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Harmonic and Full-harmonic Structures on a Differentiable Manifold

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Introduction

Let Ω be a bounded domain in the d-dimensional euclidean space ($d \geqq 2$). G. Stampacchia [17] (also, C. B. Morrey Jr. [14] and O. A. Ladyzhenskaya and N. N. Ural'tzeva [9]) discussed properties of solutions of a second order elliptic partial differential equation on Ω of the form

$$Lu \equiv -\sum_{i,j} \frac{\partial}{\partial x_j}\left(g_{ij} \frac{\partial u}{\partial x_i} + b_j u \right) + \sum_i a_i \frac{\partial u}{\partial x_i} + qu = 0 \tag{1}$$

with not necessarily continuous coefficients. In fact, Stampacchia only assumed that coefficients g_{ij}, a_i, b_j and q are measurable functions on Ω satisfying the following conditions (2) and (3):

$$\sum g_{ij}\xi_i\xi_j \geqq \nu |\xi|^2 \quad \text{for some } \nu > 0 \quad \text{and} \quad |g_{ij}| \leqq M. \tag{2}$$

(3) $a_i \in \boldsymbol{L}^d(\Omega)$, $b_j \in \boldsymbol{L}^r(\Omega)$, $q \in \boldsymbol{L}^{r/2}(\Omega)$ for $r > d$. (Cf. [9] and [14], in which it is assumed that $a_i \in \boldsymbol{L}^r(\Omega)$. In case $d = 2$, this assumption may be necessary; the paper [17] primarily concerns the case $d \geqq 3$.)

On the ground of Stampacchia's work, R.-M. and M. Hervé [7] developed a theory of superharmonic functions associated with the equation (1), under an additional condition:

$$q - \sum_j \frac{\partial b_j}{\partial x_j} \geqq 0 \quad \text{and} \quad q - \sum_i \frac{\partial a_i}{\partial x_i} \geqq 0 \quad \text{in the distribution sense.} \tag{4}$$

In fact, they showed that the continuous solutions of (1) form a harmonic space on Ω in the sense of M. Brelot [1] and then constructed the corresponding Green function on Ω.

In this paper, we take a connected C^1-manifold Ω and consider a contravariant tensor (g^{ij}), contravariant vectors (a^i) and (b^j) and a function q on Ω which locally satisfy conditions (2) and (3). Our differential equation may be written as

$$Lu \equiv \Delta u - \sum_i a^i \frac{\partial u}{\partial x_i} + \frac{1}{\sqrt{G}} \sum_j \frac{\partial}{\partial x_j}(\sqrt{G}\, b_j u) - qu = 0 \tag{1'}$$

with

$$\Delta u = \frac{1}{\sqrt{G}} \sum_j \frac{\partial}{\partial x_j}\left(\sqrt{G}\, g^{ij} \frac{\partial u}{\partial x_i}\right),$$

where G is the determinant of $(g_{ij})=(g^{ij})^{-1}$. Without assuming any condition corresponding to (4), we shall show that the continuous solutions of (1′) form a harmonic space on Ω (called the L-harmonic structure). Construction of the corresponding Green function and the integral representation of superharmonic functions associated with (1′) are also discussed on a subdomain ω of Ω, following the lines of [7]—without the assumption (4), but with a certain restriction on the domain ω (§3).

The best part of this paper is devoted to discussions of a full-harmonic structure (in the sense of [11]) which is determined by a general boundary condition and is subordinate to the L-harmonic structure. Its original model is the theory of full-superharmonic functions associated with the classical harmonic functions and the Kuramochi boundary, for which boundary condition is given as vanishing normal derivatives (cf. [3] and [10]). There is also a work by S. Itô [8], which is intended to give a generalization of the theory of Kuramochi boundary in the case where the harmonic structure is given by an elliptic partial differential equation on a manifold. We shall consider a general boundary condition $(\boldsymbol{R}, B)$ determined by a class $\boldsymbol{R}$ of L-harmonic functions and a bilinear form B on $\boldsymbol{R}\times\boldsymbol{R}$. This idea of boundary condition is a generalization of that given in [12].

We shall show (§4) that, on an "end" on Ω satisfying certain conditions, we can define a full-harmonic structure in terms of condition $(\boldsymbol{R}, B)$. Then (in §5), we construct the corresponding Green function, extending the methods given in [3], [10] and [12]. With this Green function we can apply the general theory given in [11] and obtain an integral representation theorem for full-superharmonic functions associated with our full-harmonic structure. For this integral representation, we consider an ideal boundary. As was remarked in [11], in the classical case this ideal boundary can be the Martin boundary or the Kuramochi boundary according as the choice of boundary condition. Thus we may generalize the known theorems on these ideal boundaries to our case. In this paper we give two of such theorems (§6). The first of them is a characterization of "minimal" points in terms of the reduced function and the other is on an equivalence of two types of "thinness" at ideal boundary points (cf. M. Brelot [2] for the case of Martin boundary).

§ 1. Preliminaries

1.1. *Metric tensor.*

As a base space Ω, we take a connected non-compact C^1-manifold of dimension $d \geqq 2$. We consider a symmetric covariant tensor (g_{ij}) on Ω which satisfies the following condition (G):

(G): On each relatively compact coordinate neighborhood U in Ω, each g_{ij} is a bounded measurable function on U and there exists $\lambda > 0$ such that

$$\lambda \sum_{i=1}^{d} \xi_i^2 \leqq \sum_{i,j=1}^{d} g_{ij}(x) \xi_i \xi_j$$

for all $x \in U$ and real numbers $\xi_1, \cdots, \xi_d$.

A manifold Ω with such a metric tensor is a locally compact metrizable space, and hence it is countable at infinity.

Let $G(x)$ be the determinant of (g_{ij}) on each coordinate neighborhood. Then $dV = \sqrt{G} dx_1 \cdots dx_d$ defines a positive measure on Ω. For any open set ω in Ω, let $\boldsymbol{L}^p(\omega)$ (resp. $\boldsymbol{L}^p_{\mathrm{loc}}(\omega)$) $(p \geqq 1)$ be the space of p-th power summable real functions on ω with respect to the measure dV. We consider the usual norm $\|\cdot\|_{p,\omega}$ on $\boldsymbol{L}^p(\omega)$: $\|f\|_{p,\omega}^p = \int_\omega |f|^p dV$.

For $f, g \in \boldsymbol{L}^1_{\mathrm{loc}}(\omega)$, we write $f \leqq g$ or $f = g$ on ω' for an open set $\omega' \subset \omega$ if it is so almost everywhere on ω' with respect to dV.

We denote by $\boldsymbol{C}^1(\omega)$ the space of continuously differentiable functions on ω and by $\boldsymbol{C}^1_0(\omega)$ the subspace consisting of functions with compact supports in ω.

For a set A in Ω, its closure in Ω will be denoted by $\bar{A}$ and its boundary in Ω by ∂A.

1.2. *The spaces $\boldsymbol{D}(\omega)$ and $\boldsymbol{D}_0(\omega)$.*

Let ω be an open set in Ω. Given $f \in \boldsymbol{C}^1(\omega)$,

$$D_\omega[f] = \int_\omega \sum g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} dV$$

is well-defined, where (g^{ij}) is the inverse matrix of (g_{ij}). Let $\boldsymbol{C}^1_D(\omega) = \{f \in \boldsymbol{C}^1(\omega); D_\omega[f] < \infty\}$. Obviously $\boldsymbol{C}^1_0(\omega) \subset \boldsymbol{C}^1_D(\omega)$.

Now let ω be a domain (connected open set) and let ω' be a relatively compact domain such that $\bar{\omega}' \subset \omega$. For $f \in \boldsymbol{C}^1_D(\omega)$, we define

$$\|f\|_{D,\omega,\omega'} = [D_\omega[f] + \|f\|_{2,\omega'}^2]^{1/2}$$

and

$\boldsymbol{D}(\omega)$ = the completion of $\boldsymbol{C}^1_D(\omega)$ with respect to the norm $\|f\|_{D,\omega,\omega'}$,
$\boldsymbol{D}_0(\omega)$ = the closure of $\boldsymbol{C}^1_0(\omega)$ in $\boldsymbol{D}(\omega)$.

For any $f \in \boldsymbol{D}(\omega)$, $D_\omega[f]$, $\|f\|_{D,\omega,\omega'}$ are well-defined. In case ω is a relatively compact coordinate neighborhood, then the space $\boldsymbol{D}(\omega)$ (resp. $\boldsymbol{D}_0(\omega)$) may be identified with the Sobolev space $H^1(\omega)$ (resp. $H_0^1(\omega)$) (see [17] for this notation). Thus, for $f \in \boldsymbol{D}(\omega)$, we have (cf. [5])

(i) f is identified with a function in $\boldsymbol{L}^2_{\mathrm{loc}}(\omega)$;

(ii) In each coordinate neighborhood, grad $f=(\partial f/\partial x_1, \cdots, \partial f/\partial x_d)$ is defined almost everywhere to be a covariant vector on ω and $|\mathrm{grad}\ f|=(\sum g^{ij}(\partial f/\partial x_i)(\partial f/\partial x_j))^{1/2}$ belongs to $\boldsymbol{L}^2(\omega)$; $D_\omega[f]=\int_\omega |\mathrm{grad}\ f|^2 dV$.

For any f, $g \in \boldsymbol{D}(\omega)$,

$$D_\omega[f, g]=\int_\omega \sum g^{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}dV$$

is well-defined and $\boldsymbol{D}(\omega)$ is a Hilbert space with respect to the inner product $D_\omega[f, g]+\int_{\omega'} fg dV$ (cf. [5]). Also we have

LEMMA 1.1. *If $f_n \in \boldsymbol{D}(\omega)$ and $D_\omega[f_n]\to 0$, then there exist constants c_n such that $f_n+c_n\to 0$ in $\boldsymbol{D}(\omega)$.*

Using this lemma, we obtain

LEMMA 1.2. *If ω' is a relatively compact domain such that $\bar{\omega}'\subset\omega$, then the injection map of $\boldsymbol{D}(\omega)$ into $\boldsymbol{L}^2(\omega')$ is continuous.*

LEMMA 1.3. *If $1 \notin \boldsymbol{D}_0(\omega)$, then the norm $\|f\|_{D,\omega,\omega'}$ is equivalent to the norm $D_\omega[f]^{1/2}$ on $\boldsymbol{D}_0(\omega)$.*

Remark that if ω is relatively compact, then $1 \notin \boldsymbol{D}_0(\omega)$.

From the definition of $\boldsymbol{D}_0(\omega)$, we easily have

LEMMA 1.4. *If $\omega\subset\omega_1$ and $f \in \boldsymbol{D}_0(\omega)$, then*

$$f^{\#}=\begin{cases} f & \text{on } \omega \\ 0 & \text{on } \omega_1-\omega \end{cases}$$

defines an element in $\boldsymbol{D}_0(\omega_1)$.

1.3. *The space $\boldsymbol{D}_{\mathrm{loc}}(\omega)$ and lattice structures.*

For an open set ω in Ω, we define

$$\boldsymbol{D}_{\mathrm{loc}}(\omega)=\left\{f \in \boldsymbol{L}^2_{\mathrm{loc}}(\omega);\ \begin{array}{l}\text{for any relatively compact domain}\\ \omega' \text{ such that } \bar{\omega}'\subset\omega,\ f|\omega' \in \boldsymbol{D}(\omega')\end{array}\right\},$$

where $f|\omega'$ denotes the restriction of f to ω'. For $f \in \boldsymbol{D}_{\mathrm{loc}}(\omega)$, grad f is defined on ω as a covariant vector.

For a domain ω, we easily have

LEMMA 1.5. *If $f \in \boldsymbol{D}_{\mathrm{loc}}(\omega)$ and $\varphi \in \boldsymbol{C}_0^1(\omega)$, then $\varphi f \in \boldsymbol{D}_0(\omega)$.*

COROLLARY. *If* $f \in \boldsymbol{D}_{\mathrm{loc}}(\omega)$ *and* $f=0$ *outside a compact set in* ω, *then* $f \in \boldsymbol{D}_0(\omega)$.

The following results can be seen by using the corresponding results on euclidean spaces (see [5]; also cf. [3]):

LEMMA 1.6. (a) *If* $f \in \boldsymbol{D}_{\mathrm{loc}}(\omega)$, *then* $f^+=\max(f, 0)$ (*hence* $f^-=-\min(f, 0)$) *belongs to* $\boldsymbol{D}_{\mathrm{loc}}(\omega)$ *and*

$$\operatorname{grad} f^+ = \begin{cases} \operatorname{grad} f & \text{a.e. } on \quad \{x \in \omega; f(x)>0\} \\ 0 & \text{a.e. } on \quad \{x \in \omega; f(x) \leqq 0\}. \end{cases}$$

(b) *If* $f, g \in \boldsymbol{D}(\omega)$, *then* $\max(f, g)$, $\min(f, g) \in \boldsymbol{D}(\omega)$ *and*

$$D_\omega[\max(f, g)]+D_\omega[\min(f, g)]=D_\omega[f]+D_\omega[g].$$

LEMMA 1.7. (a) *For* $f \in \boldsymbol{D}(\omega)$, $f \in \boldsymbol{D}_0(\omega)$ *if and only if* $|f| \in \boldsymbol{D}_0(\omega)$.
(b) *If* $f \in \boldsymbol{D}(\omega)$, $f \geqq 0$ *on* ω *and if* $f \leqq g$ *outside a compact set in* ω *for some* $g \in \boldsymbol{D}_0(\omega)$, *then* $f \in \boldsymbol{D}_0(\omega)$.

§2. Equation $\boldsymbol{L}u=0$ and its solutions

2.1. *Equation* $Lu=0$ *and L-harmonic functions.*

Now we consider two contravariant vectors $a=(a^i)$ and $b=(b^j)$ on Ω and a function q on Ω. We assume

$$|a| \in \boldsymbol{L}^d_{\mathrm{loc}}(\Omega), \qquad |b| \in \boldsymbol{L}^r_{\mathrm{loc}}(\Omega) \quad \text{and} \quad q \in \boldsymbol{L}^{r/2}_{\mathrm{loc}}(\Omega)$$

for some $r>d$ (if $d=2$, then we furthermore assume that $|a| \in \boldsymbol{L}^r_{\mathrm{loc}}(\Omega)$), where

$$|a|=(\sum g_{ij}a^ia^j)^{1/2} \qquad \text{and} \qquad |b|=(\sum g_{ij}b^ib^j)^{1/2}.$$

We formally consider the equation

$$Lu=\Delta u-\sum_{i=1}^d a^i\frac{\partial u}{\partial x_i}+\frac{1}{\sqrt{G}}\sum_{j=1}^d \frac{\partial}{\partial x_j}(\sqrt{G}\, b^j u)-qu=0,$$

where

$$\Delta u=\frac{1}{\sqrt{G}}\sum_{j=1}^d \frac{\partial}{\partial x_j}\Big(\sum_{i=1}^d \sqrt{G}\, g^{ij}\frac{\partial u}{\partial x_i}\Big).$$

If ω is a relatively compact domain in Ω and if $f, g \in \boldsymbol{D}(\omega)$, then

$$A_{L,\omega}[f, g]\equiv D_\omega[f, g]+\int_\omega\Big(\sum_{i=1}^d a^i\frac{\partial f}{\partial x_i}g+\sum_{j=1}^d b^j f\frac{\partial g}{\partial x_j}+qfg\Big)dV$$

is well-defined and Soblev's lemma implies that the mapping $(f, g)\to A_{L,\omega}[f, g]$ is continuous on $\boldsymbol{D}(\omega)\times\boldsymbol{D}(\omega)$ (cf., e.g., [17]).

If ω is any domain and $u \in \boldsymbol{D}_{\mathrm{loc}}(\omega)$, then $A_{L,\omega}[u, \varphi]$ is defined for any $\varphi \in \boldsymbol{C}_0^1(\omega)$ by $A_{L,\omega}[u, \varphi]=A_{L,\omega'}[u, \varphi]$, where ω' is a relatively compact domain such that $\bar{\omega}' \subset \omega$ and the support of φ is contained in ω'.

$u \in \boldsymbol{D}_{\mathrm{loc}}(\omega)$ is called a solution of $Lu=0$ on ω if $A_{L,\omega}[u, \varphi]=0$ for all $\varphi \in \boldsymbol{C}_0^1(\omega)$. $u \in \boldsymbol{D}_{\mathrm{loc}}(\omega)$ is called an L-supersolution on ω if $A_{L,\omega}[u, \varphi] \geqq 0$ for any $\varphi \in C_0^1(\omega)$ with $\varphi \geqq 0$ on ω.

It is known (e.g., [17] for $d \geqq 3$; [9], [14] for $d \geqq 2$) that any solution of $Lu=0$ is equal to a locally Hölder continuous function almost everywhere. Thus, we call a continuous solution of $Lu=0$ on a domain ω an *L-harmonic function* on ω. If ω is an open set and u is L-harmonic on each component of ω, then we say that u is L-harmonic on ω. The set of all L-harmonic functions on ω will be denoted by $\mathcal{H}_L(\omega)$. By definition, we easily have

PROPOSITION 2.1. *Each $\mathcal{H}_L(\omega)$ is a real linear space and $\mathfrak{H}_L=\{\mathcal{H}_L(\omega)\}_{\omega.\mathrm{open}}$ is a sheaf of continuous functions.*

2.2. *Minimum principles on an L-adapted ball.*

A domain ω will be called an L-adapted domain (cf. [7]) if it is relatively compact and $A_{L,\omega}$ is coercive on $\boldsymbol{D}_0(\omega)$, i.e., there is $\lambda>0$ such that

$$A_{L,\omega}[f, f] \geqq \lambda D_\omega[f]$$

for all $f \in \boldsymbol{D}_0(\omega)$. By Lemma 1.4, we see that any subdomain of an L-adapted domain is L-adapted.

By Théorème 3.1 of [17] (for $d=2$, we must modify its proof—see Theorem 5.1 of [9]), we have

LEMMA 2.1. *For any $x \in \Omega$, there exists an L-adapted coordinate neighborhood of x.*

The following lemma is proved in [7] (Lemma 1, *a*)):

LEMMA 2.2. *Let ω be an L-adapted domain and $u \in \boldsymbol{D}(\omega)$ be an L-supersolution on ω. If $u \geqq g$ outside a compact set in ω for some $g \in \boldsymbol{D}_0(\omega)$, then $u \geqq 0$ on ω.*

We shall say that a domain U in Ω is a ball if there is a coordinate neighborhood U' such that $\bar{U} \subset U'$ and U is expressed as $\{x;\ |x|<r\}$ with respect to the coordinate.

Using Théorème 3.3 and Théorème 7.3 of [17] (Theorem 5.2 and Theorem 14.1 of [9] if $d=2$; also cf. [14] and the proof of Theorem 1 in [7]), we have

PROPOSITION 2.2. *If U is an L-adapted ball in Ω, then for any $\varphi \in \boldsymbol{C}^1(\partial U)$, there exists a unique $u \in \boldsymbol{C}(\bar{U})$ such that $u=\varphi$ on ∂U and u is L-harmonic on U. Furthermore $u \in \boldsymbol{D}(U)$. Here $\boldsymbol{C}^1(\partial U)$ is the set of the restrictions of $g \in \boldsymbol{C}^1(U')$ to ∂U for some $U' \supset \bar{U}$ and $\boldsymbol{C}(\bar{U})$ is the set of all continuous functions* *on* $\bar{U}$.

The function u in the above proposition will be denoted by $H_{\varphi}^{L,U}$

PROPOSITION 2.3. *Let U be an L-adapted ball in Ω. If u is an L-supersolution on U and if* $\liminf_{x\to\xi, x\in U} u(x) \geqq 0$ *for all* $\xi \in \partial U$, *then* $u \geqq 0$ *on U.*

PROOF. Let $u_0 = H_1^{L,U}$. By Proposition 2.2, u_0 is continuous on $\bar{U}$ and $u_0 = 1$ on ∂U. Hence there is a compact set K_0 in U such that $u_0(x) \geqq 1/2$ for $x \in U - K_0$. For any $\varepsilon > 0$, there exists a compact set $K \supset K_0$ such that $u(x) > -\varepsilon$ on $U-K$. Then $u + 2\varepsilon u_0 \geqq 0$ on $U-K$. Let ω be a domain such that $K \subset \omega \subset \bar{\omega} \subset U$. Then $u + 2\varepsilon u_0 \in \boldsymbol{D}(\omega)$. Hence, by Lemma 2.2 (taking $g=0$), we have $u + 2\varepsilon u_0 \geqq 0$ on ω. Since ω can be taken to contain any point in U and since ε is arbitrary, we have $u \geqq 0$ on U.

2.3. *L-harmonic structure.*

Now we prove our main theorem in this section: (Cf. Théorème 1 of [7])

THEOREM 2.1. $\mathfrak{H}_L = \{\mathcal{H}_L(\omega)\}_{\omega:\text{open}}$ *satisfies Axioms* 1, 2 *and* 3 *of* M. Brelot [1], *so that it defines a structure of harmonic space on Ω.*

We shall call $\mathfrak{H}_L$ the L-harmonic structure.

