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Asymptotic Expansions of Some Test Criteria for Homogeneity of Variances and Covariance Matrices From Normal Populations

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CONTENTS

0. Introduction and summary. It is very difficult to obtain exact distributions of most of the test statistics used in the multivariate analysis. Therefore, some statisticians have hitherto obtained asymptotic expansion of test criteria instead of the exact distributions. This paper is divided into five parts and deals with the problem of asymptotic expansions of test criteria on variances and covariances in normal populations. In Part I we compare a modified likelihood ratio test (Bartlett [3]) with the asymptotically UMP invariant test (Lehmann [13]) for testing homogeneity of variances of k

normal populations under the local alternativs. Comparison of the two tests is done by first expanding asymptotically the test statistics under the assumption of local alternatives approaching the null hypothesis. Such an asymptotic expansion in other problem was first used by Sugiura [24]. In our previous paper [27], the two tests were compared under fixed alternative. Part II deals with the asymptotic expansions of sphericity test criterion by using the result of Part I under local alternatives and with numerical power. Part III is concerned with multivariate Bartlett's test, the limiting distribution of which under fixed alternative was shown to be normal by Sugiura [23]. We shall give the asymptotic expansion in Part III. Part IV is to give the asymptotic expansions of some test criteria for testing the equality of two covariance matrices. Some desirable properties of the above tests were discussed by Anderson and Das Gupta [2] and Giri [6]. Part V treats a modified likelihood ratio test for eigenvalues and eigenvectors of covariance matrix. The unbiasedness and the asymptotic expansion of the test criterion will be given.

PART I. BARTLETT'S TEST AND LEHMANN'S TEST

1. Preliminaries. Let $X_{i1}, X_{i2},\cdots, X_{iN_i}$ be a random sample from a normal distribution with mean μ_i and variance $\sigma_i^2 (i=1, 2, \cdots,\mathrm{k})$. For testing the hypothesis $H: \sigma_1^2=\cdots=\sigma_k^2$ against all alternatives $K: \sigma_i^2 \neq \sigma_j^2$ for some i and j $(i \neq j)$ with unspecified μ_i, the L test criterion due to Lehmann [13] is given by

$$(1.1) \qquad L=\frac{1}{2}\sum_{\alpha=1}^{k}\mathrm{n}_\alpha\{\log(S_\alpha/n_\alpha)-n^{-1}\sum_{\beta=1}^{k}n_\beta\log(S_\beta/n_\beta)\}^2,$$

where $S_j=\sum_{\alpha=1}^{N_j}(X_{j\alpha}-\bar{X}_j)^2$ with $\bar{X}_j=N_j^{-1}\sum_{\alpha=1}^{N_j}X_{j\alpha}$ and $n_j=N_j-1$ with $n=\sum_{\alpha=1}^{k}n_\alpha$. The M test criterion due to Bartlett [3], without correction factor, is given by

$$(1.2) \qquad M=n\log(\sum_{\alpha=1}^{k}S_\alpha/n)-\sum_{\alpha=1}^{k}n_\alpha\log(S_\alpha/n_\alpha)$$

with the same notation as above. The L (or M) test rejects the hypothesis H, when the observed value of L(or M) is larger than a preassigned constant.

In our previous paper [27], the L test and the M test were compared from the following points: (1) Unbiasedness (2) Limiting distribution under local alternatives (3) Asymptotic expansion under fixed alternative. The M test has also been investigated by many authors. (Mahalanobis [14], Nayer [18], Bishop and Nair [4], Nair [17], Hartley [10], Thompson and Merrington [28], Box [5], Pearson [19], Harsaae [9], etc.) However as far as the author is aware, Sugiura and Nagao [27] are the first ones to consider the L test.

We could not compute the power by using (3) of our previous paper [27] when the alternative hypothesis is near to the null hypothesis. We shall give the asymptotic expansions of the L test and M test under specical sequences of alternatives converging to the null hypothesis with the rate of $n^{-\delta}$, where n means the total degrees of freedom. As we have already seen in our previous paper [27], the limiting distributions of the L (or M) are (a) normal for $0<\delta<\frac{1}{2}$ (b) non-central χ^2 for $\delta=\frac{1}{2}$ (c) χ^2 for $\delta>\frac{1}{2}$.

2. Asymptotic expansions of Lehmann's test under local alternatives.

2. *a. Asymptotic distribution for* $\delta=\frac{1}{4}$. Without loss of generality, we may assume $\sigma_1^2=1$ under the alternative hypotheses. First we shall consider the sequence of alternatives $K_a : \sigma_\alpha^2 = 1+n^{-\frac{1}{4}}\theta_\alpha$ $(\alpha=1, 2, \cdots, k)$. By the fact that the statistic $T_\alpha=[(S_\alpha/\sigma_\alpha^2)-n_\alpha]/\sqrt{2n_\alpha}$ has asmptotically the standard normal distribution as n_α tends to infinity, the statistic L is expressed in terms of $T_1, T_2, \cdots, T_k$ as

$$(2.a.1)\ L=\frac{n}{2}\sum_{\alpha=1}^k\rho_\alpha(\tilde{\sigma}_\alpha-\tilde{\sigma})^2+\sqrt{2n}\sum_{\alpha=1}^k\sqrt{\rho_\alpha}(\tilde{\sigma}_\alpha-\tilde{\sigma})T_\alpha+\sum_{\alpha=1}^k(\tilde{\sigma}-\tilde{\sigma}_\alpha+1)T_\alpha^2$$
$$-(\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,T_\alpha)^2+O_p(n^{-\frac{1}{2}}),$$

which was used in getting the asymptotic expansion of the L under the fixed alternative by Sugiura and Nagao [27], where $\tilde{\sigma}_\alpha=\log\sigma_\alpha^2$ and $\tilde{\sigma}=\sum_{\alpha=1}^k\rho_\alpha\cdot\log\sigma_\alpha^2$ with fixed $\rho_\alpha=n_\alpha/n$. Under K_a, we can rewrite the expression of L in (2.a.1) as

$$(2.a.2)\qquad L=\frac{n^{\frac{1}{2}}}{2}\sum_{\alpha=1}^k\rho_\alpha(\theta_\alpha-\tilde{\theta})^2-\frac{n^{\frac{1}{4}}}{2}\sum_{\alpha=1}^k\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^2-\tilde{\theta}_2)+n^{\frac{1}{4}}l_0(T)$$
$$+l_1(T)+n^{-\frac{1}{4}}l_2(T)+O_p(n^{-\frac{1}{2}}),$$

where, putting $\tilde{\theta}_\beta=\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^\beta$ with $\tilde{\theta}_1=\tilde{\theta}$,

$$l_0(T)=\sum_{\alpha=1}^k\sqrt{2\rho_\alpha}(\theta_\alpha-\tilde{\theta})T_\alpha,$$

$$(2.a.3)\qquad l_1(T)=\sum_{\alpha=1}^k T_\alpha^2-(\sum_{\alpha=1}^k\sqrt{\rho_\alpha}T_\alpha)^2-\frac{1}{2}\sum_{\alpha=1}^k\sqrt{2\rho_\alpha}(\theta_\alpha^2-\tilde{\theta}_2)\mathrm{T}_\alpha$$
$$+\frac{1}{8}\sum_{\alpha=1}^k\rho_\alpha(\theta_\alpha^2-\tilde{\theta}_2)^2+\frac{1}{3}\sum_{\alpha=1}^k\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^3-\tilde{\theta}_3),$$

$$l_2(T)=\frac{1}{3}\sum_{\alpha=1}^k\sqrt{2\rho_\alpha}(\theta_\alpha^3-\tilde{\theta}_3)T_\alpha-\sum_{\alpha=1}^k(\theta_\alpha-\tilde{\theta})T_\alpha^2$$

$$-\frac{1}{4}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^4-\tilde{\theta}_4)-\frac{1}{6}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha^2-\tilde{\theta}_2)(\theta_\alpha^3-\tilde{\theta}_3).$$

From the expression of (2.a.2), we can see that the statistic $n^{-\frac{1}{4}}L'$ converges in law to the normal distribution with mean zero and variance $\tau_L^2=2\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})^2$, where $L'=L-\frac{n^{\frac{1}{2}}}{2}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})^2+\frac{1}{2}n^{\frac{1}{4}}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^2-\tilde{\theta}_2)$, which was shown in our previous paper [27].

Now we shall derive, more precisely, the distribution of the L test. The characteristic function of $n^{-\frac{1}{4}}L'/\tau_L(\tau_L>0)$ is expressed as

$$C_L(t)=E[e^{itl_0(T)/\tau_L}\{1+n^{-\frac{1}{4}}l_1(T)it/\tau_L+n^{-\frac{1}{2}}[l_2(T)it/\tau_L+\frac{1}{2}l_1^2(T)(it/\tau_L)^2]\}]+O(n^{-\frac{3}{4}}). \tag{2.a.4}$$

By using the formulae (3.a.6) given in Section 3.a, we obtain each term of (2.a.4). Letting $a_\alpha=\sqrt{2\rho_\alpha}(\theta_\alpha-\tilde{\theta})it/\tau_L$ in $l_0(T)$, we have

$$E[e^{itl_0(T)/\tau_L}]=e^{-\frac{t^2}{2}}\left\{1+n^{-\frac{1}{2}}\frac{\sqrt{2}}{3}\sum_{\alpha=1}^{k}a_\alpha^3/\sqrt{\rho_\alpha}\right\}+O(n^{-1}). \tag{2.a.5}$$

Noting $\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}a_\alpha=0$, we can write each expectation in (2.a.4) as

$$E[l_1(T)e^{itl_0(T)/\tau_L}]=e^{-\frac{t^2}{2}}\left\{\sum_{\alpha=1}^{k}a_\alpha^2+k-1-\frac{1}{2}\sum_{\alpha=1}^{k}\sqrt{2\rho_\alpha}(\theta_\alpha^2-\tilde{\theta}_2)a_\alpha+\frac{1}{8}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha^2-\tilde{\theta}_2)^2+\frac{1}{3}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^3-\tilde{\theta}_3)\right\}+O(n^{-\frac{1}{2}}), \tag{2.a.6}$$

$$E[l_2(T)e^{itl_0(T)/\tau_L}]=e^{-\frac{t^2}{2}}\left\{\frac{1}{3}\sum_{\alpha=1}^{k}\sqrt{2\rho_\alpha}(\theta_\alpha^3-\tilde{\theta}_3)a_\alpha-\sum_{\alpha=1}^{k}(\theta_\alpha-\tilde{\theta})a_\alpha^2-\sum_{\alpha=1}^{k}(\theta_\alpha-\tilde{\theta})-\frac{1}{4}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^4-\tilde{\theta}_4)-\frac{1}{6}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha^2-\tilde{\theta}_2)(\theta_\alpha^3-\tilde{\theta}_3)\right\}+O(n^{-\frac{1}{2}}), \tag{2.a.7}$$

and

$$E[l_1^2(T)e^{itl_0(T)/\tau_L}]=e^{-\frac{t^2}{2}}\ [\{\sum_{\alpha=1}^{k}a_\alpha^2\}^2-\sum_{\alpha=1}^{k}\sqrt{2\rho_\alpha}(\theta_\alpha^2-\tilde{\theta}_2)a_\alpha \tag{2.a.8}$$

$$\sum_{\alpha=1}^{k} a_\alpha^2 + 2\Big\{k+1+\frac{1}{3}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^3-\tilde{\theta}_3)$$

$$+\frac{1}{8}\sum_{\alpha=1}^{k}(\theta_\alpha^2-\tilde{\theta}_2)^2\Big\}\sum_{\alpha=1}^{k}a_\alpha^2+\frac{1}{2}\{\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}(\theta_\alpha^2-\tilde{\theta}_2)a_\alpha\}^2$$

$$-\Big\{k+1+\frac{1}{8}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha^2-\tilde{\theta}_2)^2+\frac{1}{3}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^3-\tilde{\theta}_3)\Big\}$$

$$\cdot\sum_{\alpha=1}^{k}\sqrt{2\rho_\alpha}(\theta_\alpha^2-\tilde{\theta}_2)\,a_\alpha+k^2-1+\Big\{\frac{1}{8}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha^2-\tilde{\theta}_2)^2$$

$$+\frac{1}{3}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^3-\tilde{\theta}_3)\Big\}^2+\frac{1}{4}(k+1)\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha^2-\tilde{\theta}_2)^2$$

$$+\frac{2}{3}(k-1)\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^3-\tilde{\theta}_3)\Big]+O(n^{-\frac{1}{2}}).$$

Thus the characteristic function of $n^{-\frac{1}{4}}L'/\tau_L$ can be expanded asymptotically as follows:

$$C_L(t)=e^{-\frac{t^2}{2}}\Big[1+n^{-\frac{1}{4}}\Big\{[k-1+\frac{1}{8}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha^2-\tilde{\theta}_2)^2 \tag{2.a.9}$$

$$+\frac{1}{3}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^3-\tilde{\theta}_3)]\frac{\mathrm{it}}{\tau_L}-\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^2-\tilde{\theta}_2)\Big(\frac{\mathrm{it}}{\tau_L}\Big)^2$$

$$+\tau_L^2\Big(\frac{\mathrm{it}}{\tau_L}\Big)^3\Big\}-n^{-\frac{1}{2}}\Big\{\sum_{\alpha=0}^{2}g_{2\alpha+1}\Big(\frac{\mathrm{it}}{\tau_L}\Big)^{2\alpha+1}-\sum_{\alpha=1}^{3}g_{2\alpha}\Big(\frac{\mathrm{it}}{\tau_L}\Big)^{2\alpha}\Big\}\Big]$$

$$+O(n^{-\frac{3}{4}}),$$

where the coefficients g_α are given by (2.a.10).

$$g_1=\sum_{\alpha=1}^{k}(\theta_\alpha-\tilde{\theta})+\frac{1}{4}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^4-\tilde{\theta}_4)+\frac{1}{6}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha^2-\tilde{\theta}_2)(\theta_\alpha^3-\tilde{\theta}_3),$$

$$g_2=\frac{1}{3}(k+1)\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^3-\tilde{\theta}_3)+\frac{1}{2}(k^2-1)+\frac{1}{2}\Big\{\frac{1}{8}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha^2$$

$$-\tilde{\theta}_2)^2$$

$$+\frac{1}{3}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^3-\tilde{\theta}_3)\Big\}^2+\frac{1}{8}(k+1)\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha^2-\tilde{\theta}_2)^2,$$

$$(2.a.10)\quad g_3=\frac{2}{3}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})^3+\Big\{k+1+\frac{1}{8}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha^2-\tilde{\theta}_2)^2$$

$$+\frac{1}{3}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^3-\tilde{\theta}_3)\Big\}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^2-\tilde{\theta}_2),$$

$$g_4=\Big\{k+1+\frac{1}{3}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^3-\tilde{\theta}_3)+\frac{1}{8}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha^2-\tilde{\theta}_2)^2\Big\}\tau_L^2$$

$$+\frac{1}{2}\{\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^2-\tilde{\theta}_2)\}^2,$$

$$g_5=\tau_L^2\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^2-\tilde{\theta}_2),\qquad g_6=\frac{\tau_L^4}{2}.$$

Inverting this characteristic function, we have the following theorem.

THEOREM 2.a. *Under the sequence of alternatives* K_α: $\sigma_\alpha^2=1+n^{-\frac{1}{4}}\theta_\alpha(\alpha=1, 2, \cdots, k)$, *the distribution of the statistic* $L'=L-\frac{n^{\frac{1}{2}}}{2}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})^2+\frac{n^{\frac{1}{4}}}{2}\cdot\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^2-\tilde{\theta}_2)$ *is expanded asymptotically as follows*:

$$(2.a.11)\quad P(L'/n^{\frac{1}{4}}\tau_L\leq x)=\Phi(x)-\mathrm{n}^{-\frac{1}{4}}\Big\{\Big[(k-1)+\frac{1}{8}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha^2-\tilde{\theta}_2)^2$$

$$+\frac{1}{3}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^3-\tilde{\theta}_3)\Big]\Phi^{(1)}(x)/\tau_L$$

$$+\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})(\theta_\alpha^2-\tilde{\theta}_2)\Phi^{(2)}(x)/\tau_L^2+\Phi^{(3)}(x)/\tau_L\Big\}$$

$$+n^{-\frac{1}{2}}\sum_{\alpha=1}^{6}g_\alpha\Phi^{(\alpha)}(x)/\tau_L^\alpha+O(n^{-\frac{3}{4}}),$$

where $\tau_L^2=2\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})^2$, $\tilde{\theta}_\beta=\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^\beta$ *with* $\tilde{\theta}_1=\tilde{\theta}$ *and* $\Phi^{(j)}(x)$ *means the j-th derivative of the standard normal distribution function* $\Phi(x)$.
The coefficients g_α *are given by* (2.a.10).

2.b. *Asymptotic distribution for* $\delta=\frac{1}{2}$. The statistic L can be rewritten as follows:

$$(2.b.1)\quad L=\frac{1}{2}\Big[\sum_{\alpha=1}^{k}n_\alpha\{\log(S_\alpha/n_\alpha)\}^2-\frac{1}{n}\{\sum_{\beta=1}^{k}n_\beta\log(S_\beta/n_\beta)\}^2\Big].$$

By expressing the L with T_α $(\alpha=1, 2, \cdots, \mathrm{k})$ as in Section 2.a, we gave the limiting distribution under the sequence K_b : $\sigma_\alpha^2=1+n^{-\frac{1}{2}}\theta_\alpha\,(\alpha=1, 2, \cdots, k)$ in our previous paper [27]. However we can not obtain the asymptotic expansion of the L by the same method.

Instead, we consider the distribution of $y_\alpha=\sqrt{\frac{n_\alpha}{2}}\log S_\alpha/n_\alpha$, which is given by (2.b.2).

$$\text{(2.b.2)} \qquad c_{n_\alpha}\sigma_\alpha^{-n_\alpha}\exp\left[\sqrt{\frac{n_\alpha}{2}}y_\alpha-\frac{n_\alpha}{2\sigma_\alpha^2}\exp\left(\sqrt{\frac{2}{n_\alpha}}y_\alpha\right)\right], \qquad -\infty<y_\alpha<\infty,$$

where $c_n=(n/2)^{-\frac{1}{2}(n-1)}\left\{\Gamma\left[\frac{n}{2}\right]\right\}^{-1}$. Then we can express the characteristic function of L as

$$\text{(2.b.3)} \qquad C_L(t)=\left\{\prod_{\alpha=1}^{k} c_{n_\alpha}\sigma_\alpha^{-n_\alpha}\right\}\int\exp[\mathrm{it}\textstyle\sum_{\alpha=1}^{k}y_\alpha^2-\mathrm{it}(\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,y_\alpha)^2]$$

$$\cdot\exp\left[\sum_{\alpha=1}^{k}\left(\sqrt{\frac{n_\alpha}{2}}y_\alpha-\frac{n_\alpha}{2\sigma_\alpha^2}\exp\sqrt{\frac{2}{n_\alpha}}y_\alpha\right)\right]dy_1\cdots dy_k.$$

Under the sequence K_b : $\sigma_\alpha^2=1+n^{-\frac{1}{2}}\theta_\alpha(\alpha=1, 2, \cdots, k)$, the second enponential part in the above integrand is expanded asymptotically for large n as follows:

$$\text{(2.b.4)} \qquad \sum_{\alpha=1}^{k}\frac{n_\alpha}{2\sigma_\alpha^2}\exp\sqrt{\frac{2}{n_\alpha}}\mathrm{y}_\alpha=\frac{1}{2}\sum_{\alpha=1}^{k}\frac{n_\alpha}{\sigma_\alpha^2}+\frac{\sqrt{n}}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\mathrm{y}_\alpha+\frac{1}{2}\sum_{\alpha=1}^{k}y_\alpha^2$$

$$-\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha y_\alpha+n^{-\frac{1}{2}}\left(\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha^2 y_\alpha\right.$$

$$-\frac{1}{2}\sum_{\alpha=1}^{k}\theta_\alpha y_\alpha^2+\left.\frac{1}{3\sqrt{2}}\sum_{\alpha=1}^{k}\frac{y_\alpha^3}{\sqrt{\rho_\alpha}}\right)+n^{-1}\left(\frac{1}{2}\sum_{\alpha=1}^{k}\theta_\alpha^2 y_\alpha^2\right.$$

$$-\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha^3 y_\alpha-\frac{1}{3\sqrt{2}}\sum_{\alpha=1}^{k}\frac{\theta_\alpha}{\sqrt{\rho_\alpha}}y_\alpha^3$$

$$\left.+\frac{1}{12}\sum_{\alpha=1}^{k}\frac{y_\alpha^4}{\rho_\alpha}\right)+O(n^{-\frac{3}{2}}).$$

Therefore we have

$$C_L(t)=\left\{\prod_{\alpha=1}^{k} c_{n_\alpha}\sigma_\alpha^{-n_\alpha}\exp\left[-\frac{n_\alpha}{2\sigma_\alpha^2}\right]\right\}\int\exp\Big[\mathrm{it}\textstyle\sum_{\alpha=1}^{k}y_\alpha^2-\mathrm{it}(\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,y_\alpha)^2 \tag{2.b.5}$$

$$\frac{1}{2}\sum_{\alpha=1}^{k}y_\alpha^2+\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha \mathrm{y}_\alpha\Big]\{1+\mathrm{n}^{-\frac{1}{2}}l_1(y)+n^{-1}l_2(y)$$

$$+O(n^{-\frac{3}{2}})\Big\}dy_1\cdots dy_k,$$

where the functions $l_1(y)$ and $l_2(y)$ are given below.

$$l_1(y)=\frac{1}{2}\sum_{\alpha=1}^{k}\theta_\alpha y_\alpha^2-\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha^2 y_\alpha-\frac{1}{3\sqrt{2}}\sum_{\alpha=1}^{k}y_\alpha^3/\sqrt{\rho_\alpha}, \tag{2.b.6}$$

$$l_2(y)=\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha^3 y_\alpha-\frac{1}{2}\sum_{\alpha=1}^{k}\theta_\alpha^2 y_\alpha^2+\frac{1}{3\sqrt{2}}\sum_{\alpha=1}^{k}\frac{\theta_\alpha y_\alpha^3}{\sqrt{\rho_\alpha}} \tag{2.b.7}$$

$$-\frac{1}{12}\sum_{\alpha=1}^{k}\frac{y_\alpha^4}{\rho_\alpha}+\frac{1}{2}l_1^2(y).$$

After some calculation, the exponential part of the integrand in (2.b.5) can be written as follows:

$$-\frac{1}{2}(y-\eta)'\Sigma^{-1}(y-\eta)+\frac{1}{4}\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^2+\frac{\mathrm{it}}{2(1-2\mathrm{it})}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})^2, \tag{2.b.8}$$

where $\Sigma=(\sigma_{\alpha\beta})$ with $\sigma_{\alpha\beta}=(\delta_{\alpha\beta}-2\mathrm{it}\sqrt{\rho_\alpha\rho_\beta})(1-2\mathrm{it})^{-1}$, $\eta=(\eta_1,\cdots,\eta_k)'$ with $\eta_\alpha=\sqrt{\frac{\rho_\alpha}{2}}(\theta_\alpha-2\mathrm{it}\tilde{\theta})(1-2\mathrm{it})^{-1}$ and $y=(y_1,\cdots,y_k)'$. The symbol $\delta_{\alpha\beta}$ denotes Kronecker delta. Hence (2.b.5) is expressed as $C_L(t)=C_{L_1}(t)\;C_{L_2}(t)$, where $C_{L_1}(t)$ and $C_{L_2}(t)$ are given as follows:

$$C_{L_1}(t)=\left\{\prod_{\alpha=1}^{k} c_{n_\alpha}\sigma_\alpha^{-n_\alpha}\quad\exp\left[-\frac{n_\alpha}{2\sigma_\alpha^2}\right]\right\}\exp\left[\frac{1}{4}\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^2\right](2\pi)^{\frac{k}{2}}|\Sigma|^{\frac{1}{2}} \tag{2.b.9}$$

$$\cdot\exp\left[\frac{\mathrm{it}}{2(1-2\mathrm{it})}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})^2\right],$$

$$C_{L_2}(t)=E[1+n^{-\frac{1}{2}}l_1(y)+n^{-1}l_2(y)+O(n^{-\frac{3}{2}})], \tag{2.b.10}$$

where E denotes the expectation with respect to the k-variate normal distribution with mean vector η and covariance matrix Σ. So we shall first calculate the first factor (2.b.9). The symmetric matrix Σ has a simple characteristic root equal to 1 and (k-1)-ple root equal to (1-2it)$^{-1}$. Applying Stirling's formula $\log\Gamma(x)=\log\sqrt{2\pi}+\left(x-\frac{1}{2}\right)\log x-x+\frac{1}{12x}+O(x^{-2})$ to the coefficient c_{n_α} in (2.b.9), we get

$$C_{L_1}(t)=(1-2\mathrm{it})^{-\frac{f}{2}}\exp\left[\frac{2\mathrm{it}}{(1-2\mathrm{it})}\delta_L^2\right]\left\{1+\frac{n^{-\frac{1}{2}}}{3}(\nu_3+3\tilde{\theta}\nu_2+\tilde{\theta}^3)\right.$$
$$+n^{-1}\left(\frac{1}{18}\nu_3^2+\frac{\tilde{\theta}^2}{2}\nu_2^2+\frac{\tilde{\theta}^6}{18}+\frac{\tilde{\theta}}{3}\nu_2\nu_3+\frac{\tilde{\theta}^3}{9}\nu_3+\frac{\tilde{\theta}^4}{3}\nu_2-\frac{3}{8}\nu_4\right.$$
$$\left.\left.-\frac{3}{2}\tilde{\theta}\nu_3-\frac{9}{4}\tilde{\theta}^2\nu_2-\frac{3}{8}\tilde{\theta}^4-\frac{\tilde{\rho}}{6}\right)\right\}+O(n^{-\frac{3}{2}}), \tag{2.b.11}$$

where $\delta_L^2=\frac{1}{4}\sum_{\alpha=1}^k\rho_\alpha(\theta_\alpha-\tilde{\theta})^2$, $\nu_\beta=\sum_{\alpha=1}^k\rho_\alpha(\theta_\alpha-\tilde{\theta})^\beta$, f$=k-1$ and $\tilde{\rho}=\sum_{\alpha=1}^k\rho_\alpha^{-1}$. In order to calculate the second factor $C_{L_2}(t)$, the moments with respect to y_α are needed.

$$E[y_\alpha]=\eta_\alpha, \qquad E[y_\alpha y_\beta]=\sigma_{\alpha\beta}+\eta_\alpha\eta_\beta, \tag{2.b.12}$$
$$E[y_\alpha^2 y_\beta]=\sigma_{\alpha\alpha}\eta_\beta+2\sigma_{\alpha\beta}\eta_\alpha+\eta_\alpha^2\eta_\beta,$$
$$E[y_\alpha^3 y_\beta]=3\sigma_{\alpha\alpha}\sigma_{\alpha\beta}+3\sigma_{\alpha\alpha}\eta_\alpha\eta_\beta+3\sigma_{\alpha\beta}\eta_\alpha^2+\eta_\alpha^3\eta_\beta,$$
$$E[y_\alpha^2 y_\beta^2]=\sigma_{\alpha\alpha}\sigma_{\beta\beta}+\sigma_{\alpha\alpha}\eta_\beta^2+2\sigma_{\alpha\beta}^2+4\sigma_{\alpha\beta}\eta_\alpha\eta_\beta+\sigma_{\beta\beta}\eta_\alpha^2+\eta_\alpha^2\eta_\beta^2,$$
$$E[y_\alpha^3 y_\beta^2]=6\sigma_{\alpha\alpha}\sigma_{\alpha\beta}\eta_\beta+3\sigma_{\alpha\alpha}\sigma_{\beta\beta}\eta_\alpha+3\sigma_{\alpha\alpha}\eta_\alpha\eta_\beta^2+6\sigma_{\alpha\beta}^2\eta_\alpha$$
$$+6\sigma_{\alpha\beta}\eta_\alpha^2\eta_\beta+\sigma_{\beta\beta}\eta_\alpha^3+\eta_\alpha^3\eta_\beta^2,$$
$$E[y_\alpha^3 y_\beta^3]=9\sigma_{\alpha\alpha}\sigma_{\alpha\beta}\sigma_{\beta\beta}+9\sigma_{\alpha\alpha}\sigma_{\alpha\beta}\eta_\beta^2+9\sigma_{\alpha\alpha}\sigma_{\beta\beta}\eta_\alpha\eta_\beta$$
$$+3\sigma_{\alpha\alpha}\eta_\alpha\eta_\beta^3+6\sigma_{\alpha\beta}^3+18\sigma_{\alpha\beta}^2\eta_\alpha\eta_\beta+9\sigma_{\alpha\beta}\sigma_{\beta\beta}\eta_\alpha^2$$
$$+9\sigma_{\alpha\beta}\eta_\alpha^2\eta_\beta^2+3\sigma_{\beta\beta}\eta_\alpha^3\eta_\beta+\eta_\alpha^3\eta_\beta^3.$$

Putting $a_{\alpha\beta}=(\delta_{\alpha\beta}-\sqrt{\rho_\alpha\rho_\beta})$, $b_{\alpha\beta}=\sqrt{\rho_\alpha\rho_\beta}$, $c_\alpha=\sqrt{\rho_\alpha}(\theta_\alpha-\tilde{\theta})/\sqrt{2}$ and $d_\alpha=\sqrt{\rho_\alpha}\tilde{\theta}/\sqrt{2}$, we can express $\sigma_{\alpha\beta}$ and η_α as $\sigma_{\alpha\beta}=a_{\alpha\beta}(1-2\mathrm{it})^{-1}+b_{\alpha\beta}$ and $\eta_\alpha=c_\alpha(1-2\mathrm{it})^{-1}+d_\alpha$. Thus we obtain the following table by substituting the above $\sigma_{\alpha\beta}$ and η_α into each termin (2.b.12), which is essential in order to simplify (2.b.10).

Table 1

power of $(1-2it)^{-1}$ term	3	2	1	0
η_α			c_α	d_α
$\sigma_{\alpha\beta}$			$a_{\alpha\beta}$	$b_{\alpha\beta}$
$\sigma_{\alpha\beta}^2$		$a_{\alpha\beta}^2$	$2a_{\alpha\beta}b_{\alpha\beta}$	$b_{\alpha\beta}^2$
$\sigma_{\alpha\beta}^3$	$a_{\alpha\beta}^3$	$3b_{\alpha\beta}a_{\alpha\beta}^2$	$3b_{\alpha\beta}^2a_{\alpha\beta}$	$b_{\alpha\beta}^3$
$\eta_\alpha\eta_\beta$		$c_\alpha c_\beta$	$c_\alpha d_\beta+d_\alpha c_\beta$	$d_\alpha d_\beta$
$\sigma_{\alpha\alpha}\eta_\beta$		$a_{\alpha\alpha}c_\beta$	$a_{\alpha\alpha}d_\beta+b_{\alpha\alpha}c_\beta$	$b_{\alpha\alpha}d_\beta$
$\sigma_{\alpha\beta}\eta_\alpha$		$a_{\alpha\beta}c_\alpha$	$a_{\alpha\beta}d_\alpha+c_\alpha b_{\alpha\beta}$	$b_{\alpha\beta}d_\alpha$
$\sigma_{\alpha\alpha}\sigma_{\alpha\beta}$		$a_{\alpha\alpha}a_{\alpha\beta}$	$a_{\alpha\alpha}b_{\alpha\beta}+b_{\alpha\alpha}a_{\alpha\beta}$	$b_{\alpha\alpha}b_{\alpha\beta}$
$\sigma_{\alpha\alpha}\sigma_{\beta\beta}$		$a_{\alpha\alpha}a_{\beta\beta}$	$a_{\alpha\alpha}b_{\beta\beta}+b_{\alpha\alpha}a_{\beta\beta}$	$b_{\alpha\alpha}b_{\beta\beta}$
$\eta_\alpha^2\eta_\beta$	$c_\alpha^2c_\beta$	$c_\alpha^2d_\beta+2c_\alpha c_\beta d_\alpha$	$2c_\alpha d_\alpha d_\beta+d_\alpha^2c_\beta$	$d_\alpha^2d_\beta$
$\sigma_{\alpha\beta}\eta_\alpha^2$	$a_{\alpha\beta}c_\alpha^2$	$b_{\alpha\beta}c_\alpha^2+2a_{\alpha\beta}c_\alpha d_\alpha$	$2b_{\alpha\beta}c_\alpha d_\alpha+a_{\alpha\beta}d_\alpha^2$	$b_{\alpha\beta}d_\alpha^2$
$\sigma_{\alpha\alpha}\eta_\beta^2$	$a_{\alpha\alpha}c_\beta^2$	$b_{\alpha\alpha}c_\beta^2+2a_{\alpha\alpha}c_\beta d_\beta$	$2b_{\alpha\alpha}c_\beta d_\beta+a_{\alpha\alpha}d_\beta^2$	$b_{\alpha\alpha}d_\beta^2$
$\sigma_{\beta\beta}\eta_\alpha^2$	$a_{\beta\beta}c_\alpha^2$	$c_\alpha^2b_{\beta\beta}+2a_{\beta\beta}c_\alpha d_\alpha$	$2b_{\beta\beta}c_\alpha d_\alpha+a_{\beta\beta}d_\alpha^2$	$b_{\beta\beta}d_\alpha^2$
$\sigma_{\alpha\beta}^2\eta_\alpha$	$a_{\alpha\beta}^2c_\alpha$	$a_{\alpha\beta}^2d_\alpha+2a_{\alpha\beta}b_{\alpha\beta}c_\alpha$	$2a_{\alpha\beta}b_{\alpha\beta}d_\alpha+b_{\alpha\beta}^2c_\alpha$	$b_{\alpha\beta}^2d_\alpha$
$\sigma_{\alpha\alpha}\eta_\alpha\eta_\beta$	$a_{\alpha\alpha}c_\alpha c_\beta$	$b_{\alpha\alpha}c_\alpha c_\beta+a_{\alpha\alpha}c_\alpha d_\beta+a_{\alpha\alpha}c_\beta d_\alpha$	$b_{\alpha\alpha}c_\alpha d_\beta+b_{\alpha\alpha}d_\alpha c_\beta+a_{\alpha\alpha}d_\alpha d_\beta$	$b_{\alpha\alpha}d_\alpha d_\beta$
$\sigma_{\alpha\beta}\eta_\alpha\eta_\beta$	$a_{\alpha\beta}c_\alpha c_\beta$	$b_{\alpha\beta}c_\alpha c_\beta+a_{\alpha\beta}c_\alpha d_\beta+a_{\alpha\beta}c_\beta d_\alpha$	$b_{\alpha\beta}c_\alpha d_\beta+b_{\alpha\beta}c_\beta d_\alpha+a_{\alpha\beta}d_\alpha d_\beta$	$b_{\alpha\beta}d_\alpha d_\beta$
$\sigma_{\alpha\alpha}\sigma_{\alpha\beta}\eta_\beta$	$a_{\alpha\alpha}a_{\alpha\beta}c_\beta$	$a_{\alpha\alpha}a_{\alpha\beta}d_\beta+a_{\alpha\alpha}b_{\alpha\beta}c_\beta+a_{\alpha\beta}b_{\alpha\alpha}c_\beta$	$a_{\alpha\alpha}b_{\alpha\beta}d_\beta+a_{\alpha\beta}b_{\alpha\alpha}d_\beta+b_{\alpha\alpha}b_{\alpha\beta}c_\beta$	$b_{\alpha\alpha}b_{\alpha\beta}d_\beta$
$\sigma_{\alpha\alpha}\sigma_{\beta\beta}\eta_\alpha$	$a_{\alpha\alpha}a_{\beta\beta}c_\alpha$	$a_{\alpha\alpha}a_{\beta\beta}d_\alpha+a_{\alpha\alpha}b_{\beta\beta}c_\alpha+a_{\beta\beta}b_{\alpha\alpha}c_\alpha$	$a_{\alpha\alpha}b_{\beta\beta}d_\alpha+a_{\beta\beta}b_{\alpha\alpha}d_\alpha+b_{\alpha\alpha}b_{\beta\beta}c_\alpha$	$b_{\alpha\alpha}b_{\beta\beta}d_\alpha$
$\sigma_{\alpha\alpha}\sigma_{\alpha\beta}\sigma_{\beta\beta}$	$a_{\alpha\alpha}a_{\alpha\beta}a_{\beta\beta}$	$a_{\alpha\alpha}a_{\beta\beta}b_{\alpha\beta}+a_{\alpha\alpha}a_{\alpha\beta}b_{\beta\beta}+a_{\alpha\beta}b_{\alpha\alpha}a_{\beta\beta}$	$a_{\alpha\alpha}b_{\beta\beta}b_{\alpha\beta}+a_{\beta\beta}b_{\alpha\alpha}b_{\alpha\beta}+a_{\alpha\beta}b_{\alpha\alpha}b_{\beta\beta}$	$b_{\alpha\alpha}b_{\beta\beta}b_{\alpha\alpha}$

