{(q+1)p+2q^2+5q+5\}\phi^4\Big].$$

The first factor in (8.4) can be expanded by the formula (2.14) as follows:

$$(8.6)\qquad \left|I-\frac{2}{n}\Theta\right|^{\frac{n}{2}}=\{\mathrm{etr}(-\Theta)\}\Big[1-\frac{1}{n}\mathrm{tr}\,\Theta^2+\frac{1}{6n^2}\{3(\mathrm{tr}\,\Theta^2)^2-8\mathrm{tr}\,\Theta^3\}$$
$$+O(n^{-3})\Big]$$

By Lemma 5, we finally obtain the following asymptotic formula for the moment generating function of the Pillai's criterion:

$$(8.7)\quad M(t)=(1-2t)^{-pq/2}\Big\{\mathrm{etr}\Big(\frac{2t}{1-2t}\Theta\Big)\Big\}\Big[1-\frac{1}{4n}\{pq(p+q+1)+4\mathrm{tr}\,\Theta^2$$

$$-2pq(p+q+1)(1-2t)^{-1}+(pq(p+q+1)-4(q+p+1)\text{tr }\Theta$$
$$-8\text{tr }\Theta^2)(1-2t)^{-2}+4(p+q+1)\text{tr }\Theta\cdot(1-2t)^{-3}+4\text{tr }\Theta^2\cdot(1-2t)^{-4}\}$$
$$+\frac{1}{96n^2}\{\sum_{\alpha=0}^{8}\tilde{A}_\alpha(\Theta)(1-2t)^{-\alpha}\}+O(n^{-3})\Big],$$

where the coefficients $\tilde{A}_\alpha(\Theta)(\alpha=0, 1,\cdots, 8)$ are given by (8.8).

(8.8) $$\tilde{A}_0(\Theta)=pqh_0+24pq(p+q+1)\text{tr }\Theta^2-128\text{tr }\Theta^3+48(\text{tr }\Theta^2)^2,$$

$$\tilde{A}_1(\Theta)=-pqh_1-48pq(p+q+1)\text{tr }\Theta^2,$$

$$\tilde{A}_2(\Theta)=pqh_2-2h_1\text{tr }\Theta+96(\text{tr }\Theta)^2-24\{qp^2+q(q+1)p-4\}\text{tr }\Theta^2$$
$$-96(p+q+1)(\text{tr }\Theta)\text{tr }\theta^2-192(\text{tr }\Theta^2)^2,$$

$$\tilde{A}_3(\Theta)=-pqh_3+4h_2\text{ tr }\Theta+96\{qp^2+(q^2+q+4)p+4(q+1)\}\text{tr }\Theta^2$$
$$+96(p+q+1)(\text{tr }\Theta)\text{tr }\Theta^2+640\text{tr }\Theta^3,$$

$$\tilde{A}_4(\Theta)=pqh_4-6h_3\text{tr }\Theta+48\{p^2+2(q+1)p+q^2+2q-3\}(\text{tr }\Theta)^2$$
$$-24\{qp^2+(q^2+q+12)p+4(3q+5)\}\text{tr }\Theta^2$$
$$+192(p+q+1)(\text{tr }\Theta)\text{tr }\Theta^2+288(\text{tr }\Theta^2)^2,$$

$$\tilde{A}_5(\Theta)=8h_4\text{tr }\Theta-96\{p^2+2(q+1)p+q^2+2q+3\}(\text{tr }\Theta)^2$$
$$-48\{qp^2+(q^2+q+12)p+4(3q+4)\}\text{tr }\Theta^2$$
$$-192(p+q+1)(\text{tr }\Theta)\text{tr }\Theta^2-768\text{tr }\Theta^3,$$

$$\tilde{A}_6(\Theta)=48\{p^2+2(q+1)p+q^2+2q+7\}(\text{tr }\Theta)^2+24\{qp^2+(q^2+q+20)p$$
$$+4(5q+8)\}\text{tr }\Theta^2-96(p+q+1)(\text{tr }\Theta)\text{tr }\Theta^2-128\text{tr }\Theta^3-192(\text{tr }\Theta^2)^2,$$

$$\tilde{A}_7(\Theta)=96(p+q+1)(\text{tr }\Theta)\text{tr }\Theta^2+384\text{tr }\Theta^3,$$

$$\tilde{A}_8(\Theta)=48(\text{tr }\Theta^2)^2,$$

with $h_\alpha(\alpha=0, 1,\cdots, 4)$ defined by (5.10). Inverting this moment generating function, we obtain the following

THEOREM 8.1. *Under the sequence of alternatives* K: $P=\frac{2}{n}\Theta$, *the distribution of the Pillai's criterion for testing the independence between two sets of variates with* p *components and* q *components* $(p\leq q)$ *can be expressed asymptotically as follows:*

$$(8.9) \quad P(V<z)=P(\chi_f^2(\delta^2)<z)-\frac{1}{4n}\{(pq(p+q+1)+4\mathrm{tr}\,\Theta^2)P(\chi_f^2(\delta^2)<z)$$

$$-2pq(p+q+1)P(\chi_{f+2}^2(\delta^2)<z)+(pq(p+q+1)-4(p+q+1)\mathrm{tr}\,\Theta-8\mathrm{tr}\,\Theta^2)$$

$$\cdot P(\chi_{f+4}^2(\delta^2)<z)+4(p+q+1)\mathrm{tr}\,\Theta\cdot P(\chi_{f+6}^2(\delta^2)<z)+4\mathrm{tr}\,\Theta^2\cdot P(\chi_{f+8}^2(\delta^2)<z)\}$$

$$+\frac{1}{96n^2}\{\sum_{\alpha=0}^{8}\tilde{A}_\alpha(\theta)P(\chi_{f+2\alpha}^2(\delta^2)<z)\}+O(n^{-3}),$$

where $f=pq$, $\delta^2=\mathrm{tr}\,\Theta$ *and the coefficients* $\tilde{A}_\alpha(\Theta)\,(\alpha=0,1,\dots,8)$ *are given by* (8.8).

It is worthwhile to note that the asymptotic formula (8.9) can be specialized to some interesting cases, namely, asymptotic expansions of the distributions of the multiple correlation coefficient when $p=1$, and the correlation coefficient when $p=q=1$, under the assumption $\rho^2=(2/n)\theta$.

8.4. *Limiting distributions when* $\gamma>1$. In this section we consider the limiting distributions of the Pillai's criterion $V=n\mathrm{tr}\,R$ under the sequences of alternatives K_γ: $P=2n^{-\gamma}\Theta$ with $\gamma>1$. The distribution of R is given by Constantine [6], in the following form:

$$(8.10) \quad \pi^{\frac{p^2}{2}}\Gamma_p\left(\frac{n}{2}\right)\left\{\Gamma_p\left(\frac{q}{2}\right)\Gamma_p\left(\frac{n-q}{2}\right)\Gamma_p\left(\frac{p}{2}\right)\right\}^{-1}|I-P|^{\frac{n}{2}}|R|^{(q-p-1)/2}$$

$$\cdot|I-R|^{(n-q-p-1)/2}\prod_{i<j}^{p}(r_i^2-r_j^2)\sum_{k=0}^{\infty}\sum_{(\kappa)}\left\{\left(\frac{n}{2}\right)_\kappa\left(\frac{n}{2}\right)_\kappa\Big/\left(\frac{q}{2}\right)_\kappa\right\}$$

$$\cdot\{C_\kappa(R)\,C_\kappa(P)/k!\,C_\kappa(I)\}\,dR\,, \qquad (1>r_1^2>r_2^2>\cdots>r_p^2>0).$$

Putting $nR=W$, then we can get the following limiting distribution of W under the assumption $P=2n^{-\gamma}\Theta(\gamma>1)$.

$$(8.11) \quad 2^{-\frac{pq}{2}}\pi^{\frac{p^2}{2}}\left\{\Gamma_p\left(\frac{q}{2}\right)\Gamma_p\left(\frac{p}{2}\right)\right\}^{-1}\left\{\mathrm{etr}\left(-\frac{1}{2}W\right)\right\}|W|^{(q-p-1)/2}\prod_{i<j}(w_i^2-w_j^2)\,dW\,,$$

$$(w_1^2>w_2^2>\cdots>w_p^2>0)$$

From (8.11) we see that the characteristic function of tr W is $(1-2it)^{-pq/2}$ Therefore we have the following

THEOREM 8.2. *Under the sequence of alternatives* K_γ: $P=2n^{-\gamma}\Theta$ *for* $\gamma>1$, *the limiting distribution of the Pillai's criterion* $V=n\mathrm{tr}\,R$ *for testing the hypothesis of independence is the* χ^2 *distribution with* pq *degrees of freedom.*

8.5. *Numerical example.* By putting $\Theta=0$ and $\delta=0$ in (8.9), we can see that asymptotic expansion of the null distribution of the Pillai's criterion for independence agrees with the formula (5.11) in the case of multivariate linear hypothesis when m is replaced by n. This result also follows from the result (e.g. Anderson [2]) that the distribution of R under hypothesis is the same as that of the characteristic roots of $S_h(S_h+S_e)^{-1}$, where the random matrices S_h and S_e are independently distributed as the Wishart distributions $W_p(q, \Sigma)$ and $W_p(n-q, \Sigma)$, respectively.

Example 8.1. When $N=87, p=2$ and $q=3$, the formula (5.11) after changing m to n gives the approximate 5% point as 12.3365. Pillai and Jayachandran [31] gave the exact 5% point as 12.33642 (computed from their Table 8. Upper 5% points of $V^{(2)}$ $m=0$ and $n=40$). For the alternative hypotheses K_1: $\rho_1^2=0.005$, $\rho_2^2=0.005$ and K_2: $\rho_1^2=0.001$, $\rho_2^2=0.05$, the following approximate powers are computed by the formula (8.10), based on our 5% point.

Approximate power of n tr R,

	Neglecting terms of order			Exact
	$O(n^{-1})$	$O(n^{-2})$	$O(n^{-3})$	
K_1	0.09400	0.08536	0.08505	0.0850549
K_2	0.30971	0.29246	0.29079	0.292

This shows that our approximate powers give good approximation to the corresponding exact values due to Pillai and Jayachandran [30].

9. Asymptotic non-null distributions of the Hotelling's criterion under local alternatives

9.1. *Asymptotic distribution when* $\gamma=1$. The Hotelling's criterion for testing the hypothesis of independence between two sets of variates was defined in Section 8.1 as $T_0^2=m\text{tr}\ S_{12}S_{22}^{-1}S_{21}(S_{11}-S_{12}S_{22}^{-1}S_{21})^{-1}=m\ \text{tr}\ R(I-R)^{-1}$. First we obtain the characteristic function of T_0^2 under the alternative K in a form covenient for our asymptotic approximation. By reducing the test for independence to that of the linear hypothesis as in Anderson and Das Gupta [3], Sugiura and Fujikoshi [39], etc., we see that the conditional characteristic function of T_0^2 for given $Y(n\times q)$ is obtained from (6.1) by making the substitutions $n\to m$ and $\Omega\to\frac{1}{2}(I-P)^{-\frac{1}{2}}B'Y'YB(I-P)^{-\frac{1}{2}}$, where the rows of Y are independently normally distributed with mean zero and covariance matrix I_q, and $B(q\times p)$ is given by

$$B'=\begin{pmatrix} \rho_1 & 0 \cdots 0 & 0 \cdots 0 \\ 0 & \rho_2 \cdots 0 & 0 \cdots 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 \cdots \rho_p & 0 \cdots 0 \end{pmatrix}. \tag{9.1}$$

Taking expectation with respect to $Y'Y$ by the Wishart distribution $W_q(n, I_q)$ with the formula (1.10), we can write the characteristic function of T_0^2 as (9.2).

$$(9.2)\qquad C(t)=\Gamma_p\Big(\frac{m+q}{2}\Big)\Big\{\Big(\frac{m}{2}\Big)^{pq/2}\Gamma_p\Big(\frac{m}{2}\Big)\Big\}^{-1}\Big\{\Gamma_p\Big(\frac{q}{2}\Big)\Big\}^{-1}\int_{S>0}\Big|I+\frac{2}{m}S\Big|^{-(m+q)/2}$$

$$\cdot\{\mathrm{etr}(2itS)\}\,|S|^{(q-p-1)/2}{}_1F_1\Big(\frac{m+q}{2};\frac{q}{2};2it(I-P)^{-1}PS\Big)dS\,.$$

Now we will derive the distribution of T_0^2 under the same sequence of alternatives K: $P=\frac{2}{n}\Theta$ as in the case of the Pillai's criterion. Using the formulas (1.6) and (1.8), we rewrite ${}_1F_1$ in (9.2) as

$$(9.3)\qquad \Gamma_p\Big(\frac{q}{2}\Big)\frac{2^{p(p-1)/2}}{(2\pi i)^{p(p+1)/2}}\int_{\Re(T)=X_0>0}(\mathrm{etr}\ T)\,|T|^{-\frac{q}{2}}\Big|I-\frac{2it}{n}\Delta^{\frac{1}{2}}T^{-1}\Delta^{\frac{1}{2}}S\Big|^{-\frac{m+q}{2}}dT$$

for any fixed S and large n with $\Delta=2\Big(I-\frac{2}{n}\Theta\Big)^{-1}\Theta$. Considering the expansions of $\Big|I+\frac{2}{m}S\Big|^{-(m+q)/2}$ and $\Big|I-\frac{2it}{n}\Delta^{\frac{1}{2}}T^{-1}\Delta^{\frac{1}{2}}S\Big|^{-(m+q)/2}$ with respect to m by using (2.14), we have the following approximation for $C(t)$:

$$(9.4)\qquad \Big\{\Gamma_p\Big(\frac{m+q}{2}\Big)\Big/\Big(\frac{m}{2}\Big)^{pq/2}\Gamma_p\Big(\frac{m}{2}\Big)\Big\}\{C_1(t)+C_2(t)\}\,,$$

where $C_1(t)$ is the expression obtained by replacing $n\to m$ and $\Omega\to\Theta$ in (6.9) and (6.11) and $C_2(t)$ is defined, with the notation $\mathcal{A}_\Theta[\quad]$ in (6.12), by

$$(9.5)\qquad C_2(t)=\mathcal{A}_\Theta\Big[\frac{1}{m}\{4it\mathrm{tr}\,\Theta^{\frac{3}{2}}T^{-1}\Theta^{\frac{1}{2}}S+(2it)^2\mathrm{tr}(\Theta^{\frac{1}{2}}T^{-1}\Theta^{\frac{1}{2}}S)^2\}$$

$$+\frac{1}{m^2}\{6it\mathrm{tr}\,\Theta^{\frac{5}{2}}T^{-1}\Theta^{\frac{1}{2}}S+2it\mathrm{tr}\,\Theta^{\frac{3}{2}}T^{-1}\Theta^{\frac{3}{2}}S-4itq\mathrm{tr}\,\Theta^{\frac{3}{2}}T^{-1}\Theta^{\frac{1}{2}}S$$

$$+2(2it)^2(\mathrm{tr}\,\Theta^{\frac{3}{2}}T^{-1}\Theta^{\frac{1}{2}}S)^2+4(2it)^2\mathrm{tr}\,\Theta^{\frac{1}{2}}T^{-1}\Theta^{\frac{1}{2}}S\Theta^{\frac{3}{2}}T^{-1}\Theta^{\frac{1}{2}}S$$

$$-(2it)^2q\,\mathrm{tr}(\Theta^{\frac{1}{2}}T^{-1}\Theta^{\frac{1}{2}}S)^2-4itq(\mathrm{tr}\,\Theta^{\frac{3}{2}}T^{-1}\Theta^{\frac{1}{2}}S)\mathrm{tr}\ S$$

$$+2(2it)^3\mathrm{tr}(\Theta^{\frac{1}{2}}T^{-1}\Theta^{\frac{1}{2}}S)^2\,\mathrm{tr}\,\Theta^{\frac{3}{2}}T^{-1}\Theta^{\frac{1}{2}}S+\frac{4}{3}(2it)^3\mathrm{tr}(\Theta^{\frac{1}{2}}T^{-1}\Theta^{\frac{1}{2}}S)^3$$

$$+4it(\mathrm{tr}\,\Theta^{\frac{3}{2}}T^{-1}\Theta^{\frac{1}{2}}S)\mathrm{tr}\ S^2-(2it)^2q\,\mathrm{tr}(\Theta^{\frac{1}{2}}T^{-1}\Theta^{\frac{1}{2}}S)^2\mathrm{tr}\ S$$

$$+\frac{1}{2}(2it)^4(\mathrm{tr}(\Theta^{\frac{1}{2}}T^{-1}\Theta^{\frac{1}{2}}S)^2)^2+(2it)^2\mathrm{tr}(\Theta^{\frac{1}{2}}T^{-1}\Theta^{\frac{1}{2}}S)^2\mathrm{tr}S^2\}+O(m^{-3})\Big]\,.$$