To prove Theorem 2.1, we prepare the following two results, which are essentially given in [17] (in case $d=2$, we must modify the proofs).

PROPOSITION 2.4. (Harnack's inequality; see [17], Théorème 8.1) *If ω is a relatively compact domain in a coordinate neighborhood in Ω and K is a compact set in ω, then there exists* $\lambda > 0$ *such that*

$$\max_{x\in K} u(x) \leqq \lambda \min_{x\in K} u(x)$$

for all $u \in \mathcal{H}_L(\omega)$ *such that* $u \geqq 0$ *on ω.*

LEMMA 2.3. (See Lemme 5.2 in [17] and Proposition 2 in [7]) *Let ω be a relatively compact domain in a coordinate neighborhood in Ω and let ω′ be a domain such that* $\bar{\omega}' \subset \omega$. *Then there exists* $\alpha > 0$ *such that*

$$D_{\omega'}[u] \leqq \alpha \int_{\omega} u^2 dV$$

for any $u \in \mathcal{H}_L(\omega)$.

Remark that Lemme 5.2 in [17] is valid without the assumption (5.2) in [17].

PROOF of Theorem 2.1. Our Proposition 2.1 is nothing but Axiom 1 of [1] for $\mathfrak{H}_L$. To show Axiom 2, let U be any L-adapted ball in Ω and we shall prove that U is regular with respect to $\mathfrak{H}_L$. Given $\varphi \in \boldsymbol{C}(\partial U)$, we can choose $\varphi_n \in \boldsymbol{C}^1(\partial U)$ such that $\varphi_n \to \varphi$ uniformly on ∂U as $n \to \infty$. Let $u_n = H_{\varphi_n}^{L,U}$ on $\bar{U}$.

Put $\varepsilon_n=\max_{\xi\in\partial U}|\varphi_n(\xi)-\varphi(\xi)|$ and $M=\sup_{x\in U}H_1^{L,U}(x)$. Then $\varepsilon_n\to 0$ $(n\to\infty)$ and $M<\infty$. By Proposition 2.3, we have

$$|u_n(x)-u_m(x)|\leqq(\varepsilon_n+\varepsilon_m)H_1^{L,U}(x)\leqq(\varepsilon_n+\varepsilon_m)M,$$

so that $\{u_n\}$ converges uniformly on $\bar{U}$. Let $u=\lim_{n\to\infty}u_n$. Then $u\in \boldsymbol{C}(\bar{U})$ and $u=\varphi$ on ∂U. By Lemma 2.3, we also see that $D_{\omega'}[u_n-u_m]\to 0(n, m\to\infty)$ for any domain ω' such that $\bar{\omega}'\subset U$. It follows that $u\in \boldsymbol{D}_{\mathrm{loc}}(U)$ and $D_{\omega'}[u_n-u]\to 0$ for any such ω'. Hence $A_{L,U}[u, \psi]=\lim_{n\to\infty}A_{L,U}[u_n, \psi]=0$ for any $\psi\in \boldsymbol{C}_0^1(U)$, so that $u\in\mathcal{H}_L(U)$. By Proposition 2.3, such u is uniquely determined by φ, and $u\geqq 0$ whenever $\varphi\geqq 0$. Thus Axiom 2 of [1] for $\mathfrak{H}_L$ is verified. Finally, Lemma 1 in [4] shows that our Proposition 2.4 implies Axiom 3 of [1] for $\mathfrak{H}_L$. Thus the theorem is completely proved.

REMARK. We can similarly show that an L-adapted domain ω is regular with respect to $\mathfrak{H}_L$ if its boundary $\partial\omega$ satisfies the following condition (A) (cf. [9]; ω is of type S in [11] or [17]):

(A): For any coordinate ball $U=\{|x|<r_0\}$ with center on $\partial\omega$, there are two constants α and ρ with $0<\alpha<1$ and $0<\rho\leqq r_0/2$ such that for any $\xi\in\partial\omega\cap\left\{|x|<\frac{r_0}{2}\right\}$ and for any r with $0<r\leqq\rho$ we have $\mathrm{mes}\{B(\xi, r)-\omega\}\geqq\alpha$ $\mathrm{mes}B(\xi, r)$, where $B(\xi, r)=\{|x-\xi|<r\}$ and "mes" means the Lebesgue measure, with respect to the given coordinate of U.

Superharmonic functions and potentials with respect to the L-harmonic structure will be called L-superharmonic functions and L-potentials, respectively.

PROPOSITION 2.5. *Any lower semicontinuous L-supersolution is L-superharmonic.*

PROOF. Let v be a lower semicontinuous L-supersolution on ω. For any L-adapted ball U such that $\bar{U}\subset\omega$ and for any continuous function $\varphi\in \boldsymbol{C}(\partial U)$ such that $\varphi\leqq v$ on ∂U, $v-H_\varphi^{L,U}$ is an L-supersolution on U and $\liminf_{x\to\xi, x\in U}\{v(x)-H_\varphi^{L,U}(x)\}\geqq 0$ for all $\xi\in\partial U$. Hence, by Proposition 2.3, $v\geqq H_\varphi^{L,U}$ on U. It follows that v is L-superharmonic on ω.

COROLLARY. (cf. [7]) *If* $q-(1/\sqrt{G})\sum_i\partial(\sqrt{G}\,b^i)/\partial x_i\geqq 0$ *in the distribution sense on* ω, *then the constant function* 1 *is L-superharmonic on* ω.

2.4. *Domains on which minimum principle holds.*

We say that the minimum principle holds on a domain ω if $\liminf_{x\to\partial^*\omega}u(x)\geqq 0$ implies $u\geqq 0$ for any L-superharmonic function u on ω, where $\liminf_{x\to\partial^*\omega}u(x)\geqq 0$ means that given $\varepsilon>0$ there is a compact set K in

ω such that $u(x) \geqq -\varepsilon$ on $\omega - K$. It is known (see [1], Theorem 3, (ii) and its footnote) that if there is an L-superharmonic function v on ω such that $\inf_{x \in \omega} v(x) > 0$, then the minimum principle holds on ω.

PROPOSITION 2.6. *If one of the following conditions is satisfied, then the minimum principle holds on* ω:

(a) *There exists an L-adapted domain* ω_0 *such that* $\bar{\omega} \subset \omega_0$.

(b) $q - (1/\sqrt{G}) \sum_i \partial(\sqrt{G}\, b^i)/\partial x_i \geqq 0$ *in the distribution sense on* ω.

PROOF. If (a) is satisfied, then, by Lax-Milgram's theorem, there is $g \in \boldsymbol{D}_0(\omega_0)$ such that $A_{L,\omega_0}[g, \varphi] = A_{L,\omega_0}[1, \varphi]$ for all $\varphi \in \boldsymbol{D}_0(\omega_0)$. Then $u = 1 - g$ is L-harmonic on ω_0 and Lemma 2.2 implies that $u \geqq 0$ on ω_0. Using Harnack's inequality, $u > 0$ on ω_0, so that $\inf_{x \in \omega} u(x) > 0$. Hence the minimum principle holds on ω. If (b) is satisfied, then the corollary to Proposition 2.5 guarantees the existence of an L-superharmonic function with positive infimum on ω.

We say that the *weak* minimum principle holds on a domain ω if for an L-superharmonic function u on ω, $u \geqq 0$ outside a compact set in ω implies $u \geqq 0$ on ω. The following result is given in [4] (Theorem 2):

PROPOSITION 2.7. *If there exists a positive L-superharmonic function on* ω, *then the weak minimum principle holds on* ω.

2.5. *Proportionality of L-potentials with point supports.*

Given an L-superharmonic function v on a domain ω, the complement (in ω) of the largest open set in which v is L-harmonic is called the support of v (see [1], [6]). It is known ([6]) that if ω admits a positive L-potential, then there exists at least one L-potential whose support is equal to $\{y\}$ for each $y \in \omega$.

Given $y \in \omega$, let

$$\boldsymbol{P}_y^{L,\omega} = \left\{ u \in \mathscr{H}_L(\omega - \{y\});\ \begin{array}{l} u \geqq 0 \text{ on } \omega - \{y\} \text{ and there are an } L\text{-potential } p \text{ and} \\ \text{a compact set } K \text{ in } \omega \text{ such that } u \leqq p \text{ on } \omega - K \end{array} \right\}.$$

By the above remark, $\boldsymbol{P}_y^{L,\omega}$ is not empty if ω admits a positive L-potential.

PROPOSITION 2.8. *Suppose* ω *is a domain which admits a positive L-potential on it. Then elements in* $\boldsymbol{P}_y^{L,\omega}$ *are proportional to each other, and they are equal to L-potentials on* ω *with support* $\{y\}$.

The proof of this proposition may be carried out in the same way as that of Proposition 4 of [7]. Remark that the minimum principle used in the proof in [7] may be replaced by the weak minimum principle, which holds on our domain ω (Proposition 2.7).

§ 3. L-semiadapted domains and L-Green functions

3.1. *L-semiadapted domain.*

Given an equation $Lu=0$ as in the previous section, we consider the function $Q_L=|a|^2+|b|^2+|q|$. Q_L is a non-negative function on Ω and belongs to $\boldsymbol{L}_{\mathrm{loc}}^{d/2}(\Omega)$. Let

$$\boldsymbol{D}_L(\omega)=\left\{f\in\boldsymbol{D}(\omega);\int_{\omega}Q_L f^2 dV<\infty\right\}.$$

If $Q_L=0$ on ω, or if ω is relatively compact, then $\boldsymbol{D}_L(\omega)=\boldsymbol{D}(\omega)$. In case $Q_L\neq 0$ on ω, then $\boldsymbol{D}_L(\omega)$ is a Hilbert space with respect to the inner product

$$D_{L,\omega}[f, g]=D_{\omega}[f, g]+\int_{\omega}Q_L f g\, dV.$$

In this case, we define $\|f\|_{L,\omega}=D_{L,\omega}[f,f]^{1/2}$. In case $Q_L=0$ on ω, we also denote by $\|f\|_{L,\omega}$ any one of $\|f\|_{D,\omega,\omega'}$. Thus, in any case, $\boldsymbol{D}_L(\omega)$ is a Hilbert space with the norm $\|f\|_{L,\omega}$.

We denote by $\boldsymbol{D}_{L,0}(\omega)$ the closure of $\boldsymbol{C}_0^1(\omega)$ in $\boldsymbol{D}_L(\omega)$, which coincides with the space $\boldsymbol{D}_L(\omega)\cap\boldsymbol{D}_0(\omega)$.

Obviously, $A_{L,\omega}$ is defined to be a continuous bilinear form on $\boldsymbol{D}_L(\omega)\times\boldsymbol{D}_L(\omega)$. A domain ω in Ω will be called *L-semiadapted* if

(i) $Q_L=0$ on ω and $1\notin\boldsymbol{D}_0(\omega)$,

or

(ii) $Q_L\neq 0$ on ω and $A_{L,\omega}$ is coercive on $\boldsymbol{D}_{L,0}(\omega)$, i.e., there is $\lambda>0$ such that $A_{L,\omega}[f,f]\geqq\lambda\|f\|_{L,\omega}^2$ for all $f\in\boldsymbol{D}_{L,0}(\omega)$.

It is easy to see that an L-adapted domain is L-semiadapted. Obviously, any subdomain of an L-semiadapted domain is L-semiadapted. (In particular, any relatively compact subdomain of an L-semiadapted domain is L-adapted.) Remark that if $|a|=|b|=0$ and $q\geqq 0$, $q\neq 0$ on ω, then ω is L-semiadapted.

LEMMA 3.1. *Let ω be an L-semiadapted domain and let ω' be any relatively compact domain such that $\bar{\omega}'\subset\omega$. The restriction mapping $\boldsymbol{D}_L(\omega)\to\boldsymbol{L}^2(\omega')$ is continuous, i.e., there is $M>0$ such that*

$$\int_{\omega'}f^2\,dV\leqq M\|f\|_{L,\omega}^2$$

for all $f\in\boldsymbol{D}_L(\omega)$.

PROOF. If $Q_L=0$ on ω, then this lemma is nothing but Lemma 1.2. Suppose $Q_L\neq 0$ on ω and suppose the lemma is not true. Then there would exist $f_n\in\boldsymbol{D}_L(\omega)$ such that $\int_{\omega'}f_n^2 dV=1$ and $\|f_n\|_{L,\omega}\to 0\ (n\to\infty)$. By Lemma

1.1, we find constants c_n such that $\int_{\omega_1}(f_n+c_n)^2dV\to 0$ for any relatively compact domain ω_1 such that $\bar{\omega}_1\subset\omega$. Taking a subsequence, we may assume $f_n+c_n\to 0$ a.e. on ω as well as $f_n\to 0$ a.e. on the set $\{x\in\omega; Q_L>0\}$. Since the last set is of positive measure, $c_n\to 0$, and hence $\int_{\omega'}f_n^2dV\to 0$, a contradiction.

3.2. *Function $h_f^{L,\omega}$ and the space $\boldsymbol{H}_L(\omega)$.*

Let ω be an L-semiadapted domain. Then, given $f\in\boldsymbol{D}_L(\omega)$, Lax-Milgram's theorem implies that there exists a unique L-harmonic function $h_f^{L,\omega}$ on ω such that $f-h_f^{L,\omega}\in\boldsymbol{D}_{L,0}(\omega)$. Obviously, the mapping $f\to h_f^{L,\omega}$ is linear.

LEMMA 3.2. *Let ω be an L-semiadapted domain and let $u\in\boldsymbol{D}_L(\omega)$ be L-harmonic on ω. If there is $g\in\boldsymbol{D}_{L,0}(\omega)$ such that $u\geqq g$ outside a compact set in ω, then $u\geqq 0$ on ω.*

PROOF. (cf. the proof of Lemme 1, a) in [7]) Since $0\leqq u^-\leqq g^-$ and $g^-\in\boldsymbol{D}_{L,0}(\omega)$, $u^-\in\boldsymbol{D}_{L,0}(\omega)$. Hence $A_{L,\omega}[u, u^-]=0$. It follows from Lemma 1.6 that $A_{L,\omega}[u^-, u^-]=0$. Since $A_{L,\omega}$ is coercive on $\boldsymbol{D}_{L,0}(\omega)$, $u^-=0$. Hence $u\geqq 0$.

COROLLARY. *Let ω be an L-semiadapted domain. If $f\in\boldsymbol{D}_L(\omega)$ and $f\geqq 0$ outside a compact set in ω, then $h_f^{L,\omega}\geqq 0$ on ω.*

PROOF. Apply the above lemma to $u=h_f^{L,\omega}$ and $g=h_f^{L,\omega}-f$.

Let $\boldsymbol{H}_L(\omega)=\{u\in\boldsymbol{D}_L(\omega);$ solution of $Lu=0$ on $\omega\}$. We may identify it with $\mathcal{H}_L(\omega)\cap\boldsymbol{D}_L(\omega)$. For an L-semiadapted domain ω, $\boldsymbol{H}_L(\omega)\cap\boldsymbol{D}_{L,0}(\omega)=\{0\}$. If $f\in\boldsymbol{D}_L(\omega)$, then $h_f^{L,\omega}\in\boldsymbol{H}_L(\omega)$ and the mapping $f\to h_f^{L,\omega}$ is continuous from $\boldsymbol{D}_L(\omega)$ into $\boldsymbol{H}_L(\omega)$.

Given $u, v\in\boldsymbol{H}_L(\omega)$, let $u\underset{\omega}{\vee}v=h^{L,\omega}_{\max(u,v)}$ and $u\underset{\omega}{\wedge}v=h^{L,\omega}_{\min(u,v)}$. Then, $u\underset{\omega}{\vee}v$, $u\underset{\omega}{\wedge}v\in\boldsymbol{H}_L(\omega)$ and $u+v=u\underset{\omega}{\vee}v+u\underset{\omega}{\wedge}v$. $\boldsymbol{H}_L(\omega)$ is a vector lattice with respect to these operations. By Lemma 3.2, we have

$$u\underset{\omega}{\wedge}v\leqq\min(u, v)\leqq\max(u, v)\leqq u\underset{\omega}{\vee}v.$$

We write $u\underset{\omega}{\perp}v$ if $[(u\underset{\omega}{\vee}0)-(u\underset{\omega}{\wedge}0)]\underset{\omega}{\wedge}[(v\underset{\omega}{\vee}0)-(v\underset{\omega}{\wedge}0)]=0$. Note that $(u\underset{\omega}{\vee}0)\underset{\omega}{\perp}(u\underset{\omega}{\wedge}0)$ for any $u\in\boldsymbol{H}_L(\omega)$.

3.3. *Green operator $G^{L,\omega}$ and existence of positive potentials.*

By the same method as Lemme 3 and Proposition 5 of [7], we obtain the following extension of Lemma 3.2.:

PROPOSITION 3.1. *Let ω be an L-semiadapated domain and let u be an L-supersolution on ω. If there is $g\in\boldsymbol{D}_{L,0}(\omega)$ such that $u\geqq g$ outside a compact* *set in ω, then $u \geqq 0$ on ω.*

COROLLARY. *If ω is an L-semiadpted domain and if g is a lower semicontinuous L-supersolution belonging to $\boldsymbol{D}_{L,0}(\omega)$, then g is an L-potential.*

PROPOSITION 3.2. *Let ω be an L-semiadapted domain. For any bounded measurable function ψ with compact support in ω, there exists a continuous function g belonging to $\boldsymbol{D}_{L,0}(\omega)$ such that*

$$(*) \qquad A_{L,\omega}[g, \varphi] = \int_\omega \psi\varphi dV$$

for all $\varphi \in \boldsymbol{D}_{L,0}(\omega)$. Furthermore, if $\psi \geqq 0$, then g is an L-potential on ω.

PROOF. By Lemma 3.1, the linear functional $f \to \int_\omega f\psi dV$ is continuous on $\boldsymbol{D}_{L,0}(\omega)$. Hence, by Lax-Milgram's theorem, there is $g \in \boldsymbol{D}_{L,0}(\omega)$ such that (*) is satisfied for all $\varphi \in \boldsymbol{D}_{L,0}(\omega)$. On each coordinate neighborhood, g is a solution of the equation $Lu = \psi$, so that, by Théorème 7.3 of [17] (also see [7], p. 310; [9], Theorem 14.1; [14], Theorem 4.7), g may be taken to be continuous. If $\psi \geqq 0$, then $A_{L,\omega}[g, \varphi] = \int_\omega \psi\varphi dV \geqq 0$ for $\varphi \geqq 0$. Hence g is an L-supersolution on ω. Then the above corollary implies that g is an L-potential.

The function g in the above proposition will be denoted by $G^{L,\omega}(\psi)$. Obviously, the mapping $\psi \to G^{L,\omega}(\psi)$ is linear and non-negative. If $\psi \geqq 0$ and $\psi \neq 0$, then $G^{L,\omega}(\psi)$ cannot vanish identically, so that it is a positive L-potential. Thus we have:

COROLLARY 1. *An L-semiadapted domain admits a positive L-potential on it.*

COROLLARY 2. *The weak minimum principle holds on an L-semiadapted domain.*

3.4. *L-Green function $g_y^{L,\omega}$.*

In the sequel of this section (§3), we shall assume that $|a| \in \boldsymbol{L}^r_{\mathrm{loc}}(\Omega)$, so that $|a|, |b|, |q|^{1/2} \in \boldsymbol{L}^r_{\mathrm{loc}}(\Omega)$ $(r > d)$. Under this assumption, the adjoint equation

$$L^*w \equiv \Delta w - \sum_j b^j \frac{\partial w}{\partial x_j} + \frac{1}{\sqrt{G}} \sum_i \frac{\partial}{\partial x_i}(\sqrt{G}\, a^i w) - qw = 0$$

can be treated in the same way as the equation $Lu = 0$ and we obtain L^*-harmonic structure on Ω. We remark that an L-semiadapted domain is also L^*-semiadapted, $\boldsymbol{D}_{L^*}(\omega) = \boldsymbol{D}_L(\omega)$ and $A_{L^*,\omega}[f, g] = A_{L,\omega}[g, f]$ for $f, g \in \boldsymbol{D}_L(\omega)$.

We may apply the arguments in sections 5 and 6 of [7] to our case and obtain

THEOREM 3.1. *Let ω be an L-semiadapted domain and let $y \in \omega$. Then there exists a unique positive potential, denoted by $g_y^{L,\omega}$, having the following property: If $\{\psi_k\}$ is a sequence of non-negative bounded measurable functions on ω such that the supports of ψ_k are compact and decrease to the point set $\{y\}$ and $\int \psi_k dV=1$ for each k, then $G^{L,\omega}(\psi_k)$ tends to $g_y^{L,\omega}$ locally uniformly on $\omega-\{y\}$. Furthermore, $g_y^{L,\omega} \in \boldsymbol{L}^1_{\mathrm{loc}}(\omega)$ and for any bounded measurable function ψ with compact support in ω, we have*

$$\int_\omega g_y^{L,\omega}(x)\psi(x)dV(x)=[G^{L^*,\omega}(\psi)](y).$$

Remark that Proposition 2.8 plays an essential role in proving the above theorem. The function $g_y^{L,\omega}$ may be called the L-Green function of ω with pole at y.