Table 1 (continued)

power of $(1-2it)^{-1}$ term	6	5	4	3	2	1	0
$\eta_\alpha^3\eta_\beta$			$c_\alpha^3c_\beta$	$3c_\alpha^2c_\beta d_\alpha+c_\alpha^3d_\beta$	$3c_\alpha^2d_\alpha d_\beta+3c_\alpha c_\beta d_\alpha^2$	$3c_\alpha d_\alpha^2d_\beta+c_\beta d_\alpha^3$	$d_\alpha^3d_\beta$
$\eta_\alpha^2\eta_\beta^2$			$c_\alpha^2c_\beta^2$	$2c_\alpha^2c_\beta d_\beta+2c_\alpha c_\beta^2d_\alpha$	$c_\alpha^2d_\beta^2+4c_\alpha c_\beta d_\alpha d_\beta+c_\beta^2d_\alpha^2$	$2c_\alpha d_\alpha d_\beta^2+2c_\beta d_\alpha^2d_\beta$	$d_\alpha^2d_\beta^2$
$\sigma_{\beta\beta}\eta_\alpha^3$			$a_{\beta\beta}c_\alpha^3$	$b_{\beta\beta}c_\alpha^3+3a_{\beta\beta}c_\alpha^2d_\alpha$	$3b_{\beta\beta}c_\alpha^2d_\alpha+3a_{\beta\beta}c_\alpha d_\alpha^2$	$3b_{\beta\beta}c_\alpha d_\alpha^2+a_{\beta\beta}d_\alpha^3$	$b_{\beta\beta}d_\alpha^3$
$\eta_\alpha^3\eta_\beta^2$		$c_\alpha^3c_\beta^2$	$2c_\alpha^3c_\beta d_\beta+3c_\alpha^2c_\beta^2d_\alpha$	$6c_\alpha^2c_\beta d_\alpha d_\beta+c_\beta^3d_\beta^2$ $+3c_\alpha c_\beta^2d_\alpha^2$	$6c_\alpha c_\beta d_\alpha^2d_\beta+c_\beta^2d_\alpha^3+3c_\alpha^2d_\alpha d_\beta^2$	$3c_\alpha d_\alpha^2d_\beta^2+2c_\beta d_\alpha^3d_\beta$	$d_\alpha^3d_\beta^2$
$\eta_\alpha^3\eta_\beta^3$	$c_\alpha^3c_\beta^3$	$3c_\alpha^3c_\beta^2d_\beta$ $+3c_\alpha^2c_\beta^3d_\alpha$	$3c_\alpha^3c_\beta d_\beta^2+9c_\alpha^2c_\beta^2d_\alpha d_\beta$ $+3c_\alpha c_\beta^3d_\alpha^2$	$9c_\alpha^2c_\beta d_\alpha d_\beta^2+c_\alpha^3d_\beta^3$ $+c_\beta^3d_\alpha^3+9c_\alpha c_\beta^2d_\alpha^2d_\beta$	$9c_\alpha c_\beta d_\alpha^2d_\beta^2+3c_\beta^2d_\alpha^3d_\beta+3c_\alpha^2d_\alpha d_\beta^3$	$3c_\alpha d_\alpha^2d_\beta^3+3c_\beta d_\alpha^3d_\beta^2$	$d_\alpha^3d_\beta^3$
$\sigma_{\alpha\alpha}\eta_\alpha\eta_\beta^2$			$a_{\alpha\alpha}c_\alpha c_\beta^2$	$a_{\alpha\alpha}c_\beta^2d_\alpha+c_\alpha c_\beta^2b_{\alpha\alpha}$ $+2a_{\alpha\alpha}c_\alpha c_\beta d_\beta$	$b_{\alpha\alpha}c_\beta^2d_\alpha+2a_{\alpha\alpha}c_\beta d_\alpha d_\beta+2b_{\alpha\alpha}c_\alpha c_\beta d_\beta$ $+a_{\alpha\alpha}c_\alpha d_\beta^2$	$2b_{\alpha\alpha}c_\beta d_\alpha d_\beta$ $+a_{\alpha\alpha}d_\alpha d_\beta^2+b_{\alpha\alpha}c_\alpha d_\beta^2$	$b_{\alpha\alpha}d_\beta^2d_\alpha$
$\sigma_{\alpha\beta}\eta_\alpha^2\eta_\beta$			$a_{\alpha\beta}c_\alpha^2c_\beta$	$a_{\alpha\beta}c_\alpha^2d_\beta+b_{\alpha\beta}c_\alpha^2c_\beta$ $+2a_{\alpha\beta}c_\alpha c_\beta d_\alpha$	$b_{\alpha\beta}c_\alpha^2d_\beta+2a_{\alpha\beta}c_\alpha d_\alpha d_\beta+2b_{\alpha\beta}c_\alpha c_\beta d_\alpha$ $+a_{\alpha\beta}c_\beta d_\alpha^2$	$2c_\alpha d_\alpha d_\beta b_{\alpha\beta}+a_{\alpha\beta}d_\alpha^2d_\beta$ $+b_{\alpha\beta}c_\beta d_\alpha^2$	$b_{\alpha\beta}d_\alpha^2d_\beta$
$\sigma_{\alpha\alpha}\sigma_{\alpha\beta}\eta_\beta^2$			$a_{\alpha\alpha}a_{\alpha\beta}c_\beta^2$	$a_{\alpha\alpha}b_{\alpha\beta}c_\beta^2+a_{\alpha\beta}b_{\alpha\alpha}c_\beta^2$ $+2a_{\alpha\alpha}a_{\alpha\beta}c_\beta d_\beta$	$b_{\alpha\alpha}b_{\alpha\beta}c_\beta^2+2a_{\alpha\alpha}b_{\alpha\beta}c_\beta d_\beta$ $+2a_{\alpha\beta}b_{\alpha\alpha}c_\beta d_\beta+a_{\alpha\alpha}a_{\alpha\beta}d_\beta^2$	$2b_{\alpha\alpha}b_{\alpha\beta}c_\beta d_\beta+a_{\alpha\alpha}b_{\alpha\beta}d_\beta^2$ $+a_{\alpha\beta}b_{\alpha\alpha}d_\beta^2$	$b_{\alpha\alpha}b_{\alpha\beta}d_\beta^2$
$\sigma_{\alpha\beta}^2\eta_\alpha\eta_\beta$			$a_{\alpha\beta}^2c_\alpha c_\beta$	$a_{\alpha\beta}^2c_\alpha d_\beta+a_{\alpha\beta}^2d_\alpha c_\beta$ $+2a_{\alpha\beta}b_{\alpha\beta}c_\alpha c_\beta$	$a_{\alpha\beta}^2d_\alpha d_\beta+2a_{\alpha\beta}b_{\alpha\beta}c_\alpha d_\beta$ $+2a_{\alpha\beta}b_{\alpha\beta}c_\beta d_\alpha+b_{\alpha\beta}^2c_\alpha c_\beta$	$2a_{\alpha\beta}b_{\alpha\beta}d_\alpha d_\beta+b_{\alpha\beta}^2c_\alpha d_\beta$ $+b_{\alpha\beta}^2c_\beta d_\alpha$	$b_{\alpha\beta}^2d_\alpha d_\beta$
$\sigma_{\alpha\beta}\sigma_{\beta\beta}\eta_\alpha^2$			$a_{\alpha\beta}a_{\beta\beta}c_\alpha^2$	$a_{\beta\beta}b_{\alpha\beta}c_\alpha^2+a_{\alpha\beta}b_{\beta\beta}c_\alpha^2$ $+2a_{\beta\beta}a_{\alpha\beta}c_\alpha d_\alpha$	$b_{\alpha\beta}b_{\beta\beta}c_\alpha^2+2a_{\beta\beta}b_{\alpha\beta}c_\alpha d_\alpha$ $+2a_{\alpha\beta}b_{\beta\beta}c_\alpha d_\alpha+a_{\alpha\beta}a_{\beta\beta}d_\alpha^2$	$2b_{\alpha\beta}b_{\beta\beta}c_\alpha d_\alpha+b_{\alpha\beta}a_{\beta\beta}d_\alpha^2$ $+a_{\alpha\beta}b_{\beta\beta}d_\alpha^2$	$b_{\alpha\beta}b_{\beta\beta}d_\alpha^2$
$\sigma_{\alpha\alpha}\eta_\alpha\eta_\beta^3$		$a_{\alpha\alpha}c_\alpha c_\beta^3$	$b_{\alpha\alpha}c_\alpha c_\beta^3+3a_{\alpha\alpha}c_\alpha c_\beta^2d_\beta$ $+a_{\alpha\alpha}c_\beta^3d_\alpha$	$3b_{\alpha\alpha}c_\alpha c_\beta^2d_\beta+b_{\alpha\alpha}c_\beta^3d_\alpha$ $+3a_{\alpha\alpha}c_\beta^2d_\alpha d_\beta+3a_{\alpha\alpha}c_\alpha c_\beta d_\beta^2$	$3c_\beta^2b_{\alpha\alpha}d_\alpha d_\beta+3b_{\alpha\alpha}c_\alpha c_\beta d_\beta^2$ $+3a_{\alpha\alpha}c_\beta d_\alpha d_\beta^2+a_{\alpha\alpha}c_\alpha d_\beta^3$	$3b_{\alpha\alpha}c_\beta d_\alpha d_\beta^2+b_{\alpha\alpha}c_\alpha d_\beta^3$ $+a_{\alpha\alpha}d_\alpha d_\beta^3$	$b_{\alpha\alpha}d_\alpha d_\beta^3$
$\sigma_{\beta\beta}\eta_\alpha^3\eta_\beta$		$a_{\beta\beta}c_\beta^3c_\beta$	$b_{\beta\beta}c_\alpha^3c_\beta+3a_{\beta\beta}c_\alpha^2c_\beta d_\alpha$ $+a_{\beta\beta}c_\alpha^3d_\beta$	$3b_{\beta\beta}c_\alpha^2c_\beta d_\alpha+b_{\beta\beta}c_\alpha^3d_\beta$ $+3a_{\beta\beta}c_\alpha^2d_\alpha d_\beta+3a_{\beta\beta}c_\alpha c_\beta d_\alpha^2$	$3b_{\beta\beta}c_\alpha^2d_\alpha d_\beta+3b_{\beta\beta}c_\alpha c_\beta d_\alpha^2$ $+3a_{\beta\beta}c_\alpha d_\alpha^2d_\beta+a_{\beta\beta}c_\beta d_\alpha^3$	$3b_{\beta\beta}c_\alpha d_\alpha^2d_\beta+b_{\beta\beta}c_\beta d_\alpha^3$ $+a_{\beta\beta}d_\alpha^3d_\beta$	$b_{\beta\beta}d_\alpha^3d_\beta$
$\sigma_{\alpha\beta}\eta_\alpha^2\eta_\beta^2$		$a_{\alpha\beta}c_\alpha^2c_\beta^2$	$c_\alpha^2c_\beta^2b_{\alpha\beta}+2a_{\alpha\beta}c_\alpha^2c_\beta d_\beta$ $+2a_{\alpha\beta}c_\alpha c_\beta^2d_\alpha$	$2b_{\alpha\beta}c_\alpha^2c_\beta d_\beta+2b_{\alpha\beta}c_\alpha c_\beta^2d_\alpha$ $+a_{\alpha\beta}c_\alpha^2d_\beta^2+a_{\alpha\beta}c_\beta^2d_\alpha^2$ $+4a_{\alpha\beta}c_\alpha c_\beta d_\alpha d_\beta$	$b_{\alpha\beta}c_\alpha^2d_\beta^2+4b_{\alpha\beta}c_\alpha c_\beta d_\alpha d_\beta+b_{\alpha\beta}c_\beta^2d_\alpha^2$ $+2a_{\alpha\beta}c_\alpha d_\alpha d_\beta^2+2a_{\alpha\beta}c_\beta d_\alpha^2d_\beta$	$2b_{\alpha\beta}c_\alpha d_\alpha d_\beta^2+2b_{\alpha\beta}c_\beta d_\alpha^2d_\beta$ $+a_{\alpha\beta}d_\alpha^2d_\beta^2$	$d_\alpha^2d_\beta^2b_{\alpha\beta}$
$\sigma_{\alpha\alpha}\sigma_{\beta\beta}\eta_\alpha\eta_\beta$			$c_\alpha c_\beta a_{\alpha\alpha}a_{\beta\beta}$	$a_{\alpha\alpha}b_{\beta\beta}c_\alpha c_\beta+a_{\beta\beta}b_{\alpha\alpha}c_\alpha c_\beta$ $+a_{\alpha\alpha}a_{\beta\beta}c_\alpha d_\beta+a_{\alpha\alpha}a_{\beta\beta}c_\beta d_\alpha$	$b_{\alpha\alpha}b_{\beta\beta}c_\alpha c_\beta+a_{\alpha\alpha}b_{\beta\beta}c_\alpha d_\beta+a_{\alpha\alpha}b_{\beta\beta}c_\beta d_\alpha$ $+a_{\beta\beta}b_{\alpha\alpha}c_\alpha d_\beta+a_{\beta\beta}b_{\alpha\alpha}c_\beta d_\alpha$ $+a_{\alpha\alpha}a_{\beta\beta}d_\alpha d_\beta$	$b_{\alpha\alpha}b_{\beta\beta}c_\alpha d_\beta+b_{\alpha\alpha}b_{\beta\beta}c_\beta d_\alpha$ $+a_{\alpha\alpha}b_{\beta\beta}d_\alpha d_\beta$ $+a_{\beta\beta}b_{\alpha\alpha}d_\alpha d_\beta$	$b_{\alpha\alpha}b_{\beta\beta}d_\alpha d_\beta$

Put $\zeta_\beta = \sum_{\alpha=1}^{k}(\theta_\alpha - \tilde{\theta})^\beta$. Then by using Table 1 and $\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}c_\alpha = 0$, we obtain the following formulae, which are also applied to the M test in the latter section.

$$E\left(\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha^2 y_\alpha\right) = \frac{1}{\sqrt{2}}(\nu_3 + 2\tilde{\theta}\nu_2)(1-2\mathrm{it})^{-1} + \frac{1}{\sqrt{2}}(\tilde{\theta}\nu_2 + \tilde{\theta}^3), \tag{2.b.13}$$

$$E\left(\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha^3 y_\alpha\right) = \frac{1}{\sqrt{2}}(\nu_4 + 3\tilde{\theta}\nu_3 + 3\tilde{\theta}^2\nu_2)(1-2\mathrm{it})^{-1} + \frac{1}{\sqrt{2}}(\tilde{\theta}\nu_3 + 3\tilde{\theta}^2\nu_2 + \tilde{\theta}^4), \tag{2.b.14}$$

$$E\left(\sum_{\alpha=1}^{k}\theta_\alpha y_\alpha^2\right) = \frac{1}{2}(\nu_3 + \tilde{\theta}\nu_2)(1-2\mathrm{it})^{-2} + \{\tilde{\theta}\nu_2 + \zeta_1 + (k-1)\tilde{\theta}\}(1-2\mathrm{it})^{-1} + \frac{\tilde{\theta}^3}{2} + \tilde{\theta}, \tag{2.b.15}$$

$$E\left(\sum_{\alpha=1}^{k}\theta_\alpha^2 y_\alpha^2\right) = \frac{1}{2}(\nu_4 + 2\tilde{\theta}\nu_3 + \tilde{\theta}^2\nu_2)(1-2\mathrm{it})^{-2} + \{\tilde{\theta}\nu_3 + 2\tilde{\theta}^2\nu_2 + \zeta_2 + 2\tilde{\theta}\zeta_1 - \nu_2 + (\mathrm{k}-1)\tilde{\theta}^2\}(1-2\mathrm{it})^{-1} + \frac{1}{2}(\tilde{\theta}^2 + 2)(\nu_2 + \tilde{\theta}^2), \tag{2.b.16}$$

$$\begin{aligned} E\left(\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha^2 y_\alpha\right)^2 = {} & \frac{1}{2}(\nu_3 + 2\tilde{\theta}\nu_2)^2(1-2\mathrm{it})^{-2} + (\tilde{\theta}\nu_2\nu_3 + 2\tilde{\theta}^2\nu_2^2 \\ & + \tilde{\theta}^3\nu_3 + 2\tilde{\theta}^4\nu_2 + \nu_4 + 4\tilde{\theta}\nu_3 + 4\tilde{\theta}^2\nu_2 - \nu_2^2)(1-2\mathrm{it})^{-1} + \frac{\tilde{\theta}^2\nu_2^2}{2} \\ & + \tilde{\theta}^4\nu_2 + \nu_2^2 + 2\tilde{\theta}^2\nu_2 + \frac{\tilde{\theta}^6}{6} + \tilde{\theta}^4, \end{aligned} \tag{2.b.17}$$

$$E\left(\sum_{\alpha=1}^{k}y_\alpha^3/\sqrt{\rho_\alpha}\right) = \frac{1}{2\sqrt{2}}\nu_3(1-2\mathrm{it})^{-3} + \frac{3}{\sqrt{2}}\left(\zeta_1 + \frac{1}{2}\tilde{\theta}\nu_2\right)(1-2\mathrm{it})^{-2} + \frac{3}{\sqrt{2}}(k-1)\tilde{\theta}(1-2\mathrm{it})^{-1} + \frac{1}{\sqrt{2}}\left(3\tilde{\theta} + \frac{1}{2}\tilde{\theta}^3\right), \tag{2.b.18}$$

$$\begin{aligned} E\left(\sum_{\alpha=1}^{k}\theta_\alpha y_\alpha^3/\sqrt{\rho_\alpha}\right) = {} & \frac{1}{2\sqrt{2}}(\nu_4 + \tilde{\theta}\nu_3)(1-2\mathrm{it})^{-3} + \frac{3}{\sqrt{2}}\left(\zeta_2 + \tilde{\theta}\zeta_1 - \nu_2 \right. \\ & \left. + \frac{1}{2}\tilde{\theta}\nu_3 + \frac{\tilde{\theta}^2}{2}\nu_2\right)(1-2\mathrm{it})^{-2} + \frac{3}{\sqrt{2}}\left\{\tilde{\theta}\zeta_1 + (k-1)\tilde{\theta}^2 + \nu_2 \right. \\ & \left. + \frac{\tilde{\theta}^2}{2}\nu_2\right\}(1-2\mathrm{it})^{-1} + \frac{1}{\sqrt{2}}\left(3\tilde{\theta}^2 + \frac{\tilde{\theta}^4}{2}\right), \end{aligned} \tag{2.b.19}$$

(2.b.20)
$$E(\textstyle\sum_{\alpha=1}^{k}\theta_\alpha y_\alpha^2\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha^2 y_\alpha)=\frac{1}{2\sqrt{2}}(\nu_3+\tilde{\theta}\nu_2)(\nu_3+2\tilde{\theta}\nu_2)(1-2\mathrm{it})^{-3}$$
$$+\frac{1}{\sqrt{2}}\Big\{2\nu_4+\nu_3\Big[\zeta_1+(k+5)\tilde{\theta}+\frac{3}{2}\tilde{\theta}\nu_2+\frac{\tilde{\theta}^3}{2}\Big]+\nu_2\Big[2\tilde{\theta}\zeta_1+2(k$$
$$+1)\tilde{\theta}^2+\frac{5}{2}\tilde{\theta}^2\nu_2+\frac{\tilde{\theta}^4}{2}-2\nu_2\Big]\Big\}(1-2\mathrm{it})^{-2}+\frac{1}{\sqrt{2}}\Big\{\tilde{\theta}\nu_3\Big(\frac{\tilde{\theta}^2}{2}+3\Big)$$
$$+\nu_2[\nu_2(\tilde{\theta}^2+2)+2\tilde{\theta}^4+(k+7)\tilde{\theta}^2+\tilde{\theta}\zeta_1]+(k-1)\tilde{\theta}^4+\zeta_1\tilde{\theta}^3\Big\}(1-2\mathrm{it})^{-1}$$
$$+\frac{\tilde{\theta}^2}{\sqrt{2}}(\nu_2+\tilde{\theta}^2)\Big(3+\frac{\tilde{\theta}^2}{2}\Big),$$

(2.b.21)
$$E(\textstyle\sum_{\alpha=1}^{k}y_\alpha^4/\rho_\alpha)=\frac{1}{4}\nu_4(1-2\mathrm{it})^{-4}+(\tilde{\theta}\nu_3-3\nu_2+3\zeta_2)(1-2\mathrm{it})^{-3}$$
$$+3\Big\{\Big(\frac{1}{2}\tilde{\theta}^2+1\Big)\nu_2+2\tilde{\theta}\zeta_1+\tilde{\rho}-2k+1\Big\}(1-2\mathrm{it})^{-2}+3(k-1)(\tilde{\theta}^2+2)$$
$$\cdot(1-2\mathrm{it})^{-1}+\frac{\tilde{\theta}^4}{4}+3\tilde{\theta}^2+3,$$

(2.b.22)
$$E(\textstyle\sum_{\alpha=1}^{k}\theta_\alpha y_\alpha^2)^2=\frac{1}{4}(\nu_3+\tilde{\theta}\nu_2)^2(1-2\mathrm{it})^{-4}+\Big\{2\nu_4+\nu_3[\tilde{\theta}\nu_2+\zeta_1+(k+3)\tilde{\theta}]$$
$$+\nu_2[(\tilde{\theta}^2-2)\nu_2+(k+1)\tilde{\theta}^2+\zeta_1\tilde{\theta}]\Big\}(1-2\mathrm{it})^{-3}+\Big\{\tilde{\theta}\nu_3\Big(\frac{1}{2}\tilde{\theta}^2+5\Big)$$
$$+\nu_2\Big[(\tilde{\theta}^2+2)\nu_2+\frac{1}{2}\tilde{\theta}^4+2\zeta_1\tilde{\theta}+(2k+3)\tilde{\theta}^2-4\Big]+\zeta_1^2+2(k+1)\tilde{\theta}\zeta_1$$
$$+(k^2-1)\tilde{\theta}^2+2\zeta_2\Big)(1-2\mathrm{it})^{-2}+\Big\{\nu_2(\tilde{\theta}^4+8\tilde{\theta}^2+4)+(k-1)\tilde{\theta}^4+\zeta_1\tilde{\theta}^3$$
$$+2(k-1)\tilde{\theta}^2+2\zeta_1\tilde{\theta}\Big\}(1-2\mathrm{it})^{-1}+\frac{\tilde{\theta}^6}{4}+3\tilde{\theta}^4+3\tilde{\theta}^2,$$

(2.b.23)
$$E(\textstyle\sum_{\alpha=1}^{k}y_\alpha^3/\sqrt{\rho_\alpha}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha^2 y_\alpha)=\frac{\nu_3}{4}(\nu_3+2\tilde{\theta}\nu_2)(1-2\mathrm{it})^{-4}+\Big\{\frac{3}{2}\nu_4$$
$$+\nu_3\Big(\frac{\tilde{\theta}^3}{4}+\tilde{\theta}\nu_2+3\tilde{\theta}+\frac{3}{2}\zeta_1\Big)+\nu_2\Big[\frac{3}{2}(\tilde{\theta}^2-1)\nu_2+3\tilde{\theta}\zeta_1\Big]\Big\}(1-2\mathrm{it})^{-3}$$
$$+\Big\{\frac{3}{2}(k+1)\tilde{\theta}\nu_3+\nu_2\Big[\frac{3}{4}(\tilde{\theta}^2+2)\nu_2+\frac{3}{4}\tilde{\theta}^4+\frac{3}{2}(2k+3)\tilde{\theta}^2$$

$$+\frac{3}{2}\tilde{\theta}\zeta_1-3k\Big]+\frac{3}{2}\zeta_1\tilde{\theta}^3+6\zeta_1\tilde{\theta}+3\zeta_2\Big\}(1-2\mathrm{it})^{-2}+\Big\{\frac{\tilde{\theta}}{4}(\tilde{\theta}^2+6)\nu_3$$

$$+\nu_2\Big[\frac{\tilde{\theta}^4}{2}+\frac{3}{2}(k+1)\tilde{\theta}^2+3(k-1)\Big]$$

$$+\frac{3\tilde{\theta}^2}{2}(k-1)(\tilde{\theta}^2+2)\Big\}(1-2\mathrm{it})^{-1}+(\nu_2+\tilde{\theta}^2)\Big(\frac{\tilde{\theta}^4}{4}+3\tilde{\theta}^2+3\Big),$$

(2.b.24) $$E\Big(\sum_{\alpha=1}^{k}y_\alpha^3/\sqrt{\rho_\alpha}\sum_{\alpha=1}^{k}\theta_\alpha y_\alpha^2\Big)=m_5(1-2\mathrm{it})^{-5}+m_4(1-2\mathrm{it})^{-4}$$

$$+m_3(1-2\mathrm{it})^{-3}+m_2(1-2\mathrm{it})^{-2}+m_1(1-2\mathrm{it})^{-1}+m_0,$$

where the coefficients m_α are given as follows:

$$m_5=\frac{\nu_3}{4\sqrt{2}}(\nu_3+\tilde{\theta}\nu_2),\quad m_4=\frac{1}{\sqrt{2}}\Big\{3\nu_4+\nu_3\Big[\frac{5}{4}\tilde{\theta}\nu_2+\frac{1}{2}(k+5)\tilde{\theta}+2\zeta_1\Big]$$

$$+\nu_2\Big[3\Big(\frac{\tilde{\theta}^2}{4}-1\Big)\nu_2+\frac{3}{2}\zeta_1\tilde{\theta}\Big]\Big\},$$

$$m_3=\frac{1}{\sqrt{2}}\Big\{\nu_3\Big[\frac{1}{4}\tilde{\theta}^3+\frac{1}{2}(3k+16)\tilde{\theta}\Big]+\nu_2\Big[\frac{3}{2}(\tilde{\theta}^2+2)\nu_2+3(k+1)\tilde{\theta}^2$$

(2.b.25)

$$+\frac{9}{2}\zeta_1\tilde{\theta}-6(k+2)\Big]+3\zeta_1^2+12\zeta_2+3(k+3)\tilde{\theta}\zeta_1\Big\},$$

$$m_2=\frac{1}{\sqrt{2}}\Big\{\frac{1}{4}\nu_3(\tilde{\theta}^3+6\tilde{\theta})+\nu_2[\tilde{\theta}^4+3(k+3)\tilde{\theta}^2+6(k+1)]+3(k^2-1)\tilde{\theta}^2$$

$$+\frac{3}{2}\zeta_1\tilde{\theta}^3+3(k+4)\zeta_1\tilde{\theta}\Big\},$$

$$m_1=\frac{1}{\sqrt{2}}\Big\{\nu_2\Big(\frac{\tilde{\theta}^4}{2}+6\tilde{\theta}^2+6\Big)+2(k-1)\tilde{\theta}^4+\frac{1}{2}\zeta_1\tilde{\theta}^3+12(k-1)\tilde{\theta}^2+3\zeta_1\tilde{\theta}\Big\},$$

$$m_0=\frac{1}{\sqrt{2}}\Big(\frac{\tilde{\theta}^6}{4}+5\tilde{\theta}^4+15\tilde{\theta}^2\Big),$$

(2.b.26) $$E\Big(\sum_{\alpha=1}^{k}y_\alpha^3/\sqrt{\rho_\alpha}\Big)^2=p_6(1-2\mathrm{it})^{-6}+p_5(1-2\mathrm{it})^{-5}+p_4(1-2\mathrm{it})^{-4}$$

$$+p_3(1-2\mathrm{it})^{-3}+p_2(1-2\mathrm{it})^{-2}+p_1(1-2\mathrm{it})^{-1}+p_0,$$

where the coefficients p_α are given as follows:

$$p_6=\frac{1}{8}\nu_3^2, \qquad p_5=\frac{3}{4}\{3\nu_4+\nu_3(\tilde{\theta}\nu_2+2\zeta_1)-3\nu_2^2\},$$

$$p_4=\frac{3}{8}\{4(k+5)\tilde{\theta}\nu_3+\nu_2[\nu_2(3\tilde{\theta}^2+6)+12\zeta_1\tilde{\theta}-24(k+2)]+12\zeta_1^2+48\zeta_2\},$$

$$p_3=\frac{1}{4}\nu_3(\tilde{\theta}^3+6\tilde{\theta})$$

$$+\frac{9}{2}\nu_2(k+1)(\tilde{\theta}^2+2)+9(k+3)\zeta_1\tilde{\theta}-9k^2-18k+12+15\tilde{\rho}, \tag{2.b.27}$$

$$p_2=\frac{3}{4}\nu_2(\tilde{\theta}^4+12\tilde{\theta}^2+12)+\frac{9}{2}(k^2-1)\tilde{\theta}^2+\frac{3}{2}\zeta_1\tilde{\theta}^3+9\zeta_1\tilde{\theta}+9(k^2-1),$$

$$p_1=\frac{3}{2}(k-1)(\tilde{\theta}^4+12\tilde{\theta}^2+12), \qquad p_0=\frac{1}{8}\tilde{\theta}^6+\frac{15}{4}\tilde{\theta}^4+\frac{45}{2}\tilde{\theta}^2+15.$$

Thus, by the above formulae, second factor $C_{L_2}(t)$ is given by

$$C_{L_2}(t)=1+n^{-\frac{1}{2}}\left\{\frac{\nu_3}{12}(1-2\mathrm{it})^{-3}+\left(\frac{1}{2}\zeta_1-\frac{1}{4}\nu_3\right)(1-2\mathrm{it})^{-2}\right. \tag{2.b.28}$$

$$\left.+\left(\frac{\nu_3}{2}+\frac{\tilde{\theta}}{2}\nu_2-\frac{1}{2}\zeta_1\right)(1-2\mathrm{it})^{-1}+\frac{\tilde{\theta}}{2}\nu_2+\frac{\tilde{\theta}^3}{3}\right\}+n^{-1}\textstyle\sum_{\alpha=0}^{6}g_{2\alpha}$$

$$\cdot(1-2\mathrm{it})^{-\alpha}+O(n^{-\frac{3}{2}}),$$

where the coefficients $g_{2\alpha}$ $(\alpha=0, 1, 2, \cdots, 6)$ are given by

$$g_{12}=\frac{1}{288}\nu_3^2, \qquad g_{10}=\frac{1}{16}\nu_4+\frac{1}{24}\zeta_1\nu_3-\frac{1}{48}\nu_3^2-\frac{1}{16}\nu_2^2,$$

$$g_8=-\frac{13}{48}\nu_4+\nu_3\left(\frac{7}{96}\nu_3+\frac{\tilde{\theta}}{24}\nu_2-\frac{1}{6}\zeta_1\right)+\frac{5}{16}\nu_2^2-\frac{1}{4}(k+2)\nu_2$$

$$+\frac{1}{8}\zeta_1^2+\frac{1}{2}\zeta_2, \qquad g_6=\frac{7}{12}\nu_4-\nu_3\left(\frac{1}{8}\nu_3+\frac{\tilde{\theta}}{12}\nu_2-\frac{3}{8}\zeta_1-\frac{\tilde{\theta}^3}{36}\right.$$

$$\left.-\frac{\tilde{\theta}}{4}\right)-\nu_2\left\{\frac{3}{4}\nu_2-\frac{\tilde{\theta}}{4}\zeta_1-\frac{3}{4}(k+2)\right\}-\frac{1}{4}k^2-\frac{1}{2}k+\frac{1}{3}$$

$$+\frac{5}{12}\tilde{\rho}-\frac{1}{4}\zeta_1^2-\frac{5}{4}\zeta_2,$$

$$(2.b.29)\quad g_4=-\frac{3}{4}\nu_4+\nu_3\Bigl(\frac{1}{8}\nu_3+\frac{\tilde{\theta}}{8}\nu_2-\frac{\tilde{\theta}^3}{12}-\frac{3}{4}\tilde{\theta}-\frac{1}{4}\zeta_1\Bigr)+\nu_2\Bigl\{\Bigl(\frac{\tilde{\theta}^2}{8}+1\Bigr)\nu_2$$

$$-\frac{1}{2}(2k+3)\Bigr\}+\frac{5}{4}\zeta_2+\frac{1}{8}\zeta_1^2+\frac{\tilde{\theta}^3}{6}\zeta_1+\frac{\tilde{\theta}}{2}\zeta_1+\frac{1}{4}(k^2+2k-2-\tilde{\rho}),$$

$$g_2=\frac{3}{4}\nu_4+\nu_3\Bigl(\frac{\tilde{\theta}}{4}\nu_2+\frac{\tilde{\theta}^3}{6}+\frac{3}{2}\tilde{\theta}\Bigr)+\nu_2\Bigl[\frac{\nu_2}{4}(\tilde{\theta}^2-3)+\frac{1}{4}\Bigl\{\frac{2}{3}\tilde{\theta}^4+3\tilde{\theta}^2$$

$$-\zeta_1\tilde{\theta}+2(k+1)\Bigr\}\Bigr]-\frac{1}{2}\zeta_2-\zeta_1\Bigl(\frac{\tilde{\theta}^3}{6}+\frac{\tilde{\theta}}{2}\Bigr),$$

$$g_0=\frac{\tilde{\theta}}{2}\nu_3+\nu_2\Bigl\{\nu_2\Bigl(\frac{\tilde{\theta}^2}{8}+\frac{1}{4}\Bigr)+\frac{\tilde{\theta}^4}{6}+\frac{3}{2}\tilde{\theta}^2\Bigr\}+\frac{1}{18}\tilde{\theta}^6+\frac{3}{8}\tilde{\theta}^4+\frac{1}{6}.$$

Multiplying the two factors (2.b.11), (2.b.28) together and arranging the terms according to the power of $(1-2\mathrm{it})^{-1}$, we can get the asymptotic formula for the characteristic function of L under K_b as

$$(2.b.30)\quad C_L(t)=(1-2\mathrm{it})^{-\frac{f}{2}}\exp\Bigl[\frac{2\mathrm{it}}{(1-2\mathrm{it})}\delta_L^2\Bigr]\Bigl[1-n^{-\frac{1}{2}}\Bigl\{\frac{\nu_3}{12}(1-2\mathrm{it})^{-3}$$

$$+\Bigl(\frac{1}{2}\zeta_1-\frac{1}{4}\nu_3\Bigr)(1-2\mathrm{it})^{-2}+\Bigl(\frac{\nu_3}{2}+\frac{\tilde{\theta}}{2}\nu_2-\frac{\zeta_1}{2}\Bigr)(1-2\mathrm{it})^{-1}$$

$$-\frac{\nu_3}{3}-\frac{\tilde{\theta}}{2}\nu_2\Bigr\}+n^{-1}\textstyle\sum_{\alpha=0}^{6}h_{2\alpha}(1-2\mathrm{it})^{-\alpha}+O(n^{-\frac{3}{2}})\Bigr],$$

where

$$h_{12}=g_{12},\qquad h_{10}=g_{10},\qquad h_8=g_8,$$

$$h_6=\frac{7}{12}\nu_4-\nu_3\Bigl(\frac{11}{72}\nu_3+\frac{\tilde{\theta}}{6}\nu_2-\frac{3}{8}\zeta_1-\frac{\tilde{\theta}}{4}\Bigr)-\nu_2\Bigl\{\frac{3}{4}\nu_2-\frac{\tilde{\theta}}{4}\zeta_1$$

$$-\frac{3}{4}(k+2)\Bigr\}-\frac{1}{4}\zeta_1^2-\frac{5}{4}\zeta_2-\frac{k^2}{4}-\frac{k}{2}+\frac{1}{3}+\frac{5}{12}\tilde{\rho},$$

$$(2.b.31)\quad h_4=-\frac{3}{4}\nu_4+\nu_3\Bigl(\frac{5}{24}\nu_3+\frac{3}{8}\tilde{\theta}\nu_2-\frac{3}{4}\tilde{\theta}-\frac{5}{12}\zeta_1\Bigr)+\nu_2\Bigl\{\Bigl(\frac{\tilde{\theta}^2}{8}+1\Bigr)\nu_2$$

$$-\frac{1}{2}(2k+3)-\frac{\tilde{\theta}}{2}\zeta_1\Bigr\}+\frac{5}{4}\zeta_2+\frac{1}{8}\zeta_1^2+\frac{\tilde{\theta}}{2}\zeta_1+\frac{1}{4}(k^2+2k$$

$$-2-\tilde{\rho}),$$

$$h_2=\frac{3}{4}\nu_4-\nu_3\left(\frac{\nu_3}{6}+\frac{5}{12}\tilde{\theta}\,\nu_2-\frac{3}{2}\tilde{\theta}-\frac{1}{6}\zeta_1\right)-\nu_2\left\{\left(\frac{\tilde{\theta}^2}{4}+\frac{3}{4}\right)\nu_2\right.$$

$$\left.-\frac{3}{4}\tilde{\theta}^2-\frac{\tilde{\theta}}{4}\zeta_1-\frac{1}{2}(k+1)\right\}-\frac{1}{2}\zeta_2-\frac{\tilde{\theta}}{2}\zeta_1,$$

$$h_0=-\frac{3}{8}\nu_4+\nu_3\left(\frac{1}{18}\nu_3+\frac{\tilde{\theta}}{6}\nu_2-\tilde{\theta}\right)+\frac{\nu_2}{8}\{(\tilde{\theta}^2+2)\nu_2-6\,\tilde{\theta}^2\}+\frac{1}{6}(1-\tilde{\rho}).$$

Inverting this characteristic function, we conclude the following theorem:

THEOREM 2.b. *Under the sequence of alternatives* K_b: $\sigma_\alpha^2=1+n^{-\frac{1}{2}}\theta_\alpha$ $(\alpha=1, 2, \cdots, k)$, *the asymptotic expansion of the L test given by* (1.1) *can be expressed as*

$$(2.b.32)\qquad P(L\leq x)=P(\chi_f^2(\delta_L^2)\leq x)-n^{-\frac{1}{2}}\left\{\frac{\nu_3}{12}P(\chi_{f+6}^2(\delta_L^2)\leq x)\right.$$

$$+\left(\frac{\zeta_1}{2}-\frac{1}{4}\nu_3\right)\mathrm{P}(\chi_{f+4}^2(\delta_L^2)\leq x)+\left(\frac{1}{2}\nu_3+\frac{\tilde{\theta}}{2}\nu_2-\frac{1}{2}\zeta_1\right)$$

$$\left.\cdot P(\chi_{f+2}^2(\delta_L^2)\leq x)-\left(\frac{1}{3}\nu_3+\frac{\tilde{\theta}}{2}\nu_2\right)P(\chi_f^2(\delta_L^2)\leq x)\right\}$$

$$+n^{-1}\textstyle\sum_{\alpha=0}^{6}h_{2\alpha}P(\chi_{f+2\alpha}^2(\delta_L^2)\leq x)+O(n^{-\frac{3}{2}}),$$

where the symbol $\chi_f^2(\delta_L^2)$ *means the non-central* χ^2 *variate with* $f=k-1$ *degrees of freedom and noncentrality parameter* $\delta_L^2=\frac{1}{4}\nu_2$. *The coefficients* $h_{2\alpha}(\alpha=0, 1, 2, \cdots, 6)$ *are given by* (2.b.31) *with* $\nu_\beta=\sum_{\alpha=1}^{k}\rho_\alpha\,(\theta_\alpha-\tilde{\theta})^\beta$, $\zeta_\beta=\sum_{\alpha=1}^{k}(\theta_\alpha-\tilde{\theta})^\beta$, $\tilde{\theta}=\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha$ *and* $\tilde{\rho}=\sum_{\alpha=1}^{k}\rho_\alpha^{-1}$.

2.c. *Asymptotic distribution for* $\delta=1$. We shall give the asymptotic expansion of the statistic L by the same method as in 2.b. under K_c: $\sigma_\alpha^2=1+n^{-1}\theta_\alpha$ $(\alpha=1, 2, \cdots, k)$. The characteristic function of L is given by (2.b.3). Under K_c, the expansion of the exponential part corresponding to (2.b.4) is given by

$$(2.c.1)\qquad \textstyle\sum_{\alpha=1}^{k}\frac{n_\alpha}{2\sigma_\alpha^2}\exp\left[\sqrt{\frac{2}{n_\alpha}}y_\alpha\right]=\frac{1}{2}\sum_{\alpha=1}^{k}\frac{n_\alpha}{\sigma_\alpha^2}+\frac{\sqrt{n}}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,y_\alpha+\frac{1}{2}\sum_{\alpha=1}^{k}y_\alpha^2$$

$$+n^{-\frac{1}{2}}\left(\frac{1}{3\sqrt{2}}\sum_{\alpha=1}^{k}\frac{y_{\alpha}^{3}}{\sqrt{\rho_{\alpha}}}-\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_{\alpha}}\theta_{\alpha}y_{\alpha}\right)$$

$$+n^{-1}\left(\frac{1}{12}\sum_{\alpha=1}^{k}\frac{y_{\alpha}^{4}}{\rho_{\alpha}}-\frac{1}{2}\sum_{\alpha=1}^{k}\theta_{\alpha}y_{\alpha}^{2}\right)+O(n^{-\frac{3}{2}}),$$

which yields the characteristic function under K_c: $\sigma_{\alpha}^{2}=1+n^{-1}\theta_{\alpha}$ $(1, 2, \dots, k)$:

$$C_L(t)=\left\{\prod_{\alpha=1}^{k}c_{n_\alpha}\sigma_{\alpha}^{-n_\alpha}\exp\left[-\frac{n_\alpha}{2\sigma_{\alpha}^{2}}\right]\right\}(2\pi)^{-\frac{k}{2}}|\Sigma|^{-\frac{1}{2}}E\left[1+n^{-\frac{1}{2}}\left(\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\right.\right.$$
$$\cdot\sqrt{\rho_{\alpha}}\theta_{\alpha}y_{\alpha}-\left.\frac{1}{3\sqrt{2}}\sum_{\alpha=1}^{k}\frac{y_{\alpha}^{3}}{\sqrt{\rho_{\alpha}}}\right)+n^{-1}\left\{\frac{1}{2}\sum_{\alpha=1}^{k}\theta_{\alpha}y_{\alpha}^{2}-\frac{1}{12}\sum_{\alpha=1}^{k}\frac{y_{\alpha}^{4}}{\rho_{\alpha}}\right.$$
$$\left.\left.+\frac{1}{4}\left(\sum_{\alpha=1}^{k}\sqrt{\rho_{\alpha}}\theta_{\alpha}y_{\alpha}-\frac{1}{3}\sum_{\alpha=1}^{k}\frac{y_{\alpha}^{3}}{\sqrt{\rho_{\alpha}}}\right)^{2}\right\}+O(n^{-\frac{3}{2}})\right], \tag{2.c.2}$$

where $\Sigma=(\sigma_{\alpha\beta})$ with $\sigma_{\alpha\beta}=(\delta_{\alpha\beta}-2\mathrm{it}\sqrt{\rho_{\alpha}\rho_{\beta}})(1-2\mathrm{it})^{-1}$ and E denotes the expectation with respect to the k-variate normal distribution with mean zeroand covariance matrix Σ. Here we have the same covariance matrix as under K_b but different mean vector.