The expansion of $\left\{\Gamma_p\left(\frac{m+q}{2}\right)/\left(\frac{m}{2}\right)^{pq/2}\Gamma_p\left(\frac{m}{2}\right)\right\}$ $C_1(t)$ hasbeen already obtained in (6.16) with n, Ω replaced by m and Θ, respectively. For simplification of $C_2(t)$, we carry out each integral $\mathcal{A}_\Theta[\quad]$ in (9.5). Using Lemmas 4, 5, 6 and 7, we obtain the following identities:

(9.6) $$\mathcal{A}_\Theta[\mathrm{tr}\,\Theta^{\frac{3}{2}}T^{-1}\Theta^{\frac{1}{2}}S]=(1-2it)^{-pq/2-1}(\mathrm{tr}\,\Theta^2)\mathrm{etr}\left(\frac{2it}{1-2it}\Theta\right),$$

$$\mathcal{A}_\Theta[\mathrm{tr}\,\Theta^{\frac{5}{2}}T^{-1}\Theta^{\frac{1}{2}}S]=(1-2it)^{-pq/2-1}(\mathrm{tr}\,\Theta^3)\mathrm{etr}\left(\frac{2it}{1-2it}\Theta\right),$$

$$\mathcal{A}_\Theta[\mathrm{tr}\,\Theta^{\frac{3}{2}}T^{-1}\Theta^{\frac{3}{2}}S+4it(\mathrm{tr}\,\Theta^{\frac{3}{2}}T^{-1}\Theta^{\frac{1}{2}}S)^2]=(1-2it)^{-pq/2-1}\Big\{\mathrm{tr}\,\Theta^3$$
$$+\frac{4it}{1-2it}(\mathrm{tr}\,\Theta^2)^2\Big\}\mathrm{etr}\left(\frac{2it}{1-2it}\Theta\right),$$

$$\mathcal{A}_\Theta[\mathrm{tr}\,(\Theta^{\frac{1}{2}}T^{-1}\Theta^{\frac{1}{2}}S)^2]=(1-2it)^{-pq/2-2}(\mathrm{tr}\,\Theta^2)\mathrm{etr}\left(\frac{2it}{1-2it}\Theta\right),$$

$$\mathcal{A}_\Theta[\mathrm{tr}\,\Theta^{\frac{1}{2}}T^{-1}\Theta^{\frac{1}{2}}S\Theta^{\frac{3}{2}}T^{-1}\Theta^{\frac{1}{2}}S]=(1-2it)^{-pq/2-2}(\mathrm{tr}\,\Theta^3)\mathrm{etr}\left(\frac{2it}{1-2it}\Theta\right),$$

$$\mathcal{A}_\Theta[(\mathrm{tr}\,\Theta^{\frac{3}{2}}T^{-1}\Theta^{\frac{1}{2}}S)\mathrm{tr}\,S]=(1-2it)^{-pq/2-2}\Big\{\left(\frac{1}{2}pq+1\right)\mathrm{tr}\,\Theta^2$$
$$+\frac{2it}{1-2it}(\mathrm{tr}\,\Theta)\mathrm{tr}\,\Theta^2\Big\}\mathrm{etr}\left(\frac{2it}{1-2it}\Theta\right),$$

$$\mathcal{A}_\Theta[\mathrm{tr}(\Theta^{\frac{1}{2}}T^{-1}\Theta^{\frac{1}{2}}S)^2\,\mathrm{tr}\Theta^{\frac{3}{2}}T^{-1}\Theta^{\frac{1}{2}}S]=(1-2it)^{-pq/2-3}(\mathrm{tr}\,\Theta^2)^2\mathrm{etr}\left(\frac{2it}{1-2it}\Theta\right),$$

$$\mathcal{A}_\Theta[\mathrm{tr}(\Theta^{\frac{1}{2}}T^{-1}\Theta^{\frac{1}{2}}S)^3]=(1-2it)^{-pq/2-3}(\mathrm{tr}\,\Theta^3)\mathrm{etr}\left(\frac{2it}{1-2it}\Theta\right),$$

$$\mathcal{A}_\Theta[(\mathrm{tr}\,\Theta^{\frac{3}{2}}T^{-1}\Theta^{\frac{1}{2}}S)\mathrm{tr}\,S^2]=(1-2it)^{-pq/2-3}\Big[\frac{1}{4}\{qp^2+(q^2+q+4)p$$
$$+4(q+1)\}\,\mathrm{tr}\,\Theta^2+\frac{2it}{1-2it}\{(p+q+1)(\mathrm{tr}\,\Theta)\mathrm{tr}\,\Theta^2+2\mathrm{tr}\,\Theta^3\}$$
$$+\left(\frac{2it}{1-2it}\right)^2(\mathrm{tr}\,\Theta^2)^2\Big]\mathrm{etr}\left(\frac{2it}{1-2it}\Theta\right),$$

$$\mathcal{A}_\Theta[\mathrm{tr}(\Theta^{\frac{1}{2}}T^{-1}\Theta^{\frac{1}{2}}S)^2\mathrm{tr}\,S]=(1-2it)^{-pq/2-3}\Big\{\left(\frac{1}{2}pq+2\right)\mathrm{tr}\,\Theta^2$$
$$+\frac{2it}{1-2it}(\mathrm{tr}\,\Theta)\,\mathrm{tr}\,\Theta^2\Big\}\mathrm{etr}\left(\frac{2it}{1-2it}\Theta\right),$$

$$\mathcal{A}_{\Theta}[(\mathrm{tr}(\Theta^{\frac{1}{2}} T^{-1}\Theta^{\frac{1}{2}} S)^2)^2] = (1-2it)^{-pq/2-4}(\mathrm{tr}\ \Theta^2)^2 \mathrm{etr}\left(\frac{2it}{1-2it}\Theta\right),$$

$$\mathcal{A}_{\Theta}[\mathrm{tr}(\Theta^{\frac{1}{2}} T^{-1}\Theta^{\frac{1}{2}} S)^2 \mathrm{tr}\ S^2] = (1-2it)^{-pq/2-4}\Big[(\mathrm{tr}\ \Theta)^2$$

$$+\frac{1}{4}\{qp^2+(q^2+q+8)p+4(2q+3)\}\mathrm{tr}\ \Theta^2 + \frac{2it}{1-2it}\{(p+q+1)(\mathrm{tr}\ \Theta)\mathrm{tr}\ \Theta^2$$

$$+4\mathrm{tr}\ \Theta^3\} + \left(\frac{2it}{1-2it}\right)^2 (\mathrm{tr}\ \Theta^2)^2\Big]\mathrm{etr}\left(\frac{2it}{1-2it}\Theta\right).$$

In the derivation of the formulas (9.6), we used Lemmas 4～7 after expressing, for example, $\mathrm{tr}\ \Theta^{\frac{3}{2}} T^{-1}\Theta^{\frac{1}{2}} S$ as $\frac{1}{2}\mathrm{tr}\ (\Theta^{\frac{3}{2}} T^{-1}\Theta^{\frac{1}{2}} + \Theta^{\frac{1}{2}} T^{-1}\Theta^{\frac{3}{2}})S$, etc., since the matrices Ω and Θ in Lemmas 4 ～7 are symmetric. The asymptotic expansion of $\Gamma_p\left(\frac{m+q}{2}\right)/\left\{\left(\frac{m}{2}\right)^{pq/2}\Gamma_p\left(\frac{m}{2}\right)\right\}$ is obtained from (6.10) by making the substitution $n\to m$. Hence we can finally write the approximation (9.4) for $C(t)$ as follows:

$$(9.7)\qquad (1-2it)^{-pq/2}\left\{\mathrm{etr}\left(\frac{2it}{1-2it}\Theta\right)\right\}\Big[1+\frac{1}{4m}\{pq(q-p-1)-4\mathrm{tr}\ \Theta^2$$

$$-2q(pq-2\mathrm{tr}\ \Theta)(1-2it)^{-1}+(pq(p+q+1)-4(p+2q+1)\mathrm{tr}\ \Theta$$

$$+8\mathrm{tr}\ \Theta^2)(1-2it)^{-2}+4((p+q+1)\mathrm{tr}\ \Theta-2\mathrm{tr}\ \Theta^2)(1-2it)^{-3}$$

$$+4\mathrm{tr}\ \Theta^2(1-2it)^{-4}\}+\frac{1}{96m^2}\left\{\sum_{\alpha=0}^{8}\tilde{B}_\alpha(\Theta)(1-2it)^{-\alpha}\right\}+O(m^{-3})\Big],$$

where the coefficients $\tilde{B}_\alpha(\Theta)(\alpha=0, 1,\cdots, 8)$ are given by (9.8).

$$(9.8)\qquad \tilde{B}_0(\Theta)=pql_0+24q\{p^2-(q-1)p+4\}\mathrm{tr}\ \Theta^2-128\mathrm{tr}\ \Theta^3+48(\mathrm{tr}\ \Theta^2)^2,$$

$$\tilde{B}_1(\Theta)=-l_1(pq-2\mathrm{tr}\ \Theta)+48pq^2\mathrm{tr}\ \Theta^2-96q(\mathrm{tr}\ \Theta)\mathrm{tr}\ \Theta^2,$$

$$\tilde{B}_2(\Theta)=pql_2-2(l_1+2l_2)\mathrm{tr}\ \Theta+48(q^2+2)(\mathrm{tr}\ \Theta)^2-24\{3qp^2-q(q-3)p$$

$$-4(2q+1)\}\mathrm{tr}\ \Theta^2+96(p+2q+1)(\mathrm{tr}\ \theta)\mathrm{tr}\ \Theta)^2-192(\mathrm{tr}\ \Theta^2)^2,$$

$$\tilde{B}_3(\Theta)=-pql_3+2(2l_2+3l_3)\mathrm{tr}\ \Theta-96(qp+2q^2+q+4)(\mathrm{tr}\ \Theta)^2+48\{qp^2$$

$$-(3q^2-q+8)p-8(3q+2)\}\mathrm{tr}\ \Theta^2-96(p-q+1)(\mathrm{tr}\ \Theta)\mathrm{tr}\ \Theta^2$$

$$+640\mathrm{tr}\ \Theta^3+192(\mathrm{tr}\ \Theta^2)^2,$$

$$\tilde{B}_4(\Theta)=pql_4-2(3l_3+4l_4)\mathrm{tr}\ \Theta+48\{p^2+2(3q+1)p+6q^2+6q+17\}$$

$$(\mathrm{tr}\ \Theta)^2+24\{qp^2+(7q^2+q+44)p+4(20q+19)\}\mathrm{tr}\ \Theta^2$$

$$-192(p+3q+1)(\mathrm{tr}\ \Theta)\mathrm{tr}\ \theta^2-1536\mathrm{tr}\ \Theta^3+96(\mathrm{tr}\ \Theta^2)^2,$$

$$\tilde{B}_5(\Theta)=8l_4\text{tr }\Theta-96\{p^2+(3q+2)p+2q^2+3q+9\}(\text{tr }\Theta)^2-48\{qp^2+(2q^2+q+24)p+8(4q+5)\}\text{tr }\Theta^2+96(4p+7q+4)(\text{tr }\Theta)\text{tr }\Theta^2+1920\text{tr }\Theta^3-384(\text{tr }\Theta^2)^2,$$

$$\tilde{B}_6(\Theta)=48\{p^2+2(q+1)p+q^2+2q+7\}(\text{tr }\Theta)^2+24\{qp^2+(q^2+q+20)p+4(5q+8)\}\text{tr }\Theta^2-96(3p+4q+3)(\text{tr }\Theta)\text{tr }\Theta^2-1280\text{tr }\Theta^3+384(\text{tr }\Theta^2)^2,$$

$$\tilde{B}_7(\Theta)=96\{(p+q+1)(\text{tr }\Theta)\text{tr }\Theta^2+4\text{tr }\Theta^3-2(\text{tr }\Theta^2)^2\},$$

$$\tilde{B}_8(\Theta)=48(\text{tr }\Theta^2)^2,$$

with l_α $(\alpha=0, 1,\cdots, 4)$ defined by (6.18). Inverting this characteristic function, we have the following theorem:

THEOREM 9.1. *Under sequence of alternatives K: $P=\frac{2}{n}\Theta$, the distribution of the Hotelling's criterion for testing the independence between two sets of variates with p components and q comopneots $(p\leq q)$ can be approximated for large m as follows*:

$$(9.9)\quad P(T_0^2<z)\simeq P(\chi_f^2(\delta^2)<z)+\frac{1}{4m}\{(pq(q-p-1)-4\text{tr }\Theta^2)P(\chi_f^2(\delta^2)<z)-2q(pq-2\text{tr }\Theta)P(\chi_{f+2}^2(\delta^2)<z)+(pq(p+q+1)-4(p+2q+1)\text{tr }\Theta+8\text{tr }\Theta^2)P(\chi_{f+4}^2(\delta^2)<z)+4((p+q+1)\text{tr }\Theta-2\text{tr }\Theta^2)P(\chi_{f+6}^2(\delta^2)<z)+4\text{tr }\Theta^2\cdot P(\chi_{f+8}^2(\delta^2)<z)\}+\frac{1}{96m^2}\left\{\sum_{\alpha=0}^{8}\tilde{B}_\alpha(\Theta)P(\chi_{f+2\alpha}^2(\delta^2)<z)\right\},$$

where $f=pq$, $m=n-q$ and the coefficients $\tilde{B}_\alpha(\Theta)$ $(\alpha=0, 1,\cdots, 8)$ are given by (9.8).

The above theorem also includes some interesting special cases as in the case of Theorem 8.1.

9.2. *Limiting distribution when $\gamma>1$.* We now consider the limiting distributions of the Hotelling's criterion $T_0^2=m\text{tr }R(I-R)^{-1}$ under the sequence of alternatives K_γ: $P=2n^{-\gamma}\Theta(\gamma>1)$. Noting that under $P=2n^{-\gamma}\Theta(\gamma>1)$,

$$(9.10)\quad \lim_{m\to\infty}{}_1F_1\left(\frac{m+q}{2};\frac{q}{2};2it(I-P)^{-1}PS\right)=1,$$

we obtain from (9.2) that the characteristic function $C(t)$ of T_0^2 under $P=2n^{-\gamma}\Theta$ tends to $(1-2it)^{-pq/2}$ as $m\to\infty$. Hence we have the following theorem:

THEOREM 9.2. *Under the sequence of alternatives K_γ: $P=2n^{-\gamma}\Theta$ for $\gamma>1$,* *the limiting distribution of the Hotelling's criterion T_0^2 for testing the hypothesis of independence is the χ^2 distribution with pq degrees of freedom.*

9.3. *Numerical example.* Asymptotic expansion of the Hotelling's T_0^2 criterion for independence under hypothesis is obtained from (6.20) by making the substitution $n \to m$, or by putting $\Theta=0$ and $\delta=0$ in (9.9). Let us consider the same example as was used for the Pillai's criterion.

Example 9.1. When $N=87$, $p=2$ and $q=3$, the formula (6.20) after changing n to m gives the approximate 5% point as 13.3433. Pillai and Jayachandran [30] gave the exact 5% point as 13.34128 (computed from their Table 8. Upper 5% points of $U^{(2)}$ $m=0$ and $n=40$). For the alternative hypotheses K_1: $\rho_1^2=0.005$, $\rho_2^2=0.005$ and K_2: $\rho_1^2=0.001$, $\rho_2^2=0.05$, the following approximate powers are computed from the formula (9.9), based on our 5% point.