The following corollaries are easy consequences of this theorem:

COROLLARY 1. $g_y^{L,\omega}(x)=g_x^{L^*,\omega}(y)$ *for any* $x, y \in \omega$ $(x \neq y)$.

COROLLARY 2. $y \rightarrow g_y^{L,\omega}(x)$ *is continuous on* $\omega-\{x\}$ *for each* $x \in \omega$.

COROLLARY 3. *For any bounded measurable function ψ with compact support in ω, we have*

$$\int_\omega g_y^{L,\omega}\psi(y)dV(y)=G^{L,\omega}(\psi).$$

Also, the following lemma is easily shown:

LEMMA 3.3. *If $\omega_1 \subset \omega$ and $y \in \omega_1$, then there is an L-harmonic function u_y on ω_1 such that*

$$g_y^{L,\omega}=g_y^{L,\omega_1}+u_y \qquad on \qquad \omega_1.$$

3.5. *Integral representation of L-potentials.*

Now that we obtained the L-Green function, the next theorem follows from Théorème 18.2 of [6]:

THEOREM 3.2. *If v is an L-potential on an L-semiadapted domain ω, then there corresponds a unique non-negative measure μ on ω such that*

$$v(x)=\int_\omega g_y^{L,\omega}(x)\mathrm{d}\mu(y)$$

for all $x \in \omega$.

From this theorem and Lemma 3.3, we see that for any L-superharmonic function v on ω, there corresponds a non-negative measure μ on ω such that if ω' is a relatively compact domain with $\bar{\omega}' \subset \omega$ then

$$v(x)=\int_{\omega'} g_y^{L,\omega}(x)\,d\mu(y)+w(x) \qquad (x \in \omega)$$

with a function w which is L-harmonic on ω'. The measure μ is called the measure associated with v (or, the associated measure of v). If v has an L-harmonic minorant on ω, then the above expression also holds for $\omega'=\omega$, which is the Riesz representation of v. Note that $v \geqq 0$ implies $w \geqq 0$.

REMARK. We may proceed to apply the arguments in sections 6 and 7 of [7] and obtain the following results:

a) An L-superharmonic function belonging to $\boldsymbol{D}_{\mathrm{loc}}(\omega)$ is an L-supersolution on ω;

b) If ω is an L-semiadapted domain, then an L-superharmonic function belonging to $\boldsymbol{D}_{L,0}(\omega)$ is an L-potential on ω;

c) If ω is an L-semiadapted domain, then an L-superharmonic function v belonging to $\boldsymbol{D}_L(\omega)$ has an L-harmonic minorant and the greatest L-harmonic minorant of v is equal to $h_v^{L,\omega}$.

§ 4. Full-harmonic structures subordinate to the L-harmonic structure and determined by boundary conditions

4.1. *Subspaces of $\boldsymbol{D}_L(\omega)$ for an end ω.*

A domain ω in Ω will be called an *end* of Ω if it is not relatively compact and its relative boundary $\partial\omega$ is compact (may be empty). Given two ends ω and ω', we shall say that ω' is a *subend* of ω if $\omega' \subset \omega$ and $\partial\omega' \cap \omega$ is compact.

In case ω is an end of Ω, we define

$$\boldsymbol{C}^1_{\partial\omega}(\omega)=\{\varphi \in \boldsymbol{C}^1(\omega);\ \varphi=0 \text{ on } V\cap\omega \text{ for a neighborhood } V \text{ of } \partial\omega\}$$

and

$$\boldsymbol{C}^1_{\beta(\omega)}(\omega)=\left\{\varphi \in \boldsymbol{C}^1(\omega);\ \begin{matrix}\varphi=0 \text{ on an open set } \omega' \text{ such that}\\ \bar{\omega}-\omega' \text{ is compact}\end{matrix}\right\}.$$

The closure of $\boldsymbol{C}^1_{\partial\omega}(\omega)\cap\boldsymbol{D}_L(\omega)$ (resp. $\boldsymbol{C}^1_{\beta(\omega)}(\omega)\cap\boldsymbol{D}_L(\omega)$) in $\boldsymbol{D}_L(\omega)$ is denoted by $\boldsymbol{D}_{L,\partial\omega}(\omega)$ (resp. $\boldsymbol{D}_{L,\beta(\omega)}(\omega)$).

Let ω' be a subend of ω and let

$$\boldsymbol{C}^1_{\beta(\omega')}(\omega)=\{\varphi \in \boldsymbol{C}^1(\omega);\ \varphi\,|\,\omega' \in \boldsymbol{C}^1_{\beta(\omega')}(\omega')\}$$

and

$$\boldsymbol{C}^1_{\beta(\omega)-\beta(\omega')}(\omega)=\left\{\varphi \in \boldsymbol{C}^1(\omega);\ \begin{matrix}\varphi=0 \text{ on an open set } \omega'' \text{ such}\\ \text{that } \bar{\omega}-\omega'-\omega'' \text{ is compact}\end{matrix}\right\}.$$

The closure of $\boldsymbol{C}^1_{\beta(\omega')}(\omega)\cap\boldsymbol{D}_L(\omega)$(resp. $\boldsymbol{C}^1_{\beta(\omega)-\beta(\omega')}(\omega)\cap\boldsymbol{D}_L(\omega)$) in $\boldsymbol{D}_L(\omega)$ is denoted by $\boldsymbol{D}_{L,\beta(\omega')}(\omega)$ (resp. $\boldsymbol{D}_{L,\beta(\omega)-\beta(\omega')}(\omega)$). If $\bar{\omega}-\omega'$ is compact, in particular if

$\omega'=\omega$, then $\boldsymbol{D}_{L,\beta(\omega')}(\omega)=\boldsymbol{D}_{L,\beta(\omega)}(\omega)$ and $\boldsymbol{D}_{L,\beta(\omega)-\beta(\omega')}(\omega)=\boldsymbol{D}_L(\omega)$. The spaces $\boldsymbol{D}_{L,\partial\omega}(\omega)$, $\boldsymbol{D}_{L,\beta(\omega)}(\omega)$, $\boldsymbol{D}_{L,\beta(\omega')}(\omega)$ and $\boldsymbol{D}_{L,\beta(\omega)-\beta(\omega')}(\omega)$ are closed subspaces of $\boldsymbol{D}_L(\omega)$ containing $\boldsymbol{D}_{L,0}(\omega)$. It is easy to see that

$$\boldsymbol{D}_{L,\beta(\omega)}(\omega)\cap\boldsymbol{D}_{L,\partial\omega}(\omega)=\boldsymbol{D}_{L,0}(\omega)$$

and

$$\boldsymbol{D}_{L,\beta(\omega')}(\omega)\cap\boldsymbol{D}_{L,\beta(\omega)-\beta(\omega')}(\omega)=\boldsymbol{D}_{L,\beta(\omega)}(\omega).$$

We can easily obtain results corresponding to Lemmas 1.4, 1.5 and 1.7 for the spaces $\boldsymbol{D}_{L,0}(\omega)$, $\boldsymbol{D}_{L,\partial\omega}(\omega)$, $\boldsymbol{D}_{L,\beta(\omega)}(\omega)$, $\boldsymbol{D}_{L,\beta(\omega')}(\omega)$ and $\boldsymbol{D}_{L,\beta(\omega)-\beta(\omega')}(\omega)$. In particular we have the following:

(a) If ω is an end, ω' is a subend of ω and $f\in\boldsymbol{D}_{L,\partial\omega'}(\omega')$, then $f^{\#}(=f$ on ω' and $=0$ on $\omega-\omega')$ belongs to $\boldsymbol{D}_{L,\beta(\omega)-\beta(\omega')}(\omega)\cap\boldsymbol{D}_{L,\partial\omega}(\omega)$.

(b) Let $\varphi\in\boldsymbol{C}^1(\omega)$ and suppose there is a compact set K in ω such that φ is constant on each component of $\omega-K$. If, furthermore, $\varphi\in\boldsymbol{C}^1_{\partial\omega}(\omega)$ (resp. $\boldsymbol{C}^1_{\beta(\omega)}(\omega)$, $\boldsymbol{C}^1_{\beta(\omega')}(\omega)$, $\boldsymbol{C}^1_{\beta(\omega)-\beta(\omega')}(\omega)$), then $f\varphi\in\boldsymbol{D}_{L,\partial\omega}(\omega)$ (resp. $\boldsymbol{D}_{L,\beta(\omega)}(\omega)$, $\boldsymbol{D}_{L,\beta(\omega')}(\omega)$, $\boldsymbol{D}_{L,\beta(\omega)-\beta(\omega')}(\omega)$) for any $f\in\boldsymbol{D}_L(\omega)$.

(c–1) If $f\in\boldsymbol{D}_L(\omega)$ and $0\leqq|f|\leqq g$ on $V\cap\omega$ for some neighborhood V of $\partial\omega$ and $g\in\boldsymbol{D}_{L,\partial\omega}(\omega)$, then $f\in\boldsymbol{D}_{L,\partial\omega}(\omega)$.

(c–2) Let ω' be a subend of ω. If $f\in\boldsymbol{D}_L(\omega)$ and $0\leqq|f|\leqq g$ on ω'' for some open set ω'' such that $\bar{\omega}'-\omega''$ is compact (resp. $\bar{\omega}-\omega'-\omega''$ is compact) and $g\in\boldsymbol{D}_{L,\beta(\omega')}(\omega)$ (resp. $g\in\boldsymbol{D}_{L,\beta(\omega)-\beta(\omega')}(\omega)$), then $f\in\boldsymbol{D}_{L,\beta(\omega')}(\omega)$ (resp. $f\in\boldsymbol{D}_{L,\beta(\omega)-\beta(\omega')}(\omega)$).

Next, we define the subspaces $\boldsymbol{H}_{L,\partial\omega}(\omega)$, $\boldsymbol{H}_{L,\beta(\omega)}(\omega)$, $\boldsymbol{H}_{L,\beta(\omega')}(\omega)$ and $\boldsymbol{H}_{L,\beta(\omega)-\beta(\omega')}(\omega)$ as the intersections of $\boldsymbol{H}_L(\omega)$ with the spaces $\boldsymbol{D}_{L,\partial\omega}(\omega)$, $\boldsymbol{D}_{L,\beta(\omega)}(\omega)$, $\boldsymbol{D}_{L,\beta(\omega')}(\omega)$ and $\boldsymbol{D}_{L,\beta(\omega)-\beta(\omega')}(\omega)$, respectively. By the continuity of the mapping $f\to A_{L,\omega}[f,\varphi]$ on $\boldsymbol{D}_L(\omega)$ for each $\varphi\in\boldsymbol{C}^1_0(\omega)$, we infer that these are closed subspaces of $\boldsymbol{D}_L(\omega)$.

4.2. *L-full-adapted end.*

An end ω is called *L-full-adapted* if it is L-semiadapted and $1\notin\boldsymbol{D}_{\partial\omega}(\omega)$ $(\equiv\boldsymbol{D}_{\triangle,\partial\omega}(\omega))$. The latter condition implies that $\partial\omega\neq\emptyset$. Obviously, any end contained in an L-full-adapted end is L-full-adapted. Remark that if K is a closed ball in Ω and if $\Omega-K$ is L-semiadapted, then $\Omega-K$ is an L-full-adapted end. Thus, for instance, if $|a|=|b|=0$ and $q\geqq0$ on Ω, then any end ω such that $\Omega-\omega$ contains an open ball is L-full-adapted.

LEMMA 4.1. *Let ω be an L-full-adapted end and let ω' be a subend of ω. Then*

$$(1)\qquad \begin{cases} \boldsymbol{H}_{L,\beta(\omega)}(\omega)\cap\boldsymbol{H}_{L,\partial\omega}(\omega)=\{0\}, \\ \boldsymbol{H}_{L,\beta(\omega')}(\omega)\cap\boldsymbol{H}_{L,\beta(\omega)-\beta(\omega')}(\omega)=\boldsymbol{H}_{L,\beta(\omega)}(\omega), \end{cases}$$

and

$$(2)\qquad \begin{cases} \boldsymbol{H}_{L,\beta(\omega)}(\omega)+\boldsymbol{H}_{L,\partial\omega}(\omega)=\boldsymbol{H}_L(\omega), \\ \boldsymbol{H}_{L,\beta(\omega')}(\omega)+\boldsymbol{H}_{L,\beta(\omega)-\beta(\omega')}(\omega)=\boldsymbol{H}_L(\omega). \end{cases}$$

PROOF. By definition

$$\boldsymbol{H}_{L,\beta(\omega)}(\omega)\cap\boldsymbol{H}_{L,\partial\omega}(\omega)=\boldsymbol{H}_L(\omega)\cap\boldsymbol{D}_{L,0}(\omega)$$

and

$$\boldsymbol{H}_{L,\beta(\omega')}(\omega)\cap\boldsymbol{H}_{L,\beta(\omega)-\beta(\omega')}(\omega)=\boldsymbol{H}_L(\omega)\cap\boldsymbol{D}_{L,\beta(\omega)}(\omega)=\boldsymbol{H}_{L,\beta(\omega)}(\omega).$$

Since ω is L-semiadapted, we have $\boldsymbol{H}_L(\omega)\cap\boldsymbol{D}_{L,0}(\omega)=\{0\}$.

Next let φ be a function given in (b) in 4.1. We can choose φ in such a way that $\varphi\in\boldsymbol{C}^1_{\beta(\omega)}(\omega)$ and $1-\varphi\in\boldsymbol{C}^1_{\partial\omega}(\omega)$ (resp. $\varphi\in\boldsymbol{C}^1_{\beta(\omega')}(\omega)$ and $1-\varphi\in\boldsymbol{C}^1_{\beta(\omega)-\beta(\omega')}(\omega)$). For $u\in\boldsymbol{H}_L(\omega)$, we have $u\varphi\in\boldsymbol{D}_{L,\beta(\omega)}(\omega)$ and $u(1-\varphi)\in\boldsymbol{D}_{L,\partial\omega}(\omega)$ (resp. $u\varphi\in\boldsymbol{D}_{L,\beta(\omega')}(\omega)$ and $u(1-\varphi)\in\boldsymbol{D}_{L,\beta(\omega)-\beta(\omega')}(\omega)$). Since ω is L-semiadapted, Lax-Milgram's theorem asserts the existence of $g\in\boldsymbol{D}_{L,0}(\omega)$ such that $A_{L,\omega}[g,\psi]=A_{L,\omega}[u\varphi,\psi]$ for all $\psi\in\boldsymbol{D}_{L,0}(\omega)$. Then $u_1\equiv u\varphi-g$ belongs to $\boldsymbol{H}_L(\omega)\cap\boldsymbol{D}_{L,\beta(\omega)}(\omega)=\boldsymbol{H}_{L,\beta(\omega)}(\omega)$ (resp. $\boldsymbol{H}_L(\omega)\cap\boldsymbol{D}_{L,\beta(\omega')}(\omega)=\boldsymbol{H}_{L,\beta(\omega')}(\omega)$). Furthermore, $u-u_1=u(1-\varphi)+g$ belongs to $\boldsymbol{H}_L(\omega)\cap\boldsymbol{D}_{L,\partial\omega}(\omega)=\boldsymbol{H}_{L,\partial\omega}(\omega)$ (resp. $\boldsymbol{H}_L(\omega)\cap\boldsymbol{D}_{L,\beta(\omega)-\beta(\omega')}(\omega)=\boldsymbol{H}_{L,\beta(\omega)-\beta(\omega')}(\omega)$). Hence we have (2).

By the above lemma, any $u\in\boldsymbol{H}_{L,\partial\omega}(\omega)$ can be decomposed into $u=u_1+u_2$ with $u_1\in\boldsymbol{H}_{L,\beta(\omega')}(\omega)\cap\boldsymbol{H}_{L,\partial\omega}(\omega)$ and $u_2\in\boldsymbol{H}_{L,\beta(\omega)-\beta(\omega')}(\omega)\cap\boldsymbol{H}_{L,\partial\omega}(\omega)$ and this decomposition is unique. We shall denote u_1 by $u_{\beta(\omega')}$ and u_2 by $u_{\beta(\omega)-\beta(\omega')}$. Obviously, the mappings $u\to u_{\beta(\omega')}$ and $u\to u_{\beta(\omega)-\beta(\omega')}$ are linear.

The spaces $\boldsymbol{H}_{L,\partial\omega}(\omega)$, $\boldsymbol{H}_{L,\beta(\omega')}(\omega)$ and $\boldsymbol{H}_{L,\beta(\omega)-\beta(\omega')}(\omega)$ are closed under operations $\underset{\omega}{\vee}$ and $\underset{\omega}{\wedge}$. Furthermore we have

LEMMA 4.2. *If* $u\in\boldsymbol{H}_{L,\beta(\omega')}(\omega)\cap\boldsymbol{H}_{L,\partial\omega}(\omega)$ *and* $v\in\boldsymbol{H}_{L,\beta(\omega)-\beta(\omega')}(\omega)\cap\boldsymbol{H}_{L,\partial\omega}(\omega)$, *then* $u\underset{\omega}{\perp}v$.

PROOF. It is enough to show the case $u\geqq 0$ and $v\geqq 0$. Then $0\leqq\min(u,v)\leqq u$ and $0\leqq\min(u,v)\leqq v$ imply that $\min(u,v)\in\boldsymbol{D}_{L,\beta(\omega')}(\omega)\cap\boldsymbol{D}_{L,\beta(\omega)-\beta(\omega')}(\omega)\cap\boldsymbol{D}_{L,\partial\omega}(\omega)=\boldsymbol{D}_{L,0}(\omega)$. Hence $u\underset{\omega}{\wedge}v=0$.

4.3. *Operator* $S_0^{L,\omega}$.

LEMMA 4.3. *Let* ω *be an L-full-adapted end. For a given* $f\in\boldsymbol{D}_L(\omega)$, *there exists a unique* $u\in\boldsymbol{H}_{L,\beta(\omega)}(\omega)$ *such that* $u-f\in\boldsymbol{D}_{L,\partial\omega}(\omega)$.

PROOF. Choose $\varphi_0\in\boldsymbol{C}^1_{\beta(\omega)}(\omega)$ such that $\varphi_0=1$ on $V\cap\omega$ for some neighborhood V of $\partial\omega$. Then $f\varphi_0\in\boldsymbol{D}_{L,\beta(\omega)}(\omega)$ and $f\varphi_0-f\in\boldsymbol{D}_{L,\partial\omega}(\omega)$. Hence $u=h^{L,\omega}_{f\varphi_0}$ belongs to $\boldsymbol{H}_{L,\beta(\omega)}(\omega)$ and $u-f\in\boldsymbol{D}_{L,\partial\omega}(\omega)$. If u_1 and u_2 both satisfy the condition of the lemma, then $u_1-u_2 \in \boldsymbol{H}_{L,\beta(\omega)}(\omega)\cap\boldsymbol{H}_{L,\partial\omega}(\omega)=\{0\}$, so that $u_1=u_2$.

The function u in the above lemma will be denoted by $S_0^{L,\omega}(f)$. Obviously, the mapping $f\rightarrow S_0^{L,\omega}(f)$ is linear. Furthermore, the above proof shows that this mapping is continuous from $\boldsymbol{D}_L(\omega)$ into $\boldsymbol{H}_{L,\beta(\omega)}(\omega)$. In case $f \in \boldsymbol{D}_L(\omega_0)$ for some $\omega_0\supset\omega$, we also write $S_0^{L,\omega}(f)$ in place of $S_0^{L,\omega}(f|\omega)$.

LEMMA 4.4. *Let ω be an L-full-adapted end and let $f \in \boldsymbol{D}_L(\omega)$. If $f\geqq g$ on $V\cap\omega$ for some $g \in \boldsymbol{D}_{L,\partial\omega}(\omega)$ and a neighborhood V of $\partial\omega$, then $S_0^{L,\omega}(f)\geqq 0$.*

PROOF. Let φ_0 be the function in the proof of the above lemma. Then $S_0^{L,\omega}(f)\geqq S_0^{L,\omega}(f)-f\varphi_0+g\varphi_0$ on $V'\cap\omega$ for a neighborhood V' of $\partial\omega$ and also outside the support of φ_0. Since $S_0^{L,\omega}(f)-f\varphi_0+g\varphi_0 \in \boldsymbol{D}_{L,0}(\omega)$, Lemma 3.2 implies that $S_0^{L,\omega}(f)\geqq 0$.