Now applying Stirling's formula to the first factor of $C_L(t)$, we get

$$\left\{\prod_{\alpha=1}^{k}c_{n_\alpha}\sigma_{\alpha}^{-n_\alpha}\exp\left[-\frac{n_\alpha}{2\sigma_{\alpha}^{2}}\right]\right\}(2\pi)^{-\frac{k}{2}}|\Sigma|^{-\frac{1}{2}}=(1-2\mathrm{it})^{-\frac{f}{2}}\left[1-\frac{n^{-1}}{12}\right.$$
$$\left.\cdot(3\nu_2+3\tilde{\theta}^{2}+2\tilde{\rho})+O(n^{-2})\right]. \tag{2.c.3}$$

Since all product moments of odd degree from normal population with mean zero vanish, we can see that the terms of order $n^{-1/2}$ and $n^{-3/2}$ of the expectation part in (2.c.2) are zero. So putting $\eta_{\alpha}=0$ and $\sigma_{\alpha\beta}=(\delta_{\alpha\beta}-2\mathrm{it}\sqrt{\rho_{\alpha}\rho_{\beta}})$ $(1-2\mathrm{it})^{-1}$ in (2.b.12), we can easily get the following formulae with the same notations as in Section 2.b.

$$E(\textstyle\sum_{\alpha=1}^{k}\theta_{\alpha}y_{\alpha}^{2})=\left\{(\mathrm{k}-1)\tilde{\theta}+\zeta_1\right\}(1-2\mathrm{it})^{-1}+\tilde{\theta},$$

$$E(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_{\alpha}}\theta_{\alpha}y_{\alpha})^{2}=\nu_2(1-2\mathrm{it})^{-1}+\tilde{\theta}^{2},$$

$$E(\textstyle\sum_{\alpha=1}^{k}y_{\alpha}^{4}/\rho_{\alpha})=(3\tilde{\rho}-6k+3)(1-2\mathrm{it})^{-2}+6(k-1)(1-2\mathrm{it})^{-1}+3, \tag{2.c.4}$$

$$E(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha y_\alpha \sum_{\alpha=1}^{k} y_\alpha^3/\sqrt{\rho_\alpha})=3\zeta_1(1-2\mathrm{it})^{-2}+3(k-1)\tilde{\theta}(1-2\mathrm{it})^{-1}+3\tilde{\theta},$$

$$E(\textstyle\sum_{\alpha=1}^{k} y_\alpha^3/\sqrt{\rho_\alpha})^2=(-9k^2-18k+12+15\tilde{\rho})(1-2\mathrm{it})^{-3}+9(k^2-1)(1$$

$$-2\mathrm{it})^{-2}+18(k-1)(1-2\mathrm{it})^{-1}+15.$$

Thus the expectation part in (2.c.2) is given by

$$\text{(2.c.5)}\qquad 1+n^{-1}\Big[-\frac{1}{12}(3k^2+6k-4-5\tilde{\rho})(1-2\mathrm{it})^{-3}+\frac{1}{4}(k^2+2k-2-\tilde{\rho}-2\zeta_1)(1$$

$$-2\mathrm{it})^{-2}+\frac{1}{4}(\nu_2+2\zeta_1)(1-2\mathrm{it})^{-1}+\frac{1}{12}(3\tilde{\theta}^2+2)\Big]+O(n^{-2}).$$

Multiplying (2.c.3) to (2.c.5), we obtain the asymptotic formula for the characteristic function (2.c.2) under K_c as follows:

$$\text{(2.c.6)}\qquad C_L(t)=(1-2\mathrm{it})^{-\frac{f}{2}}\Big[1+n^{-1}\Big\{\frac{1}{12}(5\tilde{\rho}-3k^2-6k+4)(1-2\mathrm{it})^{-3}$$

$$+\frac{1}{4}(k^2+2k-2-\tilde{\rho}-2\zeta_1)(1-2\mathrm{it})^{-2}+\frac{1}{4}(\nu_2+2\zeta_1)(1-2\mathrm{it})^{-1}$$

$$-\frac{1}{12}(3\nu_2+2\tilde{\rho}-2)\Big\}+O(n^{-2})\Big].$$

Inverting this characteristic function, we conclude the following theorem:

THEOREM 2.c. *Under the sequence of alternatives* K_c: $\sigma_\alpha^2=1+n^{-1}\theta_\alpha$ ($\alpha=1, 2, \cdots, k$), *the distribution of the L test can be expanded asymptotically with fixed* $\rho_\alpha=n_\alpha/n(\alpha=1, 2, \cdots, k)$ *as*

$$\text{(2.c.7)}\qquad P(L\leq x)=P(\chi_f^2\leq x)+\frac{n^{-1}}{12}\{(5\tilde{\rho}-3k^2-6k+4)P(\chi_{f+6}^2\leq x)$$

$$+(3k^2+6k-6-3\tilde{\rho}-6\zeta_1)P(\chi_{f+4}^2\leq x)+(3\nu_2+6\zeta_1)P(\chi_{f+2}^2\leq x)$$

$$+(2-2\tilde{\rho}-3\nu_2)P(\chi_f^2\leq x)\}+O(n^{-2}),$$

where $f=k-1$ *and* $\nu_2=\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})^2$, $\zeta_1=\sum_{\alpha=1}^{k}(\theta_\alpha-\tilde{\theta})$ *with* $\tilde{\theta}=\sum_{\alpha=1}^{k}$ $\cdot\rho_\alpha\theta_\alpha$ *and* $\tilde{\rho}=\sum_{\alpha=1}^{k}\rho_\alpha^{-1}$.

3. Asymptotic expansions of Bartlett's test under local alternatives.

3.a. *Asymptotic distribution for* $\delta=1/4$. Using the correction factor $c=1-(\sum_{\alpha=1}^{k}n_{\alpha}^{-1}-n^{-1})/3(k-1)$, we can rewrite (1.2) as

$$cM=m\log\sum_{\alpha=1}^{k}S_{\alpha}/m-\sum_{\alpha=1}^{k}m_{\alpha}\log S_{\alpha}/m_{\alpha}, \tag{3.a.1}$$

where $m_{\alpha}=cn_{\alpha}$, $m=\sum_{\alpha=1}^{k}m_{\alpha}$. As in the Ltest, we may assume $\sigma_1^2=1$ under the alternative hypotheses. Consider the sequence of alternatives K_{a}: $\sigma_{\alpha}^2=1+\theta_{\alpha}m^{-\frac{1}{4}}(\alpha=1, 2, \cdots, k)$. By the well-known result that $U_{\alpha}=[(S_{\alpha}/\sigma_{\alpha}^2)-m_{\alpha}]/\sqrt{2m_{\alpha}}$ has asymptotically the standard normal distribution as m_{α} tends to infinity, the statistic cM is expressed in terms of U_1, U_2, $\cdots$, U_k as

$$\begin{aligned}cM=&m(\log\bar{\sigma}-\tilde{\sigma})+m^{\frac{1}{2}}\sum_{\alpha=1}^{k}\sqrt{2\rho_{\alpha}}(\bar{\nu}_{\alpha}-1)U_{\alpha}+\sum_{\alpha=1}^{k}U_{\alpha}^2\\&-(\sum_{\alpha=1}^{k}\sqrt{\rho_{\alpha}}\bar{\nu}_{\alpha}U_{\alpha})^2+O_p(m^{-\frac{1}{2}}),\end{aligned} \tag{3.a.2}$$

which was given in deriving the asymptotic expansion of the cM under the fixed alternative in our previous paper [27], where $\bar{\sigma}=\sum_{\alpha=1}^{k}\rho_{\alpha}\sigma_{\alpha}^2$, $\tilde{\sigma}=\sum_{\alpha=1}^{k}\rho_{\alpha}\log\sigma_{\alpha}^2$, $\bar{\nu}_{\alpha}=\sigma_{\alpha}^2/\bar{\sigma}$.

Now we can give the expression of cM in (3.a.2) under K_a: $\sigma_{\alpha}^2=1+m^{-\frac{1}{4}}\cdot\theta_{\alpha}(\alpha=1, 2, \cdots, k)$ as

$$\begin{aligned}cM=&\frac{m^{\frac{1}{2}}}{2}(\tilde{\theta}_2-\bar{\theta}^2)-\frac{m^{\frac{1}{4}}}{3}(\tilde{\theta}_3-\bar{\theta}^3)+m^{\frac{1}{4}}q_0(U)+q_1(U)+m^{-\frac{1}{4}}q_2(U)\\&+O_p(m^{-\frac{1}{2}}),\end{aligned} \tag{3.a.3}$$

where the coefficients $q_0(U)$, $q_1(U)$ and $q_2(U)$ are given by

$$q_0(U)=\sum_{\alpha=1}^{k}\sqrt{2\rho_{\alpha}}(\theta_{\alpha}-\bar{\theta})U_{\alpha},$$

$$\begin{aligned}q_1(U)=&\sum_{\alpha=1}^{k}U_{\alpha}^2-(\sum_{\alpha=1}^{k}\sqrt{\rho_{\alpha}}U_{\alpha})^2-\bar{\theta}\sum_{\alpha=1}^{k}\sqrt{2\rho_{\alpha}}(\theta_{\alpha}-\bar{\theta})U_{\alpha}\\&+\frac{1}{4}(\tilde{\theta}_4-\bar{\theta}^4),\end{aligned} \tag{3.a.4}$$

$$q_2(U)=\tilde{\theta}^2\sum_{\alpha=1}^{k}\sqrt{2\rho_\alpha}(\theta_\alpha-\tilde{\theta})\,U_\alpha-2\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}U_\alpha\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}(\theta_\alpha-\tilde{\theta})U_\alpha$$

$$-\frac{1}{5}(\tilde{\theta}_5-\tilde{\theta}^5).$$

From the expression of (3.a.3), we can easily see that $m^{-\frac{1}{4}}M'=m^{-\frac{1}{4}}\Big\{cM-\frac{m^{\frac{1}{2}}}{2}(\tilde{\theta}_2-\tilde{\theta}^2)+\frac{m^{\frac{1}{4}}}{3}(\tilde{\theta}_3-\tilde{\theta}^3)\Big\}$ converges in law to the normal distribution with mean zero and variance $\tau_M^2=2\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})^2$, which was shown in Sugiura and Nagao[27]. Further we shall give the asymptotic expansion of the M test. The characteristic function of $m^{-\frac{1}{4}}M'/\tau_M$ is given by

$$C_M(t)=E\Big[e^{\mathrm{it}q_0(U)/\tau_M}\Big\{1+m^{-\frac{1}{4}}q_1(U)\,\mathrm{it}/\tau_M+m^{-\frac{1}{2}}\Big[q_2(U)\mathrm{it}/\tau_M$$
$$+\frac{1}{2}q_1^2(U)(\mathrm{it}/\tau_M)^2\Big]\Big\}\Big]+O(m^{-\frac{3}{4}}). \tag{3.a.5}$$

We state the following formulae which were used to obtain the asymptotic expansion of the M test under the fixed alternative in Sugiura and Nagao [27].

$$E[e^{tU_\alpha}]=e^{-\frac{t^2}{2}}\Big[1+m_\alpha^{-\frac{1}{2}}\Big\{\sqrt{2}\,\varDelta\,\rho_\alpha t+\frac{\sqrt{2}}{3}t^3\Big\}\Big]+O(m^{-1}),$$

$$E[U_\alpha e^{tU_\alpha}]=e^{-\frac{t^2}{2}}t+O(m^{-\frac{1}{2}}),$$

$$E[U_\alpha^2 e^{tU_\alpha}]=e^{-\frac{t^2}{2}}(t^2+1)+O(m^{-\frac{1}{2}}), \tag{3.a.6}$$

$$E[U_\alpha^3 e^{tU_\alpha}]=e^{\frac{t^2}{2}}(t^3+3t)+O(m^{-\frac{1}{2}}),$$

$$E[U_\alpha^4 e^{tU_\alpha}]=e^{\frac{t}{2}}(t^4+6t^2+3)+O(m^{-\frac{1}{2}}),$$

where $\varDelta=\frac{1}{2}(n-m)$. The above formulae corresponding to T_α are obtained by putting $\varDelta=0$, which were used in Section 2.a. Setting $b_\alpha=\sqrt{2\rho_\alpha}$

$\cdot(\theta_\alpha-\tilde{\theta})\mathrm{it}/\tau_M$ in $q_0(U)$, we easily have

$$E[\mathrm{e}^{\mathrm{it}q_0(U)/\tau_M}]=e^{-\frac{t^2}{2}}\left[1+m^{-\frac{1}{2}}\frac{\sqrt{2}}{3}\sum_{\alpha=1}^{k}b_\alpha^3/\sqrt{\rho_\alpha}\right]+O(m^{-1}), \tag{3.a.7}$$

$$\begin{aligned}E[q_1(U)\,e^{\mathrm{it}q_0(U)/\tau_M}]=e^{-\frac{t^2}{2}}\Big[&\sum_{\alpha=1}^{k}b_\alpha^2+k-1-\tilde{\theta}\sum_{\alpha=1}^{k}\sqrt{2\rho_\alpha}(\theta_\alpha-\tilde{\theta})b_\alpha\\ &+\frac{1}{4}(\tilde{\theta}_4-\tilde{\theta}^4)\Big]+O(m^{-\frac{1}{2}}),\end{aligned} \tag{3.a.8}$$

$$\begin{aligned}E[q_2(U)e^{\mathrm{it}q_0(U)/\tau_M}]=e^{-\frac{t^2}{2}}&\left\{\tilde{\theta}^2\sum_{\alpha=1}^{k}\sqrt{2\rho_\alpha}(\theta_\alpha-\tilde{\theta})b_\alpha-\frac{1}{5}(\tilde{\theta}_5-\tilde{\theta}^5)\right\}\\ &+O(m^{-\frac{1}{2}}),\end{aligned} \tag{3.a.9}$$

$$\begin{aligned}E[q_1^2(U)\,e^{\mathrm{it}q_0(U)/\tau_M}]=e^{-\frac{t^2}{2}}\Big[&\{\textstyle\sum_{\alpha=1}^{k}b_\alpha^2\}^2-2\tilde{\theta}\sum_{\alpha=1}^{k}\sqrt{2\rho_\alpha}(\theta_\alpha-\tilde{\theta})b_\alpha\\ &\cdot\textstyle\sum_{\alpha=1}^{k}b_\alpha^2+2\left\{k+1+\frac{1}{4}(\tilde{\theta}_4-\tilde{\theta}^4)\right\}\sum_{\alpha=1}^{k}b_\alpha^2+2\tilde{\theta}^2\\ &\cdot\{\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}(\theta_\alpha-\tilde{\theta})b_\alpha\}^2-2\tilde{\theta}\left\{k+1+\frac{1}{4}(\tilde{\theta}_4-\tilde{\theta}^4)\right\}\\ &\cdot\textstyle\sum_{\alpha=1}^{k}\sqrt{2\rho_\alpha}(\theta_\alpha-\tilde{\theta})b_\alpha+k^2-1+2\tilde{\theta}^2\sum_{\alpha=1}^{k}\rho_\alpha\\ &\cdot(\theta_\alpha-\tilde{\theta})^2+\frac{1}{16}(\tilde{\theta}_4-\tilde{\theta}^4)^2+\frac{1}{2}(k-1)(\tilde{\theta}_4-\tilde{\theta}^4)\Big]\\ &+O(m^{-\frac{1}{2}}).\end{aligned} \tag{3.a.10}$$

Thus the characteristic function of $m^{-\frac{1}{4}}M'/\tau_M$ can be expanded asymptotically as

$$\begin{aligned}C_M(t)=e^{-\frac{t^2}{2}}\Big[1+m^{-\frac{1}{4}}&\left\{\left[k-1+\frac{1}{4}(\tilde{\theta}_4-\tilde{\theta}^4)\right]\frac{\mathrm{it}}{\tau_M}-\tilde{\theta}\tau_M^2(\mathrm{it}/\tau_M)^2\right.\\ &\left.+\tau_M^2(\mathrm{it}/\tau_M)^3\right\}-m^{-\frac{1}{2}}\{\textstyle\sum_{\alpha=0}^{2}g_{2\alpha+1}(\mathrm{it}/\tau_M)^{2\alpha+1}\\ &-\textstyle\sum_{\alpha=1}^{3}g_{2\alpha}(\mathrm{it}/\tau_M)^{2\alpha}\}\Big]+O(m^{-\frac{3}{4}}),\end{aligned} \tag{3.a.11}$$

where

$$g_1=\frac{1}{5}(\tilde\theta_5-\tilde\theta^5),\quad g_2=\frac{1}{2}(k^2-1)+\frac{1}{32}(\tilde\theta_4-\tilde\theta^4)^2+\frac{1}{4}(k-1)(\tilde\theta_4-\tilde\theta^4)$$

$$+\frac{3}{2}\tilde\theta^2\tau_M^2,\quad g_3=\tilde\theta\left\{k+1+\frac{1}{4}(\tilde\theta_4-\tilde\theta^4)\right\}\tau_M^2-\frac{4}{3}\textstyle\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde\theta)^3, \tag{3.a.12}$$

$$g_4=\left\{k+1+\frac{1}{4}(\tilde\theta_4-\tilde\theta^4)\right\}\tau_M^2+\frac{\tilde\theta^2}{2}\tau_M^4,\quad g_5=\tilde\theta\tau_M^4,\quad g_6=\frac{\tau_M^4}{2}.$$

Inverting this characteristic function, we have the following theorem:

THEOREM 3.a. *Under the sequence of alternatives* K_a: $\sigma_\alpha^2=1+m^{-\frac{1}{4}}\theta_\alpha$ $(\alpha=1, 2, \cdots, k)$, *the distribution of the statistic* $M'=cM-\frac{m^{\frac{1}{2}}}{2}(\tilde\theta_2-\tilde\theta^2)+\frac{m^{\frac{1}{4}}}{3}\cdot(\tilde\theta_3-\tilde\theta^3)$ *is expanded asymptotically for large* $m=cn$ *with* $c=1-(\sum_{\alpha=1}^{k}\cdot n_\alpha^{-1}-n^{-1})/3(k-1)$ *as*

$$P(m^{-\frac{1}{4}}M'/\tau_M\leq x)=\Phi(x)-m^{-\frac{1}{4}}\left\{\left[k-1+\frac{1}{4}(\tilde\theta_4-\tilde\theta^4)\right]\Phi^{(1)}(x)/\tau_M\right.$$
$$\left.+\tilde\theta\Phi^{(2)}(x)+\tau_M^{-1}\Phi^{(3)}(x)\right\}+m^{-\frac{1}{2}}\textstyle\sum_{\alpha=1}^{6}g_\alpha\Phi^{(\alpha)}(x)/\tau_M^\alpha$$
$$+O(m^{-\frac{3}{4}}), \tag{3.a.13}$$

where $\tau_M^2=2\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde\theta)^2$ *and the coefficients* g_α *are given by* (3.a.12).

3.b. *Asymptotic distribution for* $\delta=\frac{1}{2}$. In Sugiura and Nagao [27], we obtained the limiting distribution of the statistic M by the method in 3.a, by which, however, we could not obtain the asymptotic expansion of M by the same method.

So we put $Y_\alpha=\sqrt{\frac{m_\alpha}{2}}(\log S_\alpha/m_\alpha-\log\sigma_\alpha^2)$ $(\alpha=1, 2, \cdots, k)$, which has asymptotically normal distribution with mean zero and variance 1.
Then we can express (3.a.1) in terms of the statistics Y_α $(\alpha=1,2,\cdots,k)$ under K_b: $\sigma_\alpha^2=1+m^{-\frac{1}{2}}\theta_\alpha(\alpha=1, 2, \cdots, k)$ as follows :

$$cM = q_0(Y) + m^{-\frac{1}{2}} q_1(Y) + O_p(m^{-1}), \tag{3.b.1}$$

where

$$q_0(Y) = \sum_{\alpha=1}^{k}\left(Y_\alpha + \sqrt{\frac{\rho_\alpha}{2}}\theta_\alpha\right)^2 - \left\{\sum_{\alpha=1}^{k}\left(\sqrt{\rho_\alpha}\, Y_\alpha + \frac{1}{\sqrt{2}}\rho_\alpha\theta_\alpha\right)\right\}^2,$$

$$\begin{aligned} q_1(Y) &= \sum_{\alpha=1}^{k}(\sqrt{2}Y_\alpha + \sqrt{\rho_\alpha}\theta_\alpha)\left(\frac{1}{\sqrt{\rho_\alpha}}Y_\alpha^2 + \sqrt{2}\theta_\alpha Y_\alpha\right) - \sum_{\alpha=1}^{k}(\sqrt{2\rho_\alpha}\, Y_\alpha + \rho_\alpha\theta_\alpha) \\ &\cdot \sum_{\alpha=1}^{k}(Y_\alpha^2 + \sqrt{2\rho_\alpha}\theta_\alpha Y_\alpha) + \frac{1}{3}\{\sum_{\alpha=1}^{k}(\sqrt{2\rho_\alpha}\, Y_\alpha + \rho_\alpha\theta_\alpha)\}^3 \\ &- \frac{1}{3}\sum_{\alpha=1}^{k}\rho_\alpha\left(\sqrt{\frac{2}{\rho_\alpha}}Y_\alpha + \theta_\alpha\right)^3. \end{aligned} \tag{3.b.2}$$

Thus we can give the characteristic function of cM as (3.b.3).

$$C_M(t) = E\left[e^{itq_0(Y)}\{1 + m^{-\frac{1}{2}} itq_1(Y)\}\right] + O(m^{-1}). \tag{3.b.3}$$

In order to calculate (3.b.3), we require the distribution of $z_\alpha = \sqrt{\frac{m_\alpha}{2}}\log S_\alpha / m_\alpha$, which is given by

$$c'_{m_\alpha}\sigma_\alpha^{-m_\alpha - 2\varDelta_\alpha}\exp\left[\sqrt{\frac{m_\alpha}{2}}z_\alpha + \varDelta_\alpha\sqrt{\frac{2}{m_\alpha}}z_\alpha - \frac{m_\alpha}{2\sigma_\alpha^2}\exp\sqrt{\frac{2}{m_\alpha}}z_\alpha\right], \tag{3.b.4}$$

$$-\infty < z_\alpha < \infty,$$

where $c'_{m_\alpha} = \left(\frac{m_\alpha}{2}\right)^{\frac{1}{2}(m_\alpha - 1) + \varDelta_\alpha}\left\{\Gamma\left[\frac{m_\alpha}{2} + \varDelta_\alpha\right]\right\}^{-1}$ and $\varDelta_\alpha = \frac{1}{2}(n_\alpha - m_\alpha)$. We shall note that the distributions (2.b.2) and (3.b.4) are the same when $\varDelta_\alpha = 0$. Thus we can rewrite $q_0(Y)$ and $q_1(Y)$ in terms of z_α ($\alpha = 1, 2, \cdots, k$) under K_b. Then we have

$$\begin{aligned} q_0'(z) &= \sum_{\alpha=1}^{k} z_\alpha^2 - (\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}z_\alpha)^2 + \frac{m^{-\frac{1}{2}}}{\sqrt{2}}\{\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha^2 z_\alpha \\ &- \sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}z_\alpha\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^2\} + O(m^{-1}), \end{aligned} \tag{3.b.5}$$

(3.b.6) $\quad q_1'(z) = \sqrt{2}\sum_{\alpha=1}^{k}\left(\frac{1}{3\sqrt{\rho_\alpha}}z_\alpha^3 - \frac{1}{2}\sqrt{\rho_\alpha}\theta_\alpha^2 z_\alpha\right) - \sum_{\alpha=1}^{k}\sqrt{2}\rho_\alpha z_\alpha\sum_{\alpha=1}^{k}\Big(z_\alpha^2$

$$-\frac{\rho_\alpha}{2}\theta_\alpha^2\Big)+\frac{2\sqrt{2}}{3}\{\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}z_\alpha\}^3+O(m^{-\frac{1}{2}}).$$

Hence the expected value $E[^{\mathrm{it}q_0(Y)}]$ is given by

$$\text{(3.b.7)}\qquad E[e^{\mathrm{it}q_0'(z)}]=\Big\{\prod_{\alpha=1}^{k}c_{m_\alpha}'\,\sigma_\alpha^{-m_\alpha-2\varDelta_\alpha}\Big\}\int\exp\Big[\mathrm{it}\textstyle\sum_{\alpha=1}^{k}z_\alpha^2-\mathrm{it}(\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\mathrm{z}_\alpha)^2$$

$$+\frac{m^{-\frac{1}{2}}}{\sqrt{2}}\mathrm{it}\{\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha^2z_\alpha-\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^2\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}z_\alpha\}$$

$$+O(m^{-1})\Big]\exp\Big[\textstyle\sum_{\alpha=1}^{k}\sqrt{\frac{m_\alpha}{2}}z_\alpha+\sum_{\alpha=1}^{k}\varDelta_\alpha\sqrt{\frac{2}{m_\alpha}}z_\alpha$$

$$\textstyle\sum_{\alpha=1}^{k}\frac{m_\alpha}{2\sigma_\alpha^2}\exp\sqrt{\frac{2}{m_\alpha}}z_\alpha\Big]dz_1\cdots dz_k.$$

Under K_b: $\sigma_\alpha^2=1+m^{-\frac{1}{2}}\theta_\alpha(\alpha=1,2,\cdots,k)$, the second exponential part in the above integrand is expanded asymptotically for large m as (3.b.8).

$$\text{(3.b.8)}\qquad \textstyle\sum_{\alpha=1}^{k}\frac{m_\alpha}{2\sigma_\alpha^2}\exp\sqrt{\frac{2}{m_\alpha}}z_\alpha=\frac{1}{2}\sum_{\alpha=1}^{k}\frac{m_\alpha}{\sigma_\alpha^2}+\frac{\sqrt{m}}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}z_\alpha+\frac{1}{2}\sum_{\alpha=1}^{k}z_\alpha^2$$

$$-\frac{1}{\sqrt{2}}\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha z_\alpha+m^{-\frac{1}{2}}\Big(\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha^2z_\alpha$$

$$-\frac{1}{2}\textstyle\sum_{\alpha=1}^{k}\theta_\alpha z_\alpha^2+\frac{1}{3\sqrt{2}}\sum_{\alpha=1}^{k}\frac{z_\alpha^3}{\sqrt{\rho_\alpha}}\Big)+O(m^{-1}).$$

Thus we can rewrite (3.b.7) as follows:

$$\text{(3.b.9)}\qquad E[e^{\mathrm{it}q_0'(z)}]=\Big\{\prod_{\alpha=1}^{k}c_{m_\alpha}'\sigma_\alpha^{-m_\alpha-2\varDelta_\alpha}\exp\Big[-\frac{m_\alpha}{2\sigma_\alpha^2}\Big]\Big\}\int\exp\Big[\mathrm{it}\textstyle\sum_{\alpha=1}^{k}z_\alpha^2$$

$$-\mathrm{it}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}z_\alpha)^2-\frac{1}{2}\sum_{\alpha=1}^{k}z_\alpha^2+\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha z_\alpha\Big]$$

$$\Big\{1+m^{-\frac{1}{2}}\Big(\frac{\mathrm{it}}{\sqrt{2}}\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha^2z_\alpha-\frac{\mathrm{it}}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}z_\alpha\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^2$$

$$+\frac{1}{2}\textstyle\sum_{\alpha=1}^{k}\theta_\alpha z_\alpha^2-\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha^2\,z_\alpha-\frac{1}{3\sqrt{2}}\sum_{\alpha=1}^{k}\frac{z_\alpha^3}{\sqrt{\rho_\alpha}}$$

$$+\sqrt{2}\,\varDelta\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}z_\alpha\Big)+O(m^{-1})\Big\}dz_1\cdots dz_k,$$

where $\Delta=\frac{1}{2}(n-m)$. Since the exponential part of the integrand in (3.b.9) is the same as in the case of L, we can express, with the same notations as in Section 2.b.

$$-\frac{1}{2}(z-\eta)'\Sigma^{-1}(z-\eta)+\frac{1}{4}\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^2+\frac{\mathrm{it}}{2(1-2\mathrm{it})}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})^2. \tag{3.b.10}$$

Hence we can put $E[e^{\mathrm{it}q_0'(z)}]=E_1(t)E_2(t)$, where

$$E_1(t)=\left\{\prod_{\alpha=1}^{k}c'_{m_\alpha}\sigma_\alpha^{-m_\alpha-2\Delta_\alpha}\exp\left[-\frac{m_\alpha}{2\sigma_\alpha^2}\right]\right\}\exp\left[\frac{1}{4}\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^2\right](2\pi)^{-\frac{k}{2}}|\Sigma|^{-\frac{1}{2}} \tag{3.b.11}$$
$$\cdot\exp\left[\frac{\mathrm{it}}{2(1-2\mathrm{it})}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})^2\right],$$

$$E_2(t)=E\left\{1+m^{-\frac{1}{2}}\left(\frac{\mathrm{it}}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha^2z_\alpha-\frac{\mathrm{it}}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}z_\alpha\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^2\right.\right. \tag{3.b.12}$$
$$+\frac{1}{2}\sum_{\alpha=1}^{k}\theta_\alpha z_\alpha^2-\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\theta_\alpha^2z_\alpha-\frac{1}{3\sqrt{2}}\sum_{\alpha=1}^{k}\frac{z_\alpha^3}{\sqrt{\rho_\alpha}}$$
$$\left.\left.+\sqrt{2}\Delta\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}z_\alpha\right)+O(m^{-1})\right\}.$$

The symbol E in (3.b.12) denotes the expectation of $(z_1,\cdots,z_k)$ with respect to the k-variate normal distribution with mean η and covariance matrix Σ.

Since $|\Sigma|^{-\frac{1}{2}}=(1-2\mathrm{it})^{-\frac{f}{2}}$, the first factor (3.b.11) is calculated as follows:

$$(1-2\mathrm{it})^{-\frac{f}{2}}\exp\left[\frac{2\mathrm{it}}{(1-2\mathrm{it})}\delta_M^2\right]\left\{1+m^{-\frac{1}{2}}\left(\frac{1}{3}\nu_3+\tilde{\theta}\nu_2+\frac{1}{3}\tilde{\theta}^3-\Delta\tilde{\theta}\right)\right. \tag{3.b.13}$$
$$\left.+O(m^{-1})\right\},$$

where $\delta_M^2=\frac{1}{4}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})^2$, $\nu_\beta=\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})^\beta$, $f=k-1$ and $\tilde{\theta}=\sum_{\alpha=1}^{k}\cdot\rho_\alpha\theta_\alpha$. The first term in (3.b.13) shows that cM is asymptotically non-central χ^2 with f degrees of freedom and the noncentrality parameter δ_M^2 under K_b, which is well-known. By using (2.b.13), (2.b.15), (2.b.18) and

$$(3.\text{b}.14)\qquad E\left(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\, z_\alpha\right)=\frac{1}{\sqrt{2}}\tilde{\theta},$$

we can obtain the expectation in (3.b.12) as follows.

$$(3.\text{b}.15)\qquad 1+m^{-\frac{1}{2}}\left\{-\frac{1}{12}\nu_3(1-2\text{it})^{-3}+\left(\frac{1}{4}\nu_3-\frac{1}{2}\zeta_1\right)(1-2\text{it})^{-2}\right.$$
$$\left.+\left(\frac{1}{2}\zeta_1-\frac{1}{4}\nu_3\right)(1-2\text{it})^{-1}-\frac{1}{4}\nu_3-\tilde{\theta}\nu_2-\frac{\tilde{\theta}^3}{3}+\varDelta\tilde{\theta}\right\}+O(m^{-1}),$$

where $\zeta_\beta=\sum_{\alpha=1}^{k}(\theta_\alpha-\tilde{\theta})^\beta$. Thus, multiplying (3.b.13) with (3.b.15), we have

$$(3.\text{b}.16)\qquad E[e^{\text{it}q_0(Y)}]=(1-2\text{it})^{-\frac{f}{2}}\exp\left[\frac{2\text{it}}{(1-2\text{it})}\delta_M^2\right]\left[1+m^{-\frac{1}{2}}\left\{-\frac{\nu_3}{12}(1-2\text{it})^{-3}\right.\right.$$
$$+\left(\frac{\nu_3}{4}-\frac{\zeta_1}{2}\right)(1-2\text{it})^{-2}+\left(\frac{1}{2}\zeta_1-\frac{1}{4}\nu_3\right)(1-2\text{it})^{-1}$$
$$\left.\left.+\frac{1}{12}\nu_3\right\}+O(m^{-1})\right].$$

Also we have

$$(3.\text{b}.17)\qquad E[\text{it}q_1'(z)e^{\text{it}q_0'(z)}]=\text{it}E[q_1'(z)]+O(m^{-\frac{1}{2}}),$$

where the expectation in the left is under the distribution (3.b.4) and the expectation in the right is under the normal distribution with mean η and covariance $\varSigma$. Using Table 1, we can easily get the following formulae:

$$(3.\text{b}.18)\qquad E\left(\textstyle\sum_{\alpha=1}^{k}z_\alpha^2\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\, z_\alpha\right)=\frac{1}{2\sqrt{2}}\tilde{\theta}\nu_2(1-2\text{it})^{-2}+\frac{1}{\sqrt{2}}(k-1)\tilde{\theta}(1-2\text{it})^{-1}$$
$$+\frac{3}{\sqrt{2}}\tilde{\theta}+\frac{1}{2\sqrt{2}}\tilde{\theta}^3,$$

$$(3.\text{b}.19)\qquad E\left(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\, z_\alpha\right)^3=\frac{3}{\sqrt{2}}\tilde{\theta}+\frac{\tilde{\theta}^3}{2\sqrt{2}}.$$

By using (2.b.13), (2.b.18) and the above formulae, we have

$$(3.\text{b}.20)\qquad E[\text{it}q_1'(z)e^{\text{it}q_0'(z)}]=(1-2\text{it})^{-\frac{f}{2}}\exp\left[\frac{2\text{it}}{(1-2\text{it})}\delta_M^2\right]\left\{\frac{1}{12}\nu_3(1-2\text{it})^{-3}\right.$$

$$+\left(\frac{1}{2}\zeta_1-\frac{1}{12}\nu_3\right)(1-2\mathrm{it})^{-2}-\left(\frac{1}{4}\nu_3+\frac{\tilde{\theta}}{2}\nu_2+\frac{1}{2}\zeta_1\right)$$

$$\cdot(1-2\mathrm{it})^{-1}+\frac{\nu_3}{4}+\frac{\tilde{\theta}}{2}\nu_2+O(m^{-\frac{1}{2}})\Big\}.$$

Hence the characteristic function of cM under the sequence of alternatives K_b : $\sigma_\alpha^2=1+m^{-\frac{1}{2}}\theta_\alpha$ $(\alpha=1, 2, \cdots, k)$ is given by

$$C_M(t)=(1-2\mathrm{it})^{-\frac{f}{2}}\exp\left[\frac{2\mathrm{it}}{1-2\mathrm{it}}\delta_M^2\right]\Big[1+m^{-\frac{1}{2}}\Big\{\frac{1}{6}\nu_3(1-2\mathrm{it})^{-2} \tag{3.b.21}$$

$$-\frac{1}{2}(\nu_3+\tilde{\theta}\nu_2)(1-2\mathrm{it})^{-1}+\frac{1}{3}\nu_3+\frac{\tilde{\theta}}{2}\nu_2\Big\}+O(m^{-1})\Big].$$

Inverting this characteristic function, we have the following theorem:

THEOREM 3.b. *Under the sequence of alternatives* K_b: $\sigma_\alpha^2=1+m^{-\frac{1}{2}}\theta_\alpha$ $(\alpha=1, 2, \cdots, k)$, *the asymptotic expansion of the M test given by* (3.a.1) *can be expressed as follows*:

$$P(cM\leq x)=P(\chi_f^2(\delta_M^2)\leq x)+m^{-\frac{1}{2}}\Big\{\frac{\nu_3}{6}P(\chi_{f+4}^2(\delta_M^2)\leq x) \tag{3.b.22}$$

$$-\left(\frac{\nu_3}{2}+\frac{\tilde{\theta}}{2}\nu_2\right)P(\chi_{f+2}^2(\delta_M^2)\leq x)+\left(\frac{\nu_3}{3}+\tilde{\theta}\frac{\nu_2}{2}\right)P(\chi_f^2(\delta_M^2)\leq x)\Big\}$$

$$+O(m^{-1}),$$

where $\delta_M^2=\frac{1}{4}\sum_{\alpha=1}^k\rho_\alpha(\theta_\alpha-\tilde{\theta})^2$ *and* $c=1-(\sum_{\alpha=1}^k n_\alpha^{-1}-n^{-1})/3(k-1)$, $m=cn$ *and* $\nu_\beta=\sum_{\alpha=1}^k\rho_\alpha(\theta_\alpha-\tilde{\theta})^\beta$.

3. c. *Asymptotic distribution for* $\delta=1$. In this section, we shall give the asymptotic expansion of the M test under the sequence of alternatives K_c: $\sigma_\alpha^2=1+m^{-1}\theta_\alpha$ $(\alpha=1, 2, \cdots, k)$. By the method used in Section 3.a, we can not get the asymptotic expansion though we can get the limiting distribution as in Sugiura and Nagao [27].

