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Note on the Span of Certain Manifolds

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§1. Introduction

For a real vector bundle ξ, we denote by *Span* ξ the maximum number of the linearly independent cross-sections of ξ. Especially, we denote *Span* $M = \textit{Span}\ \tau M$, where τM is the tangent bundle of a C^∞-manifold M.

In this note, we prove the following theorem, which is the conjecture of D. Sjerve [4, p. 104, (4.6)].

Theorem 1. *Let π denote any finite group of odd order, not necessarily abelian, acting freely as diffeomorphisms on some standard sphere S^n, and $M^n = S^n/\pi$ be the orbit manifold. Then*

$$\textit{Span}\ M^n = \textit{Span}\ S^n$$

holds for $n \neq 7$.

Also, we shall give counter examples to the following conjecture of E. Thomas [7, p. 655, Conjecture 5] by $S^1 \times P_n(C)$ and the mod 3 standard lens space $L^3(3)$, where $n = u \cdot 2^{2+4d} - 1$ (u: odd, $d \geqq 1$) and $P_n(C)$ is the complex n-dimensional projective space.

Conjecture of E. Thomas: *Let M be a compact n-manifold, n odd, and let k be a positive integer such that $k \leqq \textit{Span}\ S^n$. If $w_1 M = \cdots = w_k M = 0$, then Span $M \geqq k$, where $w_i M$ is the i-th Stiefel-Whitney class of M.*

§2. Proof of Theorem 1

Theorem 2. [5, p. 551], [6, p. 53]. *Let ξ^n be an orientable n-dimensional real vector bundle over an n-dimensional complex X. Then,*

$$\textit{Span}\ \xi^n < \textit{Span} S^n \textit{ implies } \textit{Span}\ (\xi^n \oplus 1) = 1 + \textit{Span}\ \xi^n,$$

where $\xi^n \oplus 1$ is the Whitney sum of ξ^n and 1-dimensional trivial bundle over X.

Proof. Put $k = \text{Span}\ (\xi^n \oplus 1)$, then there exists an $(n+1-k)$-dimensional vector bundle η over X such that $\xi^n \oplus 1 = \eta \oplus (k-1) \oplus 1$. So, by [6, Theorem 1], *Span* $(\eta \oplus (k-1)) = \textit{Span}\ \xi^n$. This implies *Span*$(\xi^n \oplus 1) \leqq 1 + \textit{Span}\ \xi^n$. And, *Span* $(\xi^n \oplus 1) \geqq 1 + \textit{Span}\ \xi^n$ is clear. *q.e.d.*

Next, we notice that the following theorem holds for the odd-dimensional manifold of Theorem 1. This theorem is Theorem A in [3, p. 545] where π

is assumed to be the cyclic group of odd prime order, and is proved by the same methods.

THEOREM 3. *Let* p: $S^{2n+1} \longrightarrow M^{2n+1}$ *be the projection map. If* $\xi \in \widetilde{KO}(M^{2n+1}) \cap Ker\ p^*$, *then* $g.dim\ \xi \leqq 2[n/2]+1$ *for the geometric dimension* $g.dim\ \xi$ *of* ξ.

PROOF. As in [3], we consider the following lifting problem of ξ.

$$\begin{array}{ccccc} & & & & BSO(2[n/2]+1) \\ & & & & \downarrow \\ S^{2n+1} & \xrightarrow{p} & M^{2n+1} & \xrightarrow{\xi} & BSO(d) \end{array}$$

where $BSO(m)$ is the classifying space of orientable m-dimensional real vector bundles, and $BSO(2[n/2]+1) \longrightarrow BSO(d)$ is the fiber bundle induced by the inclusion map $SO(2[n/2]+1) \longrightarrow SO(d)$ for some sufficiently large integer d. The fiber V of this bundle consists of the orthonormal $(d-2[n/2]-1)$-frames in the d-dimensional Euclidean space.

The i-th homotopy group $\pi_i(V)$ of V consists of 2-primary components for $i \leqq 2n$, and the order of π is odd by the assumption. So, by [4, p. 98, (2. 1)], $H^i(M^{2n+1}; \pi_{i-1}(V)) = 0$ for $0 \leqq i \leqq 2n$. Thus the last obstruction $\theta(\xi)$ to lifting ξ is the element of $H^{2n+1}(M^{2n+1}; \pi_{2n}(V))$. The last obstruction to lifting $\xi \circ p$ is $p^*\theta(\xi)$, and this is zero by the assumption $\xi \in \widetilde{KO}(M^{2n+1}) \cap Ker\ p^*$. As, $p^*: H^{2n+1}(M^{2n+1}; \pi_{2n}(V)) \longrightarrow H^{2n+1}(S^{2n+1}; \pi_{2n}(V))$ is an isomorphism, $\theta(\xi)$ is zero. *q.e.d.*

Proof of Theorem 1.

Theorem 1 is clear for n even, because $Span\ S^n = 0$.

By Theorem 3, $Span\ (\tau M^{2n+1} \oplus 1) = 2n+2-g.\ dim\ (\tau M^{2n+1}-2n-1) \geqq 2n+1-2[n/2]$. But, for $n \neq 0, 1, 3$, $2n+1-2[n/2] \geqq 1+Span\ S^{2n+1}$ *by* [1, p. 603]. So, by Theorem 2, $Span\ M^{2n+1} \geqq Span\ S^{2n+1}$. On the other hand, $Span\ M^{2n+1} \leqq Span\ S^{2n+1}$ is clear since $\tau S^{2n+1} = p^*\tau M^{2n+1}$. *q.e.d.*

§ 3. Counter examples

LEMMA 4. $Span(S^1 \times P_n(C)) = 1+2v$, *where* $n+1 = u \cdot 2^v$ (u: odd).

PROOF. $\tau(S^1 \times P_n(C)) = p_1^*\tau_1 \oplus p_2^*\tau_2 = p_2^*(1 \oplus \tau_2)$, where p_i is the projection map onto the i-th factor and $\tau_1 = \tau S^1$, $\tau_2 = \tau(P_n(C))$. So

$$Span\ (S^1 \times P_n(C)) = Span(p_2^*(1 \oplus \tau_2)).$$

As $p_2 \circ i$ is the identity map for the inclusion map $i: P_n(C) \longrightarrow S^1 \times P_n(C)$,

$$Span(S^1 \times P_n(C)) = Span(1 \oplus \tau_2) = 2n+1-g.\ dim\ (\tau_2 - 2n).$$

By [2, p. 69], $g.dim(\tau_2 - 2n) = 2n - 2v$, and we have the lemma. *q.e.d.*

EXAMPLE 1. $S^1 \times P_n(C)$, where $n = u \cdot 2^{2+4d} - 1$ (u: odd, $d \geqq 1$).
By the above lemma and [1, p. 603],

$$Span(S^1 \times P_n(C)) = 5 + 8d,$$

$$Span\ S^{1+2n} = 7 + 8d.$$

The easy calculations for the Stiefel-Whitney classes of $S^1 \times P_n(C)$ show that

$$w_i(S^1 \times P_n(C)) = 0 \text{ for } 1 \leqq i \leqq 2(2^{2+4d} - 1).$$

It is clear that $2(2^{2+4d} - 1) \geqq 7 + 8d = Span\ S^{1+2n}$. *q.e.d.*

EXAMPLE 2. The mod 3 standard lens space $L^3(3) = S^7/Z_3$.
By [8, p. 14], $Span\ L^3(3) = 5$ and $w_i L^3(3) = 0$ for $1 \leqq i \leqq 6$. *q.e.d.*

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