Approximate power of $n\,\mathrm{tr}\, R(I-R)^{-1}$,

	Including up to			Exact
	first term	second term	third term	
K_1	0.06861	0.08425	0.08484	0.0848456
K_2	0.25548	0.29363	0.29318	0.293

This shows that our approximate powers also give good approximation to the corresponding exact values due to Pillai and Jayachandran [30].

PART IV. OTHER ASYMPTOTIC DISTRIBUTIONS

10. The determinant of a non-central Wishart distribution. Using the expression of moments due to Constantine [6], Fujikoshi [9] obtained asymptotic expansion of the distribution of $|S|$ up to order $n^{-\frac{3}{2}}$ for the non-central Wishart matrix nS having $W_p(n, \Sigma, \Omega)$, when the non-centrality matrix Ω may be regarded as a constant matrix with respect to n. Here we shall show the asymptotic expansion of the distribution of the statistic $\mu=\sqrt{\frac{n}{2p}}\left\{\log\frac{|S|}{|\Sigma|}-\frac{2}{\sqrt{n}}\mathrm{tr}\,\Theta\right\}$ instead of $|S|$ under the assumption that $\Omega=\sqrt{n}\,\Theta$. The characteristic function of μ can be expressed as (10.1) (Fujikoshi [9]).

$$(10.1)\qquad \left\{\left(\frac{2}{n}\right)^{itp\sqrt{n}/\sqrt{2p}}\Gamma_p\left(\frac{n}{2}+\frac{it\sqrt{n}}{\sqrt{2p}}\right)\Big/\Gamma_p\left(\frac{n}{2}\right)\right\}$$

$$\cdot \left\{\operatorname{etr}\left(-\frac{2it}{\sqrt{2p}}\Theta\right)\right\} {}_1F_1\left(-\frac{it\sqrt{n}}{\sqrt{2p}}; \frac{n}{2}; -\sqrt{n}\Theta\right)$$

The first factor is the same as (3.7) in Fujikoshi [9] with $it/\sqrt{2p}$ for it. By Lemma 2, the second factor can be expressed as (10.2).

$$(10.2)\quad 1-\frac{2it}{\sqrt{2p}\sqrt{n}}\operatorname{tr}\Theta^2+\frac{1}{pn}\left\{\frac{4}{3}\sqrt{2p}\,it\operatorname{tr}\Theta^3-2(it)^2\operatorname{tr}\Theta^2+(it)^2(\operatorname{tr}\Theta^2)^2\right\}$$

$$-\frac{1}{p\sqrt{2pn}\sqrt{n}}\left\{2pit(2\operatorname{tr}\Theta^4-(\operatorname{tr}\Theta)^2-\operatorname{tr}\Theta^2)+\frac{8}{3}\sqrt{2p}\,(it)^2(\operatorname{tr}\Theta^2\right.$$

$$\left.-3)\operatorname{tr}\Theta^3+\frac{2}{3}(it)^3(\operatorname{tr}\Theta^2-6)(\operatorname{tr}\Theta^2)^2\right\}+O(n^{-2}).$$

Hence we have,

THEOREM 10.1. *Let nS have the non-central Wishart distribution with n degrees of freedom and the non-centrality matrix Ω. Under the assumption that $\Omega=\sqrt{n}\,\Theta$, the distribution of $|S|$ can be expanded asymptotically as follows:*

$$(10.3)\quad P\left\{\sqrt{\frac{n}{2p}}\left(\log\frac{|S|}{|\Sigma|}-\frac{2}{\sqrt{n}}\operatorname{tr}\Theta\right)<z\right\}=\Phi(z)+\frac{1}{3\sqrt{2pn}}\{3(q+2\operatorname{tr}\Theta^2)\Phi'(z)$$

$$+\Phi^{(3)}(z)\}+\frac{1}{12pn}\Big[-16\sqrt{2p}\operatorname{tr}\Theta^3\Phi'(z)+3\{q(q+2)+4(q-2)\operatorname{tr}\Theta^2$$

$$+4(\operatorname{tr}\Theta^2)^2\}\Phi^{(2)}(z)+2(q+1+2\operatorname{tr}\Theta^2)\Phi^{(4)}(z)+\frac{1}{3}\Phi^{(6)}(z)\Big]$$

$$+\frac{1}{36p\sqrt{2p}\,n\sqrt{n}}\Big[3p\{p(2p^2+3p-1)-24(\operatorname{tr}\Theta)^2-24\operatorname{tr}\Theta^2+48\operatorname{tr}\Theta^4\}$$

$$\cdot\Phi'(z)-48\sqrt{2p}\{(q-6)\operatorname{tr}\Theta^3+2(\operatorname{tr}\Theta^2)\operatorname{tr}\Theta^3\}\Phi^{(2)}(z)+3\{q(q+2)(q+4)$$

$$+6q(q-2)\operatorname{tr}\Theta^2+12(q-4)(\operatorname{tr}\Theta^2)^2+8(\operatorname{tr}\Theta^2)^3\}\Phi^{(3)}(z)-16\sqrt{2p}\operatorname{tr}\Theta^3$$

$$\cdot\Phi^{(4)}(z)+3\left\{\frac{1}{5}(5q^2+20q+12)+4(q-1)\operatorname{tr}\Theta^2+4(\operatorname{tr}\Theta^2)^2\right\}\Phi^{(5)}(z)$$

$$+(q+2+2\operatorname{tr}\Theta^2)\Phi^{(7)}(z)+\frac{1}{9}\Phi^{(9)}(z)\Big]+O(n^{-2}),$$

where $q=p(p+1)/2$ and *$\Phi^{(r)}(z)$ denotes the r-th derivative of the standard normal distribution function $\Phi(z)$.*

When $\Omega=n\Theta$, Fujikoshi [9] obtained the limitting distribution of $|S|$, but could not derive its asymptotic expansion.

11. The trace of a non-central Wishart matrix. In this section we give asymptotic expansion of the distribution of the statistic tr $R^{-1}S$ which is a generalization of tr S and tr $\Sigma^{-1}S$, where $R(p\times p)$ is any fixed real symmetric matrix with $|R|\neq 0$. From the result of Anderson [1] we can write the characteristic function of tr $R^{-1}S$ as follows:

$$(11.1)\quad \mathrm{E}[\mathrm{e}^{it\mathrm{tr}\,R^{-1}S}]=\left|I-\frac{2it}{n}\Sigma^{\frac{1}{2}}R^{-1}\Sigma^{\frac{1}{2}}\right|^{-\frac{n}{2}}\mathrm{etr}\left\{-\Omega+\left(I-\frac{2it}{n}\Sigma^{\frac{1}{2}}R^{-1}\Sigma^{\frac{1}{2}}\right)^{-1}\Omega\right\}$$

First we assume that the nonce-ntrality matrix $\Omega=O(1)$ with respect to n. Putting $\tau=\sqrt{2\mathrm{tr}(R^{-1}\Sigma)^2}$, the characteristic function of $\xi=(\sqrt{n}/\tau)(\mathrm{tr}\,R^{-1}S-\mathrm{tr}\,R^{-1}\Sigma)$ can be given by (11.2).

$$(11.2)\quad \left|I-\frac{2it}{\tau\sqrt{n}}\Sigma^{\frac{1}{2}}R^{-1}\Sigma^{\frac{1}{2}}\right|^{-\frac{n}{2}}\left\{\mathrm{etr}\left(-it\frac{\sqrt{n}}{\tau}R^{-1}\Sigma\right)\right\}$$

$$\cdot\mathrm{etr}\left\{-\Omega+\left(I-\frac{2it}{\tau\sqrt{n}}\Sigma^{\frac{1}{2}}\cdot R^{-1}\Sigma^{\frac{1}{2}}\right)^{-1}\Omega\right\}.$$

The first factor can be expanded as follows:

$$(11.3)\quad \mathrm{e}^{-\frac{t^2}{2}}\left[1+\frac{4}{3\sqrt{n}}\left(\frac{it}{\tau}\right)^3\mathrm{tr}\,\Delta^3+\frac{2}{n}\left\{\left(\frac{it}{\tau}\right)^4\mathrm{tr}\Delta^4+\frac{4}{9}\left(\frac{it}{\tau}\right)^6(\mathrm{tr}\,\Delta^3)^2\right\}\right.$$

$$+\frac{8}{n\sqrt{n}}\left\{\frac{2}{5}\left(\frac{it}{\tau}\right)^5\mathrm{tr}\Delta^5+\frac{1}{3}\left(\frac{it}{\tau}\right)^7(\mathrm{tr}\,\Delta^3)\mathrm{tr}\,\Delta^4+\frac{4}{81}\left(\frac{it}{\tau}\right)^9(\mathrm{tr}\,\Delta^3)^3\right\}$$

$$\left.+O(n^{-2})\right],$$

where $\Delta=\Sigma^{\frac{1}{2}}R^{-1}\Sigma^{\frac{1}{2}}$. The second factor can be expanded as (11.4).

$$(11.4)\quad 1+\frac{2it}{\sqrt{n}\,\tau}\mathrm{tr}\,\Delta\Omega+\frac{2}{n}\left(\frac{it}{\tau}\right)^2\{2\mathrm{tr}\,\Delta^2\Omega+(\mathrm{tr}\,\Delta\Omega)^2\}$$

$$+\frac{8}{n\sqrt{n}}\left(\frac{it}{\tau}\right)^3\left\{\mathrm{tr}\,\Delta^3\Omega+(\mathrm{tr}\,\Delta\Omega)\mathrm{tr}\,\Delta^2\Omega+\frac{1}{6}(\mathrm{tr}\,\Delta\Omega)^3\right\}+O(n^{-2}),$$

which gives the following theorem:

THEOREM 11.1 *Let nS have the non-central Wishart distribution $W_p(n,\Sigma,\Omega)$. Under the assumption that $\Omega=O(1)$ with respect to n, we have the following*:

$$(11.5)\quad P\left\{\frac{\sqrt{n}}{\tau}(\mathrm{tr}\,R^{-1}S-\mathrm{tr}\,R^{-1}\Sigma)<z\right\}=\Phi(z)-\frac{2}{\sqrt{n}\,\tau}\left\{(\mathrm{tr}\,\Delta\Omega)\Phi'(z)+\frac{2}{3\tau^2}\mathrm{tr}\,\Delta^3\right.$$

$$\left.\cdot\Phi^{(3)}(z)\right\}+\frac{2}{n\tau^2}\left[\{2\mathrm{tr}\,\Delta^2\Omega+(\mathrm{tr}\,\Delta\Omega)^2\}\Phi^{(2)}(z)+\frac{1}{3\tau^2}\{3\mathrm{tr}\,\Delta^4\right.$$

$$+4(\mathrm{tr}\ \Delta^3)\mathrm{tr}\ \Delta\Omega\}\Phi^{(4)}(z)+\frac{4}{9\tau^4}(\mathrm{tr}\ \Delta^3)^2\Phi^{(6)}(z)\Big]-\frac{4}{3n\sqrt{n}\tau^3}\Big[\{6\mathrm{tr}\ \Delta^3\Omega$$

$$+6(\mathrm{tr}\ \Delta\Omega)\mathrm{tr}\ \Delta^2\Omega+(\mathrm{tr}\ \Delta\Omega)^3\}\Phi^{(3)}(z)+\frac{1}{5\tau^2}\{12\mathrm{tr}\ \Delta^5+15(\mathrm{tr}\ \Delta^4)$$

$$\cdot\mathrm{tr}\ \Delta\Omega+20(\mathrm{tr}\ \Delta^3)\mathrm{tr}\ \Delta^2\Omega+10\mathrm{tr}\ \Delta^3(\mathrm{tr}\ \Delta\Omega)^2\}\Phi^{(5)}(z)+\frac{2}{3\tau^4}\{3(\mathrm{tr}\ \Delta^3)$$

$$\cdot\mathrm{tr}\ \Delta^4+2(\mathrm{tr}\ \Delta^3)^2\mathrm{tr}\ \Delta\Omega\}\Phi^{(7)}(z)+\frac{8}{27\tau^6}(\mathrm{tr}\ \Delta^3)^3\Phi^{(9)}(z)\Big]+O(n^{-2}),$$

where $\tau=\sqrt{2\mathrm{tr}\,(R^{-1}\Sigma)^2}$ *and* $\Delta=\Sigma^{\frac{1}{2}}R^{-1}\Sigma^{\frac{1}{2}}$.

Finally considering the characteristic function of the statistic $\eta=\{n/2\mathrm{tr}\,\Delta^2(I+4\Theta)\}^{\frac{1}{2}}\{\mathrm{tr}\ R^{-1}S-\mathrm{tr}\ \Delta(I+2\Theta)\}$ under the assumption $\Omega=n\Theta$, based on (11.1), gives:

THEOREM 11.2. *Under the assumption that* $\Omega=n\Theta$, *the distribution of* $\mathrm{tr}\ R^{-1}S$ *can be expanded asymptotically as* (11.6).

(11.6)
$$P\Big[\frac{\sqrt{n}}{\sigma}\{\mathrm{tr}\ R^{-1}S-\mathrm{tr}\ \Delta(I+2\Theta)\}<z\Big]=\Phi(z)-\frac{4}{3\sqrt{n}\sigma^3}\{\mathrm{tr}\ \Delta^3(I+6\Theta)\}\Phi^{(3)}(z)$$

$$+\frac{2}{n\sigma^4}\Big[\{\mathrm{tr}\ \Delta^4(I+8\Theta)\}\Phi^{(4)}(z)+\frac{4}{9\sigma^2}\{\mathrm{tr}\ \Delta^3\ (I+6\Theta)\}^2\Phi^{(6)}(z)\Big]$$

$$-\frac{8}{n\sqrt{n}\sigma^5}\Big[\frac{2}{5}\{\mathrm{tr}\ \Delta^5(I+10\Theta)\}\Phi^{(5)}(z)+\frac{1}{3\sigma^2}\mathrm{tr}\ \Delta^3(I+6\Theta)\{\mathrm{tr}\ \Delta^4(I$$

$$+8\Theta)\}\Phi^{(7)}(z)+\frac{4}{81\sigma^4}\{\mathrm{tr}\ \Delta^3(I+6)\Theta\}^3\Phi^{(9)}(z)\Big]+O(n^{-2}),$$

where $\sigma=\{2\mathrm{tr}\ \Delta^2(I+4\Theta)\}^{\frac{1}{2}}$ and $\Delta=\Sigma^{\frac{1}{2}}R^{-1}\Sigma^{\frac{1}{2}}$.

The asymptotic expansions of the distributions of $\mathrm{tr}\ S$ and $\mathrm{tr}\ \Sigma^{-1}S$ can be obtained immediately by putting $R=I$ and Σ in the above two theorems, according to $\Omega=O(1)$ or $\Omega=O(n)$.