Since S_α has $\sigma_\alpha^2\chi_{n_\alpha}^2$, the characteristic function of cM is given by

$$(3.\text{c}.1)\qquad C_M(t)=K\int(\textstyle\sum_{\alpha=1}^k S_\alpha)^{mit}\displaystyle\prod_{\alpha=1}^k S_\alpha^{\frac{m_\alpha}{2}(1-2it)+\Delta_\alpha-1}\exp\left[-\frac{S_\alpha}{2\sigma_\alpha^2}\right]dS,$$

where $\Delta_\alpha=\frac{1}{2}(n_\alpha-m_\alpha)$ and the coefficient K is given by

$$(3.\text{c}.2)\qquad K=\left\{\prod_{\alpha=1}^k\Gamma\left[\frac{m_\alpha}{2}+\Delta_\alpha\right]2^{\frac{m_\alpha}{2}+\Delta_\alpha}\sigma_\alpha{}^{m_\alpha+2\Delta_\alpha}\right\}^{-1}\left(\prod_{\alpha=1}^k m_\alpha^{itm_\alpha}\right)m^{-itm}.$$

Since $\sigma_\alpha^2=1+m^{-1}\theta_\alpha$ $(\alpha=1, 2, \cdots, k)$, we can put $\sigma_\alpha^{-2}=1-m^{-1}A_\alpha$ $(\alpha=1, 2, \cdots, k)$, where

$$(3.\text{c}.3)\qquad A_\alpha=\theta_\alpha-m^{-1}\theta_\alpha^2+m^{-2}\theta_\alpha^3+O(m^{-3}).$$

Thus we have

$$(3.\text{c}.4)\qquad C_M(t)=K\int(\textstyle\sum_{\alpha=1}^k S_\alpha)^{mit}\displaystyle\prod_{\alpha=1}^k S_\alpha^{\frac{m_\alpha}{2}(1-2it)+\Delta_\alpha-1}\exp\left[-\frac{S_\alpha}{2}+\frac{A_\alpha}{2m}S_\alpha\right]dS_\alpha$$

$$=K\sum_{l_1}\cdots\sum_{l_k}\frac{\prod_{\alpha=1}^k A_\alpha^{l_\alpha}}{\prod_{\alpha=1}^k l_\alpha!}(2m)^{-\sum_{\alpha=1}^k l_\alpha}\int(\textstyle\sum_{\alpha=1}^k S_\alpha)^{mit}$$

$$\cdot\prod_{\alpha=1}^k S_\alpha^{\frac{m_\alpha}{2}(1-2it)+l_\alpha+\Delta_\alpha-1}\exp\left[-\frac{1}{2}S_\alpha\right]dS_\alpha,$$

where the range of summation is $l_\alpha=0,1,2,\cdots$ $(\alpha=1, 2,\cdots,k)$.
By the reproductive property of χ^2 distributions, we can rewrite the above integral, obtaining

$$(3.\text{c}.5)\qquad C_M(t)=K\sum_{l_1}\cdots\sum_{l_k}\frac{\prod_{\alpha=1}^k A_\alpha^{l_\alpha}}{\prod_{\alpha=1}^k l_\alpha!}(2m)^{-\sum_{\alpha=1}^k l_\alpha}\left\{\prod_{\alpha=1}^k\Gamma\left[\frac{m_\alpha}{2}(1-2it)+l_\alpha+\Delta_\alpha\right]\right\}$$

$$\left\{\Gamma\left[\frac{m}{2}(1-2it)+\textstyle\sum_{\alpha=1}^k l_\alpha+\Delta\right]\right\}^{-1}\displaystyle\int_0^\infty S^{\frac{m}{2}+\sum_{\alpha=1}^k l_\alpha+\Delta-1}$$

$$\cdot\exp\left[-\frac{S}{2}\right]dS$$

$$=K\sum_{l_1}\cdots\sum_{l_k}\frac{\prod_{\alpha=1}^k A_\alpha^{l_\alpha}}{\prod_{\alpha=1}^k l_\alpha!}(2m)^{-\sum_{\alpha=1}^k l_\alpha}\left\{\prod_{\alpha=1}^k\Gamma\left[\frac{m_\alpha}{2}(1-2it)+l_\alpha+\Delta_\alpha\right]\right\}$$

$$\frac{\Gamma\left[\frac{m}{2}+\sum_{\alpha=1}^{k}l_\alpha+\Delta\right]}{\Gamma\left[\frac{m}{2}(1-2\mathrm{it})+\sum_{\alpha=1}^{k}l_\alpha+\Delta\right]}2^{\frac{m}{2}+\sum_{\alpha=1}^{k}l_\alpha+\Delta},$$

where $\Delta=\frac{1}{2}(n-m)$. Putting

$$(3.c.6)\qquad C_1(t)=\left\{\prod_{\alpha=1}^{k}\frac{\Gamma\left[\frac{m_\alpha}{2}(1-2\mathrm{it})+\Delta_\alpha\right]}{\Gamma\left[\frac{m_\alpha}{2}+\Delta_\alpha\right]}\right\}\frac{\Gamma\left[\frac{m}{2}+\Delta\right]}{\Gamma\left[\frac{m}{2}(1-2\mathrm{it})+\Delta\right]}m^{-\mathrm{it}m}\prod_{\alpha=1}^{k}m_\alpha^{\mathrm{it}m_\alpha},$$

and

$$(3.c.7)\qquad C_2(\mathrm{t})=\left\{\prod_{\alpha=1}^{k}\sigma_\alpha^{-m_\alpha-2\Delta_\alpha}\right\}\sum_{l_1}\cdots\sum_{l_k}\frac{1}{\prod_{\alpha=1}^{k}l_\alpha!}\prod_{\alpha=1}^{k}(\Delta_\alpha/m)^{l_\alpha}$$

$$\cdot\left\{\prod_{\alpha=1}^{k}\frac{\Gamma\left[\frac{m_\alpha}{2}(1-2\mathrm{it})+l_\alpha+\Delta_\alpha\right]}{\Gamma\left[\frac{m_\alpha}{2}(1-2\mathrm{it})+\Delta_\alpha\right]}\right\}\frac{\Gamma\left[\frac{m}{2}(1-2\mathrm{it})+\Delta\right]\Gamma\left[\frac{m}{2}+\sum_{\alpha=1}^{k}l_\alpha+\Delta\right]}{\Gamma\left[\frac{m}{2}(1-2\mathrm{it})+\sum_{\alpha=1}^{k}l_\alpha+\Delta\right]\Gamma\left[\frac{m}{2}+\Delta\right]},$$

we have $C_M(t)=C_1(t)C_2(t)$. The first factor $C_1(t)$ gives the characteristic function of cM under the hypothesis, which can be expanded asymptotically with fixed $\rho_\alpha=m_\alpha/m$ ($\alpha=1, 2, \cdots, k$) (Anderson [1]) as

$$(3.c.8)\qquad (1-2\mathrm{it})^{-\frac{f}{2}}[1+m^{-2}(1-k)\Delta^2\{(1-2\mathrm{it})^{-2}-1\}]+O(m^{-3}),$$

where $f=k-1$.

Under K_c: $\sigma_\alpha^2=1+m^{-1}\theta_\alpha$ ($\alpha=1, 2, \cdots, k$), the term $\prod_{\alpha=1}^{k}\sigma_\alpha^{-m_\alpha-2\Delta_\alpha}$ is given by

$$(3.c.9)\qquad \exp\left[-\frac{\tilde{\theta}}{2}+m^{-1}\left(\frac{1}{4}\tilde{\theta}_2-\Delta\tilde{\theta}\right)-m^{-2}\left(\frac{1}{6}\tilde{\theta}_3-\frac{\Delta}{2}\tilde{\theta}_2\right)+O(m^{-3})\right],$$

where $\tilde{\theta}_\beta=\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^\beta$ with $\tilde{\theta}_1=\tilde{\theta}$. To calculate the summation terms in (3.c.7), we shall use the following abbreviated notations.

(3.c.10)

$$\alpha_1(l)=\sum_{\alpha=1}^{k} l_\alpha, \quad \alpha_2(l)=\sum_{\alpha=1}^{k}(l_\alpha)_2+\sum_{\alpha\neq\beta} l_\alpha l_\beta,$$

$$\alpha_3(l)=\sum_{\alpha=1}^{k}(l_\alpha)_3+3\sum_{\alpha\neq\beta}(l_\alpha)_2 l_\beta+\sum_{\alpha\neq\beta\neq\gamma} l_\alpha l_\beta l_\gamma,$$

$$\alpha_4(l)=\sum_{\alpha=1}^{k}(l_\alpha)_4+3\sum_{\alpha\neq\beta}(l_\alpha)_2(l_\beta)_2+4\sum_{\alpha\neq\beta}(l_\alpha)_3 l_\beta$$

$$+6\sum_{\alpha\neq\beta\neq\gamma}(l_\alpha)_2 l_\beta l_\gamma+\sum_{\alpha\neq\beta\neq\gamma\neq\delta} l_\alpha l_\beta l_\gamma l_\delta,$$

$$\beta_2(l)=\sum_{\alpha=1}^{k}\frac{1}{\rho_\alpha}(l_\alpha)_2, \quad \beta_3(l)=\sum_{\alpha=1}^{k}\frac{1}{\rho_\alpha}(l_\alpha)_3+\sum_{\alpha\neq\beta}\frac{1}{\rho_\alpha}(l_\alpha)_2 l_\beta,$$

$$\beta_4(l)=\sum_{\alpha=1}^{k}\frac{1}{\rho_\alpha}(l_\alpha)_4+\sum_{\alpha\neq\beta}\frac{1}{\rho_\alpha}(l_\alpha)_2(l_\beta)_2+2\sum_{\alpha\neq\beta}\frac{1}{\rho_\alpha}(l_\alpha)_3 l_\beta$$

$$+\sum_{\alpha\neq\beta\neq\gamma}\frac{1}{\rho_\alpha}(l_\alpha)_2 l_\beta l_\gamma,$$

$$\gamma_3(l)=\sum_{\alpha=1}^{k}\frac{1}{\rho_\alpha^2}(l_\alpha)_3, \quad \gamma_4(l)=\sum_{\alpha=1}^{k}\frac{1}{\rho_\alpha^2}(l_\alpha)_4+\sum_{\alpha\neq\beta}\frac{1}{\rho_\alpha\rho_\beta}(l_\alpha)_2(l_\beta)_2,$$

where $(l_\alpha)_\beta=l_\alpha(l_\alpha-1)\cdots(l_\alpha-\beta+1)$. Thus the summation in (3.c.7) is expressed as follows:

(3.c.11)
$$\sum_{l_1}\cdots\sum_{l_k}\prod_{\alpha=1}^{k}\left(\frac{\rho_\alpha A_\alpha}{2}\right)^{e_\alpha}\frac{1}{\prod_{\alpha=1}^{k} l_\alpha!}\Big[1+m^{-1}\{(\beta_2(l)-\alpha_2(l))(1-2\mathrm{it})^{-1}$$

$$+2\Delta\alpha_1(l)+\alpha_2(l)\}+m^{-2}\sum_{\beta=0}^{2} q_\beta(l)(1-2\mathrm{it})^{-\beta}$$

$$+O(m^{-3})\Big],$$

where $q_0(l)$, $q_1(l)$ and $q_2(l)$ are given by

(3.c.12)
$$q_0(l)=\frac{1}{2}\alpha_4(l)+\frac{4}{3}\alpha_3(l)+2\Delta(\alpha_3(l)+\alpha_2(l))+2\Delta^2\alpha_2(l),$$

(3.c.13)
$$q_1(l)=\beta_4(l)-\alpha_4(l)+4(\beta_3(l)-\alpha_3(l))+2(\beta_2(l)-\alpha_2(l))$$

$$+2\Delta\{\beta_3(l)-\alpha_3(l)+2(\beta_2(l)-\alpha_2(l))\},$$

$$(3.c.14) \qquad q_2(l)=\frac{1}{2}\gamma_4(l)+\frac{4}{3}\gamma_3(l)-(\beta_4(l)+4\beta_3(l)+2\beta_2(l))+\frac{1}{2}\alpha_4(l)$$
$$+\frac{8}{3}\alpha_3(l)+2\alpha_2(l)-2\Delta(\beta_2(l)-\alpha_2(l)).$$

First we let

$$(3.c.15) \qquad \mathcal{L}(1)=\exp[B],$$

where $\mathcal{L}(f(l))=\sum_{l_1}\cdots\sum_{l_k}\left(\prod_{\alpha=1}^{k} l_\alpha!\right)^{-1}\prod_{\alpha=1}^{k}(\rho_\alpha A_\alpha/2)^{l_\alpha}f(l)$

and

$$(3.c.16) \qquad B=\frac{1}{2}\tilde{\theta}-\frac{1}{2}m^{-1}\tilde{\theta}_2+\frac{1}{2}m^{-2}\tilde{\theta}_3+O(m^{-3}).$$

To calculate (3.c.11), the following formulae are used, which can be proved easily.

$$\mathcal{L}(\alpha_1(l))=\frac{1}{2}\{\tilde{\theta}-m^{-1}\tilde{\theta}_2+O(m^{-2})\}\exp[B],$$

$$\mathcal{L}(\alpha_2(l))=\frac{1}{4}\{\tilde{\theta}^2-2m^{-1}\tilde{\theta}\tilde{\theta}_2+O(m^{-2})\}\exp[B],$$

$$(3.c.17) \qquad \mathcal{L}(\alpha_3(l))=\frac{1}{8}\{\tilde{\theta}^3+O(m^{-1})\}\exp[B],$$

$$\mathcal{L}(\alpha_4(l))=\frac{1}{16}\{\tilde{\theta}^4+O(m^{-1})\}\exp[B],$$

$$\mathcal{L}(\beta_2(l))=\frac{1}{4}\{\tilde{\theta}_2-2m^{-1}\tilde{\theta}_3+O(m^{-2})\}\exp[B],$$

$$\mathcal{L}(\beta_3(l))=\frac{1}{8}\{\tilde{\theta}\tilde{\theta}_2+O(m^{-1})\}\exp[B],$$

$$\mathcal{L}(\beta_4(l))=\frac{1}{16}\{\tilde{\theta}_2\tilde{\theta}^2++O(m^{-1})\}\exp[B],$$

$$\mathcal{L}(\gamma_3(l))=\frac{1}{8}\{\tilde{\theta}_3+O(m^{-1})\}\exp[B],$$

$$\mathcal{L}(\gamma_4(l))=\frac{1}{16}\{\tilde{\theta}_2^2+O(m^{-1})\}\exp[B].$$

By using the above formulae, we can easily calculate (3.c.11). Hence, multiplying three factors (3.c.8), (3.c.9), (3.c.11) together and arranging the terms according to the power of $(1-2\mathrm{it})^{-1}$, we obtain the asymptotic formula for the characteristic function of cM under K_c as follows:

$$C_M(t)=(1-2\mathrm{it})^{-\frac{f}{2}}\Big[1+\frac{1}{4m}(\tilde{\theta}^2-\tilde{\theta}_2)\{1-(1-2\mathrm{it})^{-1}\} \tag{3.c.18}$$

$$+\frac{1}{m^2}\sum_{\alpha=0}^{2}h_{2\alpha}(1-2\mathrm{it})^{-\alpha}+\frac{1}{m^2}(1-k)\varDelta^2\{(1-2\mathrm{it})^{-2}-1\}+O(m^{-3})\Big],$$

where the coefficients h_0, h_2, h_4 are given by

$$h_0=\frac{1}{32}(\tilde{\theta}_2-\tilde{\theta}^2)^2+\frac{1}{3}\tilde{\theta}_3-\frac{1}{2}\tilde{\theta}\tilde{\theta}_2+\frac{1}{6}\tilde{\theta}^3+\frac{\varDelta}{2}(\tilde{\theta}^2-\tilde{\theta}_2),$$

$$h_2=-\frac{1}{16}(\tilde{\theta}_2-\tilde{\theta}^2)^2+\tilde{\theta}\tilde{\theta}_2-\frac{1}{2}\tilde{\theta}_3-\frac{1}{2}\tilde{\theta}^3+\frac{1}{2}(\tilde{\theta}_2-\tilde{\theta}^2)+\varDelta(\tilde{\theta}_2-\tilde{\theta}^2), \tag{3.c.19}$$

$$h_4=\frac{1}{32}(\tilde{\theta}_2-\tilde{\theta}^2)^2+\frac{1}{6}\tilde{\theta}_3-\frac{1}{2}\tilde{\theta}\tilde{\theta}_2+\frac{1}{3}\tilde{\theta}^3+\frac{1}{2}\tilde{\theta}^2-\frac{1}{2}\tilde{\theta}_2+\frac{\varDelta}{2}(\tilde{\theta}^2-\tilde{\theta}_2)$$

Inverting this characteristic function, we have the following theorem:

THEOREM 3.c. *Under the sequence of alternatives* K_c: $\sigma_\alpha^2=1+m^{-1}\theta_\alpha$ ($\alpha=1, 2, \cdots, k$) *the distribution of the* M *test given by* (3.a.1) *can be expanded asymptotically for large* $m=cn$ *with fixed* $\rho_\alpha=n_\alpha/n$ ($\alpha=1, 2, \cdots, k$) *as*

$$P(cM\leq x)=P(\chi_f^2\leq x)+\frac{1}{4m}(\tilde{\theta}^2-\tilde{\theta}_2)\{P(\chi_f^2\leq x)-P(\chi_{f+2}^2\leq x)\} \tag{3.c.20}$$

$$+\frac{1}{m^2}\sum_{\alpha=0}^{2}h_{2\alpha}P(\chi_{f+2\alpha}^2\leq x)+\frac{1}{m^2}(1-k)\varDelta^2\ \{P(\chi_{f+4}^2\leq x)$$

$$-P(\chi_f^2\leq x)\}+O(m^{-3}),$$

where $f=k-1$ *and the correction factor* $c=1-(\sum_{\alpha=1}^{k}n_\alpha^{-1}-n^{-1})/3(k-1)$. *The coefficients* $h_{2\alpha}(\alpha=0, 1, 2)$ *are given by* (3.c.19) *with* $\varDelta=\frac{1}{2}(n-m)$, $\tilde{\theta}_\beta=\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^\beta$ *and* $\tilde{\theta}_1=\tilde{\theta}$.

The result corresponding to this theorem in a multivariate two-sample case is proved by Sugiura [24].

4. Numerical examples. We shall give some numerical values of the asymptotic power of Lehmann's test $(=L)$ and Bartlett's test $(=cM)$ when alt ernatives are near to the null hypothesis in the following special cases.

EXAMPLE 4.1. When $k=2$ and $n_1=n_2$, the L test is shown to be equivarent to the M test by our previous paper [27]. Hence the two powers should be equal within the accuracy of the percentage points. The 5% points of L and cM in case $n_1=n_2=50$ are given as 3.929 and 3.841, respectively, by our previous paper [27]. Also it was remarked by Sugiura [24] that the asymptotic powers should be computed by the different formulae according to the departure of the alternative hypothesis K from the null hypothesis and that the non-centrality parameters δ_L^2 and δ_M^2 may be used as a measure of the distance of K from the hypothesis. For the alternatives K: $\sigma_2^2=\Delta\sigma_1^2$, case 1($\Delta$ $=1.05$) is computed by the formulae (2.c.7) and (3.c.20), as well as case 2(Δ $=1.2$) and 3($\Delta=1.4$) by the formulae (2.b.32), (3.b.22) and (2.a.11), (3.a.13) respectively.

	$P(L\geq 3.929)$			$P(cM\geq 3.841)$		
Δ	1.05	1.2	1.4	1.05	1.2	1.4
non-centrality	0.0156	0.2500	1.0000	0.0155	0.2475	0.9900
first term	0.0474	0.1047	0.1673	0.0500	0.1084	0.1729
second term	0.0060	−0.0118	0.0601	0.0035	−0.0119	0.0578
third term	——	0.0044	−0.0247	−0.0003	——	−0.0200
approx. power	0.053	0.097	0.203	0.053	0.096	0.211

EXAMPLE 4.2. When $k=2$ and $n_1=4$, $n_2=20$ the exact values of the power of the M test for some alternatives have been given by Ramachandran [21] in his Table 744a. Using the 5% point of cM 3.801 in our previous paper [27], and specifying the alternatives K: $\sigma_2^2=\Delta\sigma_1^2$, we have the following approximate powers of the cM test:

$P_K(cM \geqq 3.801)$

formula	(3. *c*. 20)	(3. *b*. 22)	(3. *a*. 13)
Δ	10/9	10/7	5/2
first term	0. 0512	0. 0843	0. 0361
second term	0. 0022	−0. 0188	0. 0997
third term	−0. 0018	——	0. 0237
approx. power	0. 052	0. 066	0. 160
exact power	0. 052	0. 068	0. 158

Thus our formulae give a reasonable approximation in this case.

EXAMPLE 4.3. When $k=3$ and $n_1=50$, $n_2=100$, $n_3=150$, the approximate 5% points of L and cM are 6.109 and 5.991 respectively, which were given by our previous paper [27]. For the alternatives K: $\sigma_3^2=\sigma_2^2=\Delta\sigma_1^2$, we have the following powers.

	$P(L \geqq 6.109)$		$P(cM \geqq 5.991)$	
formula	(2. *c*. 7)	(2. *b*. 30)	(3. *c*. 20)	(3. *b*. 22)
Δ	0. 995	1. 3	0. 995	1. 3
first term	0. 04715	0. 2063	0. 05001	0. 2127
second term	0. 00249	−0. 0402	0. 00016	−0. 0668
third term	——	0. 0035	−0. 00001	——
approx. power	0. 0496	0. 170	0. 0502	0. 146

PART II. SPHERICITY TEST

5. Asymptotic expansions under local alternatives. As in Sugiura and Nagao [26], the modified LR criterion for testing the sphericity hypothesis, based on a random sample of size $N=n+1$ from a p-variate normal population, is given by

$$(5.1)\qquad \lambda^*=|S|^{n/2}\left(\frac{1}{p}\mathrm{tr}S\right)^{-np/2},$$

where S has a Wishart distribution $W_p(\Sigma, n)$. By the transformation $S\rightarrow cHSH$ for some orthogonal $p\times p$ matrix H and scalar c, we may assume $\Sigma=\Lambda=\mathrm{diag}\,(1, \lambda_2, \cdots, \lambda_p)$ without loss of generality. In this part we shall consider the sequence of alternatives K_δ: $\lambda_\alpha=1+\theta_\alpha m^{-\delta}$ for $\delta=1/2$ and $\delta=1$, where $m=\rho n$ with

$$(5.2)\qquad \rho=1-(2p^2+p+2)/6pn,$$

and derive the asymptotic expansions of $-2\rho\log\lambda^*$ directly from those of cM given in Section 3. The relationship between Bartlett's test and the sphericity test λ^* was used by Gleser [7] to prove the unbiasedness of the latter.

5.1. *Asymptotic distribution for* $\delta=1/2$. The characteristic function of $-2\rho\log\lambda^*$ is given by

$$(5.1.1)\qquad C_{\lambda^*}(t)=c_{p,n}\int p^{-pmit}|S|^{-mit}(\mathrm{tr}S)^{mpit}|S|^{\frac{1}{2}(n-p-1)}|\Lambda|^{-\frac{n}{2}}\cdot\mathrm{etr}\left(-\frac{1}{2}\Lambda^{-1}S\right)dS,$$

where the coefficient $c_{p,n}$ is given by

$$(5.1.2)\qquad c_{p,n}^{-1}=2^{\frac{np}{2}}\pi^{\frac{1}{4}p(p-1)}\prod_{\alpha=1}^{p}\Gamma\left[\frac{1}{2}(n-\alpha+1)\right].$$

Transform the variable S to D and R by $S=D^{\frac{1}{2}}RD^{\frac{1}{2}}$ such that the matrix D is diagonal and composed of the diagonal elements of S. Then

$|\partial S/\partial(D,R)|=|D|^{-\frac{1}{2}(p-1)}$ and $\mathrm{tr}\Lambda^{-1}S=\mathrm{tr}\Lambda^{-1}D$, so we have

$$
\begin{aligned}
(5.1.3)\qquad C_{\lambda^*}(t)&=c_{p,n}\int|R|^{-\frac{1}{2}(m-2itm-p-1+2\Delta)}dR\int p^{-pmit}|D|^{-mit}(\mathrm{tr}D)^{mpit}\\
&\quad\cdot|D|^{\frac{n}{2}-1}|\Lambda|^{-\frac{n}{2}}\mathrm{etr}\Big(-\frac{1}{2}\Lambda^{-1}D\Big)dD\\
&=\left\{\prod_{\alpha=1}^{p}\frac{\Gamma\Big[\frac{m}{2}(1-2\mathrm{it})+\frac{1}{2}(1-\alpha)+\Delta\Big]}{\Gamma\Big[\frac{m}{2}+\frac{1}{2}(1-\alpha)+\Delta\Big]}\right\}\frac{\Gamma^{p}\Big[\frac{m}{2}+\Delta\Big]}{\Gamma^{p}\Big[\frac{m}{2}(1-2\mathrm{it})+\Delta\Big]}C_M(t),
\end{aligned}
$$

where $\Delta=\frac{1}{2}(n-m)$ and $C_M(t)$ is given by

$$
(5.1.4)\ C_M(t)=\frac{1}{\Gamma^{p}\Big[\frac{n}{2}\Big]2^{\frac{np}{2}}}\int|D|^{-mit}(\mathrm{tr}D/p)^{mpit}|D|^{\frac{n}{2}-1}|\Lambda|^{-\frac{n}{2}}\mathrm{etr}\Big(-\frac{1}{2}\Lambda^{-1}D\Big)dD,
$$

which may be seen as the characteristic function of Bartlett's test of p-sample case with equal sample sizes N. Thus, by Stirling's formula the first factor of (5.1.3) is expanded as

$$
(5.1.5)\qquad (1-2\mathrm{it})^{-\frac{p}{4}(p-1)}+O(m^{-1}).
$$

On the other hand, replacing k, ρ_α, m_α, m and θ_α with p, p^{-1}, m, pm and $\sqrt{p}\theta_\alpha$ in (3.b.21), respectively, the characteristic function (5.1.3) is given by

$$
\begin{aligned}
(5.1.6)\qquad C_{\lambda^*}(t)&=(1-2\mathrm{it})^{-\frac{f}{2}}\exp\Big[\frac{2\mathrm{it}}{(1\text{-}2\mathrm{it})}\delta^2\Big]\Big[1+m^{-\frac{1}{2}}\Big\{\Big(\frac{1}{3}t_3-\frac{1}{2p}t_1t_2+\frac{1}{6p^2}t_1^3\Big)\\
&\quad+\Big(\frac{1}{p}t_1t_2-\frac{1}{2}t_3-\frac{1}{2p^2}t_1^3\Big)(1-2\mathrm{it})^{-1}\\
&\quad+\frac{1}{6}\Big(t_3-\frac{3}{p}t_1t_2+\frac{2}{p^2}t_1^3\Big)(1-2\mathrm{it})^{-2}\Big\}+O(m^{-1})\Big],
\end{aligned}
$$

where $\delta^2=\Big(t_2-\frac{1}{p}t_1^2\Big)/4$, $\Theta=\mathrm{diag}(0,\theta_2,\cdots,\theta_p)$, $t_j=\mathrm{t}\Theta^j$ and $f=p(p+1)/2-1$.

Inverting this characteristic function, we have the following theorem:

THEOREM 5.1. *Under the sequence of alternatives* $K_1 : \varLambda = I + m^{-\frac{1}{2}}\Theta$, *the distribution of the modified LR criterion given by* (5.1) *can be expanded asymptotically for large* $m = \rho n$ *as follows*:

$$(5.1.7) \qquad P(-2\rho\log\lambda^* \leq x) = P(\chi_f^2(\delta^2) \leq x) + m^{-\frac{1}{2}}\left\{\left(\frac{1}{3}t_3 - \frac{1}{2p}t_1t_2 + \frac{1}{6p^2}t_1^3\right)\right.$$

$$\cdot P(\chi_f^2(\delta^2) \leq x) + \left(\frac{1}{p}t_1t_2 - \frac{1}{2}t_3 - \frac{1}{2p^2}t_1^3\right)P(\chi_{f+2}^2(\delta^2) \leq x)$$

$$\left. + \frac{1}{6}\left(t_3 - \frac{3}{p}t_1t_2 + \frac{2}{p^2}t_1^3\right)P(\chi_{f+4}^2(\delta^2) \leq x)\right\} + O(m^{-1}),$$

where the correction factor ρ *is given by* (5.2) *and* $t_j = \mathrm{tr}\Theta^j$. *The symbol* $\chi_f^2(\delta^2)$ *denotes the noncentral* χ^2 *variate with* $f = p(p+1)/2 - 1$ *degrees of freedom and non-centrality parameter* $\delta^2 = \left(t_2 - \frac{1}{p}t_1^2\right)/4$.

5.2. *Asymptotic distribution for* $\delta = 1$. In this section we give the asymptotic expansion of $-2\rho\log\lambda^*$ under K_2: $\varLambda = I + m^{-1}\Theta$.
By (5.1.3) and (3.c.5), the characteristic function of $-2\rho\log\lambda^*$ is given by

$$(5.2.1) \qquad C_{\lambda^*}(t) = \frac{p^{-pmit}\Gamma\left[\frac{1}{2}pm + p\varDelta\right]}{\Gamma\left[\frac{mp}{2}(1-2\mathrm{it}) + p\varDelta\right]}\left\{\prod_{\alpha=1}^{p}\frac{\Gamma\left[\frac{m}{2}(1-2\mathrm{it}) + \frac{1}{2}(1-\alpha) + \varDelta\right]}{\Gamma\left[\frac{1}{2}(m+1-\alpha) + \varDelta\right]}\right\}$$

$$\cdot |\varLambda|^{-\frac{m}{2}-\varDelta}\sum_{l_1}\cdots\sum_{l_p}\frac{\prod_{\alpha=1}^{p}(A_\alpha/m)^{l_\alpha}}{\prod_{\alpha=1}^{p}l_\alpha!}\frac{\left\{\prod_{\alpha=1}^{p}\Gamma\left[\frac{m}{2}(1-2\mathrm{it}) + l_\alpha + \varDelta\right]\right\}}{\Gamma^p\left[\frac{m}{2}(1-2\mathrm{it}) + \varDelta\right]}$$

$$\cdot \frac{\Gamma\left[\frac{mp}{2}(1-2\mathrm{it}) + p\varDelta\right]}{\Gamma\left[\frac{mp}{2}(1-2\mathrm{it}) + \sum_{\alpha=1}^{p}l_\alpha + p\varDelta\right]}\frac{\Gamma\left[\frac{mp}{2} + \sum_{\alpha=1}^{p}l_\alpha + p\varDelta\right]}{\Gamma\left[\frac{1}{2}pm + p\varDelta\right]},$$

where $A_\alpha = \theta_\alpha - m^{-1}\theta_\alpha^2 + m^{-2}\theta_\alpha^3 + O(m^{-3})$. The first factor of $C_{\lambda^*}(t)$ in (5.2.1) is the characteristic function of $-2\rho\log\lambda^*$ under the hypothesis, which is expanded as follows (see Anderson [1]):

$$(5.2.2) \qquad (1-2\mathrm{it})^{-\frac{f}{2}}[1+\omega_2\{(1-2\mathrm{it})^{-2}-1\}]+O(m^{-3}),$$

where $f=p(p+1/2-1$ and the coefficient ω_2 is given by

$$(5.2.3) \qquad \omega_2=\frac{1}{288p^2m^2}(p+2)(p-1)(p-2)\,(2p^3+6p^2+3p+2).$$

Using the following table, which shows the relationship between Bartlett's test and the sphericity test,

Bartlett	Sphericity
m_α	m
m	pm
θ_α	$p\theta_\alpha$
Δ_α	Δ
Δ	$p\Delta$
ρ_α	p^{-1}

the second factor in (5.2.1) is expanded asymptotically as

$$(5.2.4) \qquad 1+\frac{1}{4m}\left(t_2-\frac{1}{p}t_1^2\right)\{(1-2\mathrm{it})^{-1}-1\}+\frac{1}{m^2}\sum_{\alpha=0}^{2}h'_{2\alpha}(1-2\mathrm{it})^{-\alpha}+O(m^{-3}),$$

where the coefficients h'_0, h'_2, h'_4 are given by

$$h'_0=\frac{1}{32}\left(t_2-\frac{1}{p}t_1^2\right)^2+\frac{1}{3}t_3-\frac{1}{2p}t_1t_2+\frac{1}{6p^2}t_1^3+\frac{\Delta}{2}\left(\frac{1}{p}t_1^2-t_2\right),$$

$$h'_2=-\frac{1}{16}\left(t_2-\frac{1}{p}t_1^2\right)^2+\frac{1}{p}t_1t_2-\frac{1}{2}t_3-\frac{1}{2p^2}t_1^3+\frac{1}{2}\left(\frac{1}{p}t_2-\frac{1}{p^2}t_1^2\right)$$

$$(5.2.5) \qquad +\Delta\left(t_2-\frac{1}{p}t_1^2\right),$$

$$h'_4=\frac{1}{32}\left(t_2-\frac{1}{p}t_1^2\right)^2+\frac{1}{6}t_3-\frac{1}{2p}t_1t_2+\frac{1}{3p^2}t_1^3+\frac{1}{2p^2}t_1^2-\frac{1}{2p}t_2$$

$$+\frac{\Delta}{2}\left(\frac{1}{p}t_1^2-t_2\right),$$

with $t_j = \mathrm{tr}\Theta^j$. Hence, multiplying (5.2.2) and (5.2.4), we have the following theorem:

THEOREM 5.2. *Under the sequence of alternatives* K_2: $\Lambda = I + m^{-1}\Theta$, *the distribution of the modified LR criterion given by* (5.1) *can be expanded asymptotically for large* $m = \rho n$ *as*

$$(5.2.6) \qquad P(-2\rho\log\lambda^* \leq x) = P(\chi_f^2 \leq x) + \frac{1}{4m}\Big(t_2 - \frac{1}{p}t_1^2\Big)\{P(\chi_{f+2}^2 \leq x) - P(\chi_f^2 \leq x)\} + \omega_2\{P(\chi_{f+4}^2 \leq x) - P(\chi_f^2 \leq x)\} + \frac{1}{m^2}\textstyle\sum_{\alpha=0}^{2} h'_{2\alpha} P(\chi_{f+2\alpha}^2 \leq x) + O(m^{-3}),$$

where $f = \frac{1}{2}p(p+1) - 1$ *and the correction factor* ρ *is given by* (5.2).
The coefficients h'_0, h'_2, h'_4 *and* ω_2 *are given by* (5.2.5) *and* (5.2.3), *respectively, with* $\Delta = \frac{1}{2}(n-m)$.

6. Numerical examples. Evaluating the asymptotic formula (5.2.2) for the characteristic function under the hypothesis more precisely, we obtain

$$(6.1) \qquad P(-2\rho\log\lambda^* \leq x) = P(\chi_f^2 \leq x) + \frac{\omega_2}{m^2}\{P(\chi_{f+4}^2 \leq x) - P(\chi_f^2 \leq x)\} + \frac{\omega_3}{m^3}\{P(\chi_{f+6}^2 \leq x) - P(\chi_f^2 \leq x)\} + \frac{1}{m^4}[\omega_4\{P(\chi_{f+8}^2 \leq x) - P(\chi_f^2 \leq x)\} - \omega_2^2\ \{P(\chi_{f+4}^2 \leq x) - P(\chi_f^2 \leq x)\}] + O(m^{-5}),$$

where ω_2 is defined by (5.2.3) and

$$\omega_3 = \frac{1}{720}p(6p^4 + 15p^3 - 10p^2 - 30p + 3 + 16p^{-4}) - \frac{\Delta}{12}p(p-1)(p+1)(p+2) + \frac{\Delta^2}{6}p(2p^2 + 3p - 1 - 4p^{-2}) - \frac{2}{3}\Delta^3 p(p+1-2p^{-1}),$$

$$(6.2) \qquad \omega_4 = \frac{1}{480}p(p-1)(2p^4 + 8p^3 + 3p^2 - 17p - 14) - \frac{p\Delta}{120}(6p^4 + 15p^3 - 10p^2 - 30p$$

$$+3+16p^{-4})+\frac{\Delta^2 p}{4}(p-1)(p+1)(p+2)-\frac{\Delta^3 p}{3}(2p^2+3p-1-4p^{-2})$$

$$+\Delta^4(p-1)(p+2)+\frac{\omega_2^2}{2}.$$

Applying the inverse expansion formula due to Hill and Davis [11] to the above expression, we obtain the following asymptotic formula for the $100\alpha\%$ point of $-2\rho\log\lambda^*$:

$$u+\frac{2\omega_2 u(u+f+2)}{m^2 f(f+2)}+\frac{2\omega_3 u}{m^3 f(f+2)(f+4)}\{u^2+(f+4)u+(f+4)(f+2)\}$$

$$(6.3) \qquad +\frac{1}{m^4}\Big[\frac{2\omega_4 u}{f(f+2)(f+4)(f+6)}\{u^3+(f+6)u^2+(f+6)(f+4)u$$

$$+(f+6)(f+4)(f+2)\}-\frac{\omega_2^2 u}{f^2(f+2)^2}\{u^3+(f-2)u^2+(f+2)(f-6)u$$

$$+(f+2)^2(f-2)\}\Big]+O(m^{-5}),$$

where u is defined such that $P(\chi_f^2>u)=\alpha$ and $f=p(p+1)/2-1$.

EXAMPLE 6.1. When $n=50$ and $p=2$, the asymptotic formula (6.3) gives the following approximate 5% point.

first term	5.99147
term of order m^{-2}	0.00000
term of order m^{-3}	0.00000
term of order m^{-4}	0.00000
approx. value	5.9915

When $p=2$, making use of the exact distribution under the hypothesis, as seen in Anderson [1], we can get the exact 5% point as 5.9915. Our asymptotic formula (6.3) gives good approximation for the percentage point.

Sugiura [23] gave the asymptotic expansion under the alternative K as

$$(6.4) \qquad P([-2\rho\log\lambda^*-m\log\{(\mathrm{tr}\Sigma/p)^p/|\Sigma|\}]/\sqrt{m}\tau\leq x)=\Phi(x)-m^{-\frac{1}{2}}[(2p/3\tau^3)$$

$$\cdot\Phi^{(3)}(x)\ \{-3(t_2-1)^2+2t_3-3t_2+1\}+(1/2\tau)\Phi^{(1)}(x)(p^2+p-2t_2)]$$

$$+m^{-1}\textstyle\sum_{\alpha=1}^{3}g_{2\alpha}\Phi^{(2\alpha)}(x)/\tau^{2\alpha}+O(m^{-\frac{3}{2}}),$$

where $t_j=p^{j-1}(\mathrm{tr}\Sigma^j)/(\mathrm{tr}\Sigma)^j, \tau^2=2p(t_2-1)$ and the coefficients $g_{2\alpha}$ are given by

$$g_2=\frac{1}{24}\{132t_2^2-96t_3-4(5p^2+4p+14)t_2+3p^4+6p^3+23p^2+16p+8\},$$

$$g_4=\frac{p}{3}\{(t_2-1)^2(26t_2-3p^2-3p-8)+2(t_2-1)(15t_2-14t_3)+6t_4 \tag{6.5}$$

$$+2(p+3)(p-2)t_3-(3p^2+3p-4)t_2+p^2+p+2\},$$

$$g_6=\frac{2p^2}{9}\{[3(t_2-1)^2+3t_2-3t_3]^2+1-6(t_2-1)^2+4t_3-6t_2\}.$$

EXAMPLE 6.2. When $p=2$ and $n=50$, the approximate 5% point of $-2\cdot\rho\log\lambda^*$ is given by Example 6.1 as 5.9915. The asymptotic powers should be computed by the different formulae according to the departure of the alternative hypothesis K from the hypothesis. Let us specify the alternatives K as $\lambda_1=1$; $\lambda_2=1+\Delta$. The following case 1 ($\Delta=1$) is computed by formula (6.4) due to Sugiura [23] and the case 2 ($\Delta=0.4$) and 3 ($\Delta=0.05$) by the formulae (5.1.7) and (5.2.6) respectively.

approximate powers, when $p=2$ and $n=50$

Δ	1	0.4	0.05
first term	0.4812	0.2217	0.0500
second term	0.0856	−0.0758	0.0023
third term	−0.0003	———	−0.0002
approx. power	0.567	0.146	0.052

PART III. MULTIVARIATE BARTLETT'S TEST

7. Preliminaries. In this section, we shall give fundamental formulae used in part III, IV.

THEOREM 7.1. *Let $S(p\times p)$ be distributed according to the Wishart distribution $W(\Sigma, n)$. Then for any $p\times p$ matrix B, we have*

$$E[\mathrm{tr}BS]=n\mathrm{tr}B\Sigma, \tag{7.1}$$

$$E[\mathrm{tr}BS^2B']=n\mathrm{tr}B\Sigma^2B'+n\mathrm{tr}\Sigma\mathrm{tr}B\Sigma B'+n^2\mathrm{tr}B\Sigma^2B'. \tag{7.2}$$

PROOF. Since S is distributed as XX' where the columns of $X(p\times n)$ are independent, each with normal distribution $N(0, \Sigma)$, we have

$$E[\mathrm{tr}BS]=E[\mathrm{tr}B(XX')]=(2\pi)^{-\frac{pn}{2}}|\Sigma|^{-\frac{n}{2}}\int\mathrm{tr}B(XX')\cdot\mathrm{etr}\left[-\frac{1}{2}\Sigma^{-1}XX'\right]dX. \tag{7.3}$$

Let T be an orthogonal matrix such that $T'\Sigma T=\Lambda=\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_p)$. Put $T'X=Y=(y_{i\alpha})$, so the Jacobian is 1. Thus (7.3) is rewritten as

$$(2\pi)^{-\frac{pn}{2}}|\Lambda|^{-\frac{n}{2}}\int\mathrm{tr}C(YY')\mathrm{etr}(-\frac{1}{2}\Lambda^{-1}YY')dY, \tag{7.4}$$

where $T'BT=C=(c_{ij})$. The term $\mathrm{tr}C(YY')$ in the integral (7.4) is rewritten as

$$\mathrm{tr}C(YY')=\sum_{i=1}^{p}\sum_{\alpha=1}^{n}c_{ii}y_{i\alpha}^2+\sum_{i\neq j}\sum_{\alpha=1}^{n}c_{ij}y_{i\alpha}y_{j\alpha}. \tag{7.5}$$

Since the variables $y_{i\alpha}(i=1, 2, \cdots, p, \alpha=1, 2, \cdots, n)$ are mutually independent and $y_{i\alpha}$ is a normal distribution with mean zero and variance λ_i $(\alpha=1, 2, \cdots, n)$, by taking the expectation of (7.5), we have

$$E[\mathrm{tr}BS]=\sum_{i=1}^{p}\sum_{\alpha=1}^{n}c_{ii}\lambda_i=n\sum_{i=1}^{p}c_{ii}\lambda_i=n\mathrm{tr}(C\Lambda)=n\mathrm{tr}B\Sigma. \tag{7.6}$$

By the same argument, we have

$$(7.7)\qquad E[\mathrm{tr}BS^2B']=(2\pi)^{-\frac{pn}{2}}|\Lambda|^{-\frac{n}{2}}\int \mathrm{tr}C(YY')^2\,\mathrm{etr}\left(-\frac{1}{2}\Lambda^{-1}YY'\right)dY,$$

where $T'B'BT=C=(c_{ij})$. Since the term $\mathrm{tr}C(YY')^2$ in the integrand (7.7) is rewritten as

$$(7.8)\qquad \mathrm{tr}C(YY')^2=\sum_{i=1}^{p}\sum_{\alpha=1}^{n}c_{ii}y_{i\alpha}^4+\sum_{i\neq j}\sum_{\alpha=1}^{n}c_{ij}y_{i\alpha}y_{j\alpha}^3+\sum_{i\neq j}\sum_{\alpha=1}^{n}c_{ii}y_{i\alpha}^2y_{j\alpha}^2$$

$$+\sum_{i\neq j}\sum_{\alpha=1}^{n}c_{ij}y_{i\alpha}^3y_{j\alpha}+\sum_{i\neq j\neq k}\sum_{\alpha=1}^{n}c_{ij}y_{i\alpha}y_{k\alpha}^2y_{j\alpha}$$

$$+\sum_{i=1}^{p}\sum_{\alpha\neq\beta}c_{ii}y_{i\alpha}^2y_{i\beta}^2+\sum_{i\neq j}\sum_{\alpha\neq\beta}c_{ij}y_{i\alpha}y_{j\alpha}y_{j\beta}^2$$

$$+\sum_{i\neq j}\sum_{\alpha\neq\beta}c_{ii}y_{i\alpha}y_{i\beta}y_{j\alpha}y_{j\beta}+\sum_{i\neq j}\sum_{\alpha\neq\beta}c_{ij}y_{i\alpha}^2y_{i\beta}y_{j\beta}$$

$$+\sum_{i\neq j\neq k}\sum_{\alpha\neq\beta}c_{ij}y_{i\alpha}y_{k\alpha}y_{k\beta}y_{j\beta},$$

taking the expectation, we have

$$(7.9)\qquad 3\sum_{i=1}^{p}\sum_{\alpha=1}^{n}c_{ii}\lambda_i^2+\sum_{i\neq j}\sum_{\alpha=1}^{n}c_{ii}\lambda_i\lambda_j+\sum_{i=1}^{p}\sum_{\alpha\neq\beta}c_{ii}\lambda_i^2$$

$$=n\sum_{i=1}^{p}\lambda_i\sum_{i=1}^{p}c_{ii}\lambda_i+n^2\sum_{i=1}^{p}c_{ii}\lambda_i^2+n\sum_{i=1}^{p}c_{ii}\lambda_i^2$$

$$=n\mathrm{tr}\Lambda\mathrm{tr}\Lambda C+n^2\mathrm{tr}\Lambda^2C+n\mathrm{tr}\Lambda^2C.$$

Hence we have the desired conclusion.

