J. Sci. HIROSHIMA UNIV. SER. A-I
34 (1970), 69-72

On Purely Inseparable Extensions of Algebraic Function Fields

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(Received March 30, 1970)

In this note we shall be concerned with modular purely inseparable extensions of algebraic function fields over a perfect field k of a positive charactristic p. We shall first see that such an extension has a close connection with separating transcendence bases (Proposition 1), and then give a geometric interpretation of it (Proposition 2). Then if α is a purely inseparable isogeny of a group variety G onto another one G' defined over k, we shall show that the rational function field $k(G)$ of G over k is a modular extension of $\alpha^*(k(G'))$ by using some results by P. Cartier and M. E. Sweedler in [2], [5] and [6], where α^* is the comorphism corresponding to α (Proposition 3), and from this fact we shall show the existence of a favourable system of local parameters at the unit point e of G with respect to α (Theorem and its Corollary).

1. In the sequel let k be a perfect field of a positive characteristic p exclusively.

LEMMA 1. *Let K be an algebraic function field over k and L a purely inseparable extension of exponent 1 over K such that $[L: K]=p^s$. Then there exists a separating transcendence basis $\{t_1,\cdots, t_n\}$ of L over k such that $L=K(t_1,\cdots, t_s)$ and that $\{t_1^p,\cdots, t_s^p, t_{s+1},\cdots, t_n\}$ is a separating transcendence basis of K over k.*

This result is contained in the proof of Barsotti's Theorem in §2. 3 of [1]. Therefore we omit the proof.

PROPOSITION 1. *Let K be an algebraic function field over k and L a purely inseparable extension of K such that L is isomorphic to a tensor product $K(x_1)\otimes_K\cdots\otimes K(x_s)$ of simple extensions $K(x_i)$ over K. Then the transcendental degree n is not less than s and there exist $n-s$ elements $t_{s+1},\cdots, t_n$ in K such that $\{x_1,\cdots, x_s, t_{s+1},\cdots, t_n\}$ (resp. $\{x_1^{p^{e_1}},\cdots, x_s^{p^{e_s}}, t_{s+1},\cdots, t_n\}$) is a separating transcendence basis of L over k (resp. K over k), where e_i is the exponent of x_i over K for $i=1, 2,\cdots, s$.*

PROOF. If we put $y_i=x_i^{p^{e_i-1}}$ for each $i=1, 2,\cdots, s$, $L'=K(y_1,\cdots, y_s)$ is isomorphic to $K(y_1)\otimes_K\cdots\otimes K(y_s)$ and is of exponent *1* over k. By Lemma 1, there exists a separating transcendence basis $\{t_1,\cdots, t_n\}$ of L' over k such that $\{t_1^p,\cdots, t_s^p, t_{s+1},\cdots, t_n\}$ is that of K over k. Then we can easily see that

$L'=K(t_1,\cdots,t_s)$. Since $\{t_1,\cdots,t_n\}$ is a separating transcendence basis of L' over k, there exist n derivations $D_1,\cdots,D_n$ of L' into itself over k such that $D_i(t_j)=\delta_{ij}(i,j=1,2,\cdots,n)$. Then $\{D_1,\cdots,D_n\}$ is a basis of the L'-vector space $\mathcal{D}(L'/k)$ of the derivations of L' over k and $\{D_1,\cdots,D_s\}$ is that of L' over K, since $\{t_1,\cdots,t_s\}$ is a p-basis of L' over K. Similarly let $D'_1,\cdots,D'_s$ be s derivations of L' over K such that $D'_i(y_j)=\delta_{ij}$ $(i,j=1,2,\cdots,s)$. Then $\{D'_1,\cdots,D'_s\}$ is also a basis of $\mathcal{D}(L'/K)$ over L' and hence $\{D'_1,\cdots,D'_s,D_{s+1},\cdots,D_n\}$ must be that of $\mathcal{D}(L'/k)$ over L'. From this fact we can see that the determinant

$$\begin{vmatrix} D_1(y_1),\cdots,D_1(y_s), & D_1(t_{s+1}),\cdots,D_1(t_n) \\ \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\ D_n(y_1),\cdots,D_n(y_s), & D_n(t_{s+1}),\cdots,D_n(t_n) \end{vmatrix}$$