4.4. *Space $\boldsymbol{R}$ for boundary condition.*

Let ω_0 be an L-full-adapted end and fix it throughout the rest of this section. We consider a subspace $\boldsymbol{R}$ of $\boldsymbol{H}_{L,\partial\omega_0}(\omega_0)$ which satisfies the following set of conditions:

(R)
- (i) $\boldsymbol{R}$ is a closed subspace of $\boldsymbol{H}_{L,\partial\omega_0}(\omega_0)$ closed under operations $\underset{\omega_0}{\vee}$ and $\underset{\omega_0}{\wedge}$.
- (ii) A_{L,ω_0} is coercive on $\boldsymbol{R}+\boldsymbol{D}_{L,0}(\omega_0)$, i.e., there is $\lambda_0>0$ such that $A_{L,\omega_0}[f,f]\geqq\lambda_0\|f\|_{L,\omega_0}^2$ for all $f \in \boldsymbol{R}+\boldsymbol{D}_{L,0}(\omega_0)$.
- (iii) For any subend ω of ω_0, $u_{\beta(\omega)} \in \boldsymbol{R}$ whenever $u \in \boldsymbol{R}$.

For example, $\boldsymbol{R}=\{0\}$ satisfies (R). If $|a|=|b|=0$ and $q\geqq 0$ on ω_0, then $\boldsymbol{R}=\boldsymbol{H}_{L,\partial\omega_0}(\omega_0)$ satisfies (R).

Given $\boldsymbol{R}$ satisfying (R) and a subend ω of ω_0, we define

$$\boldsymbol{R}(\omega)=\left\{ u \in \boldsymbol{H}_{L,\partial\omega}(\omega);\ \begin{array}{l}\text{there are } v \in \boldsymbol{R} \text{ and } g \in \boldsymbol{D}_{L,0}(\omega_0)\\ \text{such that } u^{\#}=v+g\end{array}\right\},$$

where $u^{\#}=u$ on ω and $=0$ on $\omega_0-\omega$. Obviously, $\boldsymbol{R}(\omega_0)=\boldsymbol{R}$. Note that $u^{\#} \in \boldsymbol{D}_{L,\beta(\omega_0)-\beta(\omega)}(\omega_0)\cap\boldsymbol{D}_{L,\partial\omega_0}(\omega_0)$. For each $u \in \boldsymbol{R}(\omega)$, the corresponding $v \in \boldsymbol{R}$ is uniquely determined and belongs to $\boldsymbol{H}_{L,\beta(\omega_0)-\beta(\omega)}(\omega_0)\cap\boldsymbol{H}_{L,\partial\omega_0}(\omega_0)$. In fact, $v=h_{u^{\#}}^{L,\omega_0}$.

Next, let ω be a subend of ω_0 and let

$$\tilde{\boldsymbol{R}}(\omega)=\left\{ u \in \boldsymbol{H}_L(\omega);\ \begin{array}{l}\text{there are } v \in \boldsymbol{R}\cap\boldsymbol{H}_{L,\beta(\omega_0)-\beta(\omega)}(\omega_0) \text{ and}\\ g \in \boldsymbol{D}_{L,\beta(\omega)}(\omega) \text{ such that } u=v+g \text{ on } \omega\end{array}\right\}.$$

For each $u \in \tilde{\boldsymbol{R}}(\omega)$, the corresponding $v \in \boldsymbol{R}\cap\boldsymbol{H}_{L,\beta(\omega_0)-\beta(\omega)}(\omega_0)$ is uniquely determined. For, if $v_1+g_1=v_2+g_2$ on ω with $v_1,\ v_2 \in \boldsymbol{R}\cap\boldsymbol{H}_{L,\beta(\omega_0)-\beta(\omega)}(\omega_0)$ and g_1,

$g_2 \in \boldsymbol{D}_{L,\beta(\omega)}(\omega)$, then $v_1 - v_2 \in \boldsymbol{H}_{L,\partial\omega_0}(\omega_0) \cap \boldsymbol{H}_{L,\beta(\omega_0)-\beta(\omega)}(\omega_0) \cap \boldsymbol{H}_{L,\beta(\omega)}(\omega_0) = \{0\}$. The mapping $u \to v$ is obviously linear.

LEMMA 4.5. (a) *$\tilde{\boldsymbol{R}}(\omega)$ is a linear subspace of $\boldsymbol{H}_L(\omega)$ and is closed under operations $\underset{\omega}{\vee}$ and $\underset{\omega}{\wedge}$. In fact, $v_1 \underset{\omega_0}{\vee} v_2$ corresponds to $u_1 \underset{\omega}{\vee} u_2$, if $v_i \in \boldsymbol{R}$ corresponds to $u_i \in \tilde{\boldsymbol{R}}(\omega)$, $i=1, 2$.*

(b) *$\boldsymbol{R}(\omega) = \tilde{\boldsymbol{R}}(\omega) \cap \boldsymbol{H}_{L,\partial\omega}(\omega)$ and is a closed subspace of $\boldsymbol{H}_{L,\partial\omega}(\omega)$.*

(c) *$A_{L,\omega}$ is coercive on $\boldsymbol{R}(\omega) + \boldsymbol{D}_{L,0}(\omega)$.*

PROOF. (a) It is obvious that $\tilde{\boldsymbol{R}}(\omega)$ is a linear subspace of $\boldsymbol{H}_L(\omega)$. Now, let $u \in \tilde{\boldsymbol{R}}(\omega)$ and $u = v + g$ on ω with $v \in \boldsymbol{R} \cap \boldsymbol{H}_{L,\beta(\omega_0)-\beta(\omega)}(\omega_0)$ and $g \in \boldsymbol{D}_{L,\beta(\omega)}(\omega)$. Let $f = u \underset{\omega}{\vee} 0 - \max(u, 0)$. Then $f \in \boldsymbol{D}_{L,0}(\omega)$, and hence $f^{\#} \in \boldsymbol{D}_{L,0}(\omega_0)$. Next let $\tilde{g} = \max(g, -v) - \max(0, -v)$ on ω. Since $|\tilde{g}| \leqq |g|$, $\tilde{g} \in \boldsymbol{D}_{L,\beta(\omega)}(\omega)$. On the other hand

$$\tilde{g} = \max(v+g, 0) - \max(v, 0) = \max(u, 0) - \max(v, 0) \quad \text{on } \omega.$$

Hence, $u \underset{\omega}{\vee} 0 - v \underset{\omega_0}{\vee} 0 = f + \tilde{g} + \max(v, 0) - v \underset{\omega_0}{\vee} 0$ on ω. Since

$$f + \tilde{g} + [\max(v, 0) - v \underset{\omega_0}{\vee} 0] \,|\, \omega \in \boldsymbol{D}_{L,\beta(\omega)}(\omega)$$

and $v \underset{\omega_0}{\vee} 0 \in \boldsymbol{R} \cap \boldsymbol{H}_{L,\beta(\omega_0)-\beta(\omega)}(\omega_0)$, we conclude that $u \underset{\omega}{\vee} 0 \in \tilde{\boldsymbol{R}}(\omega)$ and $v \underset{\omega_0}{\vee} 0$ corresponds to $u \underset{\omega}{\vee} 0$. Thus we have (a).

(b) The equality $\boldsymbol{R}(\omega) = \tilde{\boldsymbol{R}}(\omega) \cap \boldsymbol{H}_{L,\partial\omega}(\omega)$ is easily seen from the definitions. Since ω_0 is L-semiadapted, the mapping $u \to h_{u^{\#}}^{L,\omega_0}$ is continuous from $\boldsymbol{H}_{L,\partial\omega}(\omega)$ into $\boldsymbol{H}_{L,\partial\omega_0}(\omega_0)$. Since

$$\boldsymbol{R}(\omega) = \{u \in \boldsymbol{H}_{L,\partial\omega}(\omega);\ h_{u^{\#}}^{L,\omega_0} \in \boldsymbol{R}\}$$

and $\boldsymbol{R}$ is closed in $\boldsymbol{H}_{L,\partial\omega_0}(\omega_0)$, we see that $\boldsymbol{R}(\omega)$ is closed in $\boldsymbol{H}_{L,\partial\omega}(\omega)$.

(c) Since $A_{L,\omega}[f, f] = A_{L,\omega_0}[f^{\#}, f^{\#}]$, $\|f\|_{L,\omega} = \|f^{\#}\|_{L,\omega_0}$ and $f^{\#} \in \boldsymbol{R} + \boldsymbol{D}_{L,0}(\omega_0)$ for any $f \in \boldsymbol{R}(\omega) + \boldsymbol{D}_{L,0}(\omega)$, the coerciveness of $A_{L,\omega}$ on $\boldsymbol{R}(\omega) + \boldsymbol{D}_{L,0}(\omega)$ follows from that of A_{L,ω_0} on $\boldsymbol{R} + \boldsymbol{D}_{L,0}(\omega_0)$.

4.5. *Bilinear form B for boundary condition.*

Next we consider a bilinear form $B[u, v]$ on $\boldsymbol{R} \times \boldsymbol{R}$ satisfying the following set of conditions:

$$(B)\ \begin{cases} \text{(i)} & B[u, v] \text{ is continuous on } \boldsymbol{R} \times \boldsymbol{R}. \\ \text{(ii)} & B[u, u] \geqq 0 \text{ for any } u \in \boldsymbol{R}. \\ \text{(iii)} & \text{If } u, v \in \boldsymbol{R} \text{ and } u \underset{\omega}{\perp} v, \text{ then } B[u, v] = 0. \end{cases}$$

For a subend ω of ω_0, we define a bilinear form B_ω on $\tilde{\boldsymbol{R}}(\omega) \times \boldsymbol{R}(\omega)$ by

$$B_{\omega}[u_1, u_2]=B[v_1, v_2],$$

where v_1 and v_2 are functions in $\boldsymbol{R}$ corresponding to u_1 and u_2, respectively. It is easy to see that $B_{\omega}[u_1, u_2]$ is continuous on $\boldsymbol{R}(\omega)\times\boldsymbol{R}(\omega)$. Furthermore, $B_{\omega}[u, u]\geqq 0$ for any $u\in\boldsymbol{R}(\omega)$.

4.6. *Operator* $S_{\boldsymbol{R},B}^{L,\omega}$.

Let ω be a subend of ω_0. By Lemma 4.5, c) and the above observation, we see that the bilinear form $A_{L,\omega}[u_1, u_2]+B_{\omega}[u_1, u_2]$ is continuous and coercive on $\boldsymbol{R}(\omega)$. Thus, given $f\in\boldsymbol{D}_L(\omega)$, there is a unique $u_0\in\boldsymbol{R}(\omega)$ such that

$$A_{L,\omega}[u_0, u]+B_{\omega}[u_0, u]=A_{L,\omega}[S_0^{L,\omega}(f), u]$$

for all $u\in\boldsymbol{R}(\omega)$, by Lax-Milgram's theorem. The function $S_0^{L,\omega}(f)-u_0$ will be denoted by $S_{\boldsymbol{R},B}^{L,\omega}(f)$. Obviously, if $\boldsymbol{R}=\{0\}$ (and hence $B=0$), then $S_{\{0\},0}^{L,\omega}(f)$ $=S_0^{L,\omega}(f)$.

$w=S_{\boldsymbol{R},B}^{L,\omega}(f)$ is characterized by the following two conditions:

(a) $w-S_0^{L,\omega}(f)\in\boldsymbol{R}(\omega)$;

(b) $A_{L,\omega}[w, u]+B_{\omega}[w, u]=0$ for all $u\in\boldsymbol{R}(\omega)$.

Here we remark that $S_0^{L,\omega}(f)\in\tilde{\boldsymbol{R}}(\omega)$ with the corresponding function $0\in\boldsymbol{R}$, so that $w\in\tilde{\boldsymbol{R}}(\omega)$. The mapping $f\rightarrow S_{\boldsymbol{R},B}^{L,\omega}(f)$ is obviously linear. Since $f\rightarrow S_0^{L,\omega}(f)$ is continuous, it also follows from Lax-Milgram's theorem that the mapping $f\rightarrow S_{\boldsymbol{R},B}^{L,\omega}(f)$ is continuous from $\boldsymbol{D}_L(\omega)$ into $\boldsymbol{H}_L(\omega)$.

4.7. *Minimum principles.*

PROPOSITION 4.1. *Let ω be a subend of ω_0. If $f\in\boldsymbol{D}_L(\omega)$ and $f\geqq g$ on $V\cap\omega$ for some $g\in\boldsymbol{D}_{L,\partial\omega}(\omega)$ and a neighborhood V of $\partial\omega$, then $0\leqq S_0^{L,\omega}(f)$ $\leqq S_{\boldsymbol{R},B}^{L,\omega}(f)$ on ω.*

PROOF. Let $w=S_{\boldsymbol{R},B}^{L,\omega}(f)$ and $u=w-S_0^{L,\omega}(f)$. Since $S_0^{L,\omega}(f)\geqq 0$ by Lemma 4.4, $w\geqq u$. Hence $0\leqq -(w\underset{\omega}{\vee}0)\leqq -(u\underset{\omega}{\wedge}0)$. Since $-(u\underset{\omega}{\wedge}0)\in\boldsymbol{R}(\omega)$ (Lemma 4.5), $-(w\underset{\omega}{\wedge}0)\in\boldsymbol{H}_{L,\partial\omega}(\omega)$. Since $w\in\tilde{\boldsymbol{R}}(\omega)$, Lemma 4.5 implies that $(w\underset{\omega}{\wedge}0)\in\boldsymbol{R}(\omega)$. Hence

$$A_{L,\omega}[w, w\underset{\omega}{\wedge}0]+B_{\omega}[w, w\underset{\omega}{\wedge}0]=0. \tag{1}$$

Let $v\in\boldsymbol{R}\cap\boldsymbol{H}_{L,\beta(\omega_0)-\beta(\omega)}(\omega_0)$ correspond to w. Then, by Lemma 4.5 again, $v\underset{\omega_0}{\wedge}0$ corresponds to $w\underset{\omega}{\wedge}0$. Since $(v\underset{\omega_0}{\wedge}0)\perp(v\underset{\omega_0}{\vee}0)$, we have

$$B_{\omega}[w, w\underset{\omega}{\wedge}0]=B[v, v\underset{\omega_0}{\wedge}0]=B[v\underset{\omega_0}{\wedge}0, v\underset{\omega_0}{\wedge}0]\geqq 0.$$

Hence, (1) implies

$$A_{L,\omega}[w, w\underset{\omega}{\wedge}0]\leqq 0. \tag{2}$$

Now, since $w \underset{\omega}{\wedge} 0 + w^- \in \boldsymbol{D}_{L,0}(\omega)$,

$$A_{L,\omega}[w, w \underset{\omega}{\wedge} 0] = A_{L,\omega}[w, -w^-] = A_{L,\omega}[w^-, w^-].$$

Thus, (2) and the coerciveness of $A_{L,\omega}$ on $\boldsymbol{R}(\omega) + \boldsymbol{D}_{L,0}(\omega)$ imply that $w^- = 0$ on ω. Hence $w \geqq 0$. It follows that $u \geqq -S_0^{L,\omega}(f)$. Thus, $0 \geqq u \underset{\omega}{\wedge} 0 \geqq -S_0^{L,\omega}(f)$. Since $S_0^{L,\omega}(f) \in \boldsymbol{D}_{L,\beta(\omega)}(\omega)$, $u \underset{\omega}{\wedge} 0 \in \boldsymbol{H}_{L,\beta(\omega)}(\omega)$. Hence $u \underset{\omega}{\wedge} 0 \in \boldsymbol{H}_{L,\beta(\omega)}(\omega) \cap \boldsymbol{R}(\omega)$ $\subset \boldsymbol{H}_{L,\beta(\omega)}(\omega) \cap \boldsymbol{H}_{L,\partial\omega}(\omega) = \{0\}$, i.e., $u \underset{\omega}{\wedge} 0 = 0$. Therefore, $u \geqq 0$ and the proposition is proved.

PROPOSITION 4.2. *Let ω be a subend of ω_0 and let $f \in \boldsymbol{D}_L(\omega)$. If* $\liminf_{x \to \xi, x \in \omega} f(x) \geqq 0$ *for all $\xi \in \partial\omega \cap \omega_0$ and if there is $g \in \boldsymbol{D}_{L,\partial\omega_0}(\omega_0)$ such that $f \geqq g$ on $V \cap \omega$ for some neighborhood V of $\partial\omega_0$, then $S_{\boldsymbol{R};B}^{L;\omega}(f) \geqq 0$.*

PROOF. Let $\varphi_0 \in \boldsymbol{C}_0^1(\omega_0)$ be non-negative on ω_0 and equal to 1 in a neighborhood of $\partial\omega \cap \omega_0$. Then, for any $\varepsilon > 0$, $f + \varepsilon\varphi_0 \geqq 0$ on $V' \cap \omega$ for a neighborhood V' of $\partial\omega \cap \omega_0$ and $f + \varepsilon\varphi_0 \geqq g$ on $V \cap \omega$. Hence $S_{\boldsymbol{R};B}^{L;\omega}(f)$ $+ \varepsilon S_{\boldsymbol{R};B}^{L;\omega}(\varphi_0) \geqq 0$ on ω by the above proposition. Since ε is arbitrary, $S_{\boldsymbol{R};B}^{L;\omega}(f) \geqq 0$.

COROLLARY. *Let ω be a subend of ω_0. If $f_1, f_2 \in \boldsymbol{D}_L(\omega) \cap \boldsymbol{C}(\bar{\omega} \cap \omega_0)$, $f_1 = f_2$ on $\partial\omega \cap \omega_0$, and if there are $g_1, g_2 \in \boldsymbol{D}_{L,\partial\omega_0}(\omega_0)$ such that $f_1 = g_1$, $f_2 = g_2$ on $V \cap \omega$ for some neighborhood V of $\partial\omega_0$, then $S_{\boldsymbol{R};B}^{L;\omega}(f_1) = S_{\boldsymbol{R};B}^{L;\omega}(f_2)$. In particular, if f_1, $f_2 \in \boldsymbol{D}_{L,\partial\omega_0}(\omega_0) \cap \boldsymbol{C}(\omega_0)$ and $f_1 = f_2$ on $\partial\omega \cap \omega_0$, then we have the same conclusion.*

REMARK. In Proposition 4.2 and its corollary, the conditions involving $g \in \boldsymbol{D}_{L,\partial\omega_0}(\omega_0)$ is superfluous in case $\bar{\omega} \subset \omega_0$.

4.8. *Consistency.*

PROPOSITION 4.3. *If ω is a subend of ω_0 and ω' is a subend of ω, then*

$$S_{\boldsymbol{R};B}^{L;\omega'}(S_{\boldsymbol{R};B}^{L;\omega}(f)) = S_{\boldsymbol{R};B}^{L;\omega}(f) \qquad on \quad \omega'$$

for any $f \in \boldsymbol{D}_L(\omega)$.

PROOF. Let $f_0 = S_0^{L,\omega}(f)$, $f_1 = S_{\boldsymbol{R};B}^{L;\omega}(f)$ and

$$f_2 = \begin{cases} S_0^{L,\omega'}(f_1) & \text{on} \quad \omega' \\ f_1 & \text{on} \quad \omega - \omega' \end{cases}$$

Then, $f_0, f_1, f_2 \in \boldsymbol{D}_L(\omega)$. By the definition of $S_{\boldsymbol{R};B}^{L;\omega}(f)$, $(f_1 - f_0)^\# = v_0 + g_0$ with $v_0 \in \boldsymbol{R} \cap \boldsymbol{H}_{L,\beta(\omega_0)-\beta(\omega)}(\omega_0)$ and $g_0 \in \boldsymbol{D}_{L,0}(\omega_0)$. Now

$$(f_1 - f_2)^\# = v_0 + g_0 + (f_0 - f_2)^\#$$

$$=(v_0)_{\beta(\omega_0)-\beta(\omega')}+(v_0)_{\beta(\omega')}+g_0+(f_0-f_2)^{\#}.$$

It is easy to see that $(f_0-f_2)^{\#}\in \boldsymbol{D}_{L,\beta(\omega')}(\omega_0)$. Hence

$$(v_0)_{\beta(\omega')}+g_0+(f_0-f_2)^{\#}\in \boldsymbol{D}_{L,\beta(\omega')}(\omega_0).$$

Therefore $w-S_0^{L,\omega'}(f_1)=(f_1-f_2)|\omega'\in \tilde{\boldsymbol{R}}(\omega')$. Obviously $w-S_0^{L,\omega'}(f_1)\in \boldsymbol{H}_{L,\partial\omega'}(\omega')$. Hence $w-S_0^{L,\omega'}(f_1)\in \boldsymbol{R}(\omega')$.