From the formula (7.1) in Theorem 7.1, the following corollary is obtained immediately.

COROLLARY 7.2. *For any* $p\times p$ *matrix* B, *if* S_α *and* S_β *are independently distributed according to the Wishart distributions* $W(\Sigma_\alpha, n_\alpha)$ *and* $W(\Sigma_\beta, n_\beta)$ *respectively, we have*

$$(7.10)\qquad E[\mathrm{tr}BS_\alpha S_\beta]=n_\alpha n_\beta\mathrm{tr}B\Sigma_\alpha\Sigma_\beta.$$

Now we define the statistic Y as follows.

$$(7.11) \qquad Y=\frac{\Sigma^{-\frac{1}{2}}S\Sigma^{-\frac{1}{2}}-mI}{\sqrt{2m}},$$

where S has the Wishart distribution $W(\Sigma, n)$ and $m=\rho n$ with arbitrary number $\rho=1+o(1)$. The above statistic Y is a generalization of the statistic $(\chi_n^2-n)/\sqrt{2n}$. We shall now give some asymptotic formulae with respect to Y.

THEOREM 7.3. *Suppose S_α and S_β are independently distributed according to the Wishart distributions $W(\Sigma_\alpha, n_\alpha)$ and $W(\Sigma_\beta, n_\beta)$ respectively. Let Y_α and Y_β be matrices defined by* (7.11). *Then the following asymptotic formulae hold for the symmetric matrices Ω_α, Ω_β and non-singular matrices B, Φ_α and Φ_β.*

$$\begin{aligned}(7.12) \qquad E[\mathrm{etr}\Omega_\alpha Y_\alpha]=\mathrm{etr}\left[\frac{1}{2}\Omega_\alpha^2\right]\Bigg[1&+\frac{\sqrt{2}}{\sqrt{m_\alpha}}\left(\frac{1}{3}\mathrm{tr}\Omega_\alpha^3+\Delta_\alpha\mathrm{tr}\Omega_\alpha\right)\\ &+\frac{1}{m_\alpha}\left\{\frac{1}{9}(\mathrm{tr}\Omega_\alpha^3)^2+\frac{1}{2}\mathrm{tr}\Omega_\alpha^4+\frac{2}{3}\Delta_\alpha\mathrm{tr}\Omega_\alpha\mathrm{tr}\Omega_\alpha^3\right.\\ &\left.+\Delta_\alpha\mathrm{tr}\Omega_\alpha^2+\Delta_\alpha^2(\mathrm{tr}\Omega_\alpha)^2\right\}\Bigg]+O(m_\alpha^{-\frac{3}{2}}),\end{aligned}$$

$$\begin{aligned}(7.13) \qquad E[\mathrm{tr}B(\Phi_\alpha' Y_\alpha\Phi_\alpha)^2B'\mathrm{etr}(\Omega_\alpha Y_\alpha)]&=\mathrm{etr}\left[\frac{1}{2}\Omega_\alpha^2\right]\Bigg[\mathrm{tr}B(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)^2B'\\ &+\frac{1}{2}\mathrm{tr}B(\Phi_\alpha'\Phi_\alpha)^2B'+\frac{1}{2}\mathrm{tr}\Phi_\alpha'\Phi_\alpha\mathrm{tr}B(\Phi_\alpha'\Phi_\alpha)^2B'\\ &+\frac{\sqrt{2}}{\sqrt{m_\alpha}}\Big\{\mathrm{tr}B(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)(\Phi_\alpha'\Omega_\alpha^2\Phi_\alpha)B'\\ &+\mathrm{tr}B(\Phi_\alpha'\Omega_\alpha^2\Phi_\alpha)(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)B'+\frac{1}{6}\mathrm{tr}\Omega_\alpha^3\mathrm{tr}B(\Phi_\alpha'\Phi_\alpha)^2B'\\ &+\frac{1}{6}\mathrm{tr}\Omega_\alpha^3\mathrm{tr}\Phi_\alpha'\Phi_\alpha\mathrm{tr}B(\Phi_\alpha'\Phi_\alpha)B'+\frac{1}{3}\mathrm{tr}\Omega_\alpha^3\mathrm{tr}B(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)^2B'\\ &+\frac{1}{2}\mathrm{tr}B\Phi_\alpha'\Phi_\alpha(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)B'+\frac{1}{2}\mathrm{tr}B(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)\Phi_\alpha'\Phi_\alpha B'\\ &+\frac{1}{2}\mathrm{tr}\Phi_\alpha'\Phi_\alpha\mathrm{tr}B(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)B'+\frac{1}{2}\mathrm{tr}\Phi_\alpha'\Omega_\alpha\Phi_\alpha\mathrm{tr}B(\Phi_\alpha'\Phi_\alpha)B'\\ &+\Delta_\alpha\mathrm{tr}\Omega_\alpha\mathrm{tr}B(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)^2B'+\Delta_\alpha\mathrm{tr}B(\Phi_\alpha'\Phi)(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)B'\end{aligned}$$

$$+\varDelta_\alpha \mathrm{tr}B(\varPhi'_\alpha\varOmega_\alpha\varPhi_\alpha)(\varPhi'_\alpha\varPhi_\alpha)B'+\frac{\varDelta_\alpha}{2}\mathrm{tr}\varOmega_\alpha\mathrm{tr}\varPhi'_\alpha\varPhi_\alpha\mathrm{tr}B(\varPhi'_\alpha\varPhi_\alpha)B'$$

$$\left.\left.+\frac{\varDelta_\alpha}{2}\mathrm{tr}\varOmega_\alpha\mathrm{tr}B(\varPhi'_\alpha\varPhi_\alpha)^2B'\right\}\right]+O(m_\alpha^{-1}),$$

$$(7.14)\qquad E[\mathrm{tr}B(\varPhi'_\alpha Y_\alpha\varPhi_\alpha)(\varPhi'_\beta Y_\beta\varPhi_\beta)B'\mathrm{etr}(\varOmega_\alpha Y_\alpha+\varOmega_\beta Y_\beta)]=\mathrm{etr}\left[\frac{1}{2}\varOmega_\alpha^2+\frac{1}{2}\varOmega_\beta^2\right]$$

$$\left[\mathrm{tr}B(\varPhi'_\alpha\varOmega_\alpha\varPhi_\alpha)(\varPhi'_\beta\varOmega_\beta\varPhi_\beta)B'+\frac{\sqrt{2}}{\sqrt{m_\alpha}}\left\{\frac{1}{3}\mathrm{tr}\varOmega_\alpha^3\mathrm{tr}B(\varPhi'_\alpha\varOmega_\alpha\varPhi_\alpha)(\varPhi'_\beta\varOmega_\beta\varPhi_\beta)B'\right.\right.$$

$$+\mathrm{tr}B(\varPhi'_\alpha\varOmega_\alpha^2\varPhi_\alpha)(\varPhi'_\beta\varOmega_\beta\varPhi_\beta)B'+\varDelta_\alpha\mathrm{tr}\varOmega_\alpha\mathrm{tr}B(\varPhi'_\alpha\varOmega_\alpha\varPhi_\alpha)(\varPhi'_\beta\varOmega_\beta\varPhi_\beta)B'$$

$$\left.+\varDelta_\alpha\mathrm{tr}B(\varPhi'_\alpha\varPhi_\alpha)(\varPhi'_\beta\varOmega_\beta\varPhi_\beta)B'\right\}+\frac{\sqrt{2}}{\sqrt{m_\beta}}\left\{\frac{1}{3}\mathrm{tr}\varOmega_\beta^3\mathrm{tr}B(\varPhi'_\alpha\varOmega_\alpha\varPhi_\alpha)(\varPhi'_\beta\varOmega_\beta\varPhi_\beta)B'\right.$$

$$+\mathrm{tr}B(\varPhi'_\alpha\varOmega_\alpha\varPhi_\alpha)(\varPhi'_\beta\varOmega_\beta^2\varPhi_\beta)B'+\varDelta_\beta\mathrm{tr}\varOmega_\beta\mathrm{tr}B(\varPhi'_\alpha\varOmega_\alpha\varPhi_\alpha)(\varPhi'_\beta\varOmega_\beta\varPhi_\beta)B'$$

$$\left.\left.+\varDelta_\beta\mathrm{tr}B(\varPhi'_\alpha\varOmega_\alpha\varPhi_\alpha)(\varPhi'_\beta\varPhi_\beta)B'\right\}\right]+O(m_\varepsilon^{-1}),$$

where $\varDelta_\varepsilon=\frac{1}{2}(n_\varepsilon-m_\varepsilon)$, $m_\varepsilon=\rho n_\varepsilon$ $(\varepsilon=\alpha,\ \beta)$ *with* $\rho=1+o(1)$, *and* m_α, m_β *are assumed to be the same order for large* n_ε.

PROOF. Since S_α is distributed according to the Wishart distribution $W(\varSigma_\alpha,\ n_\alpha)$,

$$(7.15)\qquad E[\mathrm{etr}\varOmega_\alpha Y_\alpha]=c_{p,n_\alpha}\int\mathrm{etr}\varOmega_\alpha\frac{(\varSigma_\alpha^{-\frac{1}{2}}S_\alpha\varSigma_\alpha^{-\frac{1}{2}}-m_\alpha I)}{\sqrt{2m_\alpha}}\ |S_\alpha|^{\frac{1}{2}(n_\alpha-p-1}$$

$$\varSigma_\alpha|^{-\frac{n_\alpha}{2}}\mathrm{etr}\left[-\frac{1}{2}\varSigma_\alpha^{-1}S_\alpha\right]dS_\alpha.$$

Put $W=\varSigma_\alpha^{-\frac{1}{2}}S_\alpha\varSigma_\alpha^{-\frac{1}{2}}$, so the Jacobian is equal to $|\partial S_\alpha/\partial W|=|\varSigma_\alpha|^{-\frac{1}{2}(p+1)}$ Thus the expectation (7.15) is given by

$$(7.16)\qquad c_{p,n_\alpha}\mathrm{etr}\left[-\frac{\sqrt{m_\alpha}}{\sqrt{2}}\varOmega_\alpha\right]\int|W|^{\frac{1}{2}(n_\alpha-p-1)}\mathrm{etr}\left[-\frac{1}{2}\left(I-\frac{\sqrt{2}}{\sqrt{m_\alpha}}\varOmega_\alpha\right)W\right]dW$$

$$=\mathrm{etr}\left[-\frac{\sqrt{m_\alpha}}{\sqrt{2}}\varOmega_\alpha\right]\left|I-\frac{\sqrt{2}}{\sqrt{m_\alpha}}\varOmega_\alpha\right|^{-\frac{m_\alpha}{2}-\varDelta_\alpha},$$

where $\varDelta_\alpha=\frac{1}{2}(n_\alpha-m_\alpha)$. Since the following asymptotic formula

$$\text{(7.17)}\qquad -\log|I-Z/n|=\textstyle\sum_{r=1}^{k}n^{-r}\mathrm{tr}Z^r/r+O(n^{-k-1})$$

holds for any symmetric matrix Z, (7.12) follows from (7.16).

Next we shall prove (7.13). By the same argument as above, we may assume $\Sigma_\alpha=I$. Therefore we have

$$\text{(7.18)}\qquad E[\mathrm{tr}B(\varPhi'_\alpha Y_\alpha\varPhi_\alpha)^2B'\mathrm{etr}(\varOmega_\alpha Y_\alpha)]=c_{p,n_\alpha}\int\mathrm{tr}B\left(\varPhi'_\alpha\frac{S_\alpha-m_\alpha I}{\sqrt{2m_\alpha}}\varPhi_\alpha\right)^2B'$$

$$\cdot\mathrm{etr}\left(\varOmega_\alpha\frac{S_\alpha-m_\alpha I}{\sqrt{2m_\alpha}}\right)|S_\alpha|^{\frac{1}{2}(n_\alpha-p-1)}\mathrm{etr}\left(-\frac{1}{2}S_\alpha\right)dS_\alpha=\frac{1}{2m_\alpha}$$

$$\cdot\,\mathrm{etr}\left[-\frac{\sqrt{m_\alpha}}{\sqrt{2}}\varOmega_\alpha\right]|\varPsi_\alpha|^{\frac{n_\alpha}{2}}E[\mathrm{tr}B(\varPhi'_\alpha S_\alpha\varPhi_\alpha)^2B'-2m_\alpha\mathrm{tr}B(\varPhi'_\alpha S_\alpha\varPhi_\alpha)$$

$$\cdot(\varPhi'_\alpha\varPhi_\alpha)B'+m_\alpha^2\mathrm{tr}B(\varPhi'_\alpha\varPhi_\alpha)^2B'],$$

where $\varPsi_\alpha=(I-\sqrt{2}\varOmega_\alpha/\sqrt{m_\alpha})^{-1}$ and the expectation is taken with respect to the Wishart distribution $W(\varPsi_\alpha,\,n_\alpha)$. Therefore, by (7.1), (7.2) in Theorem 7.1, we have

$$\text{(7.19)}\qquad E[\mathrm{tr}B\,(\varPhi'_\alpha Y_\alpha\varPhi_\alpha)^2B'\mathrm{etr}\,(\varOmega_\alpha Y_\alpha)]=\frac{1}{2m_\alpha}\mathrm{etr}\left[-\frac{\sqrt{m_\alpha}}{\sqrt{2}}\varOmega_\alpha\right]|\varPsi_\alpha|^{\frac{n_\alpha}{2}}$$

$$\cdot\{n_\alpha\mathrm{tr}B(\varPhi'_\alpha\varPsi_\alpha\varPhi_\alpha)^2B'+n_\alpha\mathrm{tr}(\varPhi'_\alpha\varPsi_\alpha\varPhi_\alpha)\mathrm{tr}B(\varPhi'_\alpha\varPsi_\alpha\varPhi_\alpha)B'$$

$$+n_\alpha^2\mathrm{tr}B(\varPhi'_\alpha\varPsi_\alpha\varPhi_\alpha)^2B'-2m_\alpha n_\alpha\mathrm{tr}B(\varPhi'_\alpha\varPsi_\alpha\varPhi_\alpha)(\varPhi'_\alpha\varPhi_\alpha)B'$$

$$+m_\alpha^2\mathrm{tr}B(\varPhi'_\alpha\varPhi_\alpha)^2B'\}.$$

For any symmetric matrix Z, the following formula holds:

$$\text{(7.20)}\qquad \left(I-\frac{1}{n}Z\right)^{-1}=\textstyle\sum_{r=0}^{k}n^{-r}Z^r+O(n^{-k-1}).$$

Applying the formula (7.20) to the bracket in (7.19), we have

$$(7.21)\qquad \frac{1}{2}\mathrm{tr}B(\Phi_\alpha'\Phi_\alpha)^2B'+\frac{1}{2}\mathrm{tr}\Phi_\alpha'\Phi_\alpha\mathrm{tr}B(\Phi_\alpha'\Phi_\alpha)B'+\mathrm{tr}B(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)^2B$$

$$+\frac{\sqrt{2}}{\sqrt{m_\alpha}}\Big\{\frac{1}{2}\mathrm{tr}B(\Phi_\alpha'\Phi_\alpha)(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)B'+\frac{1}{2}\mathrm{tr}B(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)(\Phi_\alpha'\Phi_\alpha)B'$$

$$+\frac{1}{2}\mathrm{tr}\Phi_\alpha'\Phi_\alpha\mathrm{tr}B(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)B'+\frac{1}{2}\mathrm{tr}\Phi_\alpha'\Omega_\alpha\Phi_\alpha\mathrm{tr}B(\Phi_\alpha'\Phi_\alpha)B'$$

$$+\mathrm{tr}B(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)(\Phi_\alpha'\Omega_\alpha^2\Phi_\alpha)B'+\mathrm{tr}B(\Phi_\alpha'\Omega_\alpha^2\Phi_\alpha)(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)B'$$

$$+\varDelta_\alpha\mathrm{tr}B(\Phi_\alpha'\Phi_\alpha)(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)B'+\varDelta_\alpha\mathrm{tr}B(\Phi_\alpha'\Omega_\alpha\Phi_\alpha)(\Phi_\alpha'\Phi_\alpha)B'\Big\}+O(m_\alpha^{-1}).$$

Thus, multiplying (7.12), the formula (7.13) is shown. Finally we get

$$(722)\qquad E[\mathrm{tr}B(\Phi_\alpha'Y_\alpha\Phi_\alpha)(\Phi_\beta'Y_\beta\Phi_\beta)B'\mathrm{etr}(\Omega_\alpha Y_\alpha+\Omega_\beta Y_\beta)]=\mathrm{etr}\left[-\frac{\sqrt{m_\alpha}}{\sqrt{2}}\Omega_\alpha\right]$$

$$\cdot\mathrm{etr}\left[-\frac{\sqrt{m_\beta}}{\sqrt{2}}\Omega_\beta\right]|\Psi_\alpha|^{\frac{n_\alpha}{2}}|\Psi_\beta|^{\frac{n_\beta}{2}}\frac{1}{2\sqrt{m_\alpha m_\beta}}E[\mathrm{tr}B(\Phi_\alpha'S_\alpha\Phi_\alpha)(\Phi_\beta'S_\beta\Phi_\beta)B'$$

$$-m_\beta\mathrm{tr}B(\Phi_\alpha'S_\alpha\Phi_\alpha)(\Phi_\beta'\Phi_\beta)B'-m_\alpha\mathrm{tr}B(\Phi_\alpha'\Phi_\alpha)(\Phi_\beta'S_\beta\Phi_\beta)B'$$

$$+m_\alpha m_\beta\mathrm{tr}B(\Phi_\alpha'\Phi_\alpha)(\Phi_\beta'\Phi_\beta)B'],$$

where $\Psi_\alpha=\left(I-\frac{\sqrt{2}}{\sqrt{m_\alpha}}\Omega_\alpha\right)^{-1}$ and $\Psi_\beta=\left(I-\frac{\sqrt{2}}{\sqrt{m_\beta}}\Omega_\beta\right)^{-1}$ and the symbol E denotes the expectation with respect to the Wishart distributions $W(\Psi_\alpha, n_\alpha)$ and $W(\Psi_\beta, n_\beta)$. Applying (7.1) in Theorem 7.1 and Corollary 7.2 to the expectation (7.22), we have

$$(7.23)\quad \mathrm{etr}\left[-\frac{\sqrt{m_\alpha}}{\sqrt{2}}\Omega_\alpha\right]\mathrm{etr}\left[-\frac{\sqrt{m_\beta}}{\sqrt{2}}\Omega_\beta\right]|\Psi_\alpha|^{\frac{n_\alpha}{2}}|\Psi_\beta|^{\frac{n_\beta}{2}}\frac{1}{2\sqrt{m_\alpha m_\beta}}[n_\alpha n_\beta\mathrm{tr}B(\Phi_\alpha'\Psi_\alpha\Phi_\alpha)$$

$$\cdot(\Phi_\beta'\Psi_\beta\Phi_\beta)B'-n_\alpha m_\beta\mathrm{tr}B(\Phi_\alpha'\Psi_\alpha\Phi_\alpha)(\Phi_\beta'\Phi_\beta)B'-m_\alpha n_\beta\mathrm{tr}B(\Phi_\alpha'\Phi_\alpha)(\Phi_\beta'\Psi_\beta\Phi_\beta)B'$$

$$+m_\alpha m_\beta\mathrm{tr}B(\Phi_\alpha'\Phi_\alpha)(\Phi_\beta'\Phi_\beta)B'].$$

From (7.20), the bracket in (7.23) is expanded asymptotically as follows:

$$\begin{aligned}(7.24)\qquad & \mathrm{tr}B(\varPhi_\alpha'\varOmega_\alpha\varPhi_\alpha)(\varPhi_\beta'\varOmega_\beta\varPhi_\beta)B'+\frac{\sqrt{2}}{\sqrt{m_\alpha}}\mathrm{tr}B(\varPhi_\alpha'\varOmega_\alpha^2\varPhi_\alpha)(\varPhi_\beta'\varOmega_\beta\varPhi_\beta)B' \\ & +\frac{\sqrt{2}}{\sqrt{m_\beta}}\mathrm{tr}B(\varPhi_\alpha'\varOmega_\alpha\varPhi_\alpha)(\varPhi_\beta'\varOmega_\beta^2\varPhi_\beta)B'+\frac{\varDelta_\alpha\sqrt{2}}{\sqrt{m_\alpha}}\mathrm{tr}B(\varPhi_\alpha'\varPhi_\alpha)(\varPhi_\beta'\varOmega_\beta\varPhi_\beta)B' \\ & +\frac{\varDelta_\beta\sqrt{2}}{\sqrt{m_\beta}}\mathrm{tr}B(\varPhi_\alpha'\varOmega_\alpha\varPhi_\alpha)(\varPhi_\beta'\varPhi_\beta)B'+O(m_\varepsilon^{-1}).\end{aligned}$$

Making use of (7.12), we obtain the formula (7.14).

Now the limiting distribution of the statistic $Y=(y_{\alpha\beta})$ defined by (7.11) is a $\frac{1}{2}p(p+1)$ variate normal distribution with mean zero and covariance

$$(7.25)\qquad \left(\begin{array}{cc} \overbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}^{p} & \overbrace{\qquad 0 \qquad}^{\frac{p}{2}(p-1)} \\ 0 & \begin{matrix}\frac{1}{2} & & \\ & \ddots & \\ & & \frac{1}{2}\end{matrix}\end{array}\right).$$

More precisely, the limiting distribution of a variable $(y_{11},\cdots,y_{pp},y_{12},\cdots,y_{p-1,p})$ is given by

$$(7.26)\qquad c\cdot\mathrm{etr}\left[-\frac{1}{2}Y^2\right]dY,$$

where $c=(2\pi)^{-\frac{1}{4}p(p+1)}\left(\frac{1}{2}\right)^{-\frac{1}{4}p(p-1)}$and $dY=\prod_{i\leq j}dy_{ij}$. By regarding that the random variable Y is distributed according to (7.26), we have the following theorem:

THEOREM 7.4. *Let a random matrix Y be distributed according to* (7.26). *For symmetric matrix $\varOmega$ and matrices A, B and diagonal matrices $\varLambda$, K, T, $\varPsi$, the following formulae hold.*

$$(7.27)\qquad E[\mathrm{tr}AY\mathrm{etr}(\varOmega Y)]=\mathrm{etr}\left[\frac{1}{2}\varOmega^2\right]\cdot\mathrm{tr}A\varOmega,$$

$$(7.28)\qquad E[\mathrm{tr}AYBY\mathrm{etr}(\varOmega Y)]=\mathrm{etr}\left[\frac{1}{2}\varOmega^2\right]\left\{\mathrm{tr}\,A\varOmega B\varOmega+\frac{1}{2}\mathrm{tr}AB'+\frac{1}{2}\mathrm{tr}A\mathrm{tr}B\right\},$$

$$(7.29)\quad E[\mathrm{tr}AY\mathrm{tr}BY\mathrm{etr}(\Omega Y)]=\mathrm{etr}\left[\frac{1}{2}\Omega^2\right]\left\{\mathrm{tr}A\Omega\mathrm{tr}B\Omega+\frac{1}{2}\mathrm{tr}AB+\frac{1}{2}\mathrm{tr}AB'\right\},$$

$$(7.30)\quad E[\mathrm{tr}(\Lambda YKY)\mathrm{tr}AY\mathrm{etr}(\Omega Y)]=\mathrm{etr}\left[\frac{1}{2}\Omega^2\right]\Big\{\mathrm{tr}(\Lambda\Omega K\Omega)\,\mathrm{tr}A\Omega+\frac{1}{2}\mathrm{tr}\Lambda\mathrm{tr}K\mathrm{tr}A\Omega$$

$$+\frac{1}{2}\mathrm{tr}\Lambda K\mathrm{tr}A\Omega+\mathrm{tr}\Lambda AK\Omega+\mathrm{tr}\Lambda A'K\Omega\Big\},$$

$$(7.31)\quad E[\mathrm{tr}K(\Lambda Y\Lambda)^3K\mathrm{etr}(\Omega Y)]=\mathrm{etr}\left[\frac{1}{2}\Omega^2\right]\Big\{\mathrm{tr}K\,(\Lambda\Omega\Lambda)^3K+\frac{3}{2}\mathrm{tr}K(\Lambda^3\Omega\Lambda^3)K$$

$$+\frac{1}{2}\mathrm{tr}(K\Lambda)^2\mathrm{tr}\Lambda^2\Omega\Lambda^2+\mathrm{tr}\Lambda^2\mathrm{tr}K(\Lambda^2\Omega\Lambda^2)K\Big\},$$

$$(7.32)\quad E[\mathrm{tr}\Lambda YKY\mathrm{tr}\,TY\Psi Y\mathrm{etr}(\Omega Y)]=\mathrm{etr}\left[\frac{1}{2}\Omega^2\right]\Big\{\mathrm{tr}\Lambda\Omega K\Omega\mathrm{tr}\,T\Omega\Psi\Omega+\frac{1}{2}\mathrm{tr}\Lambda\mathrm{tr}K$$

$$\cdot\mathrm{tr}\,T\Omega\Psi\Omega+\frac{1}{2}\mathrm{tr}\,T\mathrm{tr}\Psi\mathrm{tr}\Lambda\Omega K\Omega+\frac{1}{2}\mathrm{tr}\,T\Psi\mathrm{tr}\Lambda\Omega K\Omega+\frac{1}{2}\mathrm{tr}\Lambda K\mathrm{tr}\,T\Omega\Psi\Omega+2\mathrm{tr}\Lambda\Psi\Omega KT\Omega$$

$$+2\mathrm{tr}\Lambda T\Omega K\Psi\Omega+\frac{1}{4}\mathrm{tr}\Lambda\mathrm{tr}K\mathrm{tr}\,T\mathrm{tr}\Psi+\mathrm{tr}\,\Lambda KT\Psi+\frac{1}{4}\mathrm{tr}\Lambda K\mathrm{tr}\,T\Psi$$

$$+\frac{1}{2}\mathrm{tr}\Lambda\Psi\mathrm{tr}KT+\frac{1}{2}\mathrm{tr}\Lambda T\mathrm{tr}K\Psi+\frac{1}{4}\mathrm{tr}\Lambda\mathrm{tr}\,K\mathrm{tr}\,T\Psi+\frac{1}{4}\mathrm{tr}\Lambda K\mathrm{tr}\,T\mathrm{tr}\Psi\Big\}.$$

PROOF. Since

$$(7.33)\quad c\cdot\mathrm{etr}(\Omega Y)\mathrm{etr}\left(-\frac{1}{2}Y^2\right)=c\cdot\mathrm{etr}\left[\frac{1}{2}\Omega^2\right]\mathrm{etr}\left[-\frac{1}{2}(Y-\Omega)^2\right],$$

we have

$$(7.34)\quad E[\mathrm{tr}AY\mathrm{etr}\Omega Y]=\mathrm{etr}\left[\frac{1}{2}\Omega^2\right]E[\mathrm{tr}AY],$$

where the expectation of the right-hand side is taken by a normal distribution with mean matrix $\Omega=(\omega_{ij})$ and covariance (7.25).
Thus we immediately get the formula (7.27).

In order to prove (7.28), we shall write $\mathrm{tr}AYBY$:

$$(7.35)\quad \mathrm{tr}AYBY=\sum_i\sum_j\sum_k\sum_l a_{ij}b_{kl}y_{jk}y_{li}=\sum_i\sum_j a_{ij}b_{ij}y_{ij}^2$$

$$+\sum_i\sum_j a_{ii}b_{jj}y_{ij}^2-\sum_i a_{ii}b_{ii}y_{ii}^2+\sum_{\substack{(j,k)\neq(l,i)\\(j,k)\neq(i,l)}}a_{ij}b_{kl}y_{ik}y_{li}.$$

Therefore, by taking the expectation with respect to a normal distribution with mean matrix Ω and covariance (7.25) and multiplying $\mathrm{etr}\left[\frac{1}{2}\Omega^2\right]$, we have

$$(7.36)\quad \mathrm{etr}\left[\frac{1}{2}\Omega^2\right]\left\{\mathrm{tr}A\Omega B\Omega+\sum_i\sum_j a_{ij}b_{ij}\frac{1+\delta_{ij}}{2}+\sum_i\sum_j a_{ii}b_{jj}\frac{1+\delta_{ij}}{2}\right.$$

$$\left.-\sum_i a_{ii}b_{ii}\right\}=\mathrm{etr}\left[\frac{1}{2}\Omega^2\right]\left\{\mathrm{tr}A\Omega B\Omega+\frac{1}{2}\mathrm{tr}AB'+\frac{1}{2}\mathrm{tr}A\mathrm{tr}B\right\}.$$

Next we consider the expectation $E[\mathrm{tr}AY\mathrm{tr}BY\mathrm{etr}(\Omega Y)]$. Then

$$(7.37)\quad \mathrm{tr}AY\mathrm{tr}BY=\sum_i\sum_j\sum_k\sum_l a_{ij}b_{kl}y_{ij}y_{kl}=\sum_i\sum_j a_{ij}b_{ij}y^2_{ij}$$

$$+\sum_i\sum_j a_{ij}b_{ji}y^2_{ij}-\sum_i a_{ii}b_{ii}y^2_{ii}+\sum_{\substack{(i,j)\neq(k,l)\\(i,j)\neq(l,k)}}a_{ij}b_{kl}y_{ij}y_{kl}.$$

By the same argument as above, we have

$$(7.38)\quad \mathrm{etr}\left[\frac{1}{2}\Omega^2\right]\left\{\mathrm{tr}A\Omega\mathrm{tr}B\Omega+\sum_i\sum_j a_{ij}b_{ij}\frac{1+\delta_{ij}}{2}+\sum_i\sum_j a_{ij}b_{ji}\frac{1+\delta_{ij}}{2}\right.$$

$$\left.-\sum_i a_{ii}b_{ii}\right\}=\mathrm{etr}\left[\frac{1}{2}\Omega^2\right]\left\{\mathrm{tr}A\Omega\mathrm{tr}B\Omega+\frac{1}{2}\mathrm{tr}AB'+\frac{1}{2}\mathrm{tr}AB\right\}.$$

Hence the formula (7.29) is proved. Similarly

$$(7.39)\quad \mathrm{tr}(\Lambda YKY)\mathrm{tr}AY=\sum_i\sum_j\lambda_ik_ja_{ij}y^3_{ij}+\sum_i\sum_j\lambda_i\kappa_ja_{ji}y^3_{ij}$$

$$-\sum_i\lambda_i\kappa_ia_{ii}y^3_{ii}+\sum_{\substack{(i,j)\neq(k,l)\\(i,j)\neq(l,k)}}\lambda_i\kappa_ja_{kl}y^2_{ij}y_{lk}.$$

Therefore we have

$$(7.40)\quad \mathrm{etr}\left[\frac{1}{2}\Omega^2\right]\left[\mathrm{tr}(\Lambda\Omega K\Omega)\mathrm{tr}A\Omega+\frac{1}{2}\sum_i\lambda_i\sum_i\kappa_i\sum_i\sum_j a_{ij}\omega_{ji}\right.$$

$$\left.+\frac{1}{2}\sum_i\lambda_i\kappa_i\sum_i\sum_j a_{ij}\omega_{ji}+\sum_i\sum_j\lambda_i\kappa_ja_{ij}\omega_{ij}+\sum_i\sum_j\lambda_i\kappa_ja_{ji}\omega_{ij}\right]$$

$$=\mathrm{etr}\left[\frac{1}{2}\Omega^2\right]\left[\mathrm{tr}(\Lambda\Omega K\Omega)\mathrm{tr}A\Omega+\frac{1}{2}\mathrm{tr}\Lambda\mathrm{tr}K\mathrm{tr}A\Omega+\frac{1}{2}\mathrm{tr}\Lambda K\mathrm{tr}A\Omega\right.$$

$$+\operatorname{tr}\Lambda AK\Omega+\operatorname{tr}\Lambda A'K\Omega\Big].$$

Moreover we require the following expectation $E[\operatorname{tr}K(\Lambda Y\Lambda)^3K\operatorname{etr}(\Omega Y)]$.

$$\text{(7.41)}\qquad \operatorname{tr}K(\Lambda Y\Lambda)^3K=\sum_i\kappa_i^2\lambda_i^6y_{ii}^3+2\sum_{i\neq j}\kappa_i^2\lambda_i^4\lambda_j^2y_{ii}y_{ij}^2+\sum_{i\neq j}\kappa_i^2\lambda_i^2\lambda_j^4y_{jj}y_{ij}^2$$

$$+\sum_{i\neq j\neq k}\kappa_i^2\lambda_i^2\lambda_j^2\lambda_k^2y_{ij}y_{jk}y_{ki}.$$

Using the formulae (2.b.12) with respect to the moment, we have

$$\text{(7.42)}\qquad \operatorname{etr}\left[\frac{1}{2}\Omega^2\right]\left[\operatorname{tr}K(\Lambda\Omega\Lambda)^3K+3\sum_i\kappa_i^2\lambda_i^6\omega_{ii}+\sum_{i\neq j}\kappa_i^2\lambda_i^4\lambda_j^2\omega_{ii}+\frac{1}{2}\sum_{i\neq j}\kappa_i^2\lambda_i^2\lambda_j^4\omega_{jj}\right]$$

$$=\operatorname{etr}\left[\frac{1}{2}\Omega^2\right]\left[\operatorname{tr}K(\Lambda\Omega\Lambda)^3K+\frac{3}{2}\operatorname{tr}K(\Lambda^3\Omega\Lambda^3)K+\frac{1}{2}\operatorname{tr}(K\Lambda)^2\operatorname{tr}\Lambda^2\Omega\Lambda^2\right.$$

$$\left.+\operatorname{tr}\Lambda^2\operatorname{tr}K(\Lambda^2\Omega\Lambda^2)K\right].$$

Finally we shall prove the formula (7.32).

$$\text{(7.43)}\qquad \operatorname{tr}\Lambda YKY\operatorname{tr}TY\Psi Y=\sum_i\sum_j\lambda_i\tau_i\kappa_j\psi_jy_{ij}^4+\sum_i\sum_j\lambda_i\psi_i\kappa_j\tau_jy_{ij}^4$$

$$-\sum_i\lambda_i\kappa_i\tau_i\psi_iy_{ii}^4+\sum_{\substack{(i,j)\neq(k,l)\\(i,j)\neq(l,k)}}\lambda_i\kappa_j\tau_k\psi_ly_{ij}^2y_{kl}^2.$$

By taking the expectation, we have

$$\text{(7.44)}\qquad \operatorname{etr}\left[\frac{1}{2}\Omega^2\right][\operatorname{tr}(\Lambda\Omega K\Omega)\operatorname{tr}(T\Omega\Psi\Omega)+\sum_i\sum_j\sum_k\sum_l\lambda_i\kappa_j\tau_k\psi_l\left\{\left(\frac{1+\delta_{kl}}{2}\right)\omega_{ij}^2\right.$$

$$\left.+\left(\frac{1+\delta_{ij}}{2}\right)\omega_{kl}^2+\left(\frac{1+\delta_{kl}}{2}\right)\left(\frac{1+\delta_{ij}}{2}\right)\right\}+2\sum_i\sum_j\lambda_i\tau_i\kappa_j\psi_j\omega_{ij}^2$$

$$+\frac{1}{2}\sum_i\sum_j\lambda_i\tau_i\kappa_j\psi_j+2\sum_i\sum_j\lambda_i\psi_i\kappa_j\tau_j\omega_{ij}^2+\frac{1}{2}\sum_i\sum_j\lambda_i\psi_i\tau_j\kappa_j$$

$$+\sum_i\lambda_i\tau_i\kappa_i\psi_i\Big].$$

Arranging this result, we have the desired formula (7.32).