12. The modified LR criterion for equality of mean vectors and covariance matrices under special alternatives. Let the $p\times 1$ vectors $x_1^{(g)}, x_2^{(g)},\cdots,x_{N_g}^{(g)}$ be a random sample from a normal population with mean vector $\mu^{(g)}$ and covariance matrix Σ_g for $g=1, 2,\cdots, q\,(q\geq 2)$, and consider the testing problem of the hypothesis H: $\mu^{(1)}=\mu^{(2)}=\cdots=\mu^{(q)}$, $\Sigma_1=\Sigma_2=\cdots=\Sigma_q$ against the alternatives K: $\mu^{(i)}\neq\mu^{(j)}$ or $\Sigma_i\neq\Sigma_j$ for some i, j $(i\neq j)$. Under the assumption $\Sigma_1=\Sigma_2=\cdots=\Sigma_q(=\Sigma)$, the h-th moment of the modified LR criterion λ for this problem was given by Tsumura and Fukutomi [41] as

$$(12.1)\quad \Big\{n^n/\prod_{g=1}^q n_g^{n_g}\Big\}^{ph/2}\prod_{g=1}^q\Big\{\Gamma_p\Big(\frac{n_g(1+h)}{2}\Big)\Gamma_p\Big(\frac{n+q-1}{2}\Big)/\Gamma_p\Big(\frac{n_g}{2}\Big)$$

$$\Gamma_p\Big(\frac{n(1+h)+q-1}{2}\Big)\Big\}\cdot{}_1F_1\Big(\frac{1}{2}nh;\frac{1}{2}\{n(1+h)+q-1\};-\Omega\Big),$$

where $n_g=N_g-1$, $n=n_1+n_2+\cdots+n_q$ and $\Omega=\frac{1}{2}\sum_{g=1}^q\Sigma^{-\frac{1}{2}}N_g(\mu^{(g)}-\bar{\mu})(\mu^{(g)}-\bar{\mu})'\Sigma^{-\frac{1}{2}}$ with $\bar{\mu}=(\sum_{g=1}^q N_g)^{-1}\sum_{g=1}^q N_g\mu^{(g)}$. Now we will derive asymptotic expansion of the distribution of $-2\rho\log\lambda$ under the assumption $\Sigma_1=\Sigma_2=\cdots=\Sigma_q$, where the correction factor ρ is given by

$$(12.2)\qquad n\rho=n-(\sum_{g=1}^q k_g^{-1}-1)\frac{2p^2+3p-1}{6(q-1)(p+3)}-\frac{p-q+2}{p+3},$$

with $nk_g=n_g$ for fixed $k_g>0$ (Anderson [2, p. 255]). Put $m=\rho n$ and let m tend to infinity instead of n. From (12.1) the characteristic function of $-2\rho\log\lambda$ can be expressed as follows:

$$(12.3)\qquad C(t)=C_1(t)\,{}_1F_1\Big(-itm;\frac{1}{2}m(1-2it)+s;-\Omega\Big),$$

where $C_1(t)$ and s are given by (12.4) and (12.5), respectively.

$$(12.4)\quad C_1(t)=\Big\{n^n/\prod_{g=1}^q n_g^{n_g}\Big\}^{-it\rho p}\prod_{g=1}^q\Big\{\Gamma_p\Big(\frac{n_g(1-2it\rho)}{2}\Big)\Gamma_p\Big(\frac{n+q-1}{2}\Big)$$

$$\cdot/\Gamma_p\Big(\frac{n_g}{2}\Big)\Gamma_p\Big(\frac{n(1-2it\rho)+q-1}{2}\Big)\Big\},$$

$$(12.5)\qquad s=\frac{1}{2}\Big\{(\sum_{g=1}^q k_g^{-1}-1)\frac{2p^2+3p-1}{6(q-1)(p+3)}+\frac{p-q+2}{p+3}+q-1\Big\}.$$

The first factor $C_1(t)$ can be expanded asymptotically (Anderson [2, p. 255]) as follows:

$$(12.6)\qquad (1-2it)^{-\frac{f}{2}}\Big[1+\frac{\beta}{m^2}\{(1-2it)^{-2}-1\}+O(m^{-3})\Big],$$

where $f=(q-1)p(p+3)/2$ and β is given by (12.7).

$$(12.7)\quad \beta=\frac{p}{288}\Big\{6(\sum_{g=1}^q k_g^{-2}-1)(p+1)(p-1)(p+2)-(\sum_{g=1}^q k_g^{-1}-1)^2\frac{(2p^2+3p-1)^2}{(q-1)(p+3)}$$

$$-12(\sum_{g=1}^q k_g^{-1}-1)\frac{(2p^2+3p-1)(p-q+2)}{p+3}-36\frac{(q-1)(p-q+2)^2}{p+3}$$

$$-12(q-1)(-2q^2+7q+3pq-2p^2-6p-4)\Big\}\,.$$

Sugiura and Fujikoshi [39] obtained the asymptotic expansion of the same type as ${}_1F_1$ in (12.3) by using Lemma 2. By the same argument, we obtain the following asymptotic formula for the characteristic function $C(t)$:

$$\text{(12.8)}\qquad C(t)=(1-2it)^{-\frac{f}{2}}\left\{\operatorname{etr}\left(\frac{2it}{1-2it}\Omega\right)\right\}\Big[1+\frac{1}{m}\{2s\operatorname{tr}\Omega(1-2it)^{-1}$$
$$-(2s\operatorname{tr}\Omega-\operatorname{tr}\Omega^2)(1-2it)^{-2}-\operatorname{tr}\Omega^2(1-2it)^{-3}\}+\frac{1}{m^2}\{\beta((1-2it)^{-2}-1)$$
$$+\sum_{\alpha=2}^{6}D_\alpha(\Omega)(1-2it)^{-\alpha}\}+O(m^{-3})\Big],$$

where the coefficients $D_\alpha(\Omega)$ $(\alpha=2, 3,\cdots, 6)$ are given by (12.9).

$$\text{(12.9)}\qquad D_2(\Omega)=2s^2(\operatorname{tr}\Omega-2)\operatorname{tr}\Omega+2s\operatorname{tr}\Omega^2,$$
$$D_3(\Omega)=4s^2\operatorname{tr}\Omega-(4s^2+1)(\operatorname{tr}\Omega)^2-(8s+1)\operatorname{tr}\Omega^2+2s(\operatorname{tr}\Omega)\operatorname{tr}\Omega^2$$
$$+\frac{4}{3}\operatorname{tr}\Omega^3,$$
$$D_4(\Omega)=(2s^2+1)(\operatorname{tr}\Omega)^2+(6s+1)\operatorname{tr}\Omega^2-4s(\operatorname{tr}\Omega)\operatorname{tr}\Omega^2-4\operatorname{tr}\Omega^3$$
$$+\frac{1}{2}(\operatorname{tr}\Omega^2)^2,$$
$$D_5(\Omega)=\frac{8}{3}\operatorname{tr}\Omega^3+2s(\operatorname{tr}\Omega)\operatorname{tr}\Omega^2-(\operatorname{tr}\Omega^2)^2,$$
$$D_6(\Omega)=\frac{1}{2}(\operatorname{tr}\Omega^2)^2,$$

which implies,

THEOREM 12.1. *Under the assumption $\Sigma_1=\Sigma_2=\cdots=\Sigma_q$, we obtain*

$$\text{(12.10)}\qquad P(-2\rho\log\lambda<z)=P(\chi_f^2(\delta^2)<z)+\frac{1}{m}\{2s\operatorname{tr}\Omega\cdot P(\chi_{f+2}^2(\delta^2)<z)$$
$$-(2s\operatorname{tr}\Omega-\operatorname{tr}\Omega^2)P(\chi_{f+4}^2(\delta^2)<z)-\operatorname{tr}\Omega^2\cdot P(\chi_{f+6}^2(\delta^2)<z)\}$$
$$+\frac{1}{m^2}\Big[\beta\{P(\chi_{f+4}^2(\delta^2)<z)-P(\chi_f^2(\delta^2)<z)\}+\sum_{\alpha=2}^{6}D_\alpha(\Omega)P(\chi_{f+2\alpha}^2(\delta^2)<z)\Big]$$
$$+O(m^{-3}),$$

where $m=\rho n$, $f=(q-1)p(p+3)/2$, $\delta^2=\operatorname{tr}\Omega$ and the coefficients s, β and $D_\alpha(\Omega)$ $(\alpha=2, 3,\cdots, 6)$ are given by (12.5), (12.7) *and* (12.9), *respectively.*

Asymptotic expansion of the null distribution of $-2\rho\log\lambda$ is given (Anderson [2]) as follows:

$$(12.11)\quad P(-2\rho\log\lambda<z)=P(\chi_f^2<z)+\frac{\beta}{m^2}\{P(\chi_{f=4}^2<z)-P(\chi_f^2<z)\}+O(m^{-3}),$$

which is also obtained by putting $\Omega=0$ and $\delta=0$ in (12.10). Hence we can evaluate the power of the test $-2\rho\log\lambda$ when $\Sigma_1=\Sigma_2=\cdots=\Sigma_q$.

Example 12.1. When case (i) $p=2$, $q=4$, $N_1=21$, $N_2=N_3=N_4=22$ and case (ii) $p=2$, $q=4$, $N_1=11$, $N_2=18$, $N_3=25$ and $N_4=33$, the approximate 5% points of $-2\rho\log\lambda$ can be computed by (12.11) as 24.4672 and 24.2075, respectively. For the alternative hypotheses K_1: $\omega_1=0$, $\omega_2=1$ and K_2: $\omega_1=2$, $\omega_2=2$ (ω_i means the characteristic root of Ω), the following approximate powers are computed by the formula (12.10), based on the above 5% points.

Approximate power of $-2\rho\log\lambda$,

Neglecting terms of order

	case (i)			case (ii)		
	$O(m^{-1})$	$O(m^{-2})$	$O(m^{-3})$	$O(m^{-1})$	$O(m^{-2})$	$O(m^{-3})$
K_1	0.1178	0.1109	0.1016	0.1245	0.1168	0.1025
K_2	0.3863	0.3519	0.3418	0.3985	0.3617	0.3458

On the other hand, the powers of the LR criterion W, the Hotteling's criterion T_0^2 and the Pillai's criterion V for this alternatives are obtained from (5.19), (6.19) and (7.3) as follows:

Approximate power

	W	T_0^2	V
K_1	0.13966	0.13990	0.13924
K_2	0.5075	0.5040	0.5107

Note that all three tests are more powerful than the modified LR criterion $-2\rho\log\lambda$, which should be expected. The exact powers of W, T_0^2 and V under K_1 are known (Pillai and Jayachandran [30]) as 0.13965, 0.13994 and 0.13926 in their Table 10, respectively. Therefore, we can also see that our approximate powers are good approximations to the exact values.

APPENDIX I. *Calculations of* $\mathrm{tr}\,(\Lambda\partial)^2|I-\Sigma|^{-b}\Big|_{\Sigma=\Lambda}$, *etc. in the proof of Lemma* 3

We used some formulas to prove Lemma 3 in Section 2. Here, we list all the necessary reductions in the computations. We use the following notations:

$\Sigma=(\sigma_{rs})$ is a $p\times p$ symmetric matrix, $\partial=\left(\frac{1}{2}(1+\delta_{rs})\frac{\partial}{\partial\sigma_{rs}}\right)$ is a matrix of differential operators with Kronecker's delta δ_{rs}, $E_{rs}=(\partial/\partial\sigma_{rs})\Sigma$, $\Lambda=\mathrm{diag}(\lambda_1, \lambda_2,\cdots, \lambda_p)$ is a diagonal matrix such that $\lambda_r\neq 1$ for all r, $F_{rs}=(I-\Lambda)^{-1}E_{rs}$, and $V=\Lambda(I-\Lambda)^{-1}$.

(1) By using the formulas $(\partial/\partial\sigma_{rs})|\Sigma|=|\Sigma|\mathrm{tr}\ \Sigma^{-1}E_{rs}$ and $(\partial/\partial\sigma_{rs})\Sigma^{-1}=-\Sigma^{-1}E_{rs}\Sigma^{-1}$, we easily obtain the following reductions:

(A.1) $$\mathrm{tr}(\Lambda\partial)^2|I-\Sigma|^{-b}\Big|_{\Sigma=\Lambda}=\frac{1}{4}\sum_r\sum_s(1+\delta_{rs})^2\lambda_r\lambda_s(\partial^2/\partial\sigma_{rs}^2)|I-\Sigma|^{-b}\Big|_{\Sigma=\Lambda}$$

$$=\frac{b}{4}|I-\Lambda|^{-b}\sum_r\sum_s(1+\delta_{rs})^2\lambda_r\lambda_s\{b(\mathrm{tr}\ F_{rs})^2+\mathrm{tr}\ F_{rs}^2\},$$

(A.2) $$\mathrm{tr}(\Lambda\partial)^3|I-\Sigma|^{-b}\Big|_{\Sigma=\Lambda}=\frac{1}{8}\sum_r\sum_s\sum_t(1+\delta_{rs})(1+\delta_{st})(1+\delta_{tr})\lambda_r\lambda_s\lambda_t$$

$$\cdot(\partial^3/\partial\sigma_{rs}\partial\sigma_{st}\partial\sigma_{tr})|I-\Sigma|^{-b}\Big|_{\Sigma=\Lambda}$$

$$=\frac{b}{8}|I-\Lambda|^{-b}\sum_r\sum_s\sum_t(1+\delta_{rs})(1+\delta_{st})(1+\delta_{tr})\lambda_r\lambda_s\lambda_t\{b^2(\mathrm{tr}\ F_{rs})$$

$$\cdot(\mathrm{tr}\ F_{st})\mathrm{tr}\ F_{tr}+3b(\mathrm{tr}\ F_{tr})\mathrm{tr}\ F_{ts}F_{sr}+2\mathrm{tr}\ F_{rs}F_{st}F_{tr}\},$$

(A.3) $$\{\mathrm{tr}(\Lambda\partial)^2\}^2|I-\Sigma|^{-b}\Big|_{\Sigma=\Lambda}=\frac{1}{16}\sum_r\sum_s\sum_t\sum_u(1+\delta_{rs})^2(1+\delta_{tu})^2$$

$$\cdot\lambda_r\lambda_s\lambda_t\lambda_u(\partial^4/\partial\sigma_{rs}^2\partial\sigma_{tu}^2)|I-\Sigma|^{-b}\Big|_{\Sigma=\Lambda}$$

$$=\frac{b}{16}|I-\Sigma|^{-b}\sum_r\sum_s\sum_t\sum_u(1+\delta_{rs})^2(1+\delta_{tu})^2\lambda_r\lambda_s\lambda_t\lambda_u[b\{b(\mathrm{tr}\ F_{rs})^2$$

$$+\mathrm{tr}\ F_{rs}^2\}\{b(\mathrm{tr}\ F_{tu})^2+\mathrm{tr}\ F_{tu}^2\}+4b^2(\mathrm{tr}\ F_{rs})(\mathrm{tr}\ F_{tu})\mathrm{tr}\ F_{rs}F_{tu}$$

$$+4b(\mathrm{tr}\ F_{rs})\mathrm{tr}\ F_{rs}F_{tu}^2+4b(\mathrm{tr}\ F_{tu})\mathrm{tr}\ F_{tu}F_{rs}^2+2b(\mathrm{tr}\ F_{rs}F_{tu})^2$$

$$+2\mathrm{tr}(F_{rs}F_{tu})^2+4\mathrm{tr}\ F_{rs}^2F_{tu}^2],$$

(A.4) $$\mathrm{tr}(\Lambda\partial)^2|I-\Sigma|^{-b}[\{\mathrm{tr}\ \Sigma(I-\Sigma)^{-1}\}^2+(2b+1)\mathrm{tr}\{\Sigma(I-\Sigma)^{-1}\}^2]\Big|_{\Sigma=\Lambda}$$

$$=\mathrm{tr}(\Lambda\partial)^2|I-\Sigma|^{-b}\Big|_{\Sigma=\Lambda}\cdot\{(\mathrm{tr}\ V)^2+(2b+1)\mathrm{tr}\ V^2\}+\frac{1}{2}\sum_r\sum_s(1+\delta_{rs})^2$$

$$\cdot\lambda_r\lambda_s(\partial/\partial\sigma_{rs})|I-\Sigma|^{-b}(\partial/\partial\sigma_{rs})[\{\mathrm{tr}\ \Sigma(I-\Sigma)^{-1}\}^2+(2b+1)\mathrm{tr}\{\Sigma$$

$$\cdot(I-\Sigma)^{-1}\}^2]\Big|_{\Sigma=\Lambda}+|I-\Lambda|^{-b}\mathrm{tr}(\Lambda\partial)^2[\{\mathrm{tr}\ \Sigma(I-\Sigma)^{-1}\}^2$$

$$+(2b+1)\,\mathrm{tr}\{\Sigma(I-\Sigma)^{-1}\}^2]\Big|_{\Sigma=\Lambda}.$$

(2) The various summations appeared in (A.1)～(A.4) are simplified as follows:

(A.5) $$\sum_r\sum_s(1+\delta_{rs})^2\lambda_r\lambda_s(\mathrm{tr}\ F_{rs})^2=4\sum_r\Big(\frac{\lambda_r}{1-\lambda_r}\Big)^2=4\mathrm{tr}\ V^2,$$

(A.6) $$\sum_r\sum_s(1+\delta_{rs})^2\lambda_r\lambda_s\mathrm{tr}\ F_{rs}^2=4\sum_r\Big(\frac{\lambda_r}{1-\lambda_r}\Big)^2+2\sum_{1\le r<s\le p}\sum\frac{2\lambda_r\lambda_s}{(1-\lambda_r)(1-\lambda_s)}$$
$$=2\{(\mathrm{tr}\ V)^2+\mathrm{tr}\ V^2\},$$

(A.7) $$\sum_r\sum_s\sum_t(1+\delta_{rs})(1+\delta_{st})(1+\delta_{tr})\lambda_r\lambda_s\lambda_t(\mathrm{tr}\ F_{rs})(\mathrm{tr}\ F_{st})\mathrm{tr}\ F_{tr}$$
$$=8\sum_r\Big(\frac{\lambda_r}{1-\lambda_r}\Big)^3=8\mathrm{tr}\ V^3,$$

(A.8) $$\sum_r\sum_s\sum_t(1+\delta_{rs})(1+\delta_{st})(1+\delta_{tr})\lambda_r\lambda_s\lambda_t(\mathrm{tr}\ F_{rs})\mathrm{tr}\ F_{st}F_{tr}$$
$$=8\sum_r\Big(\frac{\lambda_r}{1-\lambda_r}\Big)^3+4\sum_{r\neq s}\sum\frac{\lambda_r^2\lambda_s}{(1-\lambda_r)^2(1-\lambda_s)}=4\{(\mathrm{tr}\ V)\mathrm{tr}\ V^2+\mathrm{tr}\ V^3\},$$

(A.9) $$\sum_r\sum_s\sum_t(1+\delta_{rs})(1+\delta_{st})(1+\delta_{tr})\lambda_r\lambda_s\lambda_t\mathrm{tr}\ F_{rs}F_{st}F_{tr}=8\sum_r\Big(\frac{\lambda_r}{1-\lambda_r}\Big)^3$$
$$+6\sum_{r\neq s}\sum\frac{\lambda_r^2\lambda_s}{(1-\lambda_r)^2(1-\lambda_s)}+\sum_{r\neq s}\sum_{s\neq t}\sum_{t\neq r}\frac{\lambda_r\lambda_s\lambda_t}{(1-\lambda_r)(1-\lambda_s)(1-\lambda_t)}$$
$$=(\mathrm{tr}\ V)^3+3(\mathrm{tr}\ V)\mathrm{tr}\ V^2+4\mathrm{tr}\ V^3,$$

(A.10) $$\sum_r\sum_s\sum_t\sum_u(1+\delta_{rs})^2(1+\delta_{tu})^2\{b(\mathrm{tr}\ F_{rs})^2+\mathrm{tr}\ F_{rs}^2\}\{b(\mathrm{tr}\ F_{tu})^2$$
$$+\mathrm{tr}\ F_{tu}^2\}=[\sum_r\sum_s(1+\delta_{rs})^2\lambda_r\lambda_s\{b(\mathrm{tr}\ F_{rs})^2+\mathrm{tr}\ F_{rs}^2\}]^2$$
$$=4\{(\mathrm{tr}\ V)^2+(2b+1)\mathrm{tr}\ V^2\}^2\qquad\text{(by using (A.5) and (A.6))},$$

(A.11) $$\sum_r\sum_s\sum_t\sum_u(1+\delta_{rs})^2(1+\delta_{tu})^2\lambda_r\lambda_s\lambda_t\lambda_u(\mathrm{tr}\ F_{rs})(\mathrm{tr}\ F_{tu})\mathrm{tr}\ F_{rs}F_{tu}$$
$$=16\sum_r\sum_s\frac{\lambda_r^2\lambda_s^2}{(1-\lambda_r)(1-\lambda_s)}\mathrm{tr}\ F_{rr}F_{ss}=16\sum_r\Big(\frac{\lambda_r}{1-\lambda_r}\Big)^4=16\mathrm{tr}\ V^4,$$

(A.12) $$\sum_r\sum_s\sum_t\sum_u(1+\delta_{rs})^2(1+\delta_{tu})^2\lambda_r\lambda_s\lambda_t\lambda_u(\mathrm{tr}\ F_{rs})\mathrm{tr}\ F_{rs}F_{tu}^2$$
$$=\sum_r\sum_s\sum_t\sum_u(1+\delta_{rs})^2(1+\delta_{tu})^2\lambda_r\lambda_s\lambda_t\lambda_u(\mathrm{tr}\ F_{tu})\mathrm{tr}\ F_{tu}F_{rs}^2$$

$$=4\sum_r\sum_t\sum_u(1+\delta_{tu})^2\lambda_r^2\lambda_t\lambda_u(1-\lambda_r)^{-1}\mathrm{tr}\ F_{rr}F_{tu}^2$$

$$=16\sum_r\left(\frac{\lambda_r}{1-\lambda_r}\right)^4+8\sum_{1\leq t<u\leq p}\sum\frac{\lambda_t\lambda_u}{(1-\lambda_t)(1-\lambda_u)}\left\{\left(\frac{\lambda_t}{1-\lambda_t}\right)^2+\left(\frac{\lambda_u}{1-\lambda_u}\right)^2\right\}$$

$$=8\{(\mathrm{tr}\ V)\mathrm{tr}\ V^3+\mathrm{tr}\ V^4\},$$

(A.13) $$\sum_r\sum_s\sum_t\sum_u(1+\delta_{rs})^2(1+\delta_{tu})^2\lambda_r\lambda_s\lambda_t\lambda_u(\mathrm{tr}\ F_{rs}F_{tu})^2$$

$$=16\sum_r\left(\frac{\lambda_r}{1-\lambda_r}\right)^4+16\sum_{1\leq r<s\leq p}\sum\frac{\lambda_r^2\lambda_s^2}{(1-\lambda_r)^2(1-\lambda_s)^2}=8\{(\mathrm{tr}\ V^2)^2+\mathrm{tr}\ V^4\}.$$

Noting that $\mathrm{tr}(E_{rr}E_{tu})^2=0\ (t\neq u)$ and $\mathrm{tr}(E_{rs}E_{tu})^2=2\delta_{rt}\delta_{su}(r<s,\ t<u)$, we have

(A.14) $$\sum_r\sum_s\sum_t\sum_u(1+\delta_{rs})^2(1+\delta_{tu})^2\lambda_r\lambda_s\lambda_t\lambda_u\mathrm{tr}(F_{rs}F_{tu})^2$$

$$=16\sum_r\left(\frac{\lambda_r}{1-\lambda_r}\right)^4+4\sum_{1\leq r<s\leq p}\sum\ \sum_{1\leq t<u\leq p}\sum\frac{\lambda_r\lambda_s\lambda_t\lambda_u}{(1-\lambda_r)^2(1-\lambda_s)^2}\mathrm{tr}(E_{rs}E_{tu})^2$$

$$=4\{(\mathrm{tr}\ V^2)^2+3\mathrm{tr}\ V^4\}.$$

Noting that $\mathrm{tr}\ E_{rr}^2E_{tu}^2=\delta_{rt}+\delta_{ru}\ (t\neq u)$ and $\mathrm{tr}\ E_{rs}^2E_{tu}^2=\delta_{rt}+\delta_{ru}+\delta_{st}+\delta_{su}\ (r\neq s,\ t\neq u)$, we have

(A.15) $$\sum_r\sum_s\sum_t\sum_u(1+\delta_{rs})^2(1+\delta_{tu})^2\lambda_r\lambda_s\lambda_t\lambda_u\mathrm{tr}\ F_{rs}^2F_{tu}^2$$

$$=16\sum_r\left(\frac{\lambda_r}{1-\lambda_r}\right)^4+8\sum_r\sum_{t\neq u}\sum\frac{\lambda_r^2\lambda_t\lambda_u}{(1-\lambda_r)^2(1-\lambda_t)(1-\lambda_u)}\ \mathrm{tr}\ E_{rr}^2E_{tu}^2$$

$$+\sum_{r\neq s}\sum\sum_{t\neq u}\sum\frac{\lambda_r\lambda_s\lambda_t\lambda_u}{(1-\lambda_r)(1-\lambda_s)(1-\lambda_t)(1-\lambda_u)}\mathrm{tr}\ E_{rs}^2E_{tu}^2$$

$$=4\{(\mathrm{tr}\ V)^2\mathrm{tr}\ V^2+2(\mathrm{tr}\ V)\mathrm{tr}\ V^3+\mathrm{tr}\ V^4\}.$$

(A.16) $$\sum_r\sum_s(1+\delta_{rs})^2\lambda_r\lambda_s(\partial/\partial\sigma_{rs})|I-\Sigma|^{-b}(\partial/\partial\sigma_{rs})\{\mathrm{tr}\ \Sigma(I-\Sigma)^{-1}\}^2\Big|_{\Sigma=\Lambda}$$

$$=2b|I-\Lambda|^{-b}(\mathrm{tr}\ V)\sum_r\sum_s(1+\delta_{rs})^2\lambda_r\lambda_s(\mathrm{tr}\ F_{rs})\mathrm{tr}(I-\Lambda)^{-1}F_{rs}$$

$$=8b|I-\Lambda|^{-b}(\mathrm{tr}\ V)\sum_r\lambda_r^2(1-\lambda_r)^{-3}=8b|I-\Lambda|^{-b}(\mathrm{tr}\ V)(\mathrm{tr}\ V^2+\mathrm{tr}\ V^3),$$

(A.17) $$\mathrm{tr}(\Lambda\partial)^2\{\mathrm{tr}\ \Sigma(I-\Sigma)^{-1}\}^2\Big|_{\Sigma=\Lambda}=\frac{1}{4}\sum_r\sum_s(1+\delta_{rs})^2\lambda_r\lambda_s(\partial^2/\partial\sigma_{rs}^2)$$

$$\cdot\{\mathrm{tr}\ \Sigma(I-\Sigma)^{-1}\}^2\Big|_{\Sigma=\Lambda}$$

$$=\frac{1}{2}\sum_r\sum_s(1+\delta_{rs})^2\lambda_r\lambda_s[\{\mathrm{tr}(I-\Lambda)^{-1}F_{rs}\}^2+2(\mathrm{tr}\ V)\mathrm{tr}(I-\Lambda)^{-1}F_{rs}^2]$$

$$=2\sum_r \lambda_r^2\{(1-\lambda_r)^{-4}+2(\mathrm{tr}\ V)(1-\lambda_r)^{-3}\}+2(\mathrm{tr}\ V)\sum_{1\leq r<s\leq p}\sum \frac{\lambda_r\lambda_s}{(1-\lambda_r)(1-\lambda_s)}$$

$$\cdot\left\{\frac{1}{(1-\lambda_r)}+\frac{1}{(1-\lambda_s)}\right\}=2\{\mathrm{tr}\ V^2+(\mathrm{tr}\ V)^3+(\mathrm{tr}\ V)\mathrm{tr}\ V^2+2\mathrm{tr}\ V^3$$

$$+(\mathrm{tr}\ V)^2\mathrm{tr}\ V^2+(\mathrm{tr}\ V)\mathrm{tr}\ V^3+\mathrm{tr}\ V^4\},$$

(A.18) $$\sum_r\sum_s(1+\delta_{rs})^2\lambda_r\lambda_s(\partial/\partial\sigma_{rs})|I-\Sigma|^{-b}(\partial/\partial\sigma_{rs})\mathrm{tr}\ \{\Sigma(I-\Sigma)^{-1}\}^2\Big|_{\Sigma=\Lambda}$$

$$=2b|I-\Lambda|^{-b}\sum_r\sum_s(1+\delta_{rs})^2\lambda_r\lambda_s(\mathrm{tr}\ F_{rs})\{\mathrm{tr}\ VF_{rs}+\mathrm{tr}\ V^2F_{rs}\}$$

$$=8b|I-\Lambda|^{-b}\sum_r\lambda_r^3(1-\lambda_r)^{-4}=8b|I-\Lambda|^{-b}(\mathrm{tr}\ V^3+\mathrm{tr}\ V^4),$$

(A.19) $$\mathrm{tr}(\Lambda\partial)^2\mathrm{tr}\{\Sigma(I-\Sigma)^{-1}\}^2\Big|_{\Sigma=\Lambda}=\frac{1}{4}\sum_r\sum_s(1+\delta_{rs})^2\lambda_r\lambda_s(\partial^2/\partial\sigma_{rs}^2)$$

$$\cdot\mathrm{tr}\{\Sigma(I-\Sigma)^{-1}\}^2\Big|_{\Sigma=\Lambda}$$

$$=\frac{1}{2}\sum_r\sum_s(1+\delta_{rs})^2\{\mathrm{tr}\ F_{rs}^2+\mathrm{tr}(VF_{rs})^2+4\mathrm{tr}\ VF_{rs}^2+2\mathrm{tr}\ V^2F_{rs}^2\}$$

$$=2\sum_r\frac{\lambda_r^2(1+2\lambda_r)}{(1-\lambda_r)^4}+2\sum_{1\leq r<s\leq p}\sum\frac{\lambda_r\lambda_s}{(1-\lambda_r)^3(1-\lambda_s)^3}\{1-3\lambda_r\lambda_s+\lambda_r^2\lambda_s+\lambda_r\lambda_s^2\}$$

$$=(\mathrm{tr}\ V)^2+\mathrm{tr}\ V^2+4(\mathrm{tr}\ V)\mathrm{tr}\ V^2+4\mathrm{tr}\ V^3+(\mathrm{tr}\ V^2)^2$$

$$+2(\mathrm{tr}\ V)\mathrm{tr}\ V^3+3\mathrm{tr}\ V^4.$$

(3) By inserting these results (A.5)∼(A.19) into the last expressions of the results (A.1)∼(A.4), we obtain the following desired formulas:

(A.20) $$\mathrm{tr}(\Lambda\partial)^2|I-\Sigma|^{-b}\Big|_{\Sigma=\Lambda}=\frac{b}{2}\{(\mathrm{tr}\ V)^2+(2b+1)\mathrm{tr}\ V^2\}|I-\Lambda|^{-b},$$

(A.21) $$\mathrm{tr}(\Lambda\partial)^3|I-\Sigma|^{-b}\Big|_{\Sigma=\Lambda}=\frac{b}{4}\{(\mathrm{tr}\ V)^3+3(2b+1)(\mathrm{tr}\ V)\mathrm{tr}\ V^2$$

$$+2(2b^2+3b+2)\mathrm{tr}\ V^3\}|I-\Lambda|^{-b},$$

(A.22) $$\{\mathrm{tr}(\Lambda\partial)^2\}^2|I-\Sigma|^{-b}\Big|_{\Sigma=\Lambda}=\frac{b}{4}\{b(\mathrm{tr}\ V)^4+2(2b^2+b+2)(\mathrm{tr}\ V)^2\mathrm{tr}\ V^2$$

$$+(2b+1)(2b^2+b+2)(\mathrm{tr}\ V^2)^2+8(2b+1)(\mathrm{tr}\ V)\mathrm{tr}\ V^3$$

$$+2(8b^2+10b+5)\mathrm{tr}\ V^4\}|I-\Lambda|^{-b},$$

(A.23) $$\mathrm{tr}(\Lambda\partial)^2|I-\Sigma|^{-b}[\{\mathrm{tr}\ \Sigma(I-\Sigma)^{-1}\}^2+(2b+1)\mathrm{tr}\{\Sigma(I-\Sigma)^{-1}\}^2]\Big|_{\Sigma=\Lambda}$$

$$=\frac{1}{2}\{2(2b+1)(\mathrm{tr}\ V)^2+2(2b+3)\mathrm{tr}\ V^2+4(\mathrm{tr}\ V)^3+12(2b+1)(\mathrm{tr}\ V)$$

$$\cdot\mathrm{tr}\ V^2+8(2b^2+3b+2)\mathrm{tr}\ V^3+b(\mathrm{tr}\ V)^4+2(2b^2+b+2)(\mathrm{tr}\ V)^2$$

$$\cdot\mathrm{tr}\ V^2+(2b+1)(2b^2+b+2)(\mathrm{tr}\ V^2)^2+8(2b+1)(\mathrm{tr}\ V)\mathrm{tr}\ V^3$$

$$+2(8b^2+10b+5)\mathrm{tr}\ V^4\}\,|I-\Lambda|^{-b}.$$

APPENDIX II. *Calculations of* tr $\partial^2|T-\phi\Omega^{\frac{1}{2}}\Sigma\Omega^{\frac{1}{2}}|^{-\frac{q}{2}}\Big|_{\Sigma=I}$, *etc.*