Next, we shall show that $A_{L,\omega'}[w, u]+B_{\omega'}[w, u]=0$ for all $u\in \boldsymbol{R}(\omega')$. Given $u\in \boldsymbol{R}(\omega')$, let $u^{\#}=v+g$ with $v\in \boldsymbol{R}\cap \boldsymbol{H}_{L,\beta(\omega_0)-\beta(\omega')}(\omega')$ and $g\in \boldsymbol{D}_{L,0}(\omega_0)$. Put $u_1=v-S_0^{L,\omega}(v)$ on ω. The $u_1\in \boldsymbol{H}_{L,\partial\omega}(\omega)\cap\tilde{\boldsymbol{R}}(\omega)=\boldsymbol{R}(\omega)$. Therefore

$$(*)\qquad A_{L,\omega}[f_1, u_1]+B_\omega[f_1, u_1]=0$$

by the definition of f_1. Now, $u^{\#}-u_1=g+S_0^{L,\omega}(v)$ on ω and $u^{\#}|\omega\in \boldsymbol{D}_{L,\partial\omega}(\omega)$, $u_1\in \boldsymbol{H}_{L,\partial\omega}(\omega)$. Hence $u^{\#}|\omega-u_1\in \boldsymbol{D}_{L,0}(\omega)$. Since $f_1=S_{\boldsymbol{R},B}^{L,\omega}(f)$ is L-harmonic on ω and $=w$ on ω',

$$A_{L,\omega}[f_1, u_1]=A_{L,\omega}[f_1, u^{\#}]=A_{L,\omega'}[w, u].$$

On the other hand, $f_1=v_0+g_0+f_0$ on ω with $v_0\in \boldsymbol{R}\cap \boldsymbol{H}_{L,\beta(\omega_0)-\beta(\omega)}(\omega_0)$ and $(g_0+f_0)|\omega\in \boldsymbol{D}_{L,\beta(\omega)}(\omega)$. Hence

$$B_\omega[f_1, u_1]=B[v_0, v].$$

Also, by the argument far above, we have

$$B_{\omega'}[w, u]=B[(v_0)_{\beta(\omega_0)-\beta(\omega')}, v].$$

Since $v\in \boldsymbol{H}_{L,\beta(\omega_0)-\beta(\omega')}(\omega_0)$, $B[(v_0)_{\beta(\omega')}, v]=0$ by Lemma 4.2. Hence

$$B_{\omega'}[w, u]=B[v_0, v]=B_\omega[f_1, u_1].$$

Therefore, (*) is equivalent to

$$A_{L,\omega'}[w, u]+B_{\omega'}[w, u]=0,$$

which is the required relation. Hence $w=S_{\boldsymbol{R},B}^{L,\omega'}(f_1)=S_{\boldsymbol{R},B}^{L,\omega'}(S_{\boldsymbol{R},B}^{L,\omega}(f))$.

4.9. *Full-harmonic structures associated with* $(\boldsymbol{R}, B)$.

Let ω_0, $\boldsymbol{R}$, B be as above, i.e., ω_0 is an L-full-adapted end, $\boldsymbol{R}$ is a closed subspace of $\boldsymbol{H}_{L,\partial\omega_0}(\omega_0)$ satisfying condition (R) and B is a bilinear form on $\boldsymbol{R}\times\boldsymbol{R}$ satisfying condition (B).

For an end ω contained in ω_0, an L-harmonic function u on ω is called L-full-harmonic with boundary condition $(\boldsymbol{R}, B)$, or $(L, \boldsymbol{R}, B)$-*full-harmonic*, on ω if there exist a finite number of ends $\omega_1, \cdots, \omega_k$ such that $\bar{\omega}_j\subset\omega$, $j=1, \cdots, k$, $\bar{\omega}-\bigcup_{j=1}^{k}\omega_j$ is compact and

$$S_{\boldsymbol{R},B}^{L,\omega_j}(u)=u \qquad \text{on} \quad \omega_j$$

for each $j=1, \dots, k$. Let $\tilde{\mathscr{H}}(\omega)\equiv\tilde{\mathscr{H}}_{(L,\boldsymbol{R},B)}(\omega)$ be the class of all $(L, \boldsymbol{R}, B)$-full-harmonic functions on ω. By this definition and Proposition 4.3, the following property is easily verified:

Axiom S: (i) If $u \in \tilde{\mathscr{H}}(\omega)$ and ω' is an end contained in ω, then $u|\omega' \in \tilde{\mathscr{H}}(\omega')$.

(ii) If $u \in \mathscr{H}_L(\omega)$ and if there are ends $\omega_1, \dots, \omega_k$ such that $\bar{\omega}_j\subset\omega$ and $u|\omega_j \in \tilde{\mathscr{H}}(\omega_j)$ for each $j=1, \dots, k$ and that $\bar{\omega}-\bigcup_{j=1}^{k}\omega_j$ is compact, then $u \in \tilde{\mathscr{H}}(\omega)$.

An end ω is called regular with respect to $\{\tilde{\mathscr{H}}(\omega)\}$ if $\bar{\omega}\subset\omega_0$ and for any $f \in \boldsymbol{C}(\partial\omega)$ there exists a unique $u \in \boldsymbol{C}(\bar{\omega})$ such that $u=f$ on $\partial\omega$, $u|\omega \in \tilde{\mathscr{H}}(\omega)$ and $f\geqq 0$ implies $u\geqq 0$. We have

PROPOSITION 4.4. *If ω is an end such that $\bar{\omega}\subset\omega_0$ and if $\partial\omega$ satisfies condition (A) in the Remark in* 2.3, *then ω is regular with respect to $\{\tilde{\mathscr{H}}(\omega)\}$.*

PROOF. If $\varphi \in \boldsymbol{C}^1(\partial\omega)$, then there is $f \in \boldsymbol{C}_0^1(\omega_0)$ which is equal to φ on $\partial\omega$. By the corollary to Proposition 4.2, $S_{\boldsymbol{R},B}^{L;\omega}(f)$ depends only on φ. Thus we denote it by $S^{\omega}(\varphi)$. By Théorème 7.3 of [17] (also see [9], Theorem 14.1), we see that $S^{\omega}(\varphi)$ is continuously extended to $\bar{\omega}$ by φ on $\partial\omega$, since $u=S^{\omega}(\varphi)-f$ is a solution of $Lu=-Lf$ on ω vanishing on $\partial\omega$. By Proposition 4.3, $S^{\omega}(\varphi) \in \tilde{\mathscr{H}}(\omega)$. Proposition 4.2 shows that $S^{\omega}(\varphi)\geqq 0$ if $\varphi\geqq 0$.

Next suppose $\varphi \in \boldsymbol{C}(\partial\omega)$ and $\varphi\geqq 0$. Since $\boldsymbol{C}^1(\partial\omega)$ is dense in $\boldsymbol{C}(\partial\omega)$, we find $\varphi_n \in \boldsymbol{C}^1(\partial\omega)$ such that $\varphi_n\geqq 0$ for each n and $\varphi_n\rightarrow\varphi$ uniformly on $\partial\omega$. By Proposition 4.2, we have

$$|S^{\omega}(\varphi_n-\varphi_m)|\leqq\{\sup_{\xi\in\partial\omega}|\varphi_n(\xi)-\varphi_m(\xi)|\}S^{\omega}(1).$$

Hence, $\{S^{\omega}(\varphi_n)\}$ converges uniformly on any compact set in $\bar{\omega}$. Let $u=\lim_{n\rightarrow\infty}S^{\omega}(\varphi_n)$. Then $u \in \mathscr{H}_L(\omega)$, $u\geqq 0$ on ω and u can be continuously extended to $\partial\omega$ by φ. Let ω' be any end such that $\bar{\omega}'\subset\omega$. Since $u_n=S^{\omega}(\varphi_n)$ converges to u uniformly on $\partial\omega'$,

$$u(x)=\lim_{n\rightarrow\infty}S^{\omega}(\varphi_n)(x)=\lim_{n\rightarrow\infty}S_{\boldsymbol{R},B}^{L;\omega'}(u_n)(x)$$

for any $x \in \omega'$ by Proposition 4.3. Now, Proposition 4.2 implies that $S_{\boldsymbol{R},B}^{L;\omega'}(u_n)\rightarrow S_{\boldsymbol{R},B}^{L;\omega'}(u)$ on ω'. Hence $u=S_{\boldsymbol{R},B}^{L;\omega'}(u)$ on ω'. It follows that $u \in \tilde{\mathscr{H}}(\omega)$.

Finally, we shall show that for each $\varphi \in \boldsymbol{C}(\partial\omega)$ there is at most one $u \in \boldsymbol{C}(\bar{\omega})$ such that $u=\varphi$ on $\partial\omega$ and $u|\omega \in \tilde{\mathscr{H}}(\omega)$. It is enough to prove the case $\varphi=0$. Thus suppose $u \in \boldsymbol{C}(\bar{\omega})$ vanishes on $\partial\omega$ and $u|\omega \in \tilde{\mathscr{H}}(\omega)$. There are ends $\omega_1, \dots, \omega_k$ such that $\bar{\omega}_j\subset\omega$, $j=1, \dots, k$, $\bar{\omega}-\bigcup_{j=1}^{k}\omega_j$ is compact and $S_{\boldsymbol{R},B}^{L;\omega_j}(u)=u$ on ω_j for each j. Let $\alpha=\inf_{\xi\in\partial\omega_1\cup\dots\cup\partial\omega_k}[u(\xi)/S^{\omega}(1)(\xi)]$. Since $S^{\omega}(1)>0$ on ω and $S^{\omega}(1)$, u are continuous, there is $\xi_0 \in \partial\omega_1\cup\cdots\cup\partial\omega_k$ such that $u(\xi_0)=\alpha S^{\omega}(1)(\xi_0)$. Since $u-\alpha S^{\omega}(1)\geqq 0$ on $\partial\omega_j$, Proposition 4.2 implies $u\geqq\alpha S^{\omega}(1)$ on ω_j for each j. If $\alpha<0$, then $(u-\alpha S^{\omega}(1))(x)\rightarrow-\alpha>0$ as $x\rightarrow\partial\omega$.

Hence $u-\alpha S^{\omega}(1)\geqq 0$ by the minimum principle on $\omega-\bigcup_{j=1}^{k}\bar{\omega}_j$ (cf. Proposition 2.6). Since $u(\xi_0)-\alpha S^{\omega}(1)(\xi_0)=0$, it follows that $u=\alpha S^{\omega}(1)$ on ω, which is impossible. Hence $\alpha\geqq 0$, so that $u\geqq 0$ on $\partial\omega_1\cup\cdots\cup\partial\omega_k$. It follows from Proposition 4.2 and the minimum principle on $\omega-\bigcup_{j=1}^{k}\bar{\omega}_j$ that $u\geqq 0$ on ω. By considering $-u$, we also obtain $u\leqq 0$. Hence $u=0$.

By this proposition, the following Axiom $\tilde{T}$ is immediately verified:

Axiom $\tilde{T}$: For any end $\omega\subset\omega_0$, there are a finite number of ends $\omega_1, \cdots, \omega_k$ such that each ω_j is regular with respect to $\{\tilde{\mathscr{H}}(\omega)\}$, $\bar{\omega}_j\subset\omega$ for each j and $\bar{\omega}-\bigcup_{j=1}^{k}\omega_j$ is compact.

Thus we have seen that $\tilde{\mathfrak{H}}^{\omega_0}_{(L,\boldsymbol{R},B)}\equiv\{\tilde{\mathscr{H}}_{(L,\boldsymbol{R},B)}(\omega)\}_{\omega.\mathrm{end}\subset\omega_0}$ defines a full-harmonic structure on ω_0 in the sense of [11] which is subordinate to the L-harmonic structure on Ω. The corresponding full-superharmonic functions (cf. [11]) will be called $(L, \boldsymbol{R}, B)$-*full-superharmonic.* In case ω is an open subset of ω_0 such that $\partial\omega$ is compact, a function on ω is called $(L, \boldsymbol{R}, B)$-full-harmonic (resp. $(L, \boldsymbol{R}, B)$-full-superharmonic) on ω if it is L-harmonic (resp. L-superharmonic) on any relatively component of ω and is $(L, \boldsymbol{R}, B)$-full-harmonic (resp. $(L, \boldsymbol{R}, B)$-full-superharmonic) on any end component of ω. The set of all $(L, \boldsymbol{R}, B)$-full-superharmonic functions on ω will be denoted by $\tilde{\mathcal{S}}_{(L,\boldsymbol{R},B)}(\omega)$, or simply by $\tilde{\mathcal{S}}(\omega)$. The set of all $v\in\tilde{\mathcal{S}}(\omega_0)$ which are of potential type on ω_0 (cf. [11]) is denoted by $\mathscr{P}_{(L,\boldsymbol{R},B)}$, or simply by $\mathscr{P}$. Also the set of all $v\in\mathscr{P}$ which are L-harmonic on ω_0 is denoted by $\mathscr{P}_b\equiv\mathscr{P}_{(L,\boldsymbol{R},B),b}$.

4.10. *A minimum principle for* $(L, \boldsymbol{R}, B)$-*full-superharmonic functions.*

PROPOSITION 4.5. *Let ω be an end contained in ω_0. If $v\in\tilde{\mathcal{S}}(\omega)$ and if there exists $g\in\boldsymbol{D}_{L,\partial\omega}(\omega)$ such that $v\geqq g$ on $V\cap\omega$ for some neighborhood V of $\partial\omega$, then $v\geqq 0$ on ω.*

PROOF. Choose $\varphi_0\in\boldsymbol{C}^1(\omega)\cap\boldsymbol{D}_L(\omega)$ such that $\varphi_0\equiv 1$ on $V\cap\omega$. Since $1\notin\boldsymbol{D}_{\partial\omega}(\omega)$, we see that $\varphi_0\notin\boldsymbol{D}_{L,\partial\omega}(\omega)$. It follows that $u_0\equiv S^{L,\omega}_{\boldsymbol{R},B}(\varphi_0)\notin\boldsymbol{D}_{L,\partial\omega}(\omega)$, and hence $u_0\neq 0$. Since $u_0\geqq 0$ on ω by Proposition 4.1, we have $u_0>0$ on ω.

To prove the proposition, we may assume that v is $(L, \boldsymbol{R}, B)$-full-harmonic on an open set ω' such that $\bar{\omega}'\subset\omega$ and $\bar{\omega}-\omega'$ is compact. Let ω'' be another open set consisting of a finite number of ends such that $\bar{\omega}''\subset\omega'$ and $\bar{\omega}-\omega''$ is compact, and let σ be a relatively compact neighborhood of $\partial\omega''$. Then, $\alpha=\min_{x\in\bar{\sigma}}[v(x)/u_0(x)]$ exists as a finite value. Since $v-\alpha u_0$ is $(L, \boldsymbol{R}, B)$-full-harmonic on ω' and $v-\alpha u_0\geqq 0$ on σ, $v-\alpha u_0=S^{L,\omega_\iota}_{\boldsymbol{R},B}(v-\alpha u_0)\geqq 0$ on ω_ι for each component ω_ι of ω''. Hence $v-\alpha u_0\geqq 0$ on $\bar{\omega}''$.

Now suppose $\alpha<0$. Then $v-\alpha u_0\geqq v\geqq g$ on $V\cap\omega$. Hence, applying Proposition 3.1 to each component of $\omega-\bar{\omega}''$, we conclude that $v-\alpha u_0\geqq 0$ on $\omega-\bar{\omega}''$. Thus, in this case, $v-\alpha u_0\geqq 0$ on ω. Since $v-\alpha u_0$ attains zero on $\bar{\sigma}$, we have $v=\alpha u_0$ on ω. Then $0\leqq u_0\leqq(-1/\alpha)g$ on $V\cap\omega$, which implies

$u_0 \in \boldsymbol{D}_{L,\partial\omega}(\omega)$, a contradiction. Therefore $\alpha \geqq 0$. Then $v \geqq \alpha u_0 \geqq 0$ on σ and $v \geqq g$ on $V \cap \omega$ imply, by the same arguments as above, that $v \geqq 0$ on ω.

COROLLARY 1. $\tilde{\mathcal{S}}(\omega_0) \cap \boldsymbol{D}_{L,\partial\omega_0}(\omega_0) \subset \mathcal{P}$.

PROOF. If $v \in \tilde{\mathcal{S}}(\omega_0) \cap \boldsymbol{D}_{L,\partial\omega_0}(\omega_0)$, then the above proposition implies $v \geqq 0$. If u is an $(L, \boldsymbol{R}, B)$-full-harmonic minorant of v, then the above proposition also implies that $u \leqq 0$. Hence v is of potential type.

COROLLARY 2. *Let ω be an end contained in ω_0 and let ω' be a subend of ω. Then for any $v \in \tilde{\mathcal{S}}(\omega) \cap \boldsymbol{D}_L(\omega')$* (resp. *$u \in \tilde{\mathcal{H}}(\omega) \cap \boldsymbol{D}_L(\omega')$*), *$v \geqq S_{\boldsymbol{R},B}^{L,\omega'}(v)$ on ω'* (resp. *$u = S_{\boldsymbol{R},B}^{L,\omega'}(u)$ on ω'*).

§ 5. $(L, \boldsymbol{R}, B)$-Green functions and $(L, \boldsymbol{R}, B)$-ideal boundary

In this section, we shall always assume that $|a|$, $|b|$, $|q|^{1/2} \in \boldsymbol{L}_{\mathrm{loc}}^r(\Omega)$ for some $r > d$ and ω_0 is a fixed L-full-adapted end in Ω.

5.1. *Properties of L-Green functions for an end.*

LEMMA 5.1. *Let $y \in \omega_0$ and let ω be a subend of ω_0 such that $\bar{\omega} \subset \omega_0 - \{y\}$. Then*

$$g_y^{L,\omega_0} = S_0^{L,\omega}(g_y^{L,\omega_0})$$

on ω.

PROOF. Let $\{\psi_k\}$ be a sequence of non-negative bounded measurable functions on ω_0 such that their supports are compact and decrease to $\{y\}$ and $\int \psi_k dV = 1$ for each k. We may assume that the supports of ψ_k do not intersect with $\bar{\omega}$. Since $g_k \equiv G^{L,\omega_0}(\psi_k) \in \boldsymbol{D}_{L,0}(\omega_0)$ and g_k is L-harmonic on ω, we have $g_k = S_0^{L,\omega}(g_k)$. Theorem 3.1 states that $g_k \to g_y^{L,\omega_0}$ locally uniformly on $\omega_0 - \{y\}$. Hence $S_0^{L,\omega}(g_k) \to S_0^{L,\omega}(g_y^{L,\omega_0})$ in ω (cf. the proof of Proposition 4.4). Hence we have the lemma.

LEMMA 5.2. *Let y and ω be as in the previous lemma. Then*

$$A_{L,\omega}[g_y^{L,\omega_0}, u] = -h_{u^\#}^{L^*,\omega_0}(y)$$

for any $u \in \boldsymbol{H}_{L,\partial\omega}(\omega)$, where $u^\# = u$ on ω and $= 0$ on $\omega_0 - \omega$.

PROOF. Let $\{\psi_k\}$ and $\{g_k\}$ be as in the proof of the previous lemma. For simplicity, we write $g \equiv g_y^{L,\omega_0}$.

First we shall show that $\alpha_k \equiv \|g - g_k\|_{L,\omega} \to 0$ $(k \to \infty)$. Choose $\varphi_0 \in \boldsymbol{C}_0^1(\omega_0)$ such that $\varphi_0 \equiv 1$ on a neighborhood of $\partial\omega$, $\equiv 0$ on a neighborhood of y. Since $g_k \to g$ uniformly on the support of φ_0, Lemma 2.3 implies that $\beta_k \equiv \|g\varphi_0$

$-g_k\varphi_0\|_{L,\omega}\to 0$ $(k\to\infty)$. Now,

(1) $$\alpha_k \leqq \beta_k + \|(g-g\varphi_0)-(g_k-g_k\varphi_0)\|_{L,\omega}.$$

Obviously, $(g_k-g_k\varphi_0))\,|\,\omega \in \boldsymbol{D}_{L,0}(\omega)$ and, by Lemma 5.1, we also have $(g-g\varphi_0)\,|\,\omega \in \boldsymbol{D}_{L,0}(\omega)$. Since $A_{L,\omega}$ is coercive on $\boldsymbol{D}_{L,0}(\omega)$, there is $\lambda>0$ such that

(2) $$A_{L,\omega}[f, f] \geqq \lambda\|f\|_{L,\omega}^2$$

for any $f \in \boldsymbol{D}_{L,0}(\omega)$. Also, since $g-g_k$ is L-harmonic on ω,

(3) $$A_{L,\omega}[g-g_k, f]=0$$

for $f \in \boldsymbol{D}_{L,0}(\omega)$. Thus, taking $f=[(g-g\varphi_0)-(g_k-g_k\varphi_0)]\,|\,\omega$ in (2) and (3) and applying (1), we have

$$\alpha_k \leqq \beta_k + \frac{1}{\sqrt{\lambda}} A_{L,\omega}[g\varphi_0-g_k\varphi_0, (g\varphi_0-g_k\varphi_0)-(g-g_k)]^{1/2}.$$

By the continuity of $A_{L,\omega}$ on $\boldsymbol{D}_{L,0}(\omega)\times\boldsymbol{D}_{L,0}(\omega)$, we obtain

$$\alpha_k \leqq \beta_k + M(\beta_k^2+\alpha_k\beta_k)^{1/2}$$

for some $M>0$. Since $\beta_k\to 0$ $(k\to\infty)$, this inequality implies that $\alpha_k\to 0$.