8. Asymptotic expansion under fixed alternative. Let $X_{i1}, X_{i2}, \cdots, X_{iN_i}$ be a random sample from a p-variate normal distribution with mean vector μ_i and covariance matrix Σ_i $(i=1, 2, \cdots, k)$. For testing the hypothesis H: $\Sigma_1 = \cdots = \Sigma_k$ against all alternatives K: $\Sigma_i \neq \Sigma_j$ for some i and j $(i \neq j)$ with unspecified μ_i, the multivariate Bartlett's test is given by

$$M = m\log|\textstyle\sum_{\alpha=1}^{k} S_\alpha/m| - \sum_{\alpha=1}^{k} m_\alpha \log|S_\alpha/m_\alpha|, \tag{8.1}$$

where $S_\alpha = \sum_{\beta=1}^{N_\alpha}(X_{\alpha\beta}-\bar{X}_\alpha)(X_{\alpha\beta}-\bar{X}_\alpha)'$, $\bar{X}_\alpha = N_\alpha^{-1}\sum_{\beta=1}^{N_\alpha} X_{\alpha\beta}$, $n_\alpha = N_\alpha - 1$, $m_\alpha = \rho n_\alpha, \sum_{\alpha=1}^{k} m_\alpha = m$ and $\sum_{\alpha=1}^{k} n_\alpha = n$ with correction factor

$$\rho = 1 - \Big(\textstyle\sum_{\alpha=1}^{k}\frac{1}{n_\alpha} - \frac{1}{n}\Big)\frac{2p^2+3p-1}{6(p+1)(k-1)}. \tag{8.2}$$

Using the statistic Y defined in (7.11), we can express the statistic M as follows:

$$M = m\log|\breve{\Sigma}| - \textstyle\sum_{\alpha=1}^{k} m_\alpha \log|\Sigma_\alpha| + m^{\frac{1}{2}} q_0(Y) + q_1(Y) + m^{-\frac{1}{2}} q_2(Y) + O_p(m^{-1}), \tag{8.3}$$

where the coefficients $q_0(Y)$, $q_1(Y)$ and $q_2(Y)$ are given by

$$q_0(Y) = \textstyle\sum_{\alpha=1}^{k}\sqrt{2\rho_\alpha}\,\mathrm{tr}(\Phi_\alpha\Phi'_\alpha - I)Y_\alpha,$$

$$q_1(Y) = \textstyle\sum_{\alpha=1}^{k}\mathrm{tr}\,Y_\alpha^2 - \mathrm{tr}(\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi'_\alpha Y_\alpha \Phi_\alpha)^2, \tag{8.4}$$

$$q_2(Y) = \frac{2\sqrt{2}}{3}\Big\{\mathrm{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi'_\alpha Y_\alpha \Phi_\alpha)^3 - \sum_{\alpha=1}^{k}\frac{\mathrm{tr}\,Y_\alpha^3}{\sqrt{\rho_\alpha}}\Big\},$$

with $\rho_\alpha = m_\alpha/m$, $\breve{\Sigma} = \sum_{\alpha=1}^{k} \rho_\alpha \Sigma_\alpha$ and $\Phi_\alpha = \Sigma_\alpha^{\frac{1}{2}}\,\breve{\Sigma}^{-\frac{1}{2}}$. The matrices $\Sigma_\alpha^{\frac{1}{2}}$ and $\breve{\Sigma}^{\frac{1}{2}}$ mean the symmetric matrices such that $\Sigma_\alpha^{\frac{1}{2}}\,\Sigma_\alpha^{\frac{1}{2}} = \Sigma_\alpha$ and $\breve{\Sigma}^{\frac{1}{2}}\,\breve{\Sigma}^{\frac{1}{2}} = \breve{\Sigma}$.

Now putting $M' = M - m(\log|\breve{\Sigma}| - \sum_{\alpha=1}^{k}\rho_\alpha \log|\Sigma_\alpha|)$ in (8.3), we can easily see that $M'/\sqrt{m} - \sum_{\alpha=1}^{k}\sqrt{2\rho_\alpha}\,\mathrm{tr}(\Phi_\alpha\Phi'_\alpha - I)Y_\alpha = O_p(m^{-\frac{1}{2}})$. Hence the statistic

$M'/\sqrt{m}$ converges in law to the normal distribution with mean zero and variance $\tau_M^2=2\sum_{\alpha=1}^k\rho_\alpha$ tr $(\Phi_\alpha\Phi_\alpha'-I)^2$, which was shown by Sugiura [25]. Further the characteristic function of $M'/\sqrt{m}\tau_M$ $(\tau_M>0)$ can be expressed as follows:

$$C_M(t)=E\Big[\exp(itq_0(Y)/\tau_M)\Big\{1+m^{-\frac{1}{2}}itq_1(Y)/\tau_M+m^{-1}\Big[itq_2(Y)/\tau_M \tag{8.5}$$

$$+\frac{1}{2}(it)^2q_1^2(Y)/\tau_M^2\Big]\Big\}\Big]+O(m^{-\frac{3}{2}}).$$

Applying formula (7.12) to the first term in (8.5), with the abbreviated notation $B_\alpha=\sqrt{2\rho_\alpha}(\Phi_\alpha\Phi_\alpha'-I)it/\tau_M$ in $q_0(Y)$, we have

$$E[\exp(itq_0(Y)/\tau_M)]=\exp\Big[\frac{(it)^2}{2}\Big]\Big[1+\frac{\sqrt{2}}{3}m^{-\frac{1}{2}}\sum_{\alpha=1}^k\mathrm{tr}B_\alpha^3/\sqrt{\rho_\alpha} \tag{8.6}$$

$$+m^{-1}\Big\{\frac{1}{9}(\sum_{\alpha=1}^k\mathrm{tr}B_\alpha^3/\sqrt{\rho_\alpha})^2+\frac{1}{2}\sum_{\alpha=1}^k\mathrm{tr}B_\alpha^4/\rho_\alpha$$

$$+\varDelta\sum_{\alpha=1}^k\mathrm{tr}B_\alpha^2\Big\}\Big]+O(m^{-\frac{3}{2}}),$$

where $\varDelta=\frac{1}{2}(n-m)$, and the relation $\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,\mathrm{tr}B_\alpha=0$ was used.

Next we compute the expectation $E[\exp(itq_0(Y)/\tau_M)q_1(Y)]$. Using formula (7.13) in Theorem 7.3 by putting $B=\Phi_\alpha=I$, we get

$$E[\sum_{\alpha=1}^k\mathrm{tr}Y_\alpha^2\mathrm{etr}(\sum_{\alpha=1}^kB_\alpha Y_\alpha)]=\exp\Big[\frac{(it)^2}{2}\Big]\Big[\sum_{\alpha=1}^k\mathrm{tr}B_\alpha^2+\frac{k}{2}p(p+1) \tag{8.7}$$

$$+\sqrt{2}m^{-\frac{1}{2}}\Big\{\frac{1}{3}\sum_{\alpha=1}^k\mathrm{tr}B_\alpha^3/\sqrt{\rho_\alpha}\sum_{\alpha=1}^k\mathrm{tr}B_\alpha^2+\frac{1}{6}(kp^2+kp+12)$$

$$\sum_{\alpha=1}^k\mathrm{tr}B_\alpha^3/\sqrt{\rho_\alpha}+(p+1)\sum_{\alpha=1}^k\mathrm{tr}B_\alpha/\sqrt{\rho_\alpha}\Big\}\Big]+O(m^{-1}).$$

From (7.13) and (7.14), we obtain

$$E[\mathrm{tr}(\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\Phi_\alpha'Y_\alpha\Phi_\alpha)^2\mathrm{etr}(\sum_{\alpha=1}^kB_\alpha Y_\alpha)]=\exp\Big[\frac{(it)^2}{2}\Big] \tag{8.8}$$

$$\cdot\Big[\operatorname{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi_\alpha' B_\alpha\Phi_\alpha)^2+\dfrac{1}{2}\sum_{\alpha=1}^{k}\rho_\alpha\operatorname{tr}(\Phi_\alpha'\Phi_\alpha)^2+\dfrac{1}{2}\sum_{\alpha=1}^{k}\rho_\alpha(\operatorname{tr}\Phi_\alpha'\Phi_\alpha)^2$$

$$+\sqrt{2}\,m^{-\frac{1}{2}}\Big\{\frac{1}{3}\operatorname{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi_\alpha' B_\alpha\Phi_\alpha)^2\sum_{\alpha=1}^{k}\operatorname{tr}B_\alpha^3/\sqrt{\rho_\alpha}+\dfrac{1}{6}\sum_{\alpha=1}^{k}\operatorname{tr}B_\alpha^3/\sqrt{\rho_\alpha}$$

$$\cdot[\textstyle\sum_{\alpha=1}^{k}\rho_\alpha\operatorname{tr}(\Phi_\alpha'\Phi_\alpha)^2+\sum_{\alpha=1}^{k}\rho_\alpha(\operatorname{tr}\Phi_\alpha'\Phi_\alpha)^2]+2\operatorname{tr}(\sum_{\alpha=1}^{k}\Phi_\alpha' B_\alpha^2\Phi_\alpha$$

$$\cdot\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi_\alpha' B_\alpha\Phi_\alpha)+\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\operatorname{tr}(\Phi_\alpha'\Phi_\alpha\Phi_\alpha' B_\alpha\Phi_\alpha)+\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\operatorname{tr}(\Phi_\alpha'\Phi_\alpha)$$

$$\cdot\operatorname{tr}(\Phi_\alpha' B_\alpha\Phi_\alpha)+2\Delta\operatorname{tr}(\textstyle\sum_{\alpha=1}^{k}\rho_\alpha\Phi_\alpha'\Phi_\alpha\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi_\alpha' B_\alpha\Phi_\alpha)\Big\}\Big]+O(m^{-1}).$$

Thus we have

$$\text{(8.9)}\quad E[q_1(Y)\exp(itq_0(Y)/\tau_M)]=\exp\Big[\frac{(it)^2}{2}\Big]\Big[\textstyle\sum_{\alpha=1}^{k}\operatorname{tr}B_\alpha^2$$

$$-\operatorname{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi_\alpha' B_\alpha\Phi_\alpha)^2+\dfrac{k}{2}p(p+1)-\dfrac{1}{2}\sum_{\alpha=1}^{k}\rho_\alpha\operatorname{tr}(\Phi_\alpha'\Phi_\alpha)^2$$

$$-\frac{1}{2}\textstyle\sum_{\alpha=1}^{k}\rho_\alpha(\operatorname{tr}\Phi_\alpha'\Phi_\alpha)^2+\sqrt{2}\,m^{-\frac{1}{2}}\Big\{\dfrac{1}{3}\sum_{\alpha=1}^{k}\operatorname{tr}B_\alpha^3/\sqrt{\rho_\alpha}[\sum_{\alpha=1}^{k}\operatorname{tr}B_\alpha^2$$

$$-\operatorname{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi_\alpha' B_\alpha\Phi_\alpha)^2]+\dfrac{1}{6}\sum_{\alpha=1}^{k}\operatorname{tr}B_\alpha^3/\sqrt{\rho_\alpha}[kp(p+1)+12$$

$$-\textstyle\sum_{\alpha=1}^{k}\rho_\alpha\operatorname{tr}(\Phi_\alpha'\Phi_\alpha)^2-\sum_{\alpha=1}^{k}\rho_\alpha(\operatorname{tr}\Phi_\alpha'\Phi_\alpha)^2]-2\operatorname{tr}(\sum_{\alpha=1}^{k}\Phi_\alpha' B_\alpha^2\Phi_\alpha$$

$$\cdot\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi_\alpha' B_\alpha\Phi_\alpha)+(p+1)\sum_{\alpha=1}^{k}\operatorname{tr}B_\alpha/\sqrt{\rho_\alpha}-\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\operatorname{tr}(\Phi_\alpha'\Phi_\alpha\Phi_\alpha' B_\alpha\Phi_\alpha)$$

$$-\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\operatorname{tr}(\Phi_\alpha'\Phi_\alpha)\operatorname{tr}(\Phi_\alpha' B_\alpha\Phi_\alpha)-2\Delta\operatorname{tr}(\sum_{\alpha=1}^{k}\rho_\alpha\Phi_\alpha'\Phi_\alpha\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi_\alpha' B_\alpha\Phi_\alpha)\Big\}\Big]$$

$$+O(m^{-1}).$$

As the third term in (8.5) is of order m^{-1}, we can regard the variable Y_α as the random matrix having a normal distribution with mean O and covariance (7.25). Therefore making use of (7.27), (7.28) and (7.31) after some modification, we have

$$\text{(8.10)}\quad E[\operatorname{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi_\alpha' Y_\alpha\Phi_\alpha)^3\operatorname{etr}(\sum_{\alpha=1}^{k}B_\alpha Y_\alpha)]=\exp\Big[\dfrac{(it)^2}{2}\Big]$$

$$\Big\{\mathrm{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi'_\alpha B_\alpha\Phi_\alpha)^3+\dfrac{3}{2}\textstyle\sum_{\alpha=1}^{k}\rho_\alpha\mathrm{tr}(\Phi'_\alpha\Phi_\alpha)\mathrm{tr}\,(\Phi'_\alpha\Phi_\alpha\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi'_\alpha B_\alpha\Phi_\alpha)$$

$$+\frac{3}{2}\mathrm{tr}\textstyle\sum_{\alpha=1}^{k}\rho_\alpha(\Phi'_\alpha\Phi_\alpha)^2\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi'_\alpha B_\alpha\Phi_\alpha\Big\}+O(m^{-\frac{1}{2}}).$$

Similarly, from formula (7.31) we get

$$(8.11)\qquad E[\textstyle\sum_{\alpha=1}^{k}\dfrac{\mathrm{tr}\,Y_\alpha^3}{\sqrt{\rho_\alpha}}\mathrm{etr}(\textstyle\sum_{\alpha=1}^{k}B_\alpha Y_\alpha)]=\exp\Big[\dfrac{(it)^2}{2}\Big]\Big\{\textstyle\sum_{\alpha=1}^{k}\mathrm{tr}B_\alpha^3/\sqrt{\rho_\alpha}$$

$$+\frac{3}{2}(p+1)\textstyle\sum_{\alpha=1}^{k}\mathrm{tr}B_\alpha/\sqrt{\rho_\alpha}\Big\}+O(m^{-\frac{1}{2}}).$$

Therefore we easily obtain

$$(8.12)\quad E[q_2(Y)\exp(itq_0(Y)/\tau_M)]=\exp\Big[\dfrac{(it)^2}{2}\Big]\dfrac{2\sqrt{2}}{3}\Big\{\mathrm{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi'_\alpha B_\alpha\Phi_\alpha)^3$$

$$-\textstyle\sum_{\alpha=1}^{k}\mathrm{tr}B_\alpha^3/\sqrt{\rho_\alpha}+\dfrac{3}{2}\textstyle\sum_{\alpha=1}^{k}\rho_\alpha\mathrm{tr}(\Phi'_\alpha\Phi_\alpha)\mathrm{tr}(\Phi'_\alpha\Phi_\alpha\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi'_\alpha B_\alpha\Phi_\alpha)$$

$$+\frac{3}{2}\mathrm{tr}\textstyle\sum_{\alpha=1}^{k}\rho_\alpha(\Phi'_\alpha\Phi_\alpha)^2\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi'_\alpha B_\alpha\Phi_\alpha-\dfrac{3}{2}(p+1)\textstyle\sum_{\alpha=1}^{k}\mathrm{tr}B_\alpha/\sqrt{\rho_\alpha}\Big\}$$

$$+O(m^{-\frac{1}{2}}).$$

Finally we shall compute the expectation $E[q_1^2(Y)\exp(itq_0(Y)/\tau_M)]$.
By formulae (7.28) and (7.32), we have

$$(8.13)\quad E[(\textstyle\sum_{\alpha=1}^{k}\mathrm{tr}\,Y_\alpha^2)^2\mathrm{etr}(\textstyle\sum_{\alpha=1}^{k}B_\alpha Y_\alpha)]=\exp\Big[\dfrac{(it)^2}{2}\Big]\Big\{(\mathrm{tr}\textstyle\sum_{\alpha=1}^{k}B_\alpha^2)^2$$

$$+(kp^2+kp+4)\textstyle\sum_{\alpha=1}^{k}\mathrm{tr}B_\alpha^2+\dfrac{k}{4}p(p+1)\ (kp^2+kp+4)\Big\}+O(m^{-\frac{1}{2}}).$$

By formulae in Theorem 7.4, it follows that

$$(8.14)\qquad E[\textstyle\sum_{\alpha=1}^{k}\mathrm{tr}\,Y_\alpha^2\mathrm{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\Phi'_\alpha Y_\alpha\Phi_\alpha)^2\mathrm{etr}(\textstyle\sum_{\alpha=1}^{k}B_\alpha Y_\alpha)]$$

$$=\exp\Big[\frac{(it)^2}{2}\Big]\Big\{\textstyle\sum_{\alpha=1}^{k}\mathrm{tr}B_\alpha^2\mathrm{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\Phi'_\alpha B_\alpha\Phi_\alpha)^2+\dfrac{1}{2}\textstyle\sum_{\alpha=1}^{k}\mathrm{tr}B_\alpha^2$$

$$\cdot[\textstyle\sum_{\alpha=1}^{k}\rho_\alpha\mathrm{tr}(\Phi'_\alpha\Phi_\alpha)^2+\textstyle\sum_{\alpha=1}^{k}\rho_\alpha(\mathrm{tr}\Phi'_\alpha\Phi_\alpha)^2]+\dfrac{1}{2}(kp^2+kp+8)$$

$$\mathrm{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_{\alpha}}\,\Phi_{\alpha}'B_{\alpha}\Phi_{\alpha})^{2}+\dfrac{1}{4}(kp^{2}+kp+4)\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}(\Phi_{\alpha}'\Phi_{\alpha})^{2}$$

$$+\frac{1}{4}(kp^{2}+kp+4)\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}(\mathrm{tr}\Phi_{\alpha}'\Phi_{\alpha})^{2}\Big\}+O(m^{-\frac{1}{2}}).$$

Similarly we have

$$(8.15)\qquad E[\{\mathrm{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_{\alpha}}\,\Phi_{\alpha}'Y_{\alpha}\Phi_{\alpha})^{2}\}^{2}\mathrm{etr}(\textstyle\sum_{\alpha=1}^{k}B_{\alpha}Y_{\alpha})]$$

$$=\exp\left[\frac{(it)^{2}}{2}\right]\Big[\{\mathrm{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_{\alpha}}\,\Phi_{\alpha}'B_{\alpha}\Phi_{\alpha})^{2}\}^{2}+\mathrm{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_{\alpha}}\,\Phi_{\alpha}'B_{\alpha}\Phi_{\alpha})^{2}$$

$$\{\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}(\Phi_{\alpha}'\Phi_{\alpha})^{2}+\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}(\mathrm{tr}\Phi_{\alpha}'\Phi_{\alpha})^{2}\}+4\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}(\Phi_{\alpha}'\Phi_{\alpha}$$

$$\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_{\alpha}}\,\Phi_{\alpha}'B_{\alpha}\Phi_{\alpha})^{2}+\dfrac{1}{4}\{\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}(\Phi_{\alpha}'\Phi_{\alpha})^{2}+\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}(\mathrm{tr}\Phi_{\alpha}'\Phi_{\alpha})^{2}\Big\}^{2}$$

$$+\textstyle\sum_{\alpha,\beta=1}^{k}\rho_{\alpha}\rho_{\beta}(\mathrm{tr}\Phi_{\alpha}'\Phi_{\alpha}\Phi_{\beta}'\Phi_{\beta})^{2}+\textstyle\sum_{\alpha,\beta=1}^{k}\rho_{\alpha}\rho_{\beta}\mathrm{tr}(\Phi_{\alpha}'\Phi_{\alpha}\Phi_{\beta}'\Phi_{\beta})^{2}\Big]$$

$$+O(m^{-\frac{1}{2}}).$$

Hence we get

$$(8.16)\qquad E[q_{1}^{2}(Y)\exp(itq_{0}(Y)/\tau_{M})]=\exp\left[\frac{(it)^{2}}{2}\right]\Big[\{\textstyle\sum_{\alpha=1}^{k}\mathrm{tr}B_{\alpha}^{2}-\mathrm{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_{\alpha}}\,\Phi_{\alpha}'$$

$$\cdot B_{\alpha}\Phi_{\alpha})^{2}\}^{2}+\{(kp^{2}+kp+4)-\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}(\Phi_{\alpha}'\Phi_{\alpha})^{2}-\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}(\mathrm{tr}\Phi_{\alpha}'\Phi_{\alpha})^{2}\}$$

$$\cdot\textstyle\sum_{\alpha=1}^{k}\mathrm{tr}B_{\alpha}^{2}+\{\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}(\Phi_{\alpha}'\Phi_{\alpha})^{2}+\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}\ (\mathrm{tr}\Phi_{\alpha}'\Phi_{\alpha})^{2}$$

$$-(kp^{2}+kp+8)\}\mathrm{tr}(\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_{\alpha}}\,\Phi_{\alpha}'B_{\alpha}\Phi_{\alpha})^{2}+4\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}(\Phi_{\alpha}'\Phi_{\alpha}\textstyle\sum_{\alpha=1}^{k}\sqrt{\rho_{\alpha}}\,\Phi_{\alpha}'B_{\alpha}$$

$$\cdot\Phi_{\alpha})^{2}+\frac{1}{4}\{\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}(\Phi_{\alpha}'\Phi_{\alpha})^{2}+\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}(\mathrm{tr}\Phi_{\alpha}'\Phi_{\alpha})^{2}\}^{2}$$

$$+\textstyle\sum_{\alpha,\beta=1}^{k}\rho_{\alpha}\rho_{\beta}(\mathrm{tr}\Phi_{\alpha}'\Phi_{\alpha}\Phi_{\beta}'\Phi_{\beta})^{2}+\textstyle\sum_{\alpha,\beta=1}^{k}\rho_{\alpha}\rho_{\beta}\mathrm{tr}(\Phi_{\alpha}'\Phi_{\alpha}\Phi_{\beta}'\Phi_{\beta})^{2}$$

$$+\frac{1}{4}(kp^{2}+kp+4)\{kp(p+1)-2\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}(\Phi_{\alpha}'\Phi_{\alpha})^{2}-2\textstyle\sum_{\alpha=1}^{k}\rho_{\alpha}$$

$$(\mathrm{tr}\Phi_{\alpha}'\Phi_{\alpha})^{2}\}\ \Big|+O(m^{-\frac{1}{2}}).$$

Therefore the characteristic function of $M'/\sqrt{m}\tau_{M}$ is given by

$$C_{M}(t)=\exp\left[\frac{(it)^{2}}{2}\right]\left[1+m^{-\frac{1}{2}}\left\{\frac{it}{\tau_{M}}\left[\frac{k}{2}p(p+1)-\frac{1}{2}\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}A_{\alpha}^{2}\right.\right.\right.$$
$$\left.-\frac{1}{2}\sum_{\alpha=1}^{k}\rho_{\alpha}(\mathrm{tr}A_{\alpha})^{2}\right]+\frac{(it)^{3}}{\tau_{M}^{3}}\left[\frac{4}{3}\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}(A_{\alpha}-I)^{3}+\tau_{M}^{2}\right.$$
$$\left.\left.-2\mathrm{tr}\{\sum_{\alpha=1}^{k}\rho_{\alpha}(A_{\alpha}-I)^{2}\}^{2}\right]\right\}+m^{-1}\sum_{\alpha=1}^{3}h_{2\alpha}\left(\frac{it}{\tau_{M}}\right)^{2\alpha}\Bigg]+O(m^{-\frac{3}{2}}), \tag{8.17}$$

where $\tau_{M}^{2}=2\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}(A_{\alpha}-I)^{2}$ with $A_{\alpha}=\Sigma_{\alpha}\hat{\Sigma}^{-1}$ and the coefficients h_2, h_4 and h_6 are given by

$$h_{2}=\Delta\tau_{M}^{2}-2\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}A_{\alpha}\mathrm{tr}A_{\alpha}(A_{\alpha}-I)-2\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}A_{\alpha}^{2}(A_{\alpha}-I)$$
$$-4\Delta\mathrm{tr}\sum_{\alpha=1}^{k}\rho_{\alpha}(A_{\alpha}-I)^{2}+2\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}A_{\alpha}\mathrm{tr}A_{\alpha}\sum_{\alpha=1}^{k}\rho_{\alpha}(A_{\alpha}-I)^{2}$$
$$+2\mathrm{tr}\sum_{\alpha=1}^{k}\rho_{\alpha}A_{\alpha}^{2}\sum_{\alpha=1}^{k}\rho_{\alpha}(A_{\alpha}-I)^{2}+\frac{1}{8}\Big\{\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}A_{\alpha}^{2}$$
$$+\sum_{\alpha=1}^{k}\rho_{\alpha}(\mathrm{tr}A_{\alpha})^{2}\}^{2}+\frac{1}{2}\sum_{\alpha,\beta=1}^{k}\rho_{\alpha}\rho_{\beta}(\mathrm{tr}A_{\alpha}A_{\beta})^{2}+\frac{1}{2}\sum_{\alpha,\beta=1}^{k}\rho_{\alpha}\rho_{\beta}$$
$$\mathrm{tr}(A_{\alpha}A_{\beta})^{2}+\frac{1}{8}(kp^{2}+kp+4)\{kp(p+1)-2\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}A_{\alpha}^{2}$$
$$2\sum_{\alpha=1}^{k}\rho_{\alpha}(\mathrm{tr}A_{\alpha})^{2}\}, \tag{8.18}$$

$$h_{4}=2\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}(A_{\alpha}-I)^{4}+\frac{2}{3}\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}(A_{\alpha}-I)^{3}[kp(p+1)+8$$
$$-\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}A_{\alpha}^{2}-\sum_{\alpha=1}^{k}\rho_{\alpha}(\mathrm{tr}A_{\alpha})^{2}]-8\mathrm{tr}\sum_{\alpha=1}^{k}\rho_{\alpha}(A_{\alpha}-I)^{3}$$
$$\sum_{\alpha=1}^{k}\rho_{\alpha}(A_{\alpha}-I)^{2}+\frac{8}{3}\mathrm{tr}(\sum_{\alpha=1}^{k}\rho_{\alpha}(A_{\alpha}-I)^{2})^{3}+\frac{\tau_{M}^{2}}{2}\{(kp^{2}+kp+4)$$
$$-\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}A_{\alpha}^{2}-\sum_{\alpha=1}^{k}\rho_{\alpha}(\mathrm{tr}A_{\alpha})^{2}\}+\{\sum_{\alpha=1}^{k}\rho_{\alpha}\mathrm{tr}A_{\alpha}^{2}$$

$$+\sum_{\alpha=1}^{k}\rho_\alpha(\mathrm{tr}A_\alpha)^2-(kp^2+kp+16)\}\,\mathrm{tr}\{\sum_{\alpha=1}^{k}\rho_\alpha(A_\alpha-I)^2\}^2$$

$$+4\sum_{\alpha=1}^{k}\rho_\alpha\mathrm{tr}\{A_\alpha\sum_{\alpha=1}^{k}\rho_\alpha(A_\alpha-I)^2\}^2,$$

$$h_6=\frac{8}{9}\{\sum_{\alpha=1}^{k}\rho_\alpha\mathrm{tr}(A_\alpha-I)^3\}^2+\frac{8}{3}\sum_{\alpha=1}^{k}\rho_\alpha\mathrm{tr}(A_\alpha-I)^3\Big[\frac{\tau_M^2}{2}-\mathrm{tr}\,\{\sum_{\alpha=1}^{k}\rho_\alpha$$

$$(A_\alpha-I)^2\}^2\Big]+\frac{1}{2}[\tau_M^2-2\mathrm{tr}\{\sum_{\alpha=1}^{k}\rho_\alpha(A_\alpha-I)^2\}^2]^2.$$

Inverting this characteristic function, we have the following theorem:

THEOREM 8.1. *Underithe fixed alternative* $K:\Sigma_i\neq\Sigma_j$ *for some* i, $j(i\neq j)$, *the distribution of* $M'=M-m(\log|\tilde{\Sigma}|-\sum_{\alpha=1}^{k}\rho_\alpha\log|\Sigma_\alpha|)$, *where* M *is given by* (8.1) *with* $\tilde{\Sigma}=\sum_{\alpha=1}^{k}\rho_\alpha\Sigma_\alpha$, *can be expanded asymptotically for large* $m(=\rho n)$ *as* (8.19).

$$P(M'/\sqrt{m}\tau_M\leq x)=\Phi(x)-m^{-\frac{1}{2}}\Big\{\Phi^{(1)}(x)\Big[\frac{\mathrm{k}}{2}p(p+1)-\frac{1}{2}\sum_{\alpha=1}^{k}\rho_\alpha\mathrm{tr}A_\alpha^2 \tag{8.19}$$

$$-\frac{1}{2}\sum_{\alpha=1}^{k}\rho_\alpha(\mathrm{tr}A_\alpha)^2\Big]/\tau_M+\Phi^{(3)}(x)\Big[\frac{4}{3}\sum_{\alpha=1}^{k}\rho_\alpha\mathrm{tr}(A_\alpha-I)^3$$

$$+\tau_M^2-2\mathrm{tr}\{\sum_{\alpha=1}^{k}\rho_\alpha(A_\alpha-I)^2\Big\}^2\Big]/\tau_M^3\Big\}+m^{-1}\sum_{\alpha=1}^{3}h_{2\alpha}$$

$$\cdot\Phi^{(2\alpha)}(x)/\tau_M^{2\alpha}+O(m^{-\frac{3}{2}}),$$

where $\tau_M^2=2\sum_{\alpha=1}^{k}\rho_\alpha\mathrm{tr}(A_\alpha-I)^2$ *with* $A_\alpha=\Sigma_\alpha\tilde{\Sigma}^{-1}$ *and* $\Phi^{(j)}(x)$ *means the j-th derivative of the standard normal distribution function* $\Phi(x)$. *The coefficients* $h_{2\alpha}$ *are given by* (8.18).

This theorem was obtained in case $p=1$ by Sugiura and Nagao [27].

Now we shall notice that the formula (8.19) is degenerate under the hypothesis H. In two sample problem Sugiura [24] gave the asymptotic expansion under the sequence of alternatives $\Sigma_\alpha=\Sigma+m^{-1}\Theta_\alpha$ $(\alpha=1,\ 2)$, first term of which is chi-square distribution with $\frac{1}{2}p(p+1)$ degrees of freedom.

9. Numerical examples. Under the hypothesis, we can get

$$P(M\leq x)=P(\chi_f^2\leq x)+\frac{\omega_2}{m^2}\{P(\chi_{f+4}^2\leq x)-P(\chi_f^2\leq x)\} \tag{9.1}$$

$$+\frac{\omega_3}{m^3}\{P(\chi^2_{f+6}\leq x)-P(\chi^2_f\leq x)\}+\frac{1}{m^4}[\omega_2\{P(\chi^2_{f+8}\leq x)$$

$$-P(\chi^2_f\leq x)\}-\omega_2^2\{P(\chi^2_{f+4}\leq x)-P(\chi^2_f\leq x)\}]+O(m^{-5}),$$

where $f=\frac{1}{2}(k-1)p(p+1)$. The coefficients ω_2, ω_3 and ω_4 are given by

$$(9.2)\quad \omega_2=\frac{1}{48}p(p-1)(p+1)(p+2)(\tilde{\rho}_2-1)-\frac{\Delta^2}{2}(k-1)p(p+1),$$

$$\omega_3=\frac{p}{720}(6p^4+15p^3-10p^2-30p+3)(\tilde{\rho}_3-1)-\frac{\Delta}{12}p(p-1)(p+1)(p+2)(\tilde{\rho}_2-1)$$

$$+\frac{4}{3}\Delta^3(k-1)p(p+1),$$

$$\omega_4=\frac{\omega_2^2}{2}+\frac{p}{480}(p-1)(2p^4+8p^3+3p^2-17p-14)(\tilde{\rho}_4-1)-\frac{\Delta}{120}p(6p^4$$

$$+15p^3-10p^2-30p+3)(\tilde{\rho}_3-1)+\frac{\Delta^2}{4}p(p-1)(p+1)(p+2)(\tilde{\rho}_2-1)$$

$$-3\Delta^4(k-1)p(p+1),$$

where $\Delta=\frac{1}{2}(n-m)$ and $\tilde{\rho}_\beta=\sum_{\alpha=1}^{k}\rho_\alpha^{-\beta}$. The formula (9.1) is given by Anderson [1] up to m^{-2}. Since the asymptotic null-distribution of M has the same form as in the sphericity test given by (6.1), the asymptotic formula (6.3) for the percentage point is available for our present purpose.

EXAMPLE 9.1. When case 1: $p=2$, $n_1=50$, $n_2=100$ and case 2: $p=4$, n_1 50, $n_2=100$, we have the following approximations to the 5% point.

	case 1	case 2
first term	7.8147	18.3070
term of order m^{-2}	0.0002	0.0061
term of order m^{-3}	0.0000	0.0000
term of order m^{-4}	0.0000	0.0000
approx. value	7.8149	18.3131

Evaluating the approximate power in two sample case, we may assume $\Sigma_1=\mathrm{diag}(\delta_1, \delta_2, \cdots, \delta_p)$ and $\Sigma_2=I$.

EXAMPLE 9.2. Using the 5% point of case 1 obtained in Example 9.1, we can get the approximate values of the power of the multivariate Bartlett's test from the formula (8.19).

$P_K(M\geq 7.8149)$

K term	$\delta_1=\delta_2=1.7$	$\delta_1=1.5$ $\delta_2=0.7$	$\delta_1=\delta_2=2.0$
first	0.6134	0.2462	0.8497
second	0.1071	0.1749	0.0734
third	0.0036	−0.0014	0.0068
approx. power	0.724	0.420	0.930

EXAMPLE 9.3. When $k=2$, $n_1=50$, $n_2=100$ and $p=4$, our asymptotic formula (8.19) gives the following approximate power of the multivariate Bartlett's test for the alternative K: $\delta_1=\delta_2=\delta_3=\delta_4=1.7$, based on the 5% point of case 2 obtained in Example 9.1.

$P_K(M\geq 18.3131)$

first term	second term	third term	approx. power
0.5321	0.3594	−0.0080	0.884

PART IV. SOME TEST CRITERIA FOR EQUALITY OF TWO COVARIANCE MATRICES

10. Preliminaries. Let $p\times 1$ vectors $X_{i1}, X_{i2},\cdots, X_{iN_i}$ be a random sample from p-variate normal distribution with mean vector μ_i and covariance matrix Σ_i $(i=1, 2)$. We wish to test the hypothesis H: $\Sigma_1=\Sigma_2$ against the alternatives K: $\gamma_i\geq 1$ and $\sum_{i=1}^{p}\gamma_i>p$, where γ_i means the characteristic root of $\Sigma_1\Sigma_2^{-1}(i=1, \cdots, p)$. For this problem, many test criteria are proposed, that is, $|S_2(S_1+S_2)^{-1}|^{-1}$, $\mathrm{tr}S_1S_2^{-1}$ and $\mathrm{tr}S_1(S_1+S_2)^{-1}$, where $S_i=\sum_{\alpha=1}^{N_i}(X_{i\alpha}-\bar{X}_i)\cdot(X_{i\alpha}-\bar{X}_i)'$ with $\bar{X}_i=N_i^{-1}\sum_{\alpha=1}^{N_i}X_{i\alpha}$. The hypothesis H is rejected when the observed value of these test statistics is larger than a preassigned constant. The monotonicity of the power functions of these tests was proved by Anderson and Das Gupta [2] from a more general point of view. Giri [6] also showed the test criterion $\mathrm{tr} S_1(S_1+S_2)^{-1}$ is locally best invariant. The purpose of this part is to compare the above three test criteria from the view point of asymptotic expansions of the distributions.