In this appendix we evaluate the values of tr $\partial^2|g(\Sigma)|^{-\frac{q}{2}}\Big|_{\Sigma=I}$, tr $\partial^3|g(\Sigma)|^{-\frac{q}{2}}\Big|_{\Sigma=I}$, $(\mathrm{tr}\partial^2)^2|g(\Sigma)|^{-\frac{q}{2}}\Big|_{\Sigma=I}$ and $\mathrm{tr}\partial^2|g(\Sigma)|^{-\frac{q}{2}}U_1(T,\Sigma)\Big|_{\Sigma=I}$, where $g(\Sigma)=T-\phi\Omega^{\frac{1}{2}}\Sigma\Omega^{\frac{1}{2}}$ and $U_1(T,\Sigma)$ is defined by (5.13). They are useful for the derivation of asymptotic expansions of the non-null distributions of the Pillai's criteria for multivariate linear hypothesis and independence. The following notations are used:

$$\partial=\left(\frac{1}{2}(1+\delta_{rs})\frac{\partial}{\partial\sigma_{rs}}\right)=\left(\frac{\partial}{\partial\sigma^*_{rs}}\right),\quad \frac{\partial}{\partial\sigma^*_{rs}}\Sigma=E^*_{rs}\ \text{and}$$

$$R=\Omega^{\frac{1}{2}}(T-\phi\Omega)^{-1}\Omega^{\frac{1}{2}}\ .$$

(1) The following reductions are easily obtained by using the fomulas $(\partial/\partial\sigma^*_{rs})|\Sigma|=|\Sigma|\mathrm{tr}\ \Sigma^{-1}E^*_{rs}$ and $(\partial/\partial\sigma^*_{rs})\Sigma^{-1}=-\Sigma^{-1}E^*_{rs}\Sigma^{-1}$.

(A.24) $$\mathrm{tr}\ \partial^2|g(\Sigma)|^{-\frac{q}{2}}\Big|_{\Sigma=I}=\sum_r\sum_s(\partial^2/\partial\sigma^*_{rs}\partial\sigma^*_{sr})|g(\Sigma)|^{-\frac{q}{2}}\Big|_{\Sigma=I}$$

$$=\frac{q}{2}\phi^2|T-\phi\Omega|^{-\frac{q}{2}}\sum_r\sum_s\left\{\frac{q}{2}(\mathrm{tr}\ RE^*_{rs})^2+\mathrm{tr}(RE^*_{rs})^2\right\},$$

(A.25) $$\mathrm{tr}\ \partial^3|g(\Sigma)|^{-\frac{q}{2}}\Big|_{\Sigma=I}=\sum_r\sum_s\sum_t(\partial^3/\partial\sigma^*_{rs}\partial\sigma^*_{st}\partial\sigma^*_{tr})|g(\Sigma)|^{-\frac{q}{2}}\Big|_{\Sigma=I}$$

$$=\frac{q}{2}\phi^3|T-\phi\Omega|^{-\frac{q}{2}}\sum_r\sum_s\sum_t\left\{\frac{q^2}{4}(\mathrm{tr}\ RE^*_{rs})(\mathrm{tr}\ RE^*_{st})\mathrm{tr}\ RE^*_{tr}\right.$$

$$\left.+\frac{3}{2}q(\mathrm{tr}\ RE^*_{rs})\mathrm{tr}\ RE^*_{rt}RE^*_{ts}+2\mathrm{tr}\ RE^*_{rs}RE^*_{st}RE^*_{tr}\right\},$$

(A.26) $$(\mathrm{tr}\ \partial^2)^2|g(\Sigma)|^{-\frac{q}{2}}\Big|_{\Sigma=I}=\sum_r\sum_s\sum_t\sum_u(\partial^4/\partial\sigma^{*2}_{rs}\partial\sigma^{*2}_{tu})|g(\Sigma)|^{-\frac{q}{2}}\Big|_{\Sigma=I}$$

$$=\frac{q}{2}\phi^4|T-\phi\Omega|^{-\frac{q}{2}}\Big|_{\Sigma=I}\sum_r\sum_s\sum_t\sum_u\left[\frac{q}{2}\left\{\frac{q}{2}(\mathrm{tr}\ RE^*_{rs})^2+\mathrm{tr}(RE^*_{rs})^2\right\}\right.$$

$$\cdot\left\{\frac{q}{2}(\mathrm{tr}\ RE^*_{tu})^2+\mathrm{tr}(RE^*_{tu})^2\right\}+q^2(\mathrm{tr}\ RE^*_{rs})(\mathrm{tr}\ RE^*_{tu})\mathrm{tr}\ RE^*_{rs}RE^*_{tu}$$

$$+4q(\mathrm{tr}\ RE^*_{rs})\mathrm{tr}\ RE^*_{rs}(RE^*_{tu})^2+q(\mathrm{tr}\ RE^*_{rs}RE^*_{tu})^2$$

$$\left.+4\mathrm{tr}(RE^*_{rs})^2(RE^*_{tu})^2+2\mathrm{tr}(RE^*_{rs}RE^*_{tu})^2\right].$$

(2) Let A and B be any $p\times p$ symmetric matrices. Then we have the following identities:

(A.27) $\sum_r\sum_s(\mathrm{tr}\ AE^*_{rs})\mathrm{tr}\ BE^*_{rs}=\mathrm{tr}\ AB,$

(A.28) $\sum_r\sum_s\mathrm{tr}\ AE^*_{rs}BE^*_{rs}=\frac{1}{2}\{(\mathrm{tr}\ A)\mathrm{tr}\ B+\mathrm{tr}\ AB\},$

(A.29) $\sum_r\sum_s\sum_t(\mathrm{tr}\ AF^*_{rs})(\mathrm{tr}\ AE^*_{st})\mathrm{tr}\ AE^*_{tr}=\mathrm{tr}\ A^3,$

(A.30) $\sum_r\sum_s\sum_t(\mathrm{tr}\ AE^*_{rs})\mathrm{tr}\ AE^*_{rt}AE^*_{ts}=\frac{1}{2}\{(\mathrm{tr}\ A)\mathrm{tr}A^2+\mathrm{tr}\ A^3\},$

(A.31) $\sum_r\sum_s\sum_t\mathrm{tr}\ AE^*_{rs}AE^*_{st}AE^*_{tr}=\frac{1}{8}\{(\mathrm{tr}\ A)^3+3(\mathrm{tr}\ A)\mathrm{tr}\ A^2+4\mathrm{tr}\ A^3\},$

(A.32) $\sum_r\sum_s\sum_t\sum_u(\mathrm{tr}\ AE^*_{rs})(\mathrm{tr}\ AE^*_{tu})\mathrm{tr}\ AE^*_{rs}AE^*_{tu}=\mathrm{tr}\ A^4,$

(A.33) $\sum_r\sum_s\sum_t\sum_u(\mathrm{tr}\ AE^*_{rs})\mathrm{tr}\ AE^*_{rs}(AE^*_{tu})^2=\frac{1}{2}\{(\mathrm{tr}\ A)\mathrm{tr}\ A^3+\mathrm{tr}\ A^4\},$

(A.34) $\sum_r\sum_s\sum_t\sum_u(\mathrm{tr}\ AE^*_{rs}AE^*_{tu})^2=\frac{1}{2}\{(\mathrm{tr}\ A^2)^2+\mathrm{tr}\ A^4\},$

(A.35) $\sum_r\sum_s\sum_t\sum_u\mathrm{tr}(AE^*_{rs})^2(AE^*_{tu})^2=\frac{1}{4}\{(\mathrm{tr}\ A)^2\mathrm{tr}\ A^2+2(\mathrm{tr}\ A)\mathrm{tr}\ A^3+\mathrm{tr}\ A^4\},$

(A.36) $\sum_r\sum_s\sum_t\sum_u\mathrm{tr}(AE^*_{rs}AE^*_{tu})^2=\frac{1}{4}\{(\mathrm{tr}\ A^2)^2+3\mathrm{tr}\ A^4\}.$

(3) Applying the above results (A.27)$\sim$(A.36) to the last expressions of (A.24), (A.25) and (A.26), we obtain the following formulas:

(A.37) $\mathrm{tr}\ \partial^2|g(\Sigma)|^{-\frac{q}{2}}\Big|_{\Sigma=I}=\frac{q}{4}\phi^2|T-\phi\Omega|^{-\frac{q}{2}}\{(\mathrm{tr}\ R)^2+(q+1)\mathrm{tr}\ R^2\},$

(A.38) $\mathrm{tr}\ \partial^3|g(\Sigma)|^{-\frac{q}{2}}\Big|_{\Sigma=I}=\frac{q}{8}\phi^3|T-\phi\Omega|^{-\frac{q}{2}}\{(\mathrm{tr}\ R)^3+3(q+1)(\mathrm{tr}\ R)\mathrm{tr}\ R^2$

$+(q^2+3q+4)\mathrm{tr}\ R^3\},$

(A.39) $(\mathrm{tr}\ \partial^2)^2|g(\Sigma)|^{-\frac{q}{2}}\Big|_{\Sigma=I}=\frac{q}{16}\phi^4|T-\phi\Omega|^{-\frac{q}{2}}\{q(\mathrm{tr}\ R)^4+2(q^2+q+4)(\mathrm{tr}\ R)^2$

$\cdot\mathrm{tr}\ R^2+(q+1)(q^2+q+4)(\mathrm{tr}\ R^2)^2+16(q+1)(\mathrm{tr}\ R)\mathrm{tr}\ R^3$

$$+4(2q^2+5q+5)\mathrm{tr}\ R^4\}.$$

(4) Finally we evaluate the values of $\mathrm{tr}\ \partial^2|g(\Sigma)|^{-\frac{q}{2}}U_1(\Sigma, T)\Big|_{\Sigma=I}$ which is expressed as follows:

$$\text{(A.40)}\quad \mathrm{tr}\ \partial^2|g(\Sigma)|^{-\frac{q}{2}}\Big|_{\Sigma=I}\cdot U_1(I, T)+2\sum_r\sum_s(\partial/\partial\sigma_{rs}^*)|g(\Sigma)|^{-\frac{q}{2}}(\partial/\partial\sigma_{rs}^*)\{p(p+q+1)-2\phi(p+q+1)\mathrm{tr}\ Tg(\Sigma)^{-1}+\phi^2(\mathrm{tr}\ Tg(\Sigma)^{-1})^2+\phi^2(q+1)\cdot\mathrm{tr}(Tg(\Sigma)^{-1})^2\}\Big|_{\Sigma=I}+|T-\phi\Omega|^{-\frac{q}{2}}\mathrm{tr}\ \partial^2\{p(p+q+1)-2\phi(p+q+1)\cdot\mathrm{tr}\ Tg(\Sigma)^{-1}+\phi^2(\mathrm{tr}\ Tg(\Sigma)^{-1})^2+\phi^2(q+1)\mathrm{tr}(Tg(\Sigma)^{-1})^2\}\Big|_{\Sigma=I}$$

To carry out the operations ∂ appeared in the right hand side of (A.40), we calculate more general formulas including symmetric matrix Θ, which will be used in Section 8. Put $Z=T-\phi\Omega$, then we obtain the following reductions by the same method as in the derivation of the formulas (A.37), (A.38) and (A.39).

$$\text{(A.41)}\quad 2\sum_r\sum_s(\partial/\partial\sigma_{rs}^*)|g(\Sigma)|^{-\frac{q}{2}}(\partial/\partial\sigma_{rs}^*)\mathrm{tr}\ \Theta g(\Sigma)^{-1}\Big|_{\Sigma=I}$$

$$=q\phi^2|Z|^{-\frac{q}{2}}\sum_r\sum_s(\mathrm{tr}\ \Omega^{\frac{1}{2}}Z^{-1}\Omega^{\frac{1}{2}}E_{rs}^*)\mathrm{tr}\ \Omega^{\frac{1}{2}}Z^{-1}\Theta Z^{-1}\Omega^{\frac{1}{2}}E_{rs}^*$$

$$=q\phi^2|Z|^{-\frac{q}{2}}\mathrm{tr}\ (\Omega Z^{-1})^2\Theta Z^{-1},$$

$$\text{(A.42)}\quad \sum_r\sum_s(\partial/\partial\sigma_{rs}^*)|g(\Sigma)|^{-\frac{q}{2}}(\partial/\partial\sigma_{rs}^*)(\mathrm{tr}\ \Theta g(\Sigma)^{-1})^2\Big|_{\Sigma=I}$$

$$=q\phi^2|Z|^{-\frac{q}{2}}(\mathrm{tr}\ \Theta Z^{-1})\sum_r\sum_s(\mathrm{tr}\ \Omega^{\frac{1}{2}}Z^{-1}\Omega^{\frac{1}{2}}E_{rs}^*)\mathrm{tr}\ \Omega^{\frac{1}{2}}Z^{-1}\Theta Z^{-1}\Omega^{\frac{1}{2}}E_{rs}^*$$

$$=q\phi^2|Z|^{-\frac{q}{2}}(\mathrm{tr}\ \Theta Z^{-1})\mathrm{tr}(\Omega Z^{-1})^2\Theta Z^{-1},$$

$$\text{(A.43)}\quad \sum_r\sum_s(\partial/\partial\sigma_{rs}^*)|g(\Sigma)|^{-\frac{q}{2}}(\partial/\partial\sigma_{rs}^*)\mathrm{tr}(\Theta g(\Sigma)^{-1})^2\Big|_{\Sigma=I}$$

$$=q\phi^2|Z|^{-\frac{q}{2}}\sum_r\sum_s(\mathrm{tr}\ \Omega^{\frac{1}{2}}Z^{-1}\Omega^{\frac{1}{2}}E_{rs}^*)\mathrm{tr}\ \Omega^{\frac{1}{2}}Z^{-1}\Theta Z^{-1}\Theta Z^{-1}\Omega^{\frac{1}{2}}E_{rs}^*$$

$$=q\phi^2|Z|^{-\frac{q}{2}}\mathrm{tr}(\Omega Z^{-1})^2(\Theta Z^{-1})^2,$$

$$\text{(A.44)}\quad \mathrm{tr}\ \partial^2\mathrm{tr}\ \Theta g(\Sigma)^{-1}\Big|_{\Sigma=I}=\sum_r\sum_s(\partial/\partial\sigma_{rs}^*)^2\mathrm{tr}\ \Theta g(\Sigma)^{-1}\Big|_{\Sigma=I}$$

$$=2\phi^2\sum_r\sum_s \mathrm{tr}\ \Omega^{\frac{1}{2}}Z^{-1}\Theta Z^{-1}\Omega^{\frac{1}{2}}E^*_{rs}\Omega^{\frac{1}{2}}Z^{-1}\Omega^{\frac{1}{2}}E^*_{rs}$$

$$=\phi^2\{(\mathrm{tr}\ \Omega Z^{-1})\mathrm{tr}\ \Omega Z^{-1}\Theta Z^{-1}+\mathrm{tr}(\Omega Z^{-1})^2\Theta Z^{-1}\},$$