Now we have

$$\begin{aligned} A_{L,\omega}[g_k, u] &= A_{L,\omega_0}[g_k, u^\#] \\ &= A_{L,\omega_0}[g_k, u^\# - h_{u^\#}^{L^*,\omega_0}] \\ &= \int_{\omega_0} \psi_k(u^\# - h_{u^\#}^{L^*,\omega_0})\,dV. \end{aligned}$$

Since $u^\#=0$ on the support of ψ_k and since $h_{u^\#}^{L^*,\omega_0}$ is continuous at y, we have

$$A_{L,\omega}[g_k, u] = -\int_{\omega_0} \psi_k h_{u^\#}^{L^*,\omega_0}\,dV \to -h_{u^\#}^{L^*,\omega_0}(y) \qquad (k\to\infty).$$

On the other hand, since $\|g-g_k\|_{L,\omega}\to 0$, $A_{L,\omega}[g_k, u]\to A_{L,\omega}[g, u]$. Hence we have the lemma.

5.2. *$(L, \boldsymbol{R}, B)$-Green functions.*

Now, we consider a boundary condition $(\boldsymbol{R}, B)$ satisfying conditions (R) and (B) in §4. For a point $y \in \omega_0$, the $(L, \boldsymbol{R}, B)$-Green function of ω_0 with pole at y is a function $\tilde{g}_y$ on ω_0 having the following properties:

(a) There is an L-harmonic function U_y on ω_0 such that

$$\tilde{g}_y = g_y^{L,\omega_0} + U_y \qquad \text{on} \quad \omega_0;$$

(b) $\tilde{g}_y$ is $(L, \boldsymbol{R}, B)$-full-harmonic on $\omega_0-\{y\}$;

(c) $\tilde{g}_y \in \mathcal{P}$.

It is easy to see that $\tilde{g}_y$ having the above properties is unique if it exists. Now we shall show the existence of $\tilde{g}_y$:

THEOREM 5.1. *There exists $U_y \in \boldsymbol{R}$ such that*

$$(*) \qquad A_{L,\omega_0}[U_y, v]+B[U_y, v]=h_v^{L^*,\omega_0}(y)$$

for all $v \in \boldsymbol{R}$ and $\tilde{g}_y=g_y^{L,\omega_0}+U_y$ is the $(L, \boldsymbol{R}, B)$-Green function of ω_0 with pole at y.

PROOF. First, we observe that the linear functional $v \to h_v^{L^*,\omega_0}(y)$ is continuous on $\boldsymbol{H}_{L,\partial\omega_0}(\omega_0)$. For, the coerciveness of A_{L,ω_0} (hence, of A_{L^*,ω_0}) on $\boldsymbol{D}_{L,0}(\omega_0)$ implies that the mapping $v \to h_v^{L^*,\omega_0}$ is continuous from $\boldsymbol{H}_{L,\partial\omega_0}(\omega_0)$ into $\boldsymbol{H}_{L^*,\partial\omega_0}(\omega_0)$, and our Lemma 3.1 and Corollary 5.2 plus Remarque 5.1 of [17] imply the continuity of $w \to w(y)$ on $\boldsymbol{H}_{L^*,\partial\omega_0}(\omega_0)$. Hence, from our assumption that $A_{L,\omega_0}+B$ is coercive on $\boldsymbol{R}$, it follows the existence of U_y satisfying (*).

Next, we shall show that $\tilde{g}_y=g_y^{L,\omega_0}+U_y$ is $(L, \boldsymbol{R}, B)$-full-harmonic on $\omega_0-\{y\}$. Let ω be any end such that $\bar{\omega} \subset \omega_0-\{y\}$. We consider the function $u_0=S_0^{L,\omega}(\tilde{g}_y)-\tilde{g}_y|\omega$. By Lemma 5.1, $u_0=S_0^{L,\omega}(U_y)-U_y|\omega$. Hence $u_0 \in \boldsymbol{R}(\omega)$ with corresponding function $-U_y \in \boldsymbol{R}$. For any $u \in \boldsymbol{R}(\omega)$ with corresponding $v \in \boldsymbol{R}$ (i.e., $v=h_{u^\#}^{L,\omega_0}$),

$$\begin{aligned}
&A_{L,\omega}[u_0, u]+B_\omega[u_0, u] \\
&=A_{L,\omega}[S_0^{L,\omega}(\tilde{g}_y), u]-A_{L,\omega}[g_y^{L,\omega_0}, u]-A_{L,\omega}[U_y, u]+B_\omega[u_0, u] \\
&=A_{L,\omega}[S_0^{L,\omega}(\tilde{g}_y), u]-A_{L,\omega}[g_y^{L,\omega_0}, u]-(A_{L,\omega_0}[U_y, v]+B[U_y, v]) \\
&=A_{L,\omega}[S_0^{L,\omega}(\tilde{g}_y), u]-A_{L,\omega}[g_y^{L,\omega_0}, u]-h_v^{L^*,\omega_0}(y) \\
&=A_{L,\omega}[S_0^{L,\omega}(\tilde{g}_y), u],
\end{aligned}$$

where the last equality follows from Lemma 5.2 and the equality $h_v^{L^*,\omega_0}=h_{u^\#}^{L^*,\omega_0}$. Thus, we have $\tilde{g}_y=S_{\boldsymbol{R};B}^{L;\omega}(\tilde{g}_y)$ on ω (see the definition of $S_{\boldsymbol{R};B}^{L;\omega}$ in 4.6). Therefore, $\tilde{g}_y$ is $(L, \boldsymbol{R}, B)$-full-harmonic on $\omega_0-\{y\}$.

Since U_y is L-harmonic on ω_0, it follows that $\tilde{g}_y \in \mathcal{S}(\omega_0)$. On the other hand, by Proposition 4.1 and Lemma 5.1, $-g_y^{L,\omega_0}$ is $(L, \boldsymbol{R}, B)$-full-superharmonic on $\omega_0-\{y\}$. Hence U_y is $(L, \boldsymbol{R}, B)$-full-superharmonic on $\omega_0-\{y\}$, and hence on ω_0. Since $U_y \in \boldsymbol{R} \subset \boldsymbol{H}_{L,\partial\omega_0}(\omega_0)$, Corollary 1 to Proposition 4.5 implies that $U_y \in \mathcal{P}$. Since g_y^{L,ω_0} is an L-potential on ω_0, it also follows that $\tilde{g}_y \in \mathcal{P}$.

COROLLARY. *$U_y \in \mathcal{P}_b$; in particular, $U_y \geqq 0$, so that $\tilde{g}_y \geqq g_y^{L,\omega_0}$.*

5.3. *Adjoint full-harmonic structure.*

If ω_0 is L-full-adapted, then it is also L^*-full-adapted and we have

$\boldsymbol{D}_{L^*,0}(\omega_0)=\boldsymbol{D}_{L,0}(\omega_0)$, $\boldsymbol{D}_{L^*,\partial\omega_0}(\omega_0)=\boldsymbol{D}_{L,\partial\omega_0}(\omega_0)$, etc. Given $(\boldsymbol{R}, B)$ as above, let, $\boldsymbol{R}^*=\{h_v^{L^*,\omega_0};\ v \in \boldsymbol{R}\}$ and $B^*[h_{v_1}^{L^*,\omega_0}, h_{v_2}^{L^*,\omega_0}]=B[v_2, v_1]$ for $v_1, v_2 \in \boldsymbol{R}$. Then $\boldsymbol{R}^*$ is a subspace of $\boldsymbol{H}_{L^*,\partial\omega_0}(\omega_0)$ satisfying condition (R) for L^* and B^* is a bilinear form on $\boldsymbol{R}^*\times\boldsymbol{R}^*$ satisfying condition (B). Thus we can define the $(L^*, \boldsymbol{R}^*, B^*)$-full-harmonic structure on ω_0 as the adjoint structure of the $(L, \boldsymbol{R}, B)$-full-harmonic structure.

Let $\tilde{g}_y^*=g_y^{L^*,\omega_0}+U_y^*$ be the corresponding $(L^*, \boldsymbol{R}^*, B^*)$-Green function of ω_0. By Theorem 5.1, we have

$$A_{L,\omega_0}[w, U_y^*]+B^*[U_y^*, w]=h_w^{L,\omega_0}(y)$$

for all $w \in \boldsymbol{R}^*$. If $w=h_v^{L^*,\omega_0}$ for $v \in \boldsymbol{R}$, then $h_w^{L,\omega_0}=v$ and we have

$$A_{L,\omega_0}[w, U_y^*]=A_{L,\omega_0}[v, U_y^*]=A_{L,\omega_0}[v, V_y],$$

$$B^*[U_y^*, w]=B[v, V_y],$$

where $V_y\equiv h_{U_y^*}^{L,\omega_0} \in \boldsymbol{R}$. Hence

$$A_{L,\omega_0}[v, V_y]+B[v, V_y]=v(y).$$

Letting $v=U_x$, we have

$$A_{L,\omega_0}[U_x, V_y]+B[U_x, V_y]=U_x(y).$$

Since $h_{V_y}^{L^*,\omega_0}=U_y^*$, the lefthand side is equal to $U_y^*(x)$ by Theorem 5.1. Hence we have

Theorem 5.2. *For any $x, y \in \omega_0$, $U_x(y)=U_y^*(x)$; and hence $\tilde{g}_x(y)=\tilde{g}_y^*(x)$* $(x\neq y)$.

Corollary. *The mapping $x\to\tilde{g}_x(y)$ is continuous on $\omega_0-\{y\}$ for each $y \in \omega_0$.*

5.4. *Integral representation of functions in $\mathcal{P}$.*

Theorem 5.3. *Let $v \in \mathcal{P}$ and let $v=\int g_y^{L,\omega_0}d\mu(y)+h$ be the Riesz representation of the L-superharmonic function v on ω_0* (cf. 3.5). *Then $w=\int\tilde{g}_y d\mu(y)$ belongs to $\mathcal{P}$ and $v-w \in \mathcal{P}_b$.*

Proof. Let K be a compact set in ω_0 and let

$$v_K=\int_K\tilde{g}_y d\mu(y) \qquad \text{and} \qquad u_K=\int_K U_y d\mu(y).$$

Obviously, $v_K \in \mathcal{P}$ and $v_K \uparrow w$ as $K \uparrow \omega_0$. Also, we see that $u_K \in \mathcal{H}_L(\omega_0)\cap\mathcal{S}(\omega_0)$ and $v-v_K \in \mathcal{S}(\omega_0)$. If we show that $u_K \in \mathcal{P}_b$, then the inequality $v-v_K\geqq -u_K$

implies $v-v_K\geqq 0$ (Proposition 4.5), and the rest of the proof goes in the same way as that of Theorem 3 in [10].

In order to show that $u_K\in\mathcal{P}_b$, it is enough to consider the case $U_y>0$. Let ω be an open set such that $\bar{\omega}\subset\omega_0$ and $\bar{\omega}_0-\omega$ is compcat, and let σ be a relatively compact neighborhood of $\partial\omega$ such that $\bar{\sigma}\subset\omega_0$. By continuity of the mapping $y\rightarrow U_y(x)$ and by Harnack's inequality (Proposition 2.4), there are m, $M>0$ such that $m\leqq U_y(x)\leqq M$ for all $y\in K$ and $x\in\bar{\sigma}$. Fix $y_0\in K$. Then $(M/m)U_{y_0}-U_y\geqq 0$ on $\bar{\sigma}$ for any $y\in K$. Since U_{y_0}, $U_y\in \boldsymbol{H}_{L,\partial\omega_0}(\omega_0)$, it follows from Proposition 3.1 that $(M/m)U_{y_0}\geqq U_y$ on $\omega_0-\bar{\omega}$ for any $y\in K$. Hence

$$\frac{M\mu(K)}{m}U_{y_0}\geqq\int_K U_y d\mu(y)=u_K$$

on $\omega_0-\bar{\omega}$. Since $U_{y_0}\in\mathcal{P}_b$, this inequality implies $u_K\in\mathcal{P}_b$.

COROLLARY. *If $v\in\mathcal{P}$ and if v is $(L, \boldsymbol{R}, B)$-full-harmonic on an open set $\omega\subset\omega_0$ such that $\bar{\omega}_0-\omega$ is compact, then*

$$v=\int\tilde{g}_y d\mu(y)$$

with a non-negative measure μ on $\omega_0-\omega$.

Next, let $x_0\in\omega_0$ be a fixed point and let $\gamma(y)$ be a (finite) positive continuous function on ω_0 which is equal to $1/\tilde{g}_y(x_0)$ outside a compact neighborhood of x_0 contained in ω_0. Then the function $K_y(x)=\gamma(y)\tilde{g}_y(x)$ defined for x, $y\in\omega_0$ is a *kernel* on ω_0 with respect to the $(L, \boldsymbol{R}, B)$-full-harmonic structure in the sense of [11], i.e., $K_y(x)$ satisfies conditions (i), (ii) and (iii) in 6.3 of [11] (cf. the above corollary for (iii)). Let $\Delta\equiv\Delta^{\omega_0}_{(L,\boldsymbol{R},B)}$ be the ideal boundary associated with this kernel $K_y(x)$, i.e., $\omega_0\cup\Delta\cup\partial\omega_0$ is the compactification of ω_0 for which the functions $y\rightarrow K_y(x)$ are continuously extended to Δ and points of Δ are separated by these functions. For $\eta\in\Delta$, let $K_\eta(x)=\lim_{y\rightarrow\eta, y\in\omega_0}K_y(x)$. Obviously, $K_\eta\in\mathcal{P}_b$ and $K_\eta(x_0)=1$ for any $\eta\in\Delta$.

By Theorem 8 of [11], we have the subset Δ_1 of Δ which is the image of $e(\mathcal{P}_{b,0})$ by the homeomorphism given in this theorem. In fact, $\Delta_1=\{\eta\in\Delta;\,K_\eta$ is extremal in $\mathcal{P}_b$, i.e., $u\in\mathcal{P}_b$, $K_\eta-u\in\mathcal{P}_b$ imply $u=\alpha K_\eta$ for some constant $\alpha\}$. By Theorem 7 of [11], we obtain

THEOREM 5.4. *If $u\in\mathcal{P}_b$, then there exists a unique non-negative measure μ on Δ such that $\mu(\Delta-\Delta_1)=0$ and*

$$u(x)=\int_\Delta K_\eta(x)d\mu(\eta)\qquad \textit{for}\quad x\in\omega_0.$$

Thus, combining this theorem with Theorem 5.3, we obtain the complete integral representation theorem for $v\in\mathcal{P}$:

For any $v\in\mathcal{P}$, there exists a unique non-negative measure ν on $\omega_0\cup\Delta$ such *that* $\nu(\Delta-\Delta_1)=0$ *and*

$$v(x)=\int_{\omega_0\cup\Delta}K_y(x)\,d\nu(y).$$

Here, $(d\nu)|\omega_0=(1/\gamma)d\mu$, where μ is the measure on ω_0 associated with the L-superharmonic function v.

REMARK. The ideal boundary Δ for the case $\boldsymbol{R}=\{0\}$ (i.e., $\Delta=\Delta^{\omega_0}_{(L,\{0\},0)}$) is the Martin boundary of ω_0 for the L-harmonic structure (cf. the examples at the end of [11]; also cf. Chap. X of [1]).

If $|b|=0$, then the boundary condition determined by $\boldsymbol{R}=\boldsymbol{H}_{L,\partial\omega_0}(\omega_0)$ and $B=0$ may be regarded as the condition of vanishing normal derivatives on the ideal boundary of ω_0 (cf. [12]). Thus, in this case we may say that the ideal boundary $\Delta^*\equiv\Delta^{\omega_0}_{(L^*,\boldsymbol{H}_{L,\partial\omega_0}(\omega_0),0)}$ is the Kuramochi boundary of ω_0 associated with the equation $Lu=0$ (with $|b|=0$) (cf. [8]).

5.5. *Properties of the ideal boundary* Δ.

First, we remark that $\omega_0\cup\Delta$ is a metrizable space, since ω_0 is metrizable (cf. Lemma 17 of [11]; also cf. Satz 12.1 of [3]).

For a set A in $\omega_0\cup\Delta$, its closure in $\omega_0\cup\Delta$ will be denoted by $\bar{A}^*$. The following proposition is generally true in the axiomatic theory of [11]:

PROPOSITION 5.1. *The ideal boundary* Δ *is of Stoilow type, i.e., if* ω_1 *and* ω_2 *are subends of* ω_0 *such that* $\bar{\omega}_1\cap\bar{\omega}_2=\emptyset$, *then* $\bar{\omega}_1^*\cap\bar{\omega}_2^*=\emptyset$.

Next we show

PROPOSITION 5.2. *If* w *is a positive* $(L^*, \boldsymbol{R}^*, B^*)$*-full-superharmonic function on* ω_0 *and is* $(L^*, \boldsymbol{R}^*, B^*)$*-full-harmonic on a subend* ω *of* ω_0, *then* $f=\gamma w$ *has positive continuous extension to* $\Delta\cap\bar{\omega}^*$.

PROOF. We can choose $w_1\in\mathcal{P}^*(\equiv\mathcal{P}_{(L^*,R^*,B^*)})$ which has the following two properties: (i) $w_1=w$ on an open set ω_1 such that $\bar{\omega}_1\subset\omega$ and $\bar{\omega}-\omega_1$ is compact; (ii) there is a compact set K in ω_0 such that w_1 is $(L^*, \boldsymbol{R}^*, B^*)$-full-harmonic on ω_0-K. By the corollary to Theorem 5.3, applied to the adjoint structure, we have

$$w_1(y)=\int\tilde{g}_x^*(y)\,d\mu(x)$$

for some non-negative measure μ ($\neq 0$) on K. Let $f_1=\gamma w_1$. Then

$$f_1(y)=\int K_y(x)\,d\mu(x).$$

By Harnack's principle (i.e., Axiom 3′ of Brelot [1]; also cf. Lemma 1 of [11]), we see that as $y\to\xi\in\Delta$, $K_y(x)\to K_\xi(x)$ uniformly for $x\in K$. Hence $\lim_{y\to\xi}f_1(y)$

exists and is equal to $\int K_\xi(x)d\mu(x)>0$. If $\xi\in\Delta\cap\bar{\omega}^*$, then, by virtue of the previous proposition, $\lim_{y\to\xi}f(y)=\lim_{y\to\xi}f_1(y)$. Thus we have the proposition.

COROLLARY 1. *For any* x, $x'\in\omega_0$, $f(y)=\tilde{g}_y(x)/\tilde{g}_y(x')$ *has continuous extension to* Δ.

COROLLRAY 2. *The ideal boundary* Δ *does not depend on the choice of* x_0.

5.6. *Semi-continuous extension of certain functions to* Δ.

Let u be a positive continuous function on ω_0 which is equal to some $u_1\in\mathcal{P}^*$ on a neighborhood of Δ (i.e., on an open set ω such that $\bar{\omega}_0-\omega$ is compact), where it is also $(L^*, \boldsymbol{R}^*, B^*)$-full-harmonic. Given such a function u, let

$$\mathcal{P}_u^*=\left\{\frac{w}{u};\ w\in\mathcal{P}^*\right\}.$$

Since $u=1/\gamma$ has the above property, the function $y\to K_y(x)$ belongs to $\mathcal{P}_{1/\gamma}^*$ for each $x\in\omega_0$.

Let $\{\omega_n\}$ be a decreasing sequence of open sets in ω_0 such that each ω_n is regular with respect to the $(L^*, \boldsymbol{R}^*, B^*)$-full-harmonic structure, $\bar{\omega}_{n+1}\subset\omega_n$, $\bar{\omega}_0-\omega_n$ is compact for each n and $\bigcap_n\omega_n=\emptyset$. For $w\in\mathcal{P}^*$, let

$$w_n(y)=\begin{cases} w(y) & \text{if}\quad y\in\omega_0-\omega_n \\ \int w\,d\mu_y^{*\omega_n} & \text{if}\quad y\in\omega_n \end{cases},$$

where $\mu_y^{*\omega_n}$ is the full-harmonic measure for ω_n with respect to the $(L^*, \boldsymbol{R}^*, B^*)$-full-harmonic structure (cf. [11]). Then, $w_n\in\mathcal{P}^*$ (Theorem 2 of [11]) and w_n is $(L^*, \boldsymbol{R}^*, B^*)$-full-harmonic on ω_n. Now let $f=w/u\in\mathcal{P}_n^*$ and let $f_n=w_n/u$. By Proposition 5.2, we see that f_n has continuous extension to Δ. Let $\hat{f}_n$ be the extended function on $\omega_0\cup\Delta$. Since $w_n\uparrow w$ as $n\to\infty$, $\hat{f}=\lim_{n\to\infty}\hat{f}_n$ exists, $\hat{f}$ is lower semi-continuous on $\omega_0\cup\Delta$ and $\hat{f}=f$ on ω_0. It is easy to see that the definition of $\hat{f}$ does not depend on the choice of $\{\omega_0\}$. Obviously, if $f(y)=K_y(x)$, then $\hat{f}(\xi)=K_\xi(x)$.

§ 6. Reduced functions and thin sets at ideal boundary points

In this section, we fix L and ω_0 as in §5 and boundary condition $(\boldsymbol{R}, B)$ as in §4.