11. Asymptotic expansion of $|S_2(S_1+S_2)^{-1}|$. We may assume without loss of generality that S_1 has the Wishart distribution $W(\Gamma, n_1)$ and S_2 has the Wishart distribution $W(I, n_2)$ under the alternative K, where $\Gamma=\mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_p)$ and $n_\alpha=N_\alpha-1$ $(\alpha=1, 2)$. Then the hypothesis H can be expressed by $\Gamma=I$. Put $Y_1=(\Gamma^{-\frac{1}{2}} S_1 \Gamma^{-\frac{1}{2}}-n_1 I)/\sqrt{2n_1}$ and $Y_2=(S_2-n_2 I)/\sqrt{2n_2}$, then $-\sqrt{n}$ $\log|S_2(S_1+S_2)^{-1}|$ can be expressed with the terms Y_1 and Y_2 as

$$(11.1) \qquad -\sqrt{n}\log|S_2(S_1+S_2)^{-1}|=-\sqrt{n}\log|\rho_2\tilde{\Gamma}^{-1}|+q_0(Y)+n^{-\frac{1}{2}}q_1(Y)$$
$$+n^{-1}q_2(Y)+O_p(n^{-\frac{3}{2}}),$$

where $\tilde{\Gamma}=\rho_1\Gamma+\rho_2 I$ and $n=n_1+n_2$ with the fixed $\rho_1=n_1/n$ and $\rho_2=n_2/n$. The coefficients $q_0(Y)$, $q_1(Y)$ and $q_2(Y)$ are given by

$$q_0(Y)=\sqrt{2}\{\sqrt{\rho_1}\,\mathrm{tr}\Phi_1^2 Y_1+\sqrt{\rho_2}\,\mathrm{tr}(\Phi_2^2-\rho_2^{-1}I)Y_2\},$$

$$(11.2) \qquad q_1(Y)=\rho_2^{-1}\mathrm{tr}\,Y_2^2-\mathrm{tr}(\textstyle\sum_{\alpha=1}^2\sqrt{\rho_\alpha}\,\Phi_\alpha Y_\alpha \Phi_\alpha)^2,$$

$$q_2(Y)=\frac{2\sqrt{2}}{3}\{\mathrm{tr}(\textstyle\sum_{\alpha=1}^2\sqrt{\rho_\alpha}\,\Phi_\alpha Y_\alpha \Phi_\alpha)^3-\rho_2^{-\frac{3}{2}}\mathrm{tr}\,Y_2^3\},$$

where $\Phi_1=\Gamma^{\frac{1}{2}}\tilde{\Gamma}^{-\frac{1}{2}}$ and $\Phi_2=\tilde{\Gamma}^{-\frac{1}{2}}$

Now putting $\lambda_1=-\sqrt{n}\{\log|S_2(S_1+S_2)^{-1}|-\log|\rho_2\tilde{\Gamma}^{-1}|\}$ in (11.1), we can see that the statistic λ_1 converges in law to the normal distribution with mean zero and variance $\tau_1^2=2\{\rho_1\mathrm{tr}\Phi_1^4+\rho_2\mathrm{tr}(\Phi_2^2-\rho_2^{-1}I)^2\}$, which was shown by Sugiura [25] using Siotani and Hayakawa's lemma [22]. We shall now give the asymptotic expansion. The characteristic function of λ_1/τ_1 $(\tau_1>0)$ can be expressed as

$$(11.3)\quad C_1(t)=E\Big[\exp(itq_0(Y)/\tau_1)\Big\{1+n^{-\frac{1}{2}}q_1(Y)\frac{it}{\tau_1}+n^{-1}\Big[q_2(Y)\frac{it}{\tau_1}$$

$$+\frac{q_1^2(Y)}{2}\Big(\frac{it}{\tau_1}\Big)^2\Big]\Big\}\Big]+O(n^{-\frac{3}{2}}).$$

Put $B_1=\sqrt{2\rho_1}\Phi_1^2 it/\tau_1$ and $B_2=\sqrt{2\rho_2}(\Phi_2^2-\rho_2^{-1}I)it/\tau_1$, then we have from (8.6)

$$(11.4)\quad E[\exp(itq_0(Y)/\tau_1)]=\exp\Big[\frac{(it)^2}{2}\Big]\Big[1+\frac{\sqrt{2}}{3}n^{-\frac{1}{2}}\textstyle\sum_{\alpha=1}^{2}\mathrm{tr}B_\alpha^3/\sqrt{\rho_\alpha}$$

$$+n^{-1}\Big\{\frac{1}{9}(\textstyle\sum_{\alpha=1}^{2}\mathrm{tr}\,B_\alpha^3/\sqrt{\rho_\alpha})^2+\frac{1}{2}\sum_{\alpha=1}^{2}\mathrm{tr}B_\alpha^4/\sqrt{\rho_\alpha}\Big\}\Big]+O(n^{-\frac{3}{2}}).$$

By putting $m_\alpha=n_\alpha$ in Theorem 7.3 we get

$$(11.5)\quad E[q_1(Y)\exp(itq_0(Y)/\tau_1)]=\exp\Big[\frac{(it)^2}{2}\Big]\Big[\rho_2^{-1}\mathrm{tr}B_2^2-\mathrm{tr}(\textstyle\sum_{\alpha=1}^{2}\sqrt{\rho_\alpha}\Phi_\alpha B_\alpha\Phi_\alpha)^2$$

$$+\frac{\rho_2^{-1}}{2}p(p+1)-\frac{1}{2}\textstyle\sum_{\alpha=1}^{2}\rho_\alpha\mathrm{tr}\Phi_\alpha^4-\frac{1}{2}\sum_{\alpha=1}^{2}\rho_\alpha(\mathrm{tr}\Phi_\alpha^2)^2$$

$$+\sqrt{2}\,n^{-\frac{1}{2}}\Big\{\frac{\rho_2^{-1}}{3}\mathrm{tr}\,B_2^2\textstyle\sum_{\alpha=1}^{2}\mathrm{tr}B_\alpha^3/\sqrt{\rho_\alpha}-\frac{1}{3}\sum_{\alpha=1}^{2}\mathrm{tr}B_\alpha^3/\sqrt{\rho_\alpha}\mathrm{tr}(\sum_{\alpha=1}^{2}\sqrt{\rho_\alpha}$$

$$\cdot\Phi_\alpha B_\alpha\Phi_\alpha)^2+\frac{1}{6}\textstyle\sum_{\alpha=1}^{2}\mathrm{tr}\,B_\alpha^3/\sqrt{\rho_\alpha}[\rho_2^{-1}p(p+1)-\sum_{\alpha=1}^{2}\rho_\alpha\mathrm{tr}\Phi_\alpha^4$$

$$-\textstyle\sum_{\alpha=1}^{2}\rho_\alpha(\mathrm{tr}\Phi_\alpha^2)^2]+2\rho_2^{-1}\mathrm{tr}\,B_2^3/\sqrt{\rho_2}-2\mathrm{tr}(\sum_{\alpha=1}^{2}\Phi_\alpha B_\alpha^2\Phi_\alpha$$

$$\cdot\textstyle\sum_{\alpha=1}^{2}\sqrt{\rho_\alpha}\Phi_\alpha B_\alpha\Phi_\alpha)+\rho_2^{-1}(p+1)\mathrm{tr}B_2/\sqrt{\rho_2}-\sum_{\alpha=1}^{2}\sqrt{\rho_\alpha}\mathrm{tr}(\Phi_\alpha^2 B_\alpha\Phi_\alpha^2)$$

$$-\textstyle\sum_{\alpha=1}^{2}\sqrt{\rho_\alpha}\mathrm{tr}\Phi_\alpha^2\mathrm{tr}(\Phi_\alpha B_\alpha\Phi_\alpha)\Big\}\Big]+O(n^{-1}).$$

From Theorem 7.4, we obtain the following two formulae:

$$(11.6)\quad E[q_2(Y)\exp(itq_0(Y)/\tau_1)]=\exp\Big[\frac{(it)^2}{2}\Big]\frac{2\sqrt{2}}{3}\Big\{\mathrm{tr}(\textstyle\sum_{\alpha=1}^{2}\sqrt{\rho_\alpha}\Phi_\alpha B_\alpha\Phi_\alpha)^3$$

$$-\rho_2^{-\frac{3}{2}}\mathrm{tr}B_2^3-\frac{3}{2}\rho_2^{-\frac{3}{2}}(p+1)\mathrm{tr}\,B_2+\frac{3}{2}\textstyle\sum_{\alpha=1}^{2}\rho_\alpha\mathrm{tr}\Phi_\alpha^2\mathrm{tr}\,(\Phi_\alpha^2\sum_{\alpha=1}^{2}\sqrt{\rho_\alpha}\Phi_\alpha B_\alpha\Phi_\alpha)$$

$$+\frac{3}{2}\mathrm{tr}\textstyle\sum_{\alpha=1}^{2}\rho_\alpha\Phi_\alpha^4(\sum_{\alpha=1}^{2}\sqrt{\rho_\alpha}\Phi_\alpha B_\alpha\Phi_\alpha)\Big\},$$

and

$$(11.7)\quad E[q_1^2(Y)\exp(itq_0(Y)/\tau_1)]=\exp\left[\frac{(it)^2}{2}\right][\{\mathrm{tr}(\textstyle\sum_{\alpha=1}^2\sqrt{\rho_\alpha}\varPhi_\alpha B_\alpha\varPhi_\alpha)^2$$

$$-\rho_2^{-1}\mathrm{tr}\,B_2^2\}^2+\{\mathrm{tr}(\textstyle\sum_{\alpha=1}^2\sqrt{\rho_\alpha}\varPhi_\alpha B_\alpha\varPhi_\alpha)^2-\rho_2^{-1}\mathrm{tr}B_2^2\}$$

$$\{\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\varPhi_\alpha^4+\sum_{\alpha=1}^2\rho_\alpha(\mathrm{tr}\varPhi_\alpha^2)^2-\rho_2^{-1}(p^2+p+4)\}$$

$$+4\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}(\varPhi_\alpha^2\sum_{\alpha=1}^2\sqrt{\rho_\alpha}\varPhi_\alpha B_\alpha\varPhi_\alpha)^2-4\mathrm{tr}(\varPhi_2B_2\varPhi_2)^2$$

$$+4\rho_1\rho_2^{-1}\mathrm{tr}(\varPhi_1B_1\varPhi_1)^2+\frac{1}{4}\{\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\varPhi_\alpha^4+\sum_{\alpha=1}^2\rho_\alpha(\mathrm{tr}\varPhi_\alpha^2)^2\}^2$$

$$+\textstyle\sum_{\alpha,\beta=1}^2\rho_\alpha\rho_\beta(\mathrm{tr}\varPhi_\alpha^2\varPhi_\beta^2)^2+\sum_{\alpha,\beta=1}^2\rho_\alpha\rho_\beta\mathrm{tr}\varPhi_\alpha^4\varPhi_\beta^4-\dfrac{\rho_2^{-1}}{2}p(p+1)$$

$$\cdot\{\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\varPhi_\alpha^4+\sum_{\alpha=1}^2\rho_\alpha(\mathrm{tr}\varPhi_\alpha^2)^2\}-2\{\mathrm{tr}\varPhi_2^4+(\mathrm{tr}\varPhi_2^2)^2\}$$

$$+4\rho_2^{-2}p(p+1)(p^2+p+4)].$$

Therefore the characteristic function of the statistic λ_1/τ_1 is given by

$$(11.8)\quad C_1(t)=\exp\left[\frac{(it)^2}{2}\right]\left[1+n^{-\frac{1}{2}}\left\{\frac{it}{\tau_1}\left[\frac{\rho_2^{-1}}{2}p(p+1)-\frac{1}{2}\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\varPhi_\alpha^4\right.\right.\right.$$

$$\left.-\frac{1}{2}\textstyle\sum_{\alpha=1}^2\rho_\alpha(\mathrm{tr}\varPhi_\alpha^2)^2\right]+\left(\dfrac{it}{\tau_1}\right)^3\left[\dfrac{4}{3}\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}A_\alpha^3\right.$$

$$\left.\left.\left.+2\mathrm{tr}A_2^2-2\mathrm{tr}(\textstyle\sum_{\alpha=1}^2\rho_\alpha\varPhi_\alpha A_\alpha\varPhi_\alpha)^2\right]\right\}+n^{-1}\sum_{\alpha=1}^3g_{2\alpha}\left(\dfrac{it}{\tau_1}\right)^{2\alpha}+O(n^{-\frac{3}{2}})\right],$$

where $A_1=\varPhi_1^2$ and $A_2=\varPhi_2^2-\rho_2^{-1}I$. The coefficients $g_{2\alpha}$ are given by

$$(11.9)\quad g_2=2\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\varPhi_\alpha^2\mathrm{tr}(\varPhi_\alpha^2\sum_{\alpha=1}^2\rho_\alpha\varPhi_\alpha A_\alpha\varPhi_\alpha)+2\mathrm{tr}\sum_{\alpha=1}^2\rho_\alpha\varPhi_\alpha^4$$

$$(\textstyle\sum_{\alpha=1}^2\rho_\alpha\varPhi_\alpha A_\alpha\varPhi_\alpha)-2\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}(\varPhi_\alpha^4A_\alpha)-2\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\varPhi_\alpha^2$$

$$\mathrm{tr}(\varPhi_\alpha A_\alpha\varPhi_\alpha)+\frac{1}{8}\{\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\varPhi_\alpha^4+\sum_{\alpha=1}^2\rho_\alpha(\mathrm{tr}\varPhi_\alpha^2)^2\}^2$$

$$+\frac{1}{2}\textstyle\sum_{\alpha,\beta=1}^2\rho_\alpha\rho_\beta(\mathrm{tr}\varPhi_\alpha^2\varPhi_\beta^2)^2+\dfrac{1}{2}\sum_{\alpha,\beta=1}^2\rho_\alpha\rho_\beta\mathrm{tr}\varPhi_\alpha^4\varPhi_\beta^4$$

$$-\frac{\rho_2^{-1}}{4}p(p+1)\{\sum_{\alpha=1}^{2}\rho_\alpha \mathrm{tr}\Phi_\alpha^4+\sum_{\alpha=1}^{2}\rho_\alpha(\mathrm{tr}\Phi_\alpha^2)^2\}-\{\mathrm{tr}\Phi_2^4$$

$$+(\mathrm{tr}\Phi_2^2)^2\}+\frac{\rho_2^{-2}}{8}p(p+1)(p^2+p+4),$$

$$g_4=2\sum_{\alpha=1}^{2}\rho_\alpha \mathrm{tr}A_\alpha^4-\frac{2}{3}\sum_{\alpha=1}^{2}\rho_\alpha \mathrm{tr}A_\alpha^3\{\sum_{\alpha=1}^{2}\rho_\alpha \mathrm{tr}\Phi_\alpha^4+\sum_{\alpha=1}^{2}\rho_\alpha(\mathrm{tr}\Phi_\alpha^2)^2\}$$

$$-8\mathrm{tr}\ (\sum_{\alpha=1}^{2}\rho_\alpha\Phi_\alpha A_\alpha^2\Phi_\alpha\sum_{\alpha=1}^{2}\rho_\alpha\Phi_\alpha A_\alpha\Phi_\alpha)+\frac{2}{3}\rho_2^{-1}p(p+1)\sum_{\alpha=1}^{2}\rho_\alpha \mathrm{tr}A_\alpha^3$$

$$+\frac{16}{3}\mathrm{tr}A_2^3+\frac{8}{3}\mathrm{tr}(\sum_{\alpha=1}^{2}\rho_\alpha\Phi_\alpha A_\alpha\Phi_\alpha)^3+\{\mathrm{tr}(\sum_{\alpha=1}^{2}\rho_\alpha\Phi_\alpha A_\alpha\Phi_\alpha)^2$$

$$-\mathrm{tr}A_2^2\}\{\sum_{\alpha=1}^{2}\rho_\alpha \mathrm{tr}\Phi_\alpha^4+\sum_{\alpha=1}^{2}\rho_\alpha(\mathrm{tr}\Phi_\alpha^2)^2-\rho_2^{-1}(p^2+p+4)\}$$

$$+4\sum_{\alpha=1}^{2}\rho_\alpha \mathrm{tr}(\Phi_\alpha^2\sum_{\alpha=1}^{2}\rho_\alpha\Phi_\alpha A_\alpha\Phi_\alpha)^2-4\rho_2\mathrm{tr}(\Phi_2A_2\Phi_2)^2$$

$$+4\rho_1^2\rho_2^{-1}\mathrm{tr}(\Phi_1A_1\Phi_1)^2,$$

$$g_6=\frac{8}{9}(\sum_{\alpha=1}^{2}\rho_\alpha\ \mathrm{tr}A_\alpha^3)^2+\frac{8}{3}\sum_{\alpha=1}^{2}\rho_\alpha \mathrm{tr}A_\alpha^3\{\mathrm{tr}A_2^2-\mathrm{tr}(\sum_{\alpha=1}^{2}\rho_\alpha\Phi_\alpha A_\alpha\Phi_\alpha)^2\}$$

$$+2\{\mathrm{tr}(\sum_{\alpha=1}^{2}\rho_\alpha\Phi_\alpha A_\alpha\Phi_\alpha)^2-\mathrm{tr}A_2^2\}^2.$$

Inverting this characteristic function, we obtain the following theorem:

Theorem 11.1. *Put* $\lambda_1=-\sqrt{n}\{\log|S_2(S_1+S_2)^{-1}|-\log|\rho_2\tilde{\Gamma}^{-1}|\}$ *and* $\tau_1^2=2\{\rho_1\mathrm{tr}\Phi_1^4+\rho_2\mathrm{tr}(\Phi_2^2-\rho_2^{-1}I)^2\}$. *Then under the alternative K, we have*

$$(11.10)\quad P(\lambda_1/\tau_1\leq x)=\Phi(x)-n^{-\frac{1}{2}}\Big[\Big\{\frac{\rho_2^{-1}}{2}p(p+1)-\frac{1}{2}\sum_{\alpha=1}^{2}\rho_\alpha \mathrm{tr}\Phi_\alpha^4$$

$$-\frac{1}{2}\sum_{\alpha=1}^{2}\rho_\alpha(\mathrm{tr}\Phi_\alpha^2)^2\Big\}\Phi^{(1)}(x)/\tau_1+\Big\{\frac{4}{3}\sum_{\alpha=1}^{2}\rho_\alpha \mathrm{tr}A_\alpha^3$$

$$-2\mathrm{tr}(\sum_{\alpha=1}^{2}\rho_\alpha\Phi_\alpha A_\alpha\Phi_\alpha)^2+2\mathrm{tr}A_2^2\Big\}\Phi^{(3)}(x)/\tau_1^3\Big]+n^{-1}\sum_{\alpha=1}^{3}g_{2\alpha}$$

$$\Phi^{(2\alpha)}(x)/\tau_1^{2\alpha}+O(n^{-\frac{3}{2}}),$$

where $\Phi_1=\Gamma^{-\frac{1}{2}}\tilde{\Gamma}^{-\frac{1}{2}}$, $\Phi_2=\tilde{\Gamma}^{-\frac{1}{2}}$, $A_1=\Phi_1^2$ *and* $A_2=\Phi_2^2-\rho_2^{-1}I$ *with* $\tilde{\Gamma}=\rho_1\Gamma+\rho_2 I$. *The coefficients* $g_{2\alpha}$ ($\alpha=1, 2, 3$) *are given by* (11.9).

Under the hypothesis, putting $\Gamma=I$ in (11.10), we obtain the following theorem:

THEOREM 11.2. *Put* $\lambda_1=-\sqrt{n}\{\log|S_2(S_1+S_2)^{-1}|-p\log\rho_2\}$ *and* $\tau_1^2=2p\rho_1\rho_2^{-1}$. *Then we have, under the hypothesis H,*

$$(11.11)\quad P(\lambda_1/\tau_1\leq x)=\Phi(x)-n^{\quad 2}\Big|\frac{1}{2}p(p+1)\rho_1\rho_2^{-1}\Phi^{(1)}(x)/\tau_1$$

$$+\frac{2p}{3}\rho_2^{-2}(2\rho_1\rho_2^2-2\rho_1^3+3\rho_1^2)\Phi^{(3)}(x)/\tau_1^3\Big]+n^{-1}\textstyle\sum_{\alpha=1}^{3}g_{2\alpha}'\Phi^{(2\alpha)}(x)$$

$$/\tau_1^{2\alpha}+O(n^{-\frac{3}{2}}),$$

where the coefficients $g_{2\alpha}'$ *are given by*

$$g_2'=\frac{1}{8}\rho_2^{-2}p(p+1)(\rho_1^2p^2+\rho_1^2p-4\rho_2^2+4),$$

$$(11.12)\quad g_4'=\frac{\rho_1 p}{3}\rho_2^{-3}\{p^2(-2\rho_2^3+2\rho_1^2\rho_2+2\rho_2^2-2\rho_1^2-3\rho_1\rho_2+3\rho_1)+p(-2\rho_2^3+2\rho_1^2\rho_2$$

$$+2\rho_2^2-2\rho_1^2-3\rho_1\rho_2+3\rho_1)+6\rho_2^3+6\rho_1^3-16\rho_1^2+12\rho_1\},$$

$$g_6'=\frac{2\rho_1^2}{9}p^2\rho_2^{-4}(4\rho_1^4-8\rho_1^2\rho_2^2+4\rho_2^4+12\rho_1\rho_2^2-12\rho_1^3+9\rho_1^2).$$

Theorem 11.2 was given by Sugiura [25] using different method. He also gave the asymptotic expansion under the sequence of alternatives. Now the above result differs from that of multivariate Bartlett's test, namely, the above result shows the continuity of the limiting distribution at the null hypothesis, whereas Bartlett's test has not.

12. Asymptotic expansion of $\mathrm{tr}S_1S_2^{-1}$. The test statistic $\mathrm{tr}S_1S_2^{-1}$ is expressed by the statistics Y_1 and Y_2 defined in Section 11 as

$$(12.1)\quad \sqrt{n}\,\mathrm{tr}S_1S_2^{-1}=\sqrt{n}\,\mathrm{tr}\rho_1\Gamma/\rho_2+q_0(Y)+n^{-\frac{1}{2}}q_1(Y)+n^{-1}q_2(Y)+O_p(n^{-\frac{2}{3}}),$$

where

$$q_0(Y)=\sqrt{2}\,\mathrm{tr}\Gamma(\rho_1^{\frac{1}{2}}\rho_2^{-1}Y_1-\rho_1\rho_2^{-\frac{3}{2}}Y_2), \tag{12.2}$$

$$q_1(Y)=2(\rho_1\rho_2^{-2}\mathrm{tr}\Gamma Y_2^2-\rho_1^{-\frac{1}{2}}\rho_2^{-\frac{3}{2}}\mathrm{tr}\Gamma^{\frac{1}{2}}Y_1\Gamma^{-\frac{1}{2}}Y_2),$$

$$q_2(Y)=2\sqrt{2}\,(\rho_1^{\frac{1}{2}}\rho_2^{-2}\mathrm{tr}\Gamma^{-\frac{1}{2}}Y_1\Gamma^{-\frac{1}{2}}Y_2^2-\rho_1\rho_2^{-\frac{5}{2}}\mathrm{tr}\Gamma Y_2^3).$$

Putting $\lambda_2=\sqrt{n}(\mathrm{tr}S_1S_2^{-1}-\mathrm{tr}\rho_1\rho_2^{-1}\Gamma)$ in (12.1), we can easily see that the statistic λ_2 converges in law to the normal distribution with mean zero and variance $\tau_2^2=2\rho_1\rho_2^{-3}\mathrm{tr}\Gamma^2$. Hence the characteristic function of the statistic λ_2/τ_2 is expressed as (12.3).

$$C_2(t)=E[\exp(itq_0(Y)/\tau_2)\Big[1+n^{-\frac{1}{2}}q_1(Y)\frac{it}{\tau_2}+n^{-1}\Big\{q_2(Y)\frac{it}{\tau_2} \tag{12.3}$$

$$+\frac{1}{2}q_1^2(Y)\Big(\frac{it}{\tau_2}\Big)^2\Big\}\Big]\Big]+O(n^{-\frac{3}{2}}).$$

Put $b_1=\sqrt{2\rho}\ \rho_2^{-1}it/\tau_2$ and $b_2=-\sqrt{2}\rho_1\rho_2^{-\frac{3}{2}}it/\tau_2$ in $q_0(Y)$. Then the first term is given by

$$E[\exp(itq_0(Y)/\tau_2)]=\exp\Big[\frac{(it)^2}{2}\Big]\Big[1+\frac{\sqrt{2}}{3}n^{-\frac{1}{2}}(\rho_1^{-\frac{1}{2}}b_1^3+\rho_2^{-\frac{1}{2}}b_2^3)\mathrm{tr}\Gamma^3 \tag{12.4}$$

$$+n^{-1}\Big\{\frac{1}{9}(\rho_1^{-\frac{1}{2}}b_1^3+\rho_2^{-\frac{1}{2}}b_2^3)^2(\mathrm{tr}\Gamma^3)^2+\frac{1}{2}(\rho_1^{-1}b_1^4+\rho_2^{-1}b_2^4)\mathrm{tr}\Gamma^4\Big\}\Big]$$

$$+O(n^{-\frac{3}{2}}).$$

By Theorem 7.3, we have

$$E[q_1(Y)\exp(itq_0(Y)/\tau_2)]=\exp\Big[\frac{(it)^2}{2}\Big]2\Big\{(\rho_1\rho_2^{-2}b_2^2-\rho_1^{\frac{1}{2}}\rho_2^{-\frac{3}{2}}b_1b_2)\mathrm{tr}\Gamma^3 \tag{12.5}$$

$$+\frac{\rho_1\rho_2^{-2}}{2}(p+1)\mathrm{tr}\Gamma+\sqrt{2}\,n^{-\frac{1}{2}}\Big[\frac{1}{3}(\rho_1\rho_2^{-\frac{5}{2}}b_2^5+\rho_1^{\frac{1}{2}}\rho_2^{-2}b_1^3b_2^2-\rho_1^{\frac{1}{2}}\rho_2^{-2}b_1b_2^4$$

$$-\rho_2^{-\frac{3}{2}}b_1^4b_2)(\mathrm{tr}\Gamma^3)^2+(2\rho_1\rho_2^{-\frac{5}{2}}b_2^3-\rho_2^{-\frac{3}{2}}b_1^2b_2-\rho_1^{\frac{1}{2}}\rho_2^{-2}b_1b_2^2)\mathrm{tr}\Gamma^4$$

$$+\frac{(p+1)}{6}\{\rho_1\rho_2^{-\frac{5}{2}}b_2^3+\rho_1^{\frac{1}{2}}\rho_2^{-2}b_1^3\}\mathrm{tr}\Gamma\mathrm{t}\Gamma^3+\frac{1}{2}\rho_1\rho_2^{-\frac{5}{2}}(p+2)b_2\mathrm{tr}\Gamma^2$$

$$+\frac{1}{2}\rho_1\rho_2^{-\frac{5}{2}}b_2(\mathrm{tr}\varGamma)^2\Big]\Big\}+O(n^{-1}).$$

We can calculate each expectation of n^{-1} in (12.3) using Theorem 7.4 as

$$E[q_2(Y)\exp(itq_0(Y)/\tau_2)]=\exp\left[\frac{(it)^2}{2}\right]2\sqrt{2}\Big[(\sqrt{\rho_1}\rho_2^{-2}b_1b_2^2-\rho_1\rho_2^{-\frac{5}{2}}b_2^3) \tag{12.6}$$

$$\cdot\mathrm{tr}\varGamma^4+\left\{\frac{1}{2}\rho_1^{\frac{1}{2}}\rho_2^{-2}(p+1)b_1-\rho_1\rho_2^{-\frac{5}{2}}\left(p+\frac{3}{2}\right)b_2\right\}\mathrm{tr}\varGamma^2-\frac{\rho_1}{2}\rho_2^{-\frac{5}{2}}b_2(\mathrm{tr}\varGamma)^2\Big],$$

and

$$E[q_1^2(Y)\exp(itq_0(Y)/\tau_2)]=\exp\left[\frac{(it)^2}{2}\right]4\Big\{(\rho_1^2\rho_2^{-4}b_2^4-2\rho_1^{\frac{3}{2}}\rho_2^{-\frac{7}{2}}b_1b_2^3 \tag{12.7}$$

$$+\rho_1\rho_2^{-3}b_1^2b_2^2)(\mathrm{tr}\varGamma^3)^2+(4\rho_1^2\rho_2^{-4}b_2^2+\rho_1\rho_2^{-3}b_2^2+\rho_1\rho_2^{-3}b_1^2-4\rho_1^{\frac{3}{2}}\rho_2^{-\frac{7}{2}}b_1b_2)$$

$$\cdot\mathrm{tr}\varGamma^4+[\rho_1^2\rho_2^{-4}(p+1)b_2^2-\rho_1^{\frac{3}{2}}\rho_2^{-\frac{7}{2}}(p+1)b_1b_2]\mathrm{tr}\,\varGamma\mathrm{tr}\varGamma^3+\Big[\frac{\rho_1^2}{2}\rho_2^{-4}(p+2)$$

$$+\frac{\rho_1}{2}\rho_2^{-3}\Big]\mathrm{tr}\varGamma^2+\left[\frac{\rho_1^2}{4}\rho_2^{-4}(p^2+2p+3)+\frac{1}{2}\rho_1\rho_2^{-3}\right](\mathrm{tr}\varGamma)^2\Big\}.$$

Hence the characteristic function is given by

$$C_2(t)=\exp\left[\frac{(it)^2}{2}\right]\Big[1+n^{-\frac{1}{2}}\Big\{\rho_1\rho_2^{-2}(p+1)\mathrm{tr}\varGamma it/\tau_2 \tag{12.8}$$

$$+\left(\frac{4}{3}\rho_1\rho_2^{-3}+4\rho_1^2\rho_2^{-4}+\frac{8}{3}\rho_1^3\rho_2^{-5}\right)\mathrm{tr}\varGamma^3(it/\tau_2)^3\Big\}+n^{-1}\textstyle\sum_{\alpha=1}^{3}g_{2\alpha}(it/\tau_2)^{2\alpha}$$

$$+O(n^{-\frac{3}{2}})\Big],$$

where the coefficients $g_{2\alpha}$ are given by

$$g_2=\{(3p+4)\rho_1^2\rho_2^{-4}+(2p+3)\rho_1\rho_2^{-3}\}\mathrm{tr}\varGamma^2+\Big\{\frac{1}{2}\rho_1^2\rho_2^{-4}(p^2+2p+3)$$

$$+\rho_1\rho_3^{-2}\Big\}(\mathrm{tr}\varGamma)^2,$$

$$g_4=2\{\rho_1\rho_2^{-4}+6\rho_1^2\rho_2^{-5}+10\rho_1^3\rho_2^{-6}+5\rho_1^4\rho_2^{-7}\}\mathrm{tr}\varGamma^4+(p+1)\Big\{\frac{4}{3}\rho_1^2\rho_2^{-5} \tag{12.9}$$

$$+4\rho_1^3\rho_2^{-6}+\frac{8}{3}\rho_1^4\rho_2^{-7}\Big\}\mathrm{tr}\varGamma\mathrm{tr}\varGamma^3,$$

$$g_6=\left(\frac{8}{9}\rho_1^2\rho_2^{-6}+\frac{16}{3}\rho_1^3\rho_2^{-7}+\frac{104}{9}\rho_1^4\rho_2^{-8}+\frac{32}{3}\rho_1^5\rho_2^{-9}+\frac{32}{9}\rho_1^6\rho_2^{-10}\right)(\mathrm{tr}\Gamma^3)^2,$$

which yields the following theorem:

THEOREM 12.1. *Under the alternative K, the distribution of* $\lambda_2=\sqrt{n}\{\mathrm{tr}S_1\cdot S_2^{-1}-\mathrm{tr}\rho_1\Gamma/\rho_2\}$ *can be expanded asymptotically as follows*:

$$(12.10)\quad P(\lambda_2/\tau_2\leq x)=\Phi(x)-n^{-\frac{1}{2}}\Big\{\rho_1\rho_2^{-2}(p+1)\mathrm{tr}\Gamma\Phi^{(1)}(x)/\tau_2+\Big(\frac{4}{3}\rho_1\rho_2^{-3}$$
$$+4\rho_1^2\rho_2^{-4}+\frac{8}{3}\rho_1^3\rho_2^{-5})\mathrm{tr}\Gamma^3\Phi^{(3)}(x)/\tau_2^3\Big\}+n^{-1}\textstyle\sum_{\alpha=1}^3 g_{2\alpha}\Phi^{(2\alpha)}(x)/\tau_2^{2\alpha}$$
$$+O(n^{-\frac{3}{2}}),$$

where $\tau_2^2=2\rho_1\rho_2^{-3}\mathrm{tr}\Gamma^2$ *and the coefficients* $g_{2\alpha}$ *are given by* (12.9).

Putting $\Gamma=I$ in (12.10), we obtain the following asymptotic expansion under the hypothesis.

THEOREM 12.2. *Under the hypothesis H, the distribution of* $\lambda_2=\sqrt{n}\{\mathrm{tr}S_1 S_2^{-1}-\rho_1\rho_2^{-1}p\}$ *can be expanded asymptotically as follows*:

$$(12.11)\quad P[\lambda_2/\tau_2\leq x]=\Phi(x)-n^{-\frac{1}{2}}\Big\{\rho_1\rho_2^{-2}p(p+1)\Phi^{(1)}(x)/\tau_2+p\Big(\frac{4}{3}\rho_1\rho_2^{-3}$$
$$+4\rho_1^2\rho_2^{-4}+\frac{8}{3}\rho_1^3\rho_2^{-5}\Big)\Phi^{(3)}(x)/\tau_2^3\Big\}+n^{-1}\textstyle\sum_{\alpha=1}^3 g'_{2\alpha}\Phi^{(2\alpha)}(x)/\tau_2^{2\alpha}$$
$$+O(n^{-\frac{3}{2}}),$$

where $\tau_2^2=2p\rho_1\rho_2^{-3}$ *and the coefficients* $g'_{2\alpha}$ *are given by*

$$g'_2=p\{(3p+4)\rho_1^2\rho_2^{-4}+(2p+3)\rho_1\rho_2^{-3}\}+p^2\Big\{\frac{1}{2}\rho_1^2\rho_2^{-4}(p^2+2p+3)+\rho_1\rho_2^{-3}\Big\},$$

$$g'_4=2p\{\rho_1\rho_2^{-4}+6\rho_1^2\rho_2^{-5}+10\rho_1^3\rho_2^{-6}+5\rho_1^4\rho_2^{-7}\}+p^2(p+1)\Big\{\frac{4}{3}\rho_1^2\rho_2^{-5}$$

(12.12)

$$+4\rho_1^3\rho_2^{-6}+\frac{8}{3}\rho_1^4\rho_2^{-7}\Big\},$$

$$g_6' = p^2\left(\frac{8}{9}\rho_1^2\rho_2^{-6} + \frac{16}{3}\rho_1^3\rho_2^{-7} + \frac{104}{9}\rho_1^4\rho_2^{-8} + \frac{32}{3}\rho_1^5\rho_2^{-9} + \frac{32}{9}\rho_1^6\rho_2^{-10}\right).$$

13. Asymptotic expansion of tr$S_1(S_1+S_2)^{-1}$. In this section we shall give the asymptotic expansion of the test statistic tr$S_1(S_1+S_2)^{-1}$. This statistic is expressed in terms of the random variables Y_1 and Y_2 as follows:

$$\sqrt{n}\,\mathrm{tr}S_1(S_1+S_2)^{-1} = \sqrt{n}\,\mathrm{tr}\rho_1\Phi_1^2 + + q_0(Y) + n^{-\frac{1}{2}}q_1(Y) + n^{-1}q_2(Y) \tag{13.1}$$

$$+ O_p(n^{-\frac{3}{2}}),$$

where, using the same notations as in Section 11, the coefficients $q_0(Y)$, $q_1(Y)$ and $q_2(Y)$ are given by

$$q_0(Y) = \sqrt{2}\,\{\sqrt{\rho_1}\mathrm{tr}(\Phi_1^2 - \rho_1\Phi_1^4)Y_1 - \sqrt{\rho_2}\mathrm{tr}\rho_1\Phi_1^2\Phi_2^2Y_2\},$$

$$q_1(Y) = 2\{\rho_1\mathrm{tr}\Phi_1^2(\textstyle\sum_{\alpha=1}^2\sqrt{\rho_\alpha}\Phi_\alpha Y_\alpha\Phi_\alpha)^2 - \sqrt{\rho_1}\mathrm{tr}(\Phi_1Y_1\Phi_1)$$

$$\textstyle\sum_{\alpha=1}^2\sqrt{\rho_\alpha}\Phi_\alpha Y_\alpha\Phi_\alpha\}, \tag{13.2}$$

$$q_2(Y) = 2\sqrt{2}\,\{\sqrt{\rho_1}\mathrm{tr}(\Phi_1Y_1\Phi_1)(\textstyle\sum_{\alpha=1}^2\sqrt{\rho_\alpha}\Phi_\alpha Y_\alpha\Phi_\alpha)^2$$

$$-\rho_1\mathrm{tr}\Phi_1^2(\textstyle\sum_{\alpha=1}^2\sqrt{\rho_\alpha}\Phi_\alpha Y_\alpha\Phi_\alpha)^3\}$$

Now putting $\lambda_3 = \sqrt{n}\,\{\mathrm{tr}S_1(S_1+S_2)^{-1} - \rho_1\mathrm{tr}\Phi_1^2\}$ in (13.1), we can see that the statistic λ_3 converges in law to the normal distribution with mean zero and variance $\tau_3^2 = 2\{\rho_1\mathrm{tr}(\Phi_1^2 - \rho_1\Phi_1^4)^2 + \rho_2\mathrm{tr}\rho_1^2\Phi_1^4\Phi_2^4\}$. Further we shall give the asymptotic expansion. The characteristic function of λ_3/τ_3 ($\tau_3 > 0$) can be expressed as (13.3).

$$C_3(t) = E\Big[\exp(itq_0(Y)/\tau_3)\Big\{1 + n^{-\frac{1}{2}}q_1(Y)it/\tau_3 + n^{-1}\Big[q_2(Y)it/\tau_3 \tag{13.3}$$

$$+\frac{1}{2}q_1^2(Y)(it/\tau_3)^2\Big]\Big\}\Big] + O(n^{-\frac{3}{2}}).$$

Set $B_1=\sqrt{2\rho_1}(\Phi_1^2-\rho_1\Phi_1^4)it/\tau_3$ and $B_2=-\sqrt{2\rho_2}\rho_1\Phi_1^2\Phi_2^2 it/\tau_3$.
Then from (8.6), we have

$$(13.4)\qquad E[\exp(itq_0(Y)/\tau_3)]=\exp\Big[\frac{(it)^2}{2}\Big]\Big[1+\frac{\sqrt{2}}{3}n^{-\frac{1}{2}}\textstyle\sum_{\alpha=1}^2\mathrm{tr}B_\alpha^3/\sqrt{\rho_\alpha}$$

$$+n^{-1}\Big\{\frac{1}{9}(\textstyle\sum_{\alpha=1}^2\mathrm{tr}\,B_\alpha^3/\sqrt{\rho_\alpha})^2+\frac{1}{2}\sum_{\alpha=1}^2\mathrm{tr}B_\alpha^4/\rho_\alpha\Big\}\Big]+O(n^{-\frac{3}{2}}).$$

Applying Theorem 7.3 to second term in (13.3), with the abbreviated notations $\tilde{B}_\alpha=\Phi_\alpha B_\alpha\Phi_\alpha$ $(\alpha=1, 2)$ and $\tilde{B}=\sum_{\alpha=1}^2\sqrt{\rho_\alpha}\tilde{B}_\alpha$, we obtain

$$(13.5)\qquad E[q_1(Y)\exp(itq_0(Y)/\tau_3)]=\exp\Big[\frac{(it)^2}{2}\Big]2\Big[\rho_1\mathrm{tr}\Phi_1^2\tilde{B}^2-\sqrt{\rho_1}\mathrm{tr}\tilde{B}_1\tilde{B}$$

$$+\frac{\rho_1}{2}\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^4+\frac{\rho_1}{2}\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^2\mathrm{tr}\Phi_\alpha^2-\frac{\rho_1}{2}(\mathrm{tr}\Phi_1^2)^2$$

$$-\frac{\rho_1}{2}\mathrm{tr}\,\Phi_1^4+\sqrt{2}\,n^{-\frac{1}{2}}\Big\{\frac{\rho_1}{3}\textstyle\sum_{\alpha=1}^2\mathrm{tr}B_\alpha^3/\sqrt{\rho_\alpha}[\mathrm{tr}\Phi_1^2\tilde{B}^2-\rho_1^{-\frac{1}{2}}\mathrm{tr}\tilde{B}_1\tilde{B}\,]$$

$$+\frac{\rho_1}{6}\textstyle\sum_{\alpha=1}^2\mathrm{tr}B_\alpha^3/\sqrt{\rho_\alpha}[\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_\alpha^2\mathrm{tr}\Phi_1^2\Phi_\alpha^2+\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^4$$

$$-\mathrm{tr}\Phi_1^4-(\mathrm{tr}\Phi_1^2)^2]+2\rho_1\mathrm{tr}\Phi_1^2\textstyle\sum_{\alpha=1}^2\Phi_\alpha B_\alpha^2\Phi_\alpha\tilde{B}-\mathrm{tr}(\Phi_1B_1^2\Phi_1)\tilde{B}-\sqrt{\rho_1}\mathrm{tr}\tilde{B}_1$$

$$\textstyle\sum_{\alpha=1}^2(\Phi_\alpha B_\alpha^2\Phi_\alpha)+\frac{1}{2}\rho_1\sum_{\alpha=1}^2\sqrt{\rho_\alpha}\mathrm{tr}\Phi_\alpha^2\mathrm{tr}\Phi_1^2\,\tilde{B}_\alpha+\frac{1}{2}\rho_1\sum_{\alpha=1}^2\sqrt{\rho_\alpha}\mathrm{tr}\Phi_1^2\Phi_\alpha^2$$

$$\mathrm{tr}\tilde{B}_\alpha-\sqrt{\rho_1}\mathrm{tr}\Phi_1^2\tilde{B}_1-\sqrt{\rho_1}\mathrm{tr}\tilde{B}_1\mathrm{tr}\Phi_1^2+\rho_1\textstyle\sum_{\alpha=1}^2\sqrt{\rho_\alpha}\mathrm{tr}\Phi_1^2\Phi_\alpha^2\tilde{B}_\alpha\Big\}\Big]+O(n^{-1}).$$