(A.45) $$\mathrm{tr}\ \partial^2(\mathrm{tr}\ \Theta g(\Sigma)^{-1})^2\Big|_{\Sigma=I}=\sum_r\sum_s(\partial/\partial\sigma^*_{rs})^2(\mathrm{tr}\ \Theta g(\Sigma)^{-1})^2\Big|_{\Sigma=I}$$

$$=2\phi^2\sum_r\sum_s\{(\mathrm{tr}\ \Omega^{\frac{1}{2}}Z^{-1}\Theta Z^{-1}\Omega^{\frac{1}{2}}E^*_{rs})^2+2(\mathrm{tr}\ \Theta Z^{-1})\mathrm{tr}\ \Omega^{\frac{1}{2}}Z^{-1}\Theta Z^{-1}\Omega^{\frac{1}{2}}$$

$$\cdot E^*_{rs}\Omega^{\frac{1}{2}}Z^{-1}\Omega^{\frac{1}{2}}E^*_{rs}\}=2\phi^2\{\mathrm{tr}(\Omega Z^{-1}\Theta Z^{-1})^2+(\mathrm{tr}\ \Omega Z^{-1})(\mathrm{tr}\ \Theta Z^{-1})$$

$$\cdot\mathrm{tr}\ \Omega Z^{-1}\Theta Z^{-1}+(\mathrm{tr}\ \Theta Z^{-1})\,\mathrm{tr}(\Omega Z^{-1})^2\Theta Z^{-1}\},$$

(A.46) $$\mathrm{tr}\ \partial^2\mathrm{tr}(\Theta g(\Sigma)^{-1})^2\Big|_{\Sigma=I}=\sum_r\sum_s(\partial/\partial\sigma^*_{rs})^2\mathrm{tr}(\Theta g(\Sigma)^{-1})^2\Big|_{\Sigma=I}$$

$$=2\phi^2\sum_r\sum_s\{\mathrm{tr}(\Omega^{\frac{1}{2}}Z^{-1}\Theta Z^{-1}\Omega^{\frac{1}{2}}E^*_{rs})^2+2\mathrm{tr}\ \Omega^{\frac{1}{2}}Z^{-1}\Theta Z^{-1}\Theta Z^{-1}\Omega^{\frac{1}{2}}E^*_{rs}\Omega^{\frac{1}{2}}$$

$$\cdot Z^{-1}\Omega^{\frac{1}{2}}E^*_{rs}\}=\phi^2\{(\mathrm{tr}\ \Omega Z^{-1}\Theta Z^{-1})^2+\mathrm{tr}\,(\Omega Z^{-1}\Theta Z^{-1})^2+2\,(\mathrm{tr}\ \Omega Z^{-1})$$

$$\cdot\mathrm{tr}\ \Omega Z^{-1}(\Theta Z^{-1})^2+2\mathrm{tr}(\Omega Z^{-1})^2(\Theta Z^{-1})^2\}.$$

Inserting the formula (A.37) and the identities obtained by putting $\Theta=T$ in the above equalities (A.41)～(A.46) to (A.40), we have the following formula:

(A.47) $$\mathrm{tr}\ \partial^2|g(\Sigma)|^{-\frac{q}{2}}U_1(\Sigma,T)\Big|_{\Sigma=I}=\frac{\phi^2}{4}|T-\phi\Omega|^{-\frac{q}{2}}[f_2(\mathrm{tr}\ R)^2+f_3\mathrm{tr}\ R^2$$

$$+f_4(\mathrm{tr}\ R)^3+f_5(\mathrm{tr}\ R)\mathrm{tr}\ R^2+f_6\mathrm{tr}\ R^3+\phi^4\{q(\mathrm{tr}\ R)^4+2(q^2$$

$$+q+4)(\mathrm{tr}\ R)^2\mathrm{tr}\ R^2+(q+1)(q^2+q+4)(\mathrm{tr}\ R^2)^2$$

$$+16\,(q+1)\,(\mathrm{tr}\ R)\mathrm{tr}\ R^3+4(2q^2+5q+5)\mathrm{tr}\ R^4\}],$$

where the coefficients $f_\alpha(\alpha=2,3,\cdots,6)$ are given by

(A.48) $$f_2=pq(p+q+1)-2\{qp^2+(q^2+q+4)p+4(q+1)\}\phi$$

$$+\{qp^2+(q^2+q+8)p+12\,(q+1)\}\phi^2,$$

$$f_3=pq(q+1)\,(p+q+1)-2\,(q+1)\{qp^2+(q^2+q+4)p+4\,(q+1)\}\phi$$

$$+\{q(q+1)p^2+(q+1)(q^2+q+8)p+4(2q^2+5q+5)\}\phi^2,$$

$$f_4=2\phi^2\{-q(p+q+1)+(pq+q^2+q+4)\phi\},$$

$$f_5=2\phi^2[-(q^2+q+4)(p+q+1)+\{(q^2+q+4)p+(q+1)(q^2+q+16)\}\phi],$$

$$f_6=8\phi^2[-(q+1)(p+q+1)+\{(q+1)p+2q^2+5q+5\}\phi].$$

APPENDIX III. *Tables of the upper* 5 *and* 1% *points of the Pillai's criterion*

TABLE A. *Upper* 5% *points of* tr $S_h\,(S_h+S_e)^{-1}$

(1) $n=25$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0. 332	0. 421	0. 498	0. 566	0. 628	0. 683	0. 734	0. 781	0. 824	0. 864	0. 902
3	0. 436	0. 563	0. 674	0. 774	0. 864	0. 946	1. 022	1. 092	1. 157	1. 217	1. 274
4	0. 532	0. 696	0. 841	0. 972	1. 091	1. 200	1. 300	1. 393	1. 480	1. 561	1. 636
5	0. 624	0. 824	1. 003	1. 164	1. 311	1. 447	1. 572	1. 688	1. 796	1. 897	1. 992
6	0. 712	0. 948	1. 160	1. 352	1. 527	1. 689	1. 839	1. 978	2. 108	2. 229	2. 34
7	0. 797	1. 069	1. 313	1. 535	1. 739	1. 927	2. 101	2. 264	2. 42	2. 56	2. 69

(2) $n=30$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0. 2826	0. 361	0. 430	0. 491	0. 547	0. 598	0. 645	0. 688	0. 729	0. 767	0. 803
3	0. 372	0. 483	0. 581	0. 670	0. 752	0. 827	0. 897	0. 962	1. 023	1. 080	1. 133
4	0. 455	0. 598	0. 726	0. 842	0. 949	1. 048	1. 140	1. 227	1. 307	1. 383	1. 455
5	0. 534	0. 708	0. 865	1. 009	1. 141	1. 264	1. 378	1. 486	1. 586	1. 681	1. 770
6	0. 610	0. 815	1. 001	1. 172	1. 329	1. 475	1. 612	1. 740	1. 861	1. 974	2. 081
7	0. 683	0. 920	1. 134	1. 331	1. 514	1. 684	1. 843	1. 992	2. 132	2. 264	2. 389

(3) $n=40$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0. 2179	0. 2808	0. 3367	0. 3875	0. 4342	0. 478	0. 518	0. 556	0. 592	0. 626	0. 659
3	0. 2872	0. 3760	0. 4558	0. 5291	0. 597	0. 661	0. 720	0. 776	0. 830	0. 880	0. 928
4	0. 3519	0. 4659	0. 569	0. 665	0. 754	0. 837	0. 915	0. 990	1. 060	1. 127	1. 190
5	0. 414	0. 552	0. 679	0. 796	0. 906	1. 009	1. 106	1. 198	1. 285	1. 368	1. 447
6	0. 473	0. 636	0. 786	0. 925	1. 055	1. 178	1. 293	1. 403	1. 507	1. 607	1. 701
7	0. 531	0. 718	0. 891	1. 051	1. 202	1. 344	1. 478	1. 605	1. 727	1. 842	1. 952

(4) $n=60$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0.1494	0.1943	0.2349	0.2724	0.3075	0.3407	0.3721	0.4020	0.4306	0.4579	0.4842
3	0.1974	0.2604	0.3182	0.3720	0.4226	0.4707	0.5164	0.5602	0.6021	0.6423	0.6810
4	0.2422	0.3230	0.3975	0.4673	0.5333	0.5961	0.6560	0.7134	0.7684	0.8214	0.8724
5	0.2851	0.3833	0.4744	0.5599	0.6411	0.7184	0.7923	0.8633	0.9314	0.9970	1.0603
6	0.3266	0.4420	0.5494	0.6506	0.7467	0.8385	0.9264	1.0108	1.0919	1.1701	1.2456
7	0.3670	0.4994	0.6230	0.7397	0.8508	0.9569	1.0587	1.1564	1.2506	1.3413	1.4289

(5) $n=80$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0.11362	0.14852	0.18033	0.2100	0.2380	0.2646	0.2901	0.3145	0.3381	0.3607	0.3827
3	0.15029	0.19918	0.2443	0.2867	0.3270	0.3655	0.4024	0.4380	0.4723	0.5055	0.5377
4	0.1846	0.2472	0.3053	0.3602	0.4126	0.4628	0.5110	0.5576	0.6026	0.6461	0.6884
5	0.2175	0.2934	0.3644	0.4317	0.4960	0.5577	0.6171	0.6746	0.7302	0.7840	0.8363
6	0.2493	0.3385	0.4221	0.5016	0.5777	0.6509	0.7215	0.7897	0.8559	0.9200	0.9823
7	0.2803	0.3826	0.4788	0.5704	0.6583	0.7428	0.8245	0.9035	0.9801	1.0544	1.1267

(6) $n=100$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0.09167	0.12018	0.14632	0.17083	0.19410	0.21634	0.2377	0.2583	0.2782	0.2976	0.3163
3	0.12134	0.16124	0.19826	0.23324	0.26664	0.29871	0.3296	0.3596	0.3886	0.4167	0.4441
4	0.14913	0.20014	0.24780	0.29305	0.33640	0.37815	0.4185	0.4576	0.4956	0.5325	0.5684
5	0.17575	0.23767	0.29581	0.35120	0.40438	0.45568	0.5053	0.5535	0.6004	0.6460	0.6904
6	0.20154	0.27423	0.34273	0.40814	0.47106	0.53185	0.5908	0.6480	0.7037	0.7579	0.8108
7	0.22669	0.31005	0.38881	0.46417	0.53676	0.60698	0.6751	0.7413	0.8057	0.8686	0.9299

(7) $n=130$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0.07108	0.09344	0.11405	0.13348	0.15202	0.16983	0.18703	0.20368	0.21986	0.23561	0.25096
3	0.09414	0.12540	0.15455	0.18224	0.20881	0.23444	0.25927	0.28340	0.30689	0.32980	0.35218
4	0.11575	0.15570	0.19320	0.22899	0.26342	0.29674	0.32909	0.36058	0.39128	0.42127	0.45059
5	0.13647	0.18494	0.23067	0.27443	0.31665	0.35757	0.39735	0.43612	0.47397	0.51097	0.54719
6	0.15655	0.21344	0.26729	0.31896	0.36888	0.41733	0.46449	0.51050	0.55545	0.59942	0.64248
7	0.17614	0.24136	0.30327	0.36277	0.42035	0.47629	0.53078	0.58398	0.63599	0.68690	0.73678

(8) $n=160$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0.05804	0.07643	0.09344	0.10953	0.12493	0.13977	0.15415	0.16812	0.18172	0.19500	0.20798
3	0.07689	0.10259	0.12663	0.14954	0.17159	0.19292	0.21365	0.23385	0.25358	0.27287	0.29176
4	0.09458	0.12740	0.15831	0.18790	0.21646	0.24417	0.27115	0.29749	0.32325	0.34847	0.37321
5	0.11153	0.15135	0.18903	0.22520	0.26019	0.29421	0.32738	0.35979	0.39152	0.42263	0.45315
6	0.12797	0.17470	0.21906	0.26175	0.30312	0.34338	0.38269	0.42113	0.45880	0.49575	0.53202
7	0.14402	0.19759	0.24857	0.29773	0.34542	0.39189	0.43730	0.48174	0.52531	0.56807	0.61007

(9) $n=200$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0.04663	0.06150	0.07529	0.08838	0.10094	0.11309	0.12488	0.13636	0.14758	0.15856	0.16932
3	0.06180	0.08257	0.10205	0.12067	0.13863	0.15606	0.17305	0.18964	0.20588	0.22181	0.23744
4	0.07603	0.10255	0.12759	0.15162	0.17488	0.19751	0.21960	0.24122	0.26241	0.28322	0.30367
5	0.08968	0.12185	0.15236	0.18173	0.21022	0.23798	0.26512	0.29171	0.31781	0.34345	0.36867
6	0.10292	0.14066	0.17658	0.21123	0.24490	0.27775	0.30990	0.34143	0.37239	0.40284	0.43280
7	0.11585	0.15910	0.20038	0.24028	0.27909	0.31700	0.35413	0.39056	0.42636	0.46158	0.49626

(10) $n=250$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0.03743	0.04943	0.06059	0.07120	0.08141	0.09129	0.10092	0.11031	0.11951	0.12853	0.13739
3	0.04962	0.06637	0.08212	0.09721	0.11179	0.12598	0.13983	0.15339	0.16669	0.17976	0.19261
4	0.06106	0.08245	0.10268	0.12215	0.14102	0.15943	0.17743	0.19509	0.21243	0.22949	0.24629
5	0.07204	0.09797	0.12262	0.14640	0.16952	0.19209	0.21420	0.23591	0.25726	0.27827	0.29899
6	0.08269	0.11311	0.14212	0.17018	0.19749	0.22419	0.25038	0.27611	0.30143	0.32637	0.35097
7	0.09309	0.12795	0.16129	0.19358	0.22506	0.25587	0.28610	0.31583	0.34511	0.37396	0.40242

(11) $n=350$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0.02684	0.03550	0.04357	0.05126	0.05869	0.06590	0.07293	0.07982	0.08658	0.09322	0.09976
3	0.03560	0.04767	0.05906	0.06999	0.08059	0.09092	0.10103	0.11096	0.12072	0.13033	0.13981
4	0.04381	0.05922	0.07385	0.08795	0.10166	0.11506	0.12819	0.14111	0.15383	0.16637	0.17875
5	0.05170	0.07038	0.08819	0.10542	0.12220	0.13862	0.15475	0.17063	0.18627	0.20171	0.21696
6	0.05935	0.08127	0.10223	0.12254	0.14236	0.16179	0.18089	0.19969	0.21825	0.23656	0.25467
7	0.06683	0.09194	0.11602	0.13940	0.16224	0.18465	0.20670	0.22842	0.24986	0.27104	0.29199

(12) $n=500$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0.01884	0.02495	0.03065	0.03610	0.04137	0.04649	0.05151	0.05642	0.06125	0.06601	0.07070
3	0.02500	0.03351	0.04155	0.04929	0.05680	0.06415	0.07134	0.07842	0.08540	0.09228	0.09907
4	0.03077	0.04163	0.05196	0.06194	0.07165	0.08117	0.09052	0.09972	0.10880	0.11777	0.12665
5	0.03632	0.04948	0.06206	0.07424	0.08613	0.09779	0.10927	0.12058	0.13174	0.14278	0.15371
6	0.04170	0.05714	0.07194	0.08630	0.10035	0.11414	0.12772	0.14111	0.15435	0.16745	0.18041
7	0.04695	0.06465	0.08165	0.09818	0.11436	0.13026	0.14594	0.16141	0.17671	0.19185	0.20684

TABLE B. *Upper* 1% *points of* tr $S_h(S_h+S_e)^{-1}$

(1) $n=25$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0.44	0.53	0.61	0.68	0.74	0.79	0.84	0.88	0.92	0.96	1.00
3	0.55	0.68	0.80	0.90	0.99	1.07	1.14	1.21	1.27	1.33	1.39
4	0.65	0.82	0.97	1.10	1.22	1.33	1.43	1.52	1.61	1.69	1.76
5	0.74	0.95	1.14	1.30	1.45	1.59	1.71	1.83	1.93	2.03	2.12
6	0.83	1.08	1.30	1.49	1.67	1.83	1.98	2.12	2.25	2.37	2.48
7	0.91	1.20	1.45	1.68	1.88	2.07	2.25	2.41	2.56	2.70	2.83