6.1. *Operation* $f\to f^K$ *for compact sets* K *and* $f\in\boldsymbol{D}_{L,\partial\omega_0}(\omega_0)$.

Let K be a compact set in ω_0. For any $f\in\boldsymbol{D}_{L,\partial\omega_0}(\omega_0)$, we define

$$f^K = \begin{cases} f & \text{on } K; \\ h_f^{L,\omega} & \text{on } \omega \text{ if } \omega \text{ is a relatively compact component of } \omega_0 - K; \\ S_{\boldsymbol{R},B}^{L,\omega}(f) & \text{on } \omega \text{ if } \omega \text{ is an end component of } \omega_0 - K. \end{cases}$$

By the definition of $S_{\boldsymbol{R},B}^{L,\omega}(f)$, we see that $f^K \in \boldsymbol{R}+\boldsymbol{D}_{L,0}(\omega_0)$. For simplicity, we shall write $\boldsymbol{D}^{\boldsymbol{R}}=\boldsymbol{R}+\boldsymbol{D}_{L,0}(\omega_0)$ and $\tilde{f}=h_f^{L,\omega_0}(\in \boldsymbol{R})$ for $f \in \boldsymbol{D}^{\boldsymbol{R}}$. The mapping $f \to f^K$ (resp. $f \to \widetilde{f^K}$) is linear continuous from $\boldsymbol{D}_{L,\partial\omega_0}(\omega_0)$ into $\boldsymbol{D}^{\boldsymbol{R}}$ (resp. into $\boldsymbol{R}$). If f is continuous and $f \geqq 0$ on K, then $f^K \geqq 0$ on ω_0 (see Proposition 4.2).

We shall say that a compact set K in ω_0 is regular if f^K is continuous whenever f is continuous. Remark that if ω satisfies condition (A) in the Remark in 2.3 for each component ω of $\omega_0 - K$, then K is regular. From this fact, we can show that given an open set σ there is an increasing sequence $\{K_n\}$ of regular compact sets such that $\bigcup_n \operatorname{Int}(K_n)=\sigma$, where $\operatorname{Int}(K_n)$ means the interior of K_n.

Lemma 6.1. *If $f \in \boldsymbol{D}^{\boldsymbol{R}}$, then*

$$A_{L,\omega_0}[f^K, f^K - f] + B[\widetilde{f^K}, \widetilde{f^K} - \tilde{f}] = 0.$$

Proof. If ω is a relatively compact component of $\omega_0 - K$, then $(f^K - f) | \omega \in \boldsymbol{D}_{L,0}(\omega)$. Hence $A_{L,\omega}[f^K, f^K - f] = 0$.

Next, let ω be an end component of $\omega_0 - K$. Since $S_{\boldsymbol{R},B}^{L,\omega}(f) - h_f^{L,\omega} \in \boldsymbol{R}(\omega)$ with corresponding function in $\boldsymbol{R}$ being $\widetilde{f^K}_{\beta(\omega_0)-\beta(\omega)} - \tilde{f}_{\beta(\omega_0)-\beta(\omega)}$, we have, by the definition of $S_{\boldsymbol{R},B}^{L,\omega}(f)$,

$$A_{L,\omega}[f^K, f^K - h_f^{L,\omega}] + B[\widetilde{f^K}_{\beta(\omega_0)-\beta(\omega)}, \widetilde{f^K}_{\beta(\omega_0)-\beta(\omega)} - \tilde{f}_{\beta(\omega_0)-\beta(\omega)}] = 0.$$

Since $h_f^{L,\omega} - f | \omega \in \boldsymbol{D}_{L,0}(\omega)$, $A_{L,\omega}[f^K, f^K - h_f^{L,\omega}] = A_{L,\omega}[f^K, f^K - f]$. On the other hand, by Lemmas 4.1, 4.2 and by condition (B), (iii) in 4.5, we have

$$B[\widetilde{f^K}_{\beta(\omega_0)-\beta(\omega)}, \widetilde{f^K}_{\beta(\omega_0)-\beta(\omega)} - \tilde{f}_{\beta(\omega_0)-\beta(\omega)}] = B[\widetilde{f^K}_{\beta(\omega_0)-\beta(\omega)}, \widetilde{f^K} - \tilde{f}].$$

Hence,

$$A_{L,\omega}[f^K, f^K - f] + B[\widetilde{f^K}_{\beta(\omega_0)-\beta(\omega)}, \widetilde{f^K} - \tilde{f}] = 0\,.$$

Noting that $\widetilde{f^K} = \sum_\omega \widetilde{f^K}_{\beta(\omega_0)-\beta(\omega)}$, where the sum is taken over all end components ω of $\omega_0 - K$, we obtain the lemma.

Lemma 6.2. *There is a constant $M > 0$ which is independent of K such that*

$$\|f^K\|_{L,\omega_0} \leqq M \|f\|_{L,\omega_0}$$

for all $f \in \boldsymbol{D}^{\boldsymbol{R}}$.

PROOF. By coerciveness of A_{L,ω_0} on $\boldsymbol{D}^{\boldsymbol{R}}$ (condition (R), (ii) in 4.4) and by Lemma 6.1, we have

$$\lambda_0\|f^K\|_{L,\omega_0}^2 \leqq A_{L,\omega_0}[f^K, f^K] + B[\tilde{f^K}, \tilde{f^K}]$$
$$= A_{L,\omega_0}[f^K, f] + B[\tilde{f^K}, \tilde{f}].$$

Since the mapping $f \to \tilde{f}$ is continuous from $\boldsymbol{D}^{\boldsymbol{R}}$ into $\boldsymbol{R}$, the continuity of A_{L,ω_0} and B implies

$$|A_{L,\omega_0}[f^K, f] + B[\tilde{f^K}, \tilde{f}]| \leqq M_1\|f^K\|_{L,\omega_0}\|f\|_{L,\omega_0},$$

where M_1 is independent of K. Hence, we have the lemma with $M = M_1/\lambda_0$.

6.2. *Function* φ^K *for general* φ.

Given a compact set K in ω_0, we shall define φ^K for more general functions φ on K, as a generalization of f^K defined above. We follow the arguments given in [16].

Let $\boldsymbol{C}_L(K)$ be the set of all continuous functions φ on K for which there is $f \in \boldsymbol{D}_{L,\partial\omega_0}(\omega_0)$ such that $f|K = \varphi$ and f is continuous on a neighborhood of K. Then, by virtue of the corollary to Proposition 4.2, f^K depends only on φ, so that we denote it by φ^K. Obviously, the mapping $\varphi \to \varphi^K$ is non-negative linear on $\boldsymbol{C}_L(K)$. Since $\boldsymbol{C}_L(K)$ is dense in $\boldsymbol{C}(K)$, for each $x \in \omega_0$ there is a non-negative measure $\tilde{\mu}_x^K \equiv \tilde{\mu}_x^{K,(L,\boldsymbol{R},B)}$ on K such that

$$\varphi^K(x) = \int \varphi \, d\tilde{\mu}_x^K$$

for all $\varphi \in \boldsymbol{C}_L(K)$. If $x \in K$, then $\tilde{\mu}_x^K = \delta_x$, the unit point mass at x. If $x \in \omega_0 - K$, then the measure $\tilde{\mu}_x^K$ is supported by $\partial\omega\,(\subset \partial K)$, where ω is the component of $\omega_0 - K$ containing x; furthermore, $\tilde{\mu}_x^K = \mu_x^\omega$ in case ω is regular (cf. [11] for the measure μ_x^ω).

Given a function φ on K, if it is $\tilde{\mu}_x^K$-summable for any $x \in \omega_0 - K$, then we define $\varphi^K = \varphi$ on K and

$$\varphi^K(x) = \int \varphi \, d\tilde{\mu}_x^K$$

for $x \in \omega_0 - K$. In this case φ^K is $(L, \boldsymbol{R}, B)$-full-harmonic on $\omega_0 - K$.

The following lemma can be proved in the same way as Theorem 3 of [16], using our Proposition 4.3:

LEMMA 6.3. *Let* $K \subset K'$ *and suppose* φ *is* $\tilde{\mu}_x^K$*-summable for any* $x \in \omega_0 - K$. *Then* φ^K (restricted on K') *is* $\tilde{\mu}_x^{K'}$*-summable for any* $x \in \omega_0 - K'$ *and*

$$(\varphi^K)^{K'} = \varphi^K.$$

6.3. *Properties of* v^K *for* $v \in \mathcal{P}$.

LEMMA 6.4. *Any* $v \in \mathcal{P}$ (restricted on K) *is* $\tilde{\mu}_x^K$*-summable for any* $x \in \omega_0 - K$. *If* $v, w \in \mathcal{P}$ *and* $w \geqq v$ *on* K, *then* $w \geqq v^K$ *on* ω_0; *in particular,* $v \geqq v^K$ *on* ω_0.

PROOF. We can choose $\varphi_n \in \boldsymbol{C}_0^1(\omega_0)$ such that $\varphi_n \uparrow \min(v, w)$ on a neighborhood of K. For any component ω of $\omega_0 - K$, $(\varphi_n^K - \varphi_n) | \omega \in \boldsymbol{D}_{L,\partial\omega}(\omega)$. Since $w - \varphi_n^K$ is $(L, \boldsymbol{R}, B)$-full-superharmonic on ω, Proposition 4.5 (or Proposition 3.1, if ω is relatively compact) implies that $w \geqq \varphi_n^K$. If follows that v is $\tilde{\mu}_x^K$-summable for $x \in \omega$ and $v^K \leqq w$ on ω.

LEMMA 6.5. *If* $v \in \mathcal{P}$ *and* $K \subset K'$, *then* $v^K \leqq v^{K'}$.

PROOF. By Lemma 6.3 and the above lemma, we have for any $x \in \omega_0$

$$v^K(x) = (v^K)^{K'}(x) = \int v^K d\tilde{\mu}_x^{K'} \leqq \int v \, d\tilde{\mu}_x^{K'} = v^{K'}(x).$$

LEMMA 6.6. *If* $v \in \mathcal{P}$ *and if* K *is a regular compact set, then* $v^K \in \mathcal{P}$.

PROOF. By a standard discussion (cf. e.g., the proof of Theorem 7 in [16]), we see that v^K is lower semi-continuous in case K is regular. Then, by Lemma 6.4, we see that v^K is $(L, \boldsymbol{R}, B)$-full-superharmonic on ω_0. Since $0 \leqq v^K \leqq v$ and $v \in \mathcal{P}$, $v^K \in \mathcal{P}$.

6.4. *Reduced function of* $v \in \mathcal{P}$ *for open sets.*

For $v \in \mathcal{P}$ and an open set σ in ω_0, we define

$$v_\sigma = \inf \{w \in \mathcal{P};\ w \geqq v \ \text{ on } \sigma\}.$$

Obviously, $0 \leqq v_\sigma \leqq v$ and $v_\sigma = v$ on σ. First we prove

PROPOSITION 6.1. $v_\sigma = \sup\{v^K;\ K\text{: compact} \subset \sigma\}$ *and* $v_\sigma \in \mathcal{P}$.

PROOF. Let $v_0 = \sup\{v^K;\ K\text{: compact} \subset \sigma\}$. It is easy to see that $v_0(x) = v(x)$ for $x \in \sigma$. By Lemma 6.4, we see that $v_\sigma \geqq v^K$ for any $K \subset \sigma$, and hence we have $v_\sigma \geqq v_0$. To show the converse inequality, choose an increasing sequence $\{K_n\}$ of regular compact sets in σ such that $\bigcup_n \operatorname{Int}(K_n) = \sigma$. By Lemmas 6.5 and 6.6, $\{v^{K_n}\}$ is an increasing sequence of functions in $\mathcal{P}$ and $v_0 = \lim_{n\to\infty} v^{K_n}$. Since $0 \leqq v^{K_n} \leqq v \in \mathcal{P}$, $v_0 \in \mathcal{P}$. It also follows that $v_0 \geqq v_\sigma$. Hence $v_\sigma = v_0$ and $v_\sigma \in \mathcal{P}$.

Properties of v_σ:

a) $v_1 \leqq v_2$ *on* σ *implies* $(v_1)_\sigma \leqq (v_2)_\sigma$;

b) $\sigma_1 \subset \sigma_2$ *implies* $(v_{\sigma_1})_{\sigma_2} = (v_{\sigma_2})_{\sigma_1} = v_{\sigma_1} \leqq v_{\sigma_2}$;

c) $(v_1 + v_2)_\sigma = (v_1)_\sigma + (v_2)_\sigma$.

Proofs of these properties are easy and standard. Note that, in proving c), we use Proposition 6.1 and the fact that $(v_1+v_2)^K=(v_1)^K+(v_2)^K$ for a compact set K.

LEMMA 6.7. *Let σ be an open set such that $\bar{\sigma}\subset\omega_0$ and let $v\in\mathcal{P}$. Then there exists a non-negative measure ν supported by $\bar{\sigma}^*$* (the closure of σ in $\omega_0\cup\Delta$; cf. 5.5) *such that*

$$v_\sigma(x)=\int_{\bar{\sigma}^*}K_y(x)d\nu(y)$$

for all $x\in\omega_0-\bar{\sigma}$.

The proof of this lemma is similar to that of Theorem 14 in [16] (or, the original proof by R. S. Martin [13]). Namely, first choose an increasing sequence of regular compact sets $\{K_n\}$ such that $\cup\,\mathrm{Int}(K_n)=\sigma$ and express v^{K_n} in the form

$$v^{K_n}=\int_{K_n}K_y d\nu_n(y)$$

(cf. the corollary to Theorem 5.3). By considering the values of both hand sides at a point $x_1\in\omega_0$ where $v(x_1)<\infty$, we see that $\{\nu_n(K_n)\}$ is bounded. Taking a vague limit ν of $\{\nu_n\}$ and using Proposition 6.1, we obtain Lemma 6.7.

LEMMA 6.8. *Let σ be an open set in ω_0. If $v\in\boldsymbol{D^R}\cap\mathcal{P}$, then $v_\sigma\in\boldsymbol{D^R}\cap\mathcal{P}$ and, for any increasing sequence $\{K_n\}$ of compact sets in σ such that $v^{K_n}\uparrow v_\sigma$, we have*

$$\|v^{K_n}-v_\sigma\|_{L,\omega_0}\to 0 \qquad (n\to\infty).$$

PROOF. For simplicity we write $v_n=v^{K_n}$. By Lemma 6.2, $\{v_n\}$ is bounded in $\boldsymbol{D^R}$. Hence it is weakly compact. It is easy to see that any weak limit function of $\{v_n\}$ must be equal to v_σ. It follows that $v_\sigma\in\boldsymbol{D^R}$ and $v_n\to v_\sigma$ weakly in $\boldsymbol{D^R}$. Let $n>m$. Since $(v_m)^{K_n}=v_n$, Lemma 6.1 implies

$$A_{L,\omega_0}[v_n,\ v_n-v_m]+B[\tilde{v}_n,\ \tilde{v}_n-\tilde{v}_m]=0.$$

Letting $m\to\infty$, we have

$$A_{L,\omega_0}[v_n,\ v_n-v_\sigma]+B[\tilde{v}_n,\ \tilde{v}_n-\tilde{v}_\sigma]=0.$$

Hence,

$$\begin{aligned}0&\leqq A_{L,\omega_0}[v_n-v_\sigma,\ v_n-v_\sigma]+B[\tilde{v}_n-\tilde{v}_\sigma,\ \tilde{v}_n-\tilde{v}_\sigma]\\&=A_{L,\omega_0}[v_\sigma,\ v_\sigma-v_n]+B[\tilde{v}_\sigma,\ \tilde{v}_\sigma-\tilde{v}_n]\to 0 \qquad (n\to\infty).\end{aligned}$$

From the coerciveness of A_{L,ω_0}, it follows that $\|v_n-v_\sigma\|_{L,\omega_0}\to 0$ $(n\to\infty)$.

6.5. *Reduced functions of $v \in \mathcal{P}$ for closed sets on the ideal boundarry.*

We now consider the ideal boundary Δ of ω_0 given in 5.4. Let e be a closed subset of Δ. For $v \in \mathcal{P}$, we define

$$v_e = \inf \{v_{\sigma \cap \omega_0};\ \sigma \text{ is an open neighborhood of } e \text{ in } \omega_0 \cup \Delta\}.$$

Obviously, $0 \leqq v_e \leqq v$. It is easy to see that $v_e \in \mathcal{P}_b$. Since $\omega_0 \cup \Delta$ is metrizable, we can choose a decreasing sequence $\{\sigma_n\}$ of open neighborhoods of e in $\omega_0 \cup \Delta$ such that $\cap \bar{\sigma}_n^* = e$. Then $v_{\sigma_n \cap \omega_0} \downarrow v_e$ $(n \to \infty)$.

Properties of v_e:
a) $v_1 \leqq v_2$ *implies* $(v_1)_e \leqq (v_2)_e$;
b) $e_1 \subset e_2$ *implies* $v_{e_1} \leqq v_{e_2}$;
c) $(v_1 + v_2)_e = (v_1)_e + (v_2)_e$.
These are easy to see from the corresponding properties of v_σ.

Proposition 6.2. *If $v \in \mathcal{P}$ and e is a closed subset of Δ, then there exists a non-negative measure ν on e such that*

$$v_e = \int_e K_\eta \, d\nu(\eta)$$

on ω_0.

This proposition follows from Lemma 6.7 by an argument similar to the proof of Theorem 16 of [16] or Theorem II in §3 of [13].

Also, by the same methods as Theorems 15 and 17 of [16], we have

Proposition 6.3. *If $v \in \mathcal{P}$ is expressed as*

$$v(x) = \int_{\omega_0 \cup \Delta} K_y(x) \, d\mu(y) \qquad (x \in \omega_0),$$

then for an open set σ in ω_0 and a closed subset e of Δ, we have

$$v_\sigma(x) = \int_{\omega_0 \cup \Delta} (K_y)_\sigma(x) \, d\mu(y) \qquad (x \in \omega_0)$$

and

$$v_e(x) = \int_{\omega_0 \cup \Delta} (K_y)_e(x) \, d\mu(y) \qquad (x \in \omega_0).$$

Lemma 6.9. *If $v \in \boldsymbol{D}^{\boldsymbol{R}} \cap \mathcal{P}$, then $v_e \in \boldsymbol{D}^{\boldsymbol{R}} \cap \mathcal{P}_b$ for any closed subset e of Δ and if $\{\sigma_n\}$ is a decreasing sequence of open neighborhoods of e such that $v_{\sigma_n \cap \omega_0} \downarrow v_e$, then*

$$\| v_{\sigma_n \cap \omega_0} - v_e \|_{L, \omega_0} \to 0 \qquad (n \to \infty).$$

Proof. For simplicity, let $v_n = v_{\sigma_n \cap \omega_0}$. By Lemma 6.8, $v_n \in \boldsymbol{D}^{\boldsymbol{R}} \cap \mathcal{P}$ and by Lemma 6.2, we also see that $\{v_n\}$ is bounded in $\boldsymbol{D}^{\boldsymbol{R}}$. It follows that $v_e \in \boldsymbol{D}^{\boldsymbol{R}}$ and $v_n \to v_e$ weakly in $\boldsymbol{D}^{\boldsymbol{R}}$. Now, let $n<m$ and K, K' be compact sets such that $K \subset \sigma_n \cap \omega_0$, $K' \subset \sigma_m \cap \omega_0$ and $K' \subset K$. Then, by Lemma 6.1,

$$A_{L,\omega_0}[v^{K'}, v^{K'}-v^K]+B[\widetilde{v^{K'}}, \widetilde{v^{K'}}-\widetilde{v^K}]=0.$$

Letting $K \uparrow \sigma_n \cap \omega_0$, and then $K' \uparrow \sigma_m \cap \omega_0$, it follows from Lemma 6.8 that

$$A_{L,\omega_0}[v_m, v_m-v_n]+B[\tilde{v}_m, \tilde{v}_m-\tilde{v}_n]=0.$$

Hence

$$\begin{aligned} &A_{L,\omega_0}[v_m-v_e, v_m-v_e]+B[\tilde{v}_m-\tilde{v}_e, \tilde{v}_m-\tilde{v}_e] \\ &\quad =A_{L,\omega_0}[v_m, v_n]+B[\tilde{v}_m, \tilde{v}_n]-A_{L,\omega_0}[v_e, v_m]-B[\tilde{v}_e, \tilde{v}_m] \\ &\qquad -A_{L,\omega_0}[v_m, v_e]-B[\tilde{v}_m, \tilde{v}_e]+A_{L,\omega_0}[v_e, v_e]+B[\tilde{v}_e, \tilde{v}_e] \end{aligned}$$

Since $v_m \to v_e$ weakly as $m \to \infty$, we have

$$\begin{aligned} 0 &\leqq \lim_{m\to\infty} (A_{L,\omega_0}[v_m-v_e, v_m-v_e]+B[\tilde{v}_m-\tilde{v}_e, \tilde{v}_m-\tilde{v}_e]) \\ &=(A_{L,\omega_0}[v_e, v_n]+B[\tilde{v}_e, \tilde{v}_n])-(A_{L,\omega_0}[v_e, v_e]+B[\tilde{v}_e, \tilde{v}_e]) \\ &\to 0 \qquad (n\to\infty). \end{aligned}$$

Therefore, by coerciveness of A_{L,ω_0} we have $\|v_m-v_e\|_{L,\omega_0} \to 0$ $(m\to\infty)$.