does not vanish and hence that $\{dy_1,\cdots,dy_s,dt_{s+1},\cdots,dt_n\}$ is a basis of the dual space of $\mathcal{D}(L'/k)$ over L'. This shows that $\{y_1,\cdots,y_s,t_{s+1},\cdots,t_n\}$ is a separating transcendence basis of L' over k by Proposition 2 of Chap. VII in [4]. Since L' is separably algebraic over $k(y_1,\cdots,y_s,t_{s+1},\cdots,t_n)$ and $k(y_1,\cdots,y_s,t_{s+1},\cdots,t_n)$ is a purely inseparable extension of degree $p^s=[L':K]$ over $k(y_1^p,\cdots,y_s^p,t_{s+1},\cdots,t_n)$, we see that K is separably algebraic over $k(y_1^p,\cdots,y_s^p,t_{s+1},\cdots,t_n)$. Therefore $L=K(x_1,\cdots,x_s)$ is separably algebraic over $k(x_1,\cdots,x_s,t_{s+1},\cdots,t_n)$, since $x_i^{p^{e_i}}=y_i^p$ for $i=1,2,\cdots,s$. This completes the proof. q.e.d.

COROLLARY. *Let K, L, k and $\{x_1,\cdots,x_s,t_{s+1},\cdots,t_n\}$ be as in Proposition 1. Then K and $k(x_1,\cdots,x_s,t_{s+1},\cdots,t_n)$ are linearly disjoint over $k(x_1^{p^{e_1}},\cdots,x_s^{p^{e_s}},t_{s+1},\cdots,t_n)$.*

PROPOSITION 2. *Let V and W be two algebraic varieties of dimension n defined over an algebraically closed field k and f a dominant morphism of V into W. Suppose that the rational function field $k(V)$ of V over k is isomorphic to a tensor product $K(\tau_1)\otimes_K\cdots\otimes K(\tau_s)$ of purely inseparable, simple extensions $K(\tau_i)$ of the rational function field $K=k(W)$ of W over k. Then there exists a non-empty open subset U of W satisfying the following condition: the local ring $\mathcal{O}_{x,V}$ of V at a rational point x in $f^{-1}(U)$ has a regular system $\{t_1,\cdots,t_n\}$ of parameters such that $\{t_1^{p^{e_1}},\cdots,t_s^{p^{e_s}},t_{s+1},\cdots,t_n\}$ is a regular system of parameters of the local ring $\mathcal{O}_{f(x),W}$ of W at the point $f(x)$, where e_i is the exponent of τ_i over $K=k(W)$.*

PROOF. It is well known that there exists an open subset U' of W such that U' and $f^{-1}(U')$ are non-singular. Let $\{\tau_{s+1},\cdots,\tau_n\}$ be a set of elements in $k(W)$ such that $\{\tau_1,\cdots,\tau_n\}$ satisfies the condition of Proposition 1, and U an open subset of U' such that $\{\tau_1-\tau_1(x),\cdots,\tau_n-\tau_n(x)\}$ (resp. $\{\tau_1^{p^{e_1}}-\tau_1^{p^{e_1}}(y),\cdots,\tau_s^{p^{e_s}}-\tau_s^{p^{e_s}}(y),\tau_{s+1}-\tau_{s+1}(y),\cdots,\tau_n-\tau_n(y)\}$) is a regular system of parameters of $\mathcal{O}_{x,V}$ (resp. $\mathcal{O}_{y,W}$) for any rational point x in $f^{-1}(U)$ (resp. any rational point y in U). Such an open set U exists, since $\{\tau_1,\cdots,\tau_n\}$ (resp. $\{\tau_1^{p^{e_1}},\cdots,\tau_s^{p^{e_s}},\tau_{s+1},\cdots,\tau_n\}$) is a separating transcendence basis of $k(V)$ over k (resp. $k(W)$ over k) (cf. Chap. VII in [4]). There we may put $t_i=\tau_i-\tau_i(x)$ for $i=1, 2,\cdots, n$. This completes the proof. q.e.d.

Remark. It is known that a purely inseparable extension of an algebraic function field is not necessarily modular.

2. PROPOSITION 3. *Let G, G' be group varieties defined over a perfect field k and α a purely inseparable isogeny of G onto G' defined over k. Then the rational function field $k(G)$ of G over k is a modular purely inseparable extension of $\alpha^*(k(G'))$, where α^* is the comorphism of $k(G')$ into $k(G)$ corresponding to α.*