(2) $n=30$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0.38	0.46	0.53	0.59	0.65	0.70	0.74	0.79	0.83	0.86	0.90
3	0.47	0.59	0.69	0.78	0.87	0.94	1.01	1.08	1.14	1.19	1.24
4	0.56	0.71	0.84	0.97	1.07	1.17	1.27	1.35	1.43	1.51	1.58
5	0.64	0.83	0.99	1.14	1.27	1.40	1.51	1.62	1.72	1.81	1.90
6	0.72	0.94	1.13	1.31	1.47	1.62	1.75	1.88	2.00	2.11	2.22
7	0.79	1.04	1.26	1.47	1.66	1.83	1.99	2.14	2.28	2.41	2.53

(3) $n=40$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0.293	0.361	0.420	0.473	0.521	0.57	0.61	0.64	0.68	0.71	0.74
3	0.369	0.465	0.55	0.63	0.70	0.76	0.82	0.88	0.93	0.98	1.03
4	0.438	0.561	0.67	0.77	0.86	0.95	1.03	1.10	1.17	1.24	1.30
5	0.503	0.65	0.79	0.91	1.02	1.13	1.23	1.32	1.41	1.49	1.57
6	0.565	0.74	0.90	1.04	1.18	1.30	1.42	1.53	1.64	1.74	1.83
7	0.625	0.82	1.01	1.17	1.33	1.47	1.61	1.74	1.86	1.98	2.09

(4) $n=60$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0. 2037	0. 2530	0. 2967	0. 337	0. 374	0. 408	0. 441	0. 472	0. 501	0. 529	0. 555
3	0. 2568	0. 326	0. 388	0. 445	0. 498	0. 548	0. 596	0. 641	0. 684	0. 725	0. 764
4	0. 3057	0. 393	0. 473	0. 547	0. 616	0. 682	0. 744	0. 803	0. 860	0. 914	0. 966
5	0. 352	0. 458	0. 555	0. 646	0. 731	0. 811	0. 888	0. 96	1. 03	1. 098	1. 162
6	0. 396	0. 520	0. 635	0. 741	0. 842	0. 94	1. 03	1. 12	1. 20	1. 278	1. 354
7	0. 439	0. 581	0. 712	0. 835	0. 95	1. 06	1. 17	1. 27	1. 36	1. 455	1. 544

(5) $n=80$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0. 15595	0. 1945	0. 2292	0. 2611	0. 2910	0. 319	0. 346	0. 372	0. 396	0. 419	0. 442
3	0. 1968	0. 2505	0. 299	0. 345	0. 388	0. 428	0. 467	0. 504	0. 540	0. 574	0. 607
4	0. 2345	0. 303	0. 365	0. 424	0. 479	0. 532	0. 582	0. 631	0. 677	0. 722	0. 766
5	0. 2702	0. 353	0. 429	0. 500	0. 568	0. 633	0. 695	0. 754	0. 812	0. 867	0. 921
6	0. 304	0. 401	0. 490	0. 574	0. 654	0. 731	0. 804	0. 875	0. 943	1. 009	1. 07
7	0. 338	0. 448	0. 550	0. 647	0. 739	0. 827	0. 912	0. 993	1. 07	1. 150	1. 22

(6) $n=100$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0. 12631	0. 1580	0. 1867	0. 2132	0. 2382	0. 2619	0. 2846	0. 306	0. 327	0. 347	0. 367
3	0. 15952	0. 2035	0. 2437	0. 2814	0. 317	0. 351	0. 384	0. 415	0. 445	0. 475	0. 503
4	0. 19019	0. 2460	0. 2975	0. 346	0. 392	0. 436	0. 478	0. 519	0. 559	0. 597	0. 634
5	0. 2193	0. 2867	0. 349	0. 408	0. 464	0. 518	0. 570	0. 621	0. 669	0. 716	0. 762
6	0. 2472	0. 326	0. 399	0. 469	0. 535	0. 599	0. 660	0. 720	0. 777	0. 833	0. 888
7	0. 2743	0. 364	0. 448	0. 528	0. 604	0. 678	0. 749	0. 817	0. 884	0. 949	1. 012

(7) $n=130$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0.09828	0.12328	0.14600	0.16719	0.1872	0.2064	0.2247	0.2424	0.2595	0.2761	0.2922
3	0.12419	0.15878	0.1906	0.2205	0.2490	0.2764	0.3027	0.3282	0.3529	0.3769	0.4002
4	0.14815	0.1920	0.2327	0.2711	0.3078	0.3431	0.3773	0.4103	0.4425	0.4738	0.504
5	0.1709	0.2238	0.2731	0.3198	0.3646	0.4078	0.4496	0.490	0.530	0.568	0.606
6	0.1927	0.2545	0.3123	0.3673	0.4200	0.4710	0.520	0.568	0.615	0.661	0.705
7	0.2139	0.2844	0.3506	0.4137	0.4744	0.533	0.590	0.645	0.699	0.752	0.804

(8) $n=160$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0.08043	0.10106	0.11988	0.13750	0.15422	0.1702	0.1856	0.2006	0.2150	0.2291	0.2428
3	0.10167	0.13017	0.15649	0.1813	0.2050	0.2279	0.2499	0.2713	0.2921	0.3124	0.3322
4	0.12133	0.15743	0.1910	0.2229	0.2534	0.2828	0.3114	0.3392	0.3662	0.3926	0.4184
5	0.13998	0.1835	0.2242	0.2629	0.3001	0.3361	0.3711	0.4051	0.4383	0.4707	0.5025
6	0.1579	0.2087	0.2564	0.3019	0.3457	0.3882	0.4295	0.4697	0.5090	0.5474	0.585
7	0.1753	0.2333	0.2879	0.3401	0.3905	0.4393	0.4869	0.5332	0.579	0.623	0.666

(9) $n=200$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0.06474	0.08147	0.09679	0.11116	0.12485	0.13800	0.15070	0.1630	0.1750	0.1867	0.1981
3	0.08187	0.10496	0.12633	0.14657	0.16595	0.1846	0.2028	0.2204	0.2376	0.2544	0.2709
4	0.09773	0.12695	0.15420	0.1801	0.2050	0.2292	0.2526	0.2754	0.2977	0.3196	0.3410
5	0.11278	0.14799	0.1810	0.2125	0.2429	0.2723	0.3009	0.3289	0.3563	0.3831	0.4094
6	0.12728	0.1684	0.2070	0.2440	0.2798	0.3145	0.3483	0.3813	0.4137	0.4454	0.4766
7	0.14134	0.1882	0.2325	0.2749	0.3160	0.3559	0.3948	0.4329	0.4702	0.5069	0.5429

(10) $n=250$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0. 05205	0. 06559	0. 07800	0. 08969	0. 10085	0. 11159	0. 12198	0. 13209	0. 14194	0. 15156	0. 16098
3	0. 06584	0. 08449	0. 10181	0. 11824	0. 13401	0. 14926	0. 16408	0. 17852	0. 1926	0. 2065	0. 2200
4	0. 07861	0. 10220	0. 12426	0. 14529	0. 16556	0. 18521	0. 2043	0. 2230	0. 2413	0. 2593	0. 2769
5	0. 09074	0. 11916	0. 14586	0. 17140	0. 1961	0. 2201	0. 2434	0. 2663	0. 2888	0. 3108	0. 3324
6	0. 10242	0. 13557	0. 16684	0. 19683	0. 2259	0. 2541	0. 2817	0. 3087	0. 3352	0. 3613	0. 3869
7	0. 11376	0. 15158	0. 18736	0. 2218	0. 2551	0. 2876	0. 3193	0. 3505	0. 3810	0. 4111	0. 4407

(11) $n=350$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0. 03739	0. 04718	0. 05619	0. 06469	0. 07283	0. 08069	0. 08832	0. 09575	0. 10302	0. 11014	0. 11712
3	0. 04731	0. 06079	0. 07333	0. 08527	0. 09676	0. 10790	0. 11875	0. 12936	0. 13975	0. 14996	0. 16000
4	0. 05650	0. 07354	0. 08951	0. 10477	0. 11952	0. 13386	0. 14786	0. 16157	0. 17503	0. 18826	0. 20129
5	0. 06524	0. 08575	0. 10507	0. 12360	0. 14154	0. 15903	0. 17612	0. 19289	0. 20937	0. 22559	0. 24158
6	0. 07365	0. 09757	0. 12019	0. 14194	0. 16304	0. 18363	0. 20379	0. 22358	0. 24305	0. 2622	0. 2811
7	0. 08183	0. 10911	0. 13498	0. 15991	0. 18414	0. 20781	0. 23100	0. 2538	0. 2762	0. 2983	0. 3202

(12) $n=500$

p \ q	2	3	4	5	6	7	8	9	10	11	12
2	0. 02629	0. 03320	0. 03958	0. 04562	0. 05141	0. 05701	0. 06246	0. 06778	0. 07300	0. 07811	0. 08314
3	0. 03327	0. 04278	0. 05166	0. 06012	0. 06828	0. 07621	0. 08396	0. 09154	0. 09898	0. 10631	0. 11352
4	0. 03974	0. 05176	0. 06305	0. 07387	0. 08434	0. 09454	0. 10452	0. 11431	0. 12394	0. 13343	0. 14278
5	0. 04589	0. 06036	0. 07402	0. 08714	0. 09988	0. 11231	0. 12449	0. 13646	0. 14824	0. 15986	0. 17133
6	0. 05182	0. 06869	0. 08467	0. 10007	0. 11505	0. 12968	0. 14404	0. 15816	0. 17207	0. 18580	0. 19937
7	0. 05758	0. 07682	0. 09510	0. 11275	0. 12993	0. 14675	0. 16326	0. 17952	0. 19555	0. 21138	0. 22702

Acknowledgement.

The author would like to express his deep gratitude to Prof. S. Yamamoto, University of Hiroshima, for his valuable advices and encouragements, and also Prof. N. Sugiura, University of Hiroshima, who kindly read over an early draft of this paper with valuable advices and comments.

References

[1] Anderson, T. W. (1946). The non-central Wishart distribution and certain problems of multivariate statistics. *Ann. Math. Statist.* **17** 409-431.

[2] Anderson, T. W. (1958). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

[3] Anderson, T. W. and Das Gupta, S. (1964). Monotonicity of the power functions of some tests of independence between two sets of variates. *Ann. Math. Statist.* **35** 206-208.

[4] Bagai, O. P. (1965). The distribution of the generalized variance. *Ann. Math. Statist.* **36** 120-130.

[5] Box, G. E. P. (1949). A general distribution theory for a class of likelihood criteria. *Biometrika* **36** 317-346.

[6] Constantine, A. G. (1963). Some non-central distribution problems in multivariate analysis. *Ann. Math. Statist.* **34** 1270-1285.

[7] Constantine, A. G. (1966). The distribution of Hotelling's generalized T_0^2. *Ann. Math. Statist.* **37** 215-225.

[8] Das Gupta, S., Anderson, T. W. and Mudholkar, G. S. (1964). Monotonicity of the power functions of some tests of the multivariate linear hypothesis. *Ann. Math. Statist.* **35** 200-205.

[9] Fujikoshi, Y. (1968). Asymptotic expansion of the distribution of the generalized variance in the non-central case. *J. Sci. Hiroshima Univ.* Ser. A-I **32** 293-299.

[10] Hayakawa, T. (1969). On the distribution of the latent roots of a positive definite random symmetric matrix I. *Ann. Inst. Statist. Math.* **21** 1-21.

[11] Herz, C. S. (1955). Bessel functions of matrix argument. *Ann. of Math.* **61** 474-523.

[12] Hill, G. W. and Davis, A. W. (1968). Generalized asymptotic expansions of Cornish-Fisher type. *Ann. Math. Statist.* **39** 1264-1273.

[13] Hotelling, H. (1951). A generalized T test and measure of multivariate dispersion. *Proc. Second Berkely Symp. Math. Statist. Prob.*, Univ. of California Press 23-42.

[14] Hsu, P. L. (1940). On generalized analysis of variance (I). *Biometrika* **31** 221-237.

[15] Ito, K. (1956). Asymptotic formulae for the distribution of Hotelling's generalized T_0^2 statistic. *Ann. Math. Statist.* **27** 1091-1105.

[16] Ito, K. (1960). Asymptotic formulae for the distribution of Hotelling's T_0^2 statistic. II. *Ann. Math. Statist.* **31** 1148-1153.

[17] Ito, K. (1962). A comparison of the powers of two multivariate analysis of variance tests. *Biometrika* **49** 455-462.

[18] James, A. T. (1954). Normal multivariate analysis and the orthogonal group. *Ann. Math. Statist.* **25** 40-75.

[19] James, A. T. (1960). The distribution of the latent roots of the covariance matrix. *Ann. Math. Statist.* **31** 151-158.

[20] James, Alan T. (1961). Zonal polynomials of the real positive definite symmetric matrices. *Ann. of Math.* **74** 456-469.

[21] James, Alan T. (1964). Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.* **35** 475-501.

[22] James, A. T. (1968). Calculation of zonal polynomial coefficients by use of the Laplace-Beltrami operator. *Ann. Math. Statist.* **39** 1711-1718.

[23] James, G. S. (1954). Test of linear hypothesis in univariate and multivariate analysis when the

ratios of the population variances are unknown. *Biometrika* **41** 19-43.

[24] Lawley, D. N. (1938). A generalization of Fisher's z test. *Biometrika*, **30** 180-187.

[25] Mikhail, M. N. (1965). A comparison of tests of the Wilks-Lawley hypothesis in multivariate analysis. *Biometrika* **52** 149-156.

[26] Ogawa, J. (1956). On the Masuyama's correlation coefficient of vectors. *Proc. Inst. Statist. Math.* **4** 53-60 (in Japanese).

[27] Pillai, K. C. S. (1953). On the distribution of the sum of the roots of a determinantal equation. *Ann. Math. Statist.* **24** 495 (abstract).

[28] Pillai, K. C. S. (1955). Some new test criteria in multivariate analysis. *Ann. Math. Statist.* **26** 117-121.

[29] Pillai, K. C. S. and Mijares, T. A. (1959). On the moments of the trace of a matrix and approximations to its distribution. *Ann. Math. Statist.* **30** 1135-1140.

[30] Pillai, K. C. S. and Jayachandran, K. (1967). Power comparisons of tests of two multivariate hypotheses based on four criteria. *Biometrika* **54** 195-210.

[31] Pillai, K. C. S. (1968). On the moment generating function of Pillai's $V^{(s)}$ criterion. *Ann. Math. Statist.* **39** 877-880.

[32] Posten, H. O. and Bargmann, R. E. (1964). Power of the likelihood-ratio test of the general linear hypothesis in multivariate analysis. *Biometrika* **51** 467-480.

[33] Rao, C. R. (1948). Tests of significance in multivariate analysis. *Biometrika* **35** 58-79.

[34] Roy, J. (1966). Power of the likelihood-ratio test used in analysis of dispersion. *Multivariate Analysis*, edited by P. R. Krishnaiah, Academic Press, New York. 105-127.

[35] Roy, S. N. (1957). *Some Aspects of Multivariate Analysis.* Wiley, New York.

[36] Seber, G. A. F. (1966). *The linear hypothesis.* Griffin, London.

[37] Siotani, M. (1957). Note on the utilization of the generalized Student ratio in the analysis of variance or dispersion. *Ann. Inst. Statist. Math.* **9** 157-171.

[38] Siotani, M. (1968). Some methods for asymptotic distributions in the multivariate analysis. *Mimeo Series* No. **595** Univ. of North Carolina.

[39] Sugiura, N. and Fujikoshi, Y. (1969). Asymptotic expansions of the non-null distributions of the likelihood ratio criteria for multivariate linear hypothesis and independence. *Ann. Math. Statist.* **40** 942-952.

[40] Sugiura, N. (1969). Asymptotic non-null distributions of the likelihood ratio criteria for covariance matrix under local alternatives. *Mimeo Series* No. **609** Univ. of North Carolina.

[41] Tsumura, Y. and Fukutomi, K. (1966). On the non-central distributions in multivariate analysis. (Reported in the Meeting of Math. Society of Japan. May, 1966).

[42] Wilks, S. S. (1932). Certain generalizations in the analysis of variance. *Biometrika* **24** 471-494.

Department of Mathematics
Faculty of Science
Hiroshima University