PROPOSITION 6.4. *If $v \in \boldsymbol{D}^{\boldsymbol{R}} \cap \mathcal{P}$, then for any closed set e in Δ,*

$$(v_e)_e = v_e .$$

PROOF. Let $\{\sigma_n\}$ be a decreasing sequence of open neighborhoods of e such that $v_{\sigma_n \cap \omega_0} \downarrow v_e$. Let $v_n = v_{\sigma_n \cap \omega_0}$. Lemma 6.2 shows that for any compact set K in ω_0,

$$\|(v_n-v_e)^K\|_{L,\omega_0} \leqq M\|v_n-v_e\|_{L,\omega_0}.$$

Since $(v_n-v_e)^K=(v_n)^K-(v_e)^K$ and since $(v_n)^K \to (v_n)_{\sigma_m \cap \omega_0}$ and $(v_e)^K \to (v_e)_{\sigma_m \cap \omega_0}$ in $\boldsymbol{D}^{\boldsymbol{R}}$ as $K \uparrow \sigma_m \cap \omega_0$ by Lemma 6.8, we have

$$\|(v_n)_{\sigma_m \cap \omega_0}-(v_e)_{\sigma_m \cap \omega_0}\|_{L,\omega_0} \leqq M\|v_n-v_e\|_{L,\omega_0}.$$

If $n>m$, then $(v_n)_{\sigma_m \cap \omega_0}=v_n$ by property b) of v_σ. Hence

$$0 \leqq \limsup_{n\to\infty} \|v_n-(v_e)_{\sigma_m \cap \omega_0}\|_{L,\omega_0} \leqq M \lim_{n\to\infty} \|v_n-v_e\|_{L,\omega_0}=0$$

by the above lemma. Hence $v_n \to (v_e)_{\sigma_m \cap \omega_0}$ in $\boldsymbol{D}^{\boldsymbol{R}}$ $(n\to\infty)$, which implies, again by the above lemma, $(v_e)_{\sigma_m \cap \omega_0}=v_e$. Finally, letting $m\to\infty$, we have $(v_e)_e=v_e$.

Now that we have Proposition 6.4, we can prove the following lemma in a way similar to the proof of Theorem 20 of [16]:

LEMMA 6.10. *Let e be a closed set in Δ such that for some $u \in \boldsymbol{D}^{\boldsymbol{R}} \cap \mathcal{P}$ with $u>0$, we have $u_e=0$. Then for any $v \in \mathcal{P}$, $(v_e)_e=v_e$.*

Sketch of the PROOF. We decompose v into $v=v_b+v_i$, where $v_b \in \mathcal{P}_b$ and $v_i=\int_{\omega_0} K_y d\mu(y)$. By virtue of property c) of v_e, it is enough to show that $((v_b)_e)_e=(v_b)_e$ and $((v_i)_e)_e=(v_i)_e$. To prove the first equality, we show that if $v \in \mathcal{P}_b$ then $v_{\sigma\cap\omega_0}-v_e$ is $(L, \boldsymbol{R}, B)$-full-superharmonic on ω_0 for any neighborhood σ of e, by the same argument as that in the proof of Theorem 20 of [16], and making use of the assumption $u_e=0$. Then, for another neighborhood σ' of e such that $\sigma' \subset \sigma$, properties b) and c) for v_σ imply

$$v_{\sigma\cap\omega_0}-v_e \geqq (v_{\sigma\cap\omega_0}-v_e)_{\sigma'\cap\omega_0}=(v_{\sigma\cap\omega_0})_{\sigma'\cap\omega_0}-(v_e)_{\sigma'\cap\omega_0}$$

$$=v_{\sigma'\cap\omega_0}-(v_e)_{\sigma'\cap\omega_0} \geqq 0.$$

Now letting $\sigma' \downarrow e$, and then $\sigma \downarrow e$, we obtain $(v_e)_e=v_e$.

In order to prove that $((v_i)_e)_e=(v_i)_e$, we first show that if $y \in \omega_0$, then $(K_y)_e=0$. This can be seen by the fact that $K_y \leqq \alpha u$ in a neighborhood of Δ for some α and by the assumption $u_e=0$. Then, by Proposition 6.3, we have $(v_i)_e=0$, and hence $((v_i)_e)_e=0=(v_i)_e$.

REMARK. $\boldsymbol{D}^{\boldsymbol{R}} \cap \mathcal{P}$ always contains a positive function. For, if $\boldsymbol{R}=\{0\}$ then any $G^{L,\omega_0}(\psi)$ $(\psi \geqq 0)$ belongs to $\boldsymbol{D}^{\boldsymbol{R}} \cap \mathcal{P}=\boldsymbol{D}_{L,0}(\omega_0) \cap \mathcal{P}$; if $\boldsymbol{R} \neq \{0\}$, then any U_y given in Theoerem 5.1 is positive and belongs to $\boldsymbol{R} \cap \mathcal{P}_b \subset \boldsymbol{D}^{\boldsymbol{R}} \cap \mathcal{P}$.

6.6. *Characterization of Δ_1.*

The results in the above section 6.5 allow us to prove the following characterization of the set Δ_1 (see 5.4 for the definition of Δ_1):

THEOREM 6.1.

$$\Delta_1=\{\eta \in \Delta;\, (K_\eta)_{\{\eta\}}=K_\eta\}$$

and

$$\Delta-\Delta_1=\{\eta \in \Delta;\, (K_\eta)_{\{\eta\}}=0\}.$$

For the classical harmonic structure, this result is well known in case Δ is the Martin boundary (see e.g., Theorem I in §4 of [13]) or the Kuramochi boundary (see, the corollary to Theorem 21 and Theorem 26 of [16]). For the proof of Theorem 6.1 we can essentially follow the arguments in [16].

Sketch of the PROOF of Theorem 6.1. Let $\Delta'=\{\eta \in \Delta;\, (K_\eta)_{\{\eta\}}=K_\eta\}$. Using property c) of v_e and Proposition 6.2, we easily see that $\Delta' \subset \Delta_1$ (cf. the proof of Theorem I in §4 of [13] and that of Theorem 26, 3) of [16]). On the other hand, Propositions 6.2 and 6.4, together with Lemma 6.10 imply that

$(K_\eta)_{\{\eta\}}=0$ for $\eta \in \Delta-\Delta'$ (cf. the proof of Theorem 21 of [16]). Thus, it remains to show $\Delta_1 \subset \Delta'$. Suppose $\eta \in \Delta_1$ and $\eta \notin \Delta'$. Then there is an open neighborhood σ of η such that $(K_\eta)_{\sigma\cap\omega_0} \neq K_\eta$. Let $\{\omega_n\}$ be a sequence of relatively compact open sets in ω_0 such that $\partial\omega_n \supset \partial\omega_0$, $\partial\omega_n \cap \omega_0$ is compact, $\bar{\omega}_n \cap \omega_0 \subset \omega_{n+1}$ and $\bigcup_n \omega_n = \omega_0$. If $(K_\eta)_{\omega_n} = \int K_y d\mu_n(y)$ with a positive measure μ_n on $\partial\omega_n \cap \omega_0$, then it is easy to see that $\{\mu_n(\partial\omega_n)\}$ is bounded. Since $(K_\eta)_{\omega_n} \to K_\eta$ as $n\to\infty$, it follows from the assumption $\eta \in \Delta_1$ that μ_n vaguely converges to the unit point mass δ_η at η. Let σ' be another open neighborhood of η such that $\bar{\sigma}'^* \subset \sigma$. Then $\mu_n | \bar{\sigma}'^*$ also vaguely converges to δ_η (cf. the proof of Theorem 24 in [16]). Hence, using the relation $(K_y)_{\sigma\cap\omega_0} = K_y$ for $y \in \bar{\sigma}' \cap \partial\omega_n$ and Proposition 6.3, we have

$$\begin{aligned} K_\eta &= \lim_{n\to\infty} \int_{\bar{\sigma}'\cap\partial\omega_n} K_y d\mu_n(y) \\ &= \lim_{n\to\infty} \int_{\bar{\sigma}'\cap\partial\omega_n} (K_y)_{\sigma\cap\omega_0} d\mu_n(y) \\ &\leqq \lim_{n\to\infty} ((K_\eta)_{\omega_n})_{\sigma\cap\omega_0} \leqq (K_\eta)_{\sigma\cap\omega_0}, \end{aligned}$$

which contradicts the choice of σ. Hence $\eta \in \Delta_1$ implies $\eta \in \Delta'$.

6.7. *Thin sets at ideal boundary points.*

Following Naïm [15] (Theorem 2), we give the following definition:

An open set X in ω_0 is called thin (more precisely, $(L, \boldsymbol{R}, B)$-thin) at $\xi \in \Delta$, if there is an open neighborhood σ of ξ in $\omega_0 \cup \Delta$ such that

$$(K_\xi)_{\sigma\cap X} \neq K_\xi .$$

Obviously, if X is thin at ξ and $X' \subset X$, then X' is thin at ξ. Thus, for *any* set X in ω_0, X is called thin at ξ if it is contained in an open set X' which is thin at ξ in the above sense.

If $\xi \notin \bar{X}^*$, then X is thin at ξ. Theorem 7.1 shows that ω_0 is thin at $\xi \in \Delta$ if and only if $\xi \in \Delta - \Delta_1$.

In [15], the definition of thin sets was given in terms of a class of functions instead of the above form. The equivalence of two types of definition of thin sets was generalized by M. Brelot [2] to an axiomatic theory. It can be applied to the present theory in case $\boldsymbol{R}=\{0\}$, i.e., in the case of Martin boundary. We can also establish a similar equivalence even in case $\boldsymbol{R}\neq\{0\}$; in fact we shall prove:

THEOREM 6.2. *Let u be any positive continuous function on ω_0 satisfying the condition given in* 5.6, *i.e., there are a neighborhood ω of Δ and $u_1 \in \mathcal{P}^*$ such that $u=u_1$ on $\omega\cap\omega_0$ and u is $(L^*, \boldsymbol{R}^*, B^*)$-full-harmonic on $\omega\cap\omega_0$. Let $X \subset \omega_0$ and $\xi \in \Delta\cap\bar{X}^*$. Then X is thin at ξ if and only if there is $f \in \mathcal{P}_u^*$* (see

5.6 for this class of functions) *such that*

$$(*) \qquad \liminf_{y \to \xi, y \in X} f(y) > \hat{f}(\xi).$$

To prove this theorem, we need some preparations.

LEMMA 6.11. *Let σ be an open set in ω_0 and $x \in \omega_0$. Then the function $y \to (\tilde{g}_y)_\sigma(x)$ belongs to $\mathcal{P}^*$.*

PROOF. For a compact set K in ω_0,

$$(\tilde{g}_y)^K(x) = \int \tilde{g}_y \, d\tilde{\mu}_x^K = \int \tilde{g}_z^*(y) d\tilde{\mu}_x^K(z),$$

and hence the function $y \to (\tilde{g}_y)^K(x)$ belongs to $\mathcal{P}^*$. Since $(\tilde{g}_y)_\sigma(x) = \lim_{n \to \infty} (\tilde{g}_y)^{K_n}(x)$ for some increasing equence $\{K_n\}$ of compact sets and since $(\tilde{g}_y)_\sigma(x) \leqq \tilde{g}_y(x) = \tilde{g}_x^*(y)$, we see that the function $y \to (\tilde{g}_y)_\sigma(x)$ belongs to $\mathcal{P}^*$.

COROLLARY. $(\tilde{g}_y)_\sigma(x) = (\tilde{g}_x^*)_\sigma(y)$, *where, in the right hand side, the reduced function is taken with respect to the (L^*, $\boldsymbol{R}^*$, B^*)-full-harmonic structure.*

PROOF. If $y \in \sigma$, then $(\tilde{g}_y)_\sigma(x) = \tilde{g}_y(x) = \tilde{g}_x^*(y)$. Hence, by the above lemma, $(\tilde{g}_y)_\sigma(x) \geqq (\tilde{g}_x^*)_\sigma(y)$. Starting from the adjoint structure, we obtain the converse inequality.

For an open set σ in ω_0 and a point $x \in \omega_0$, let

$$F_{\sigma,x}(y) = (K_y)_\sigma(x).$$

By the above lemma, $F_{\sigma,x} \in \mathcal{P}_{1/\gamma}^*$. We now show

LEMMA 6.12. $\hat{F}_{\sigma,x}(\xi) = (K_\xi)_\sigma(x)$ *for* $\xi \in \Delta$.

PROOF. For a compact set K, let $F_{K,x}(y) = (K_y)^K(x)$. Since

$$F_{K,x}(y) = \int K_y(z) d\tilde{\mu}_x^K(z),$$

we have

$$\hat{F}_{K,x}(\xi) = \int K_\xi(z) d\tilde{\mu}_x^K(z) = (K_\xi)^K(x)$$

for any $\xi \in \Delta$. Hence, it is enough to show that $\hat{F}_{\sigma,x}(\xi) = \sup_{K \subset \sigma} \hat{F}_{K,x}(\xi)$.

Let $w_K(y) = (\tilde{g}_y)^K(x)$ and $w(y) = (\tilde{g}_y)_\sigma(x)$. Then $w_K, w \in \mathcal{P}^*$ and $F_{K,x} = \gamma w_K$, $F_{\sigma,x} = \gamma w$. For a compact set K' in ω_0 and $u \in \mathcal{P}^*$, we shall write $u_*^{K'}(y) = \int u \, d\tilde{\mu}_y^{K', (L^*, R^*, B^*)}$. Let $f_K^{(K')} = \gamma (w_K)_*^{K'}$ and $f^{(K')} = \gamma w_*^{K'}$. By definition,

$$\hat{F}_{\sigma,x}(\xi) = \sup_{K' \subset \omega_0} \widehat{f^{(K')}}(\xi).$$

Hence, for any $\lambda<\hat{F}_{\sigma,x}(\xi)$, we find a compact set K' in ω_0 such that $\widehat{f^{(K')}}(\xi)>\lambda$. We may assume that $x\in K'$ and $\gamma(y)=1/\tilde{g}_y(x_0)$ on ω_0-K' (so that $x_0\in K'$). Choose another compact set K'' in ω_0 whose interior contains K'. Then

$$w_*^{K'}(y)-(w_K)_*^{K'}(y)=(w_*^{K'})_*^{K''}(y)-((w_K)_*^{K'})_*^{K''}(y)$$

$$=\int(w_*^{K'}-(w_K)_*^{K'})d\tilde{\mu}_y^{K'',(L^*,R^*,B^*)}.$$

If $y\in\omega_0-K''$, then the support of $\tilde{\mu}_y^{K'',(L^*,R^*,B^*)}$ is contained in $\partial K''$. Hence for $y\in\omega_0-K''$,

$$0\leqq w_*^{K'}(y)-(w_K)_*^{K'}(y)$$

$$\leqq\sup_{z\in\partial K''}\{(w_*^{K'}(z)-(w_K)_*^{K'}(z))\gamma(z)\}\int_{\partial K''}\tilde{g}_{x_0}^*(z)d\tilde{\mu}_y^{K'',(L^*,R^*,B^*)}(z)$$

$$=\sup_{z\in\partial K''}\{f^{(K')}(z)-f_K^{(K')}(z)\}\,\tilde{g}_{x_0}^*(y).$$

Thus,

$$0\leqq f^{(K')}(y)-f_K^{(K')}(y)\leqq\sup_{z\in\partial K''}\{f^{(K')}(z)-f_K^{(K')}(z)\}$$

for all $y\in\omega_0-K''$. It follows that

$$0\leqq\widehat{f^{(K')}}(\xi)-\widehat{f_K^{(K')}}(\xi)\leqq\sup_{z\in\partial K''}\{f^{(K')}(z)-f_K^{(K')}(z)\}$$

for any $\xi\in\Delta$. Since $w_K\uparrow w$, $(w_K)_*^{K'}\uparrow w_*^{K'}$ as $K\uparrow\sigma$. Since $(w_K)_*^{K'}$, $w_*^{K'}$ are both L^*-harmonic on ω_0-K'', $(w_K)_*^{K'}$ converges to $w_*^{K'}$ uniformly on $\partial K''$ as $K\uparrow\sigma$. Thus, given $\varepsilon>0$, there is $K\subset\sigma$ such that $\sup_{z\in\partial K''}\{f^{(K')}(z)-f_K^{(K')}(z)\}<\varepsilon$. For such K, we have

$$\lambda<\widehat{f^{(K')}}(\xi)\leqq\widehat{f_K^{(K')}}(\xi)+\varepsilon\leqq\hat{F}_{K,x}(\xi)+\varepsilon\leqq\sup_{K\subset\sigma}\hat{F}_{K,x}(\xi)+\varepsilon.$$

Since λ ($<F_{\sigma,x}(\xi)$) and ε (>0) are arbitrary, we have

$$\hat{F}_{\sigma,x}(\xi)=\sup_{K\subset\sigma}\hat{F}_{K,x}(\xi).$$

PROOF of Theorem 6.2. Once we obtain the above lemma, we can prove Theorem 6.2 by a method similar to that of Théorème 2 in [15]. First observe that, by virtue of Proposition 5.2, it is enough to prove the theorem in case $u=1/\gamma$. Also, we may assume that X is an open set.

If X is thin at ξ, i.e., if there is an open neighborhood σ of ξ such that $(K_\xi)_{\sigma\cap X}\neq K_\xi$, then choose $x\in\omega_0$ such that $(K_\xi)_{\sigma\cap X}(x)<K_\xi(x)$ and consider $f=F_{\sigma\cap X,x}$. Then $f\in\mathcal{P}_{1/\gamma}^*$. Since $f(y)=(K_y)_{\sigma\cap X}(x)=K_y(x)$ for $y\in\sigma\cap X$,

$$\liminf_{y\to\xi,y\in X}f(y)=\liminf_{y\to\xi,y\in X\cap\sigma}K_y(x)=K_\xi(x)>(K_\xi)_{\sigma\cap X}(x)=f(\xi),$$

where we used Lemma 6.12 in the last equality.

Conversely, suppose there is $f \in \mathcal{D}^*_{1/\gamma}$ satisfying (*) of the theorem. Choose λ such that $\liminf_{y\to\xi, y\in X} f(y) > \lambda > \hat{f}(\xi)$. Then there is an open neighborhood σ of ξ such that $f \geqq \lambda$ on $\sigma \cap X$. We may assume that $\gamma(y) = 1/\tilde{g}_y(x_0)$ for $y \in \sigma \cap \omega_0$. Let $f = \gamma w$ with $w \in \mathcal{D}^*$. Since $w(y) \geqq \lambda \tilde{g}_y(x_0) = \lambda \tilde{g}^*_{x_0}(y)$ for $y \in \sigma \cap X$, we have $w \geqq \lambda (\tilde{g}^*_{x_0})_{\sigma \cap X}$. By the corollary to Lemma 6.11, $w(y) \geqq \lambda (\tilde{g}_y)_{\sigma \cap X}(x_0)$ for all $y \in \omega_0$. Hence $f \geqq \lambda F_{\sigma \cap X, x_0}$. Therefore, again using the above lemma, we have $\hat{f}(\xi) \geqq \lambda (K_\xi)_{\sigma \cap X}(x_0)$. It follows that $(K_\xi)_{\sigma \cap X}(x_0) < 1 = K_\xi(x_0)$. Hence X is thin at ξ.

COROLLARY 1. *Let u be as in Theorem 6.2 and let $\xi \in \Delta$. Then $\xi \in \Delta_1$ if and only if $\hat{f}(\xi) = \liminf_{y\to\xi} f(y)$ for all $f \in \mathcal{D}^*_u$.*

COROLLARY 2. *The set Δ_1 and the notion of thin sets do not depend on the choice of x_0*

COROLLARY 3. *If X_1, X_2 are thin at $\xi \in \Delta$, then $X_1 \cup X_2$ is thin at ξ.*

Corresponding to the notion of $(L, \boldsymbol{R}, B)$-thinness, there is a notion of $(L, \boldsymbol{R}, B)$-fine limit at a point in Δ_1. Namely, for $\xi \in \Delta_1$, let

$$\mathcal{F}_\xi = \{\omega_0 - X;\ X \text{ is thin at } \xi\}.$$

Then $\mathcal{F}_\xi$ is a filter. Limits with respect to this filter is the $(L, \boldsymbol{R}, B)$-fine limits at ξ. By Theorem 6.2, we have

COROLLARY 4. *Any $f \in \mathcal{D}^*_u$ has an $(L, \boldsymbol{R}, B)$-fine limit $\hat{f}(\xi)$ at every $\xi \in \Delta_1$.*

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