PROOF. We use notations and results of P. Cartier [2]. If we put $N(\alpha)=N_k(\alpha)$ for convenience, $N(\alpha)$ is a cocommutative bialgebra over k and the homomorphism ω: $N(\alpha)\otimes_k k(G)\longrightarrow k(G)$ defined by $\omega(u\otimes f)=u(f)$ measures $k(G)$ to $k(G)$ in the sense of Sweedler [5], because by definition $\omega(u\otimes 1)=u(1)=\varepsilon(u)$ and $\omega(u\otimes fg)=u(fg)=\Delta(u)(f, g)$. Therefore by Lemma 2.5 in [5], $N(\alpha)(k(k(G))^{p^n})\subset k(k(G))^{p^n}=k(G)^{p^n}$, since k is perfect. On the other hand we have $k(G)^{N(\alpha)}=\{f\in k(G)\mid \omega(u\otimes f)=\varepsilon(u)f$ for any u in $N(\alpha)\}=\alpha^*(k(G'))$ and hence $\alpha^*(k(G'))$ and $k(G)^{p^n}$ are linearly disjoint for any n by Lemma 2.2. in [5]. This means that $k(G)$ is a modular extension of $\alpha_*(k(G'))$ by Theorem 1 in [6]. q.e.d.

A similar result of the following theorem was obtained for formal Lie groups by J. Dieudonné in [3] (cf. Theorem 6) and special cases of exponent one for group varieties were given by I. Barsotti in [1]. Our proof will depend on the above Proposition 3.

THEOREM. *Let G and G' be group varieties defined over an algebraically closed field k and α a purely inseparable isogeny of G onto G' defined over k. Then there exists a regular system $\{t_1,\cdots,t_n\}$ of parameters of the local ring $\mathcal{O}_{e,G}$ of G at the unit point e of G such that $\{t_1^{p^{e_1}},\cdots,t_s^{p^{e_s}}, t_{s+1},\cdots,t_n\}$ is that of the local ring $\mathcal{O}_{e',G'}$ of G' at the unit point e' of G', where $p^{e_1+\cdots e_s}$ is the degree of the rational function field $k(G)$ over the subfield $\alpha^*(k(G'))$.*

PROOF. By Proposition 3, $k(G)$ is isomorphic to a tensor product $K(\tau_1)\otimes_K\cdots\otimes K(\tau_s)$ of simple extensions $K(\tau_i)$ over K, where $K=\alpha^*(k(G'))$, and hence, by Proposition 2, there exists a rational point of G over k, at which the local ring $\mathcal{O}_{x,G}$ of G has a regular system $\{t_1',\cdots,t_n'\}$ of parameters such that $\{t_1'^{p^{e_1}},\cdots,t_s'^{p^{e_s}}, t_{s+1}',\cdots,t_n'\}$ is that of $\mathcal{O}_{\alpha(x),G'}$. Since x and e are biholomorphic by a left translation, it is easy to see that there exists a regular system $\{t_1,\cdots,t_n\}$ of parameters of $\mathcal{O}_{e,G}$ satisfying the conditions of our theorem.

Let G, G' and α be as above. Then we can define the kernel of α as an affine scheme Spec $N(\alpha)^D$, where $N(\alpha)$ is a subbialgebra of the bialgebra consisting of the left invariant semi-derivations of G (cf. [7]). We shall terminate this note by giving a relation between the structure of the field extension $k(G)$ over $k(G')$ and that of $N(\alpha)^D$ as an algebra over k.

COROLLARY. *Let G, G' and α be as in Theorem* 1. *If $k(G)$ is isomorphic to a tensor product $K(\tau_1)\otimes_K\cdots\otimes K(\tau_s)$ of simple extension $K(\tau_i)$ over $K=k(G')$, the linear dual $N(\alpha)^D$ of the bialgebra $N(\alpha)$ corresponding to the isogeny α is isomorphic to a residue ring $k[X_1,\cdots, X_s]/(X_1^{p^{e_1}},\cdots, X_s^{p^{e_s}})$ of a polynomial ring $k[X_1,\cdots, X_s]$ as algebras over k.*

PROOF. We use the same notations as in [7]. The kernel Spec $N(\alpha)^D$ of α is isomorphic to $O_{e,G}/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by the maximal ideal $\mathfrak{m}'$ of $O_{e',G'}$ (cf. Theorem 4 in [7]). However Theorem shows that $\mathfrak{a}=(t_1^{p^{e_1}},\cdots, t_s^{p^{e_s}}, t_1,\cdots, t_n)$ for a suitable choice of regular system $\{t_1,\cdots, t_n\}$ of parameters of $O_{e,G}$ and hence $O_{e,G}/\mathfrak{a}$ is isomorphic to $O_{e,G}/(t_1^{p^{e_1}},\cdots, t_s^{p^{e_s}}, t_{s+1},\cdots, t_n)$ $\cong k[t_1,\cdots, t_n]/(t_1^{p^{e_1}},\cdots, t_s^{p^{e_s}}, t_{s+1},\cdots, t_n)\cong k[X_1,\cdots, X_s]/(X_1^{p^{e_1}},\cdots, X_s^{p^{e_s}})$ as k-algebras. q.e.d.

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