Similarly, by Theorem 7.4, we have

$$(13.6)\qquad E[q_2(Y)\exp(itq_0(Y)/\tau_3)]=\exp\Big[\frac{(it)^2}{2}\Big]2\sqrt{2}\,\Big\{\sqrt{\rho_1}\mathrm{tr}\tilde{B}_1\tilde{B}^2-\rho_1\mathrm{tr}\Phi_1^2\tilde{B}^3$$

$$+\frac{\sqrt{\rho_1}}{2}\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\tilde{B}_1\Phi_\alpha^4+\rho_1\mathrm{tr}\Phi_1^4\tilde{B}+\frac{\sqrt{\rho_1}}{2}\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_\alpha^2\mathrm{tr}\Phi_\alpha^2\tilde{B}_1$$

$$+\rho_1\mathrm{tr}\Phi_1^2\mathrm{tr}\,\Phi_1^2\tilde{B}-\frac{3}{2}\rho_1\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^4\tilde{B}-\rho_1\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_\alpha^2\mathrm{tr}\Phi_1^2\Phi_\alpha^2\tilde{B}$$

$$-\frac{\rho_1}{2}\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^2\mathrm{tr}\,\Phi_\alpha^2\tilde{B}\Big\},$$

and

$$(13.7)\quad E[q_1^2(Y)\exp(itq_0(Y)/\tau_3)]=\exp\left[\frac{(it)^2}{2}\right]4\Big[\{\rho_1\mathrm{tr}\Phi_1^2\tilde{B}^2-\sqrt{\rho_1}\mathrm{tr}\tilde{B}_1\tilde{B}\}^2$$

$$+4\rho_1^2\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^4\Phi_\alpha^4\tilde{B}^2+\rho_1^2\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^4\mathrm{tr}\Phi_1^2\tilde{B}^2+\rho_1^2\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_\alpha^2$$

$$\cdot\mathrm{tr}\Phi_1^2\Phi_\alpha^2\mathrm{tr}\Phi_1^2\tilde{B}^2+4\rho_1\sqrt{\rho_1}\mathrm{tr}\Phi_1^4\tilde{B}_1\tilde{B}+\rho_1\sqrt{\rho_1}(\mathrm{tr}\Phi_1^2)^2\mathrm{tr}\tilde{B}_1\tilde{B}+\rho_1\sqrt{\rho_1}\mathrm{tr}\Phi_1^4$$

$$\cdot\mathrm{tr}\tilde{B}_1\tilde{B}+\rho_1\rho_2\mathrm{tr}\Phi_1^4\tilde{B}_2^2+\rho_1\rho_2\mathrm{tr}\Phi_2^4\tilde{B}_1^2-\rho_1^2\mathrm{tr}\tilde{B}_1^2\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^2\mathrm{tr}\Phi_\alpha^2$$

$$-\rho_1^2(\mathrm{tr}\Phi_1^2)^2\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\tilde{B}_\alpha^2-\rho_1^2\mathrm{tr}\Phi_1^4\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\tilde{B}_\alpha^2-\rho_1^2\sqrt{\rho_1}\mathrm{tr}\Phi_1^6$$

$$\cdot\mathrm{tr}\tilde{B}_1\tilde{B}-8\rho_1^2\sqrt{\rho_1}\mathrm{tr}\Phi_1^6\tilde{B}_1\tilde{B}-4\rho_1\rho_2\sqrt{\rho_1}\mathrm{tr}\Phi_1^2\Phi_2^4\tilde{B}_1\tilde{B}-\rho_1\sqrt{\rho_1\rho_2}\mathrm{tr}\tilde{B}_1\tilde{B}_2$$

$$\cdot\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^2\mathrm{tr}\Phi_\alpha^2-\rho_1\rho_2\sqrt{\rho_1}\mathrm{tr}\Phi_1^2\Phi_2^4\mathrm{tr}\tilde{B}_1\tilde{B}-4\rho_1^2\sqrt{\rho_2}\mathrm{tr}\Phi_1^6\tilde{B}_2\tilde{B}$$

$$-2\rho_1^2\sqrt{\rho_1\rho_2}\mathrm{tr}\Phi_1^2\tilde{B}_1\tilde{B}_2\{(\mathrm{tr}\Phi_1^2)^2+\mathrm{tr}\Phi_1^4\}+\frac{\rho_1^2}{4}(\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_\alpha^2\mathrm{tr}\Phi_1^2\Phi_\alpha^2$$

$$+\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^4)^2+\rho_1^2\mathrm{tr}\Phi_1^4(\sum_{\alpha=1}^2\rho_\alpha\Phi_\alpha^4)^2+\frac{\rho_1^2}{2}\sum_{\alpha,\beta=1}^2\rho_\alpha\rho_\beta$$

$$(\mathrm{tr}\Phi_1^2\Phi_\alpha^2\Phi_\beta^2)^2+\frac{\rho_1^2}{2}\textstyle\sum_{\alpha,\beta=2}^1\rho_\alpha\rho_\beta\mathrm{tr}\Phi_1^4\Phi_\alpha^2\Phi_\beta^2\mathrm{tr}\Phi_\alpha^2\Phi_\beta^2+\frac{\rho_1^2}{4}\{(\mathrm{tr}\Phi_1^2)^2$$

$$+\mathrm{tr}\Phi_1^4\}^2+\rho_1^2\{\mathrm{tr}\Phi_1^8+(\mathrm{tr}\Phi_1^4)^2\}+\frac{\rho_1\rho_2}{2}(\mathrm{tr}\Phi_1^2\Phi_2^2)^2+\frac{\rho_1\rho_2}{2}\mathrm{tr}\Phi_1^4\Phi_2^4$$

$$-\frac{\rho_1^2}{2}(\mathrm{tr}\Phi_1^2)^2\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^2\mathrm{tr}\Phi_\alpha^2-2\rho_1^2\mathrm{tr}\Phi_1^6\sum_{\alpha=1}^2\rho_\alpha\Phi_\alpha^4-\frac{\rho_1^2}{2}\mathrm{tr}\Phi_1^4\mathrm{tr}\Phi_1^2$$

$$\cdot\textstyle\sum_{\alpha=1}^2\rho_\alpha\Phi_\alpha^4-2\rho_1^2\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^4\Phi_\alpha^2\mathrm{tr}\Phi_1^2\Phi_\alpha^2-\frac{\rho_1^2}{2}\mathrm{tr}\Phi_1^4\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^2\mathrm{tr}\Phi_\alpha^2$$

$$-\frac{\rho_1^2}{2}(\mathrm{tr}\Phi_1^2)^2\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^4\Big]$$

Thus the characteristic function of λ_3/τ_3 can be expanded as follows:

$$(13.8)\quad C_3(t)=\exp\left[\frac{(it)^2}{2}\right]\Big[1+n^{-\frac{1}{2}}\Big\{\Big(\frac{it}{\tau_3}\Big)[\rho_1\textstyle\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^4+\rho_1\sum_{\alpha=1}^2\rho_\alpha$$

$$\cdot \mathrm{tr}\Phi_\alpha^2 \mathrm{tr}\Phi_1^2\Phi_\alpha^2 - \rho_1(\mathrm{tr}\Phi_1^2)^2 - \rho_1 \mathrm{tr}\Phi_1^4] + \left(\frac{it}{\tau_3}\right)^3 \Big[\frac{4}{3}\sum_{\alpha=1}^2 \rho_\alpha \mathrm{tr}A_\alpha^3$$

$$+ 4\rho_1 \mathrm{tr}\Phi_1^2\tilde{A}^2 - 4\rho_1 \mathrm{tr}\tilde{A}_1\tilde{A}]\Big\} + n^{-1}\sum_{\alpha=1}^3 g_{2\alpha}\left(\frac{it}{\tau_3}\right)^{2\alpha}\Big] + O(n^{-\frac{3}{2}}),$$

where $A_1 = \Phi_1^2 - \rho_1\Phi_1^4, A_2 = -\rho_1\Phi_1^2\Phi_2^2$, $\tilde{A}_\alpha = \Phi_\alpha A_\alpha \Phi_\alpha$ $(\alpha=1,2)$ and $\tilde{A} = \sum_{\alpha=1}^2 \rho_\alpha \tilde{A}_\alpha$. The coefficients $g_{2\alpha}$ are given by

(13.9) $$g_2 = 2\rho_1\sum_{\alpha=1}^2 \rho_\alpha \mathrm{tr}\Phi_\alpha^2 \mathrm{tr}\Phi_1^2\tilde{A}_\alpha + 2\rho_1\sum_{\alpha=1}^2 \rho_\alpha \mathrm{tr}\Phi_1^2\Phi_\alpha^2 \mathrm{tr}\tilde{A}_\alpha - 4\rho_1 \mathrm{tr}\,\Phi_1^2\tilde{A}_1$$

$$- 4\rho_1 \mathrm{tr}\Phi_1^2 \mathrm{tr}\tilde{A}_1 + 2\rho_1\sum_{\alpha=1}^2 \rho_\alpha \mathrm{tr}\Phi_\alpha^4\tilde{A}_1 + 4\rho_1 \mathrm{tr}\Phi_1^4\tilde{A} + 2\rho_1\sum_{\alpha=1}^2 \rho_\alpha$$

$$\cdot \mathrm{tr}\Phi_\alpha^2 \mathrm{tr}\Phi_\alpha^2\tilde{A}_1 + 4\rho_1\sum_{\alpha=1}^2 \rho_\alpha \mathrm{tr}\Phi_1^2\Phi_\alpha^2\tilde{A}_\alpha + 4\rho_1 \mathrm{tr}\Phi_1^2 \mathrm{tr}\Phi_1^2\tilde{A}$$

$$- 6\rho_1\sum_{\alpha=1}^2 \rho_\alpha \mathrm{tr}\Phi_1^2\Phi_\alpha^4\tilde{A} - 4\rho_1\sum_{\alpha=1}^2 \rho_\alpha \mathrm{tr}\Phi_\alpha^2 \mathrm{tr}\Phi_1^2\Phi_\alpha^2\tilde{A} - 2\rho_1\sum_{\alpha=1}^2 \rho_\alpha$$

$$\cdot \mathrm{tr}\Phi_1^2\Phi_\alpha^2 \mathrm{tr}\Phi_\alpha^2\tilde{A} + \frac{{\rho_1}^2}{2}\Big(\sum_{\alpha=1}^2 \rho_\alpha \mathrm{tr}\Phi_\alpha^2 \mathrm{tr}\Phi_1^2\Phi_\alpha^2 + \sum_{\alpha=1}^2 \rho_\alpha \mathrm{tr}\Phi_1^2\Phi_\alpha^4\Big)^2$$

$$+ 2\rho_1^2 \mathrm{tr}\Phi_1^4\ \Big(\sum_{\alpha=1}^2 \rho_\alpha \Phi_\alpha^4\Big)^2 + \rho_1^2\sum_{\alpha,\beta=1}^2 \rho_\alpha\rho_\beta(\mathrm{tr}\Phi_1^2\Phi_\alpha^2\Phi_\beta^2)^2$$

$$+ \rho_1^2\sum_{\alpha,\beta=1}^2 \rho_\alpha\rho_\beta \mathrm{tr}\Phi_1^4\Phi_\alpha^2\Phi_\beta^2 \mathrm{tr}\Phi_\alpha^2\Phi_\beta^2 + \frac{\rho_1^2}{2}\{(\mathrm{tr}\Phi_1^2)^2 + \mathrm{tr}\Phi_1^4\}^2 + 2\rho_1^2 \mathrm{tr}\Phi_1^8$$

$$+ 2\rho_1^2(\mathrm{tr}\Phi_1^4)^2 + \rho_1\rho_2(\mathrm{tr}\Phi_1^2\Phi_2^2)^2 + \rho_1\rho_2 \mathrm{tr}\Phi_1^4\Phi_2^4$$

$$- \rho_1^2\{(\mathrm{tr}\Phi_1^2)^2 + \mathrm{tr}\Phi_1^4\}\sum_{\alpha=1}^2 \rho_\alpha \mathrm{tr}\Phi_1^2\Phi_\alpha^2 \mathrm{tr}\Phi_\alpha^2 - 4\rho_1^2 \mathrm{tr}\Phi_1^6\sum_{\alpha=1}^2 \rho_\alpha \Phi_\alpha^4$$

$$- \rho_1^2\{\mathrm{tr}\Phi_1^4 + (\mathrm{tr}\Phi_1^2)^2\} \mathrm{tr}\Phi_1^2\sum_{\alpha=1}^2 \rho_\alpha \Phi_\alpha^4 - 4\rho_1^2\sum_{\alpha=1}^2 \rho_\alpha \mathrm{tr}\Phi_1^4\Phi_\alpha^2 \mathrm{tr}\Phi_1^2\Phi_\alpha^2,$$

$$g_4 = 2\sum_{\alpha=1}^2 \rho_\alpha \mathrm{tr}A_\alpha^4 + \frac{4}{3}\rho_1\sum_{\alpha=1}^2 \rho_\alpha \mathrm{tr}A_\alpha^3\Big[\sum_{\alpha=1}^2 \rho_\alpha \mathrm{tr}\Phi_\alpha^2 \mathrm{tr}\Phi_1^2\Phi_\alpha^2 + \sum_{\alpha=1}^2 \rho_\alpha$$

$$\cdot \mathrm{tr}\Phi_1^2\Phi_\alpha^4 - \mathrm{tr}\Phi_1^4 - (\mathrm{tr}\Phi_1^2)^2\Big] + 16\rho_1 \mathrm{tr}\Phi_1^2\sum_{\alpha=1}^2 \rho_\alpha \Phi_\alpha A_\alpha^2 \Phi_\alpha \tilde{A}$$

$$- 8\rho_1 \mathrm{tr}(\Phi_1 A_1^2 \Phi_1)\tilde{A} - 8\rho_1 \mathrm{tr}\tilde{A}_1\sum_{\alpha=1}^2 \rho_\alpha \Phi_\alpha A_\alpha^2 \Phi_\alpha + 8\rho_1 \mathrm{tr}\tilde{A}_1\tilde{A}^2 - 8\rho_1 \mathrm{tr}\Phi_1^2\tilde{A}^3$$

$$+16\rho_1^2\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^4\Phi_\alpha^4\tilde{A}^2+4\rho_1^2\mathrm{tr}\Phi_1^2\tilde{A}^2\{\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^4+\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_\alpha^2$$

$$\cdot\mathrm{tr}\Phi_1^2\Phi_\alpha^2\}+16\rho_1^2\mathrm{tr}\Phi_1^4\tilde{A}_1\tilde{A}+4\rho_1^2\mathrm{tr}\tilde{A}_1\tilde{A}\{(\mathrm{tr}\Phi_1^2)^2+\mathrm{tr}\Phi_1^4\}+4\rho_1\rho_2^2$$

$$\cdot\mathrm{tr}\Phi_1^4\tilde{A}_2^2+4\rho_1^2\rho_2\mathrm{tr}\Phi_2^4\tilde{A}_1^2-4\rho_1^3\mathrm{tr}\tilde{A}_1^2\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^2\mathrm{tr}\Phi_\alpha^2-4\rho_1^2\sum_{\alpha=1}^2\rho_\alpha^2$$

$$\cdot\mathrm{tr}\Phi_1^2\tilde{A}_\alpha^2\{(\mathrm{tr}\Phi_1^2)^2+\mathrm{tr}\Phi_1^4\}-4\rho_1^3\mathrm{tr}\Phi_1^6\mathrm{tr}\tilde{A}_1\tilde{A}-32\rho_1^3\mathrm{tr}\Phi_1^6\tilde{A}_1\tilde{A}$$

$$-16\rho_1^2\rho_2\mathrm{tr}\Phi_1^2\Phi_2^4\tilde{A}_1\tilde{A}-4\rho_1^2\rho_1\mathrm{tr}\tilde{A}_1\tilde{A}_2\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^2\mathrm{tr}\Phi_\alpha^2$$

$$-4\rho_1^2\rho_2\mathrm{tr}\Phi_1^2\Phi_2^4\mathrm{tr}\tilde{A}_1\tilde{A}-16\rho_1^2\rho_2\mathrm{tr}\Phi_1^6\tilde{A}_2\tilde{A}-8\rho_1^3\rho_2\mathrm{tr}\Phi_1^2\tilde{A}_1\tilde{A}_2\{(\mathrm{tr}\Phi_1^2)^2+\mathrm{tr}\Phi_1^4\},$$

$$g_6=\frac{8}{9}(\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\,A_\alpha^3)^2+\frac{16}{3}\rho_1\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\,A_\alpha^3\{\mathrm{tr}\Phi_1^2\tilde{A}^2-\mathrm{tr}\tilde{A}_1\tilde{A}\}$$

$$+8\{\rho_1\mathrm{tr}\Phi_1^2\tilde{A}^2-\rho_1\mathrm{tr}\tilde{A}_1\tilde{A}\}^2.$$

Thus inverting this characteristic function, we get the following theorem:

THEOREM 13.1. *Under the alternative K, the distribution of* $\lambda_3=\sqrt{n}\cdot\{\mathrm{tr}\,S_1(S_1+S_2)^{-1}-\rho_1\mathrm{tr}\Phi_1^2\}$ *can be expanded asymptotically as follows*:

$$(13.10)\quad P(\lambda_3/\tau_3\leq x)=\Phi(x)-n^{-\frac{1}{2}}\Big\{[\rho_1\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_1^2\Phi_\alpha^4+\rho_1\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}\Phi_\alpha^2$$

$$\cdot\mathrm{tr}\Phi_1^2\Phi_\alpha^2-\rho_1(\mathrm{tr}\Phi_1^2)^2-\rho_1\mathrm{tr}\Phi_1^4]\Phi^{(1)}(x)/\tau_3+\Big[\frac{4}{3}\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}A_\alpha^3$$

$$+4\rho_1\mathrm{tr}\Phi_1^2\tilde{A}^2-4\rho_1\mathrm{tr}\tilde{A}_1\tilde{A}\Big]\Phi^{(3)}(x)/\tau_3^3\Big\}+n^{-1}\sum_{\alpha=1}^3 g_{2\alpha}\Phi^{(2\alpha)}(x)/\tau_3^{2\alpha}$$

$$+O(n^{-\frac{3}{2}}),$$

where $\tau_3^2=2\sum_{\alpha=1}^2\rho_\alpha\mathrm{tr}A_\alpha^2$ *with* $A_1=(\Phi_1^2-\rho_1\Phi_1^4)$, $A_2=-\rho_1\Phi_1^2\Phi_2^2$ *and* $\tilde{A}_\alpha=\Phi_\alpha A_\alpha\Phi_\alpha$ $(\alpha=1, 2)$, $\tilde{A}=\sum_{\alpha=1}^2\rho_\alpha\tilde{A}_\alpha$. *The coefficients* $g_{2\alpha}$ *are given by* (13.9).

From the above theorem we obtain the following asymptotic expansion under the hypothesis.

THEOREM 13.2. *Under the hypothesis H, the distribution of* $\lambda_3=\sqrt{n}\ \{\mathrm{tr} S_1 \cdot (S_1+S_2)^{-1}-\rho_1 p\}$ *can be expanded asymptotically for large n as*

$$(13.11)\quad P(\lambda_3/\tau_3 \leq x)=\Phi(x)-n^{-\frac{1}{2}}\left[\frac{4}{3}p\rho_1\rho_2(\rho_2-\rho_1)\right]\Phi^{(3)}(x)/\tau_3^3$$

$$+n^{-1}\textstyle\sum_{\alpha=1}^{3} g'_{2\alpha}\Phi^{(2\alpha)}(x)/\tau_3^{2\alpha}+O(n^{-\frac{3}{2}}),$$

where $\tau_3^2=2p\rho_1\rho_2$ *and* $g'_2=-\rho_1\rho_2 p(p+1)$, $g'_4=2p\rho_1\rho_2(\rho_1^2+\rho_2^2-3\rho_1\rho_2)$ *and* $g'_6=8 \cdot \rho_1^2\rho_2^2(\rho_2-\rho_1)^2p^2/9$.

14. Numerical examples. Each test statistic λ_i/τ_i $(i=1,2,3)$ of one-sided tests has the following asymptotic expansion under the hypothesis.

$$(14.1)\quad P(\lambda_i/\tau_i \leq x)=\Phi(x)-\frac{1}{\sqrt{n}}\{a_1\Phi^{(1)}(x)+a_3\Phi^{(3)}(x)\}$$

$$+\frac{1}{n}\{b_2\Phi^{(2)}(x)+b_4\Phi^{(4)}(x)+b_6\Phi^{(6)}(x)\}+O(n^{-\frac{3}{2}}),$$

with the different coefficients a's and b's depending on λ_i/τ_i. Applying the general inverse expansion formula of Hill and Davis [11], we get the following asymptotic formula for the upper $\alpha\%$ point of λ_i/τ_i in terms of the upper $\alpha\%$ point u of the standard normal distribution function.

$$(14.2)\quad u+\frac{1}{\sqrt{n}}(a_1H_0(u)+a_3H_2(u))-\frac{1}{n}\Big\{b_2H_1(u)+b_4H_3(u)+b_6H_5(u)$$

$$+\frac{1}{2}a_1^2uH_0(u)^2+a_1a_3u\,H_0(u)\,H_2(u)+\frac{a_3^2}{2}uH_2(u)^2-2a_1a_3uH_0(u)$$

$$-2a_3^2uH_2(u)\Big\}+O(n^{-\frac{3}{2}}),$$

where $H_j(u)$ is defined such that $\Phi^{(j+1)}(u)=H_j(u)\Phi^{(1)}(u)$. For $j=0,\ 1,\ 2,\ 3,\ 4,\ 5,$

$$(14.3)\quad \begin{aligned} &H_0(u)=1,\ H_1(u)=-u,\ H_2(u)=u^2-1,\ H_3(u)=-u^3+3u \\ &H_4(u)=u^4-6u^2+3,\ H_5(u)=-u^5+10u^3-15u. \end{aligned}$$

The above formula was also used by Sugiura [25].

EXAMPLE 14.1. We shall consider two cases, namely, case 1: $p=2$ and $n_1=13$, $n_2=63$ case 2: $p=4$ and $n_1=50$, $n_2=100$. Asymptotic formula (14.2) for the upper 5 percent point gives:

	λ_1/τ_1		λ_2/τ_2		λ_3/τ_3	
	case 1	case 2	case 1	case 2	case 1	case 2
first term	1.6449	1.6449	1.6449	1.6449	1.6449	1.6449
second term	0.2366	0.2622	0.3592	0.5011	0.1140	0.0232
third term	0.0206	0.0322	0.0973	0.1312	-0.0388	-0.0273
approx. value	1.902	1.939	2.101	2.277	1.720	1.641
exact value	1.9069	——	2.1332	——	1.7154	——

The above exact values are given by Pillai and Jayachandran [20]. This example shows the approximate value of λ_2/τ_2 is worse than those of λ_1/τ_1 and λ_3/τ_3.

EXAMPLE 14.2. We give each approximate power of the three criteria based on the exact and the approximate percentage points in Example 14.1, which is denoted by case A and case B respectively. Let $ch_i(\Sigma_1\Sigma_2^{-1})=\gamma_i(i=1, 2,\cdots,p)$ and let the alternative hypothesis be K: $\gamma_1=1$, $\gamma_2=1.5$.

	λ_1/τ_1		λ_2/τ_2		λ_3/τ_3	
	case A	case B	case A	case B	case A	case B
first term	0.1767	0.1777	0.1516	0.1575	0.2010	0.1998
second term	0.0128	0.0127	0.0378	0.0372	-0.0130	-0.0129
third term	-0.0013	-0.0012	0.0032	0.0032	-0.0046	-0.0046
approx. power	0.188	0.189	0.193	0.198	0.183	0.182
exact power	0.186	0.186	0.192	0.192	0.194	0.194

EXAMPLE 14.3. We shall give the approximate powers in case 2 in Example 14.1.

K: $\gamma_1=\gamma_2=\gamma_3=\gamma_4=1.1$

	first term	second term	third term	approx. power
λ_1/τ_1	0.1429	0.0441	0.0040	0.1910
λ_2/τ_2	0.0921	0.0743	0.0285	0.1949
λ_3/τ_3	0.2045	-0.0087	-0.0037	0.1922

PART V. TEST CRITERION FOR EIGENVALUES AND EIGENVECTORS OF COVARIANCE MATRIX

15. Preliminaries. Let $p\times 1$ vectors $X_1, X_2,\cdots,X_N(N>p)$ be a random sample from a multivariate normal distribution with unknown mean vector μ and unknown covariance martrix Σ. Let $\theta_i(i=1,2,\cdots,p)$ be an eigenvalue of Σ and let $p\times 1$ vector $\gamma_i(i=1,2,\cdots,p)$ be an eigenvector of unit length corresponding to an eigenvalue $\theta_i(i=1,2,\cdots,p)$. From this sample we wish to test the hypothesis H: $\theta_i=\theta_{i0}$ and $\gamma_i=\gamma_{i0}(i=1,2,\cdots,k, k\leq p)$ against the alternatives K: $\theta_i\neq\theta_{i0}$ or $\gamma_j\neq\gamma_{j0}$ for some i, j, where θ_{i0} and γ_{i0} are known constant and known vector and mean μ is unspecified. The modified LR test for this problem is given by, as in Mallows [15] and R.P. Gupta [8],

$$(15.1)\qquad \Lambda=|S/n|^{\frac{n}{2}}|\Theta_0|^{-\frac{n}{2}}|\Gamma_2'S\Gamma_2/n|^{-\frac{n}{2}}\mathrm{etr}\left[-\frac{1}{2}\Theta_0^{-1}\Gamma_{10}'S\Gamma_{10}\right]e^{\frac{nk}{2}},$$

where $S=\sum_{\alpha=1}^{N}(X_\alpha-\bar{X})(X_\alpha-\bar{X})'$, $\bar{X}=N^{-1}\sum_{\alpha=1}^{N}X_\alpha$, $\Theta_0=\mathrm{diag}(\theta_{10}, \theta_{20},\cdots,\theta_{k0})$, $\Gamma_{10}=(\gamma_{10}, \gamma_{20},\cdots,\gamma_{k0})$, $n=N-1$ and a matrix Γ_2 is so chosen that $T=[\Gamma_{10}:\Gamma_2]$ is orthogonal. The statistic Λ is independent of the choice of Γ_2. In case $p=k$, we proved the unbiasedness of the test (15.1) in Sugiura and Nagao [26], the monotonicity property of which was established by Nagao [16]. In our case $k<p$, we shall prove the unbiasedness by the method in Sugiura and Nagao [26].

The acceptance region of the modified LR test is given by

$$(15.2)\qquad \omega=\left\{S\,|\,|S|^{\frac{n}{2}}|\Theta_0|^{-\frac{n}{2}}|\Gamma_2'S\Gamma_2|^{-\frac{n}{2}}\mathrm{etr}\left[-\frac{1}{2}\Theta_0^{-1}\Gamma_{10}'S\Gamma_{10}\right]\geq c_\alpha\right\},$$

where the constant c_α is determined such that the level of this test is α.

THEOREM 15.1. *For testing the hypothesis* H: $\theta_i=\theta_{i0}$, $\gamma_i=\gamma_{i0}(i=1,2,\cdots,k, k<p)$ *against the alternatives* K: $\theta_i\neq\theta_{i0}$ *or* $\gamma_j\neq\gamma_{j0}$ *for some* i, j *with unknown mean* μ, *the modified LR test having the acceptance region* (15.2) *is unbiased.*

PROOF. Under the alternative K, the statistic S has a Wishart distribution $W(\Sigma, n)$, so the probability $P_K(\omega)$ of the acceptance region ω under the alternative K is given by

$$(15.3)\qquad P_K(\omega)=c_{p,n}\int_{S\in\omega}|S|^{\frac{1}{2}(n-p-1)}|\Sigma|^{-\frac{n}{2}}\mathrm{etr}\left[-\frac{1}{2}\Sigma^{-1}S\right]dS,$$

where the coefficient $c_{p,n}$ is given by

$$(15.4)\qquad c_{p,n}^{-1}=2^{\frac{np}{2}}\pi^{p(p-1)/4}\Pi_{j=1}^{p}\Gamma\left[\frac{1}{2}(n+2-j)\right].$$

Put $A=T'ST$, where $T=(\Gamma_{10}:\Gamma_2)$. Then the Jacobian is given by $|\partial A/\partial S|=1$. So we obtain

$$(15.5)\qquad P_K(\omega)=c_{p,n}\int_{A\in\omega_1}|A|^{\frac{1}{2}(n-p-1)}|\Omega|^{-\frac{n}{2}}\mathrm{etr}\left(-\frac{1}{2}\Omega^{-1}A\right)dA,$$

where $\Omega=T'\Sigma T$ and the region ω_1 is given by

$$(15.6)\qquad \omega_1=\left\{A\,\Big|\,|A_{11}-A_{12}A_{22}^{-1}A_{21}|^{\frac{n}{2}}|\theta_0|^{-\frac{n}{2}}\mathrm{etr}\left(-\frac{1}{2}\theta_0^{-1}A_{11}\right)\geq c_\alpha\right\}.$$

The A_{ij} is submatrix obtained from A as (15.7).

$$(15.7)\qquad A=\left(\begin{array}{c:cl} \overbrace{\qquad}^{k} & \overbrace{\qquad}^{p-k} & \\ A_{11} & A_{12} & \}k \\ \hdashline A_{21} & A_{22} & \}p-k \end{array}\right).$$

R. P. Gupta [8] gave the simple proof of independence of the statistics $W=A_{11}-A_{12}A_{22}^{-1}A_{21}$ and $A_{12}A_{22}^{-1}A_{21}$ when $\Omega_{12}=0$ in his appendix, where Ω_{12} is submatrix similarly partitioned as A in (15.7). Based on Ω^{-1} expressed by Ω_{11}, Ω_{12}, Ω_{22} and translating $W=A_{11}-A_{12}A_{22}^{-1}A_{21}$, $V=A_{12}A_{22}^{-1}$, he gave the decomposition of a Wishart distribution as

$$(15.8)\quad c_{p,n}|W|^{\frac{1}{2}(n-p-1)}|\Omega_{11\cdot 2}|^{-\frac{n}{2}}\mathrm{etr}\left[-\frac{1}{2}\Omega_{11\cdot 2}^{-1}W\right]|A_{22}|^{\frac{k}{2}}\mathrm{etr}\left[-\frac{1}{2}(V-\boldsymbol{B})'\Omega_{11\cdot 2}^{-1}\right.$$
$$\left.\cdot(V-\boldsymbol{B})A_{22}\right]|A_{22}|^{\frac{1}{2}\{n-(p-k)-1\}}|\Omega_{22}|^{-\frac{n}{2}}\mathrm{etr}\left[-\frac{1}{2}\Omega_{22}^{-1}A_{22}\right],$$

where $\Omega_{11\cdot 2}=\Omega_{11}-\Omega_{12}\Omega_{22}^{-1}\Omega_{21}$ and $\boldsymbol{B}=\Omega_{12}\Omega_{22}^{-1}$.

Therefore (15.5) is rewritten as

$$(15.9)\quad P_K(\omega)=c_{p,n}\int_{(W,V,A_{22})\in\omega_2}|W|^{\frac{1}{2}(n-p-1)}|\Omega_{11\cdot2}|^{-\frac{n}{2}}\mathrm{etr}\left[-\frac{1}{2}\Omega_{11\cdot2}^{-1}W\right]|A_{22}|^{\frac{k}{2}}$$

$$\cdot\,\mathrm{etr}\left[-\frac{1}{2}(V-\boldsymbol{B})'\Omega_{11\cdot2}^{-1}(V-\boldsymbol{B})A_{22}\right]|A_{22}|^{\frac{1}{2}\{n-(p-k)-1\}}|\Omega_{22}|^{-\frac{n}{2}}$$

$$\cdot\,\mathrm{etr}\left[-\frac{1}{2}\Omega_{22}^{-1}A_{22}\right]dWdVdA_{22},$$

where

$$(15.10)\quad \omega_2=\left\{(W,\ V,\ A_{22})\,|\,|W|^{\frac{n}{2}}|\Theta_0|^{-\frac{n}{2}}\mathrm{etr}\left[-\frac{1}{2}\Theta_0^{-1}W-\frac{1}{2}\Theta_0^{-1}VA_{22}\right.\right.$$

$$\left.\left.\cdot V'\right]\geq c_\alpha\right\}.$$

Put $C=\Theta_0^{\frac{1}{2}}\Omega_{11\cdot2}^{-\frac{1}{2}}W\Omega_{11\cdot2}^{-\frac{1}{2}}\Theta_0^{\frac{1}{2}}$ and $Y=\Theta_0^{\frac{1}{2}}\Omega_{11\cdot2}^{-\frac{1}{2}}(V-\boldsymbol{B})$, so the Jacobian is given by $|\partial(W,\ V)/\partial(C,\ Y)|=|\Omega_{11\cdot2}\Theta_0^{-1}|^{\frac{1}{2}(p+1)}$

Thus (15.9) is expressed as follows:

$$(15.11)\quad P_K(\omega)=c_{p,n}\int_{(C,Y,A_{22})\in\tilde{\omega}_2}|C|^{\frac{1}{2}(n-p-1)}|\Theta_0|^{-\frac{n}{2}}\mathrm{etr}\left(-\frac{1}{2}\Theta_0^{-1}C\right)|A_{22}|^{\frac{k}{2}}$$

$$\cdot\mathrm{etr}\left[-\frac{1}{2}Y'\Theta_0^{-1}YA_{22}\right]|A_{22}|^{\frac{1}{2}\{n-(p-k)-1\}}|\Omega_{22}|^{-\frac{n}{2}}$$

$$\cdot\mathrm{etr}\left[-\frac{1}{2}\Omega_{22}^{-1}A_{22}\right]dCdYdA_{22},$$

where $\tilde{\omega}_2=\left\{(C,\ Y,\ A_{22})\,|\,(\Omega_{11\cdot2}^{\frac{1}{2}}\Theta_0^{-\frac{1}{2}}C\Theta_0^{-\frac{1}{2}}\Omega_{11\cdot2}^{\frac{1}{2}},\Omega_{11\cdot2}^{\frac{1}{2}}\Theta_0^{-\frac{1}{2}}Y+\boldsymbol{B},\ A_{22})\in\omega_2\right\}.$

So we have

$$(15.12)\quad P_H(\omega)-P_K(\omega)=c_{p,n}\left\{\int_{(C,Y,A_{22})\in\omega_2}-\int_{(C,Y,A_{22})\in\tilde{\omega}_2}\right\}|C|^{\frac{1}{2}(n-p-1)}$$

$$\cdot|\Theta_0|^{-\frac{n}{2}}\mathrm{etr}\left[-\frac{1}{2}\Theta_0^{-1}C\right]|A_{22}|^{\frac{k}{2}}\mathrm{etr}\left[-\frac{1}{2}Y'\Theta_0^{-1}\,YA_{22}\right]|A_{22}|^{\frac{1}{2}\{n-(p-k)-1\}}$$

$$\cdot\,|\Omega_{22}|^{-\frac{n}{2}}\mathrm{etr}\left[-\frac{1}{2}\Omega_{22}^{-1}A_{22}\right]dCdYdA_{22}$$

$$=c_{p,n}\left\{\int\!\!\int_{(C,Y,A_{22})\in\omega_2-\omega_2\cap\tilde{\omega}_2}-\int_{(C,Y,A_{22})\in\tilde{\omega}_2-\tilde{\omega}_2\cap\omega_2}\right\}|C|^{\frac{1}{2}(n-p-1)}$$

$$\cdot|\Theta_0|^{-\frac{n}{2}}\mathrm{etr}\left[-\frac{1}{2}\Theta_0^{-1}C\right]|A_{22}|^{\frac{k}{2}}\mathrm{etr}\left[-\frac{1}{2}Y'\Theta_0^{-1}YA_{22}\right]|A_{22}|^{-\frac{1}{2}\{n-(p-k)-1\}}$$

$$\cdot|\Omega_{22}|^{-\frac{n}{2}}\mathrm{etr}\left[-\frac{1}{2}\Omega_{22}^{-1}A_{22}\right]dCdYdA_{22}.$$

Since the integral $\int_{\omega_2}|C|^{-\frac{1}{2}(p+1)}|A_{22}|^{\frac{k}{2}}|A_{22}|^{-\frac{1}{2}\{n-(p-k)-1\}}|\Omega_{22}|^{-\frac{n}{2}}\cdot\mathrm{etr}\Big[-\frac{1}{2}$ $\cdot\Omega_{22}^{-1}A_{22}\Big]dCdYdA_{22}$ exists from (15.9) and (15.10), we have

$$P_H(\omega)-P_K(\omega)\geq c_{p,n}c_\alpha\left\{\int\!\!\int_{(C,Y,A_{22})\in\omega_2}-\int_{(C,Y,A_{22})\in\tilde{\omega}_2}\right\} \tag{15.13}$$

$$\cdot|C|^{-\frac{1}{2}(p+1)}|A_{22}|^{\frac{k}{2}}|A_{22}|^{\frac{1}{2}\{n-(p-k)-1\}}|\Omega_{22}|^{-\frac{n}{2}}$$

$$\cdot\mathrm{etr}\left[-\frac{1}{2}\Omega_{22}^{-1}A_{22}\right]dCdYdA_{22}.$$

Two integrals in (15.13) are equal by calculating the Jacobian. Hence $P_H(\omega)\geq P_K(\omega)$.

Next we shall give l-th moment of the statistic Λ given by (15.1), which is useful for obtaining the asymptotic expansion of Λ.

THEOREM 15.2. *The moment of the test* Λ *can be expressed as* (15.14).

$$E[\Lambda^l]=c_{p,n}c_{k,n+nl-p+k}^{-1}c_{p-k,n}^{-1}(e/n)^{nkl/2}(2\pi)^{k(p-k)/2} \tag{15.14}$$

$$\cdot|\Theta_0|^{-nl/2}|\Omega|^{-n/2}|\Omega_{11\cdot2}^{-1}+l\Theta_0^{-1}|^{-(n+nl)/2}|\Omega_{22}^{-1}$$

$$+\boldsymbol{B}'\Omega_{11\cdot2}^{-1}\boldsymbol{B}-\boldsymbol{B}'\Omega_{11\cdot2}^{-1}(\Omega_{11\cdot2}^{-1}+l\theta_0^{-1})^{-1}\Omega_{11\cdot2}^{-1}\boldsymbol{B}|^{-n/2},$$

where $\boldsymbol{B}=\Omega_{12}\Omega_{22}^{-1}$ *and* $c_{p,n}$ *is given by* (15.4).

PROOF. The moment of the test is given by

$$E[\Lambda^l]=c_{p,n}|\Theta_0|^{-nl/2}(e/n)^{nkl/2}\int|S|^{\frac{1}{2}(n-p-1)}|\Sigma|^{-\frac{n}{2}} \tag{15.15}$$

$$\cdot \mathrm{etr}\left[-\frac{1}{2}\Sigma^{-1}S\right]|S|^{nl/2}|\Gamma_2'S\Gamma_2|^{-nl/2}\mathrm{etr}\left[-\frac{l}{2}\Theta_0^{-1}\Gamma_{10}'S\Gamma_{10}\right]dS.$$

Put $A = T'ST$ then we can write

$$(15.16)\quad E[\Lambda^l]=c_{p,n}|\Theta_0|^{-nl/2}(e/n)^{nkl/2}\int|A|^{\frac{1}{2}(n-p-1)}|\Omega|^{-n/2}$$

$$\cdot\,\mathrm{etr}\left(-\frac{1}{2}\Omega^{-1}A\right)|A|^{nl/2}|A_{22}|^{-nl/2}\mathrm{etr}\left[-\frac{l}{2}\Theta_0^{-1}A_{11}\right]dA.$$

Expressing the Wishart distribution $W(\Omega, n)$ as (15.8), we can rewrite as follows:

$$(15.17)\quad E[\Lambda^l]=c_{p,n}|\Theta_0|^{-nl/2}(e/n)^{nkl/2}\int|W|^{\frac{1}{2}(n+nl-p-1)}|\Omega_{11\cdot2}|^{-\frac{n}{2}}$$

$$\cdot\,\mathrm{etr}\left[-\frac{1}{2}(\Omega_{11\cdot2}^{-1}+l\Theta_0^{-1})W\right]\mathrm{etr}\left[-\frac{1}{2}(V-\boldsymbol{B})'\Omega_{11\cdot2}^{-1}(V-\boldsymbol{B})A_{22}\right]$$

$$\cdot\,|A_{22}|^{\frac{1}{2})n-p-1)}|\Omega_{22}|^{-\frac{n}{2}}\mathrm{etr}\left[-\frac{1}{2}\Omega_{22}^{-1}A_{22}\right]\mathrm{etr}\left[-\frac{l}{2}\Theta_0^{-1}VA_{22}V'\right]$$

$$\cdot\,|A_{22}|^{k}dWdVdA_{22}.$$

Integrating (15.17) with respect to $W(k\times k)$ over the region $W>0$, we have

$$(15.18)\quad E[\Lambda^l]=c_{p,n}|\Theta_0|^{-nl/2}(e/n)^{nkl/2}|\Omega_{11\cdot2}|^{-n/2}$$

$$\cdot\,|\Omega_{11\cdot2}^{-1}+l\Theta_0^{-1}|^{-\frac{1}{2}(n+nl-p+k)}c_{k,n+nl-p+k}^{-1}$$

$$\cdot\int\mathrm{etr}\left[-\frac{1}{2}(V-\boldsymbol{B})'\Omega_{11\cdot2}^{-1}(V-\boldsymbol{B})A_{22}\right]|A_{22}|^{\frac{1}{2}(n+k-p-1)}|\Omega_{22}|^{-\frac{n}{2}}$$

$$\cdot\,\mathrm{etr}\left[-\frac{1}{2}\Omega_{22}^{-1}A_{22}\right]\mathrm{etr}\left[-\frac{l}{2}\Theta_0^{-1}VA_{22}V'\right]|A_{22}|^{\frac{k}{2}}dVdA_{22}.$$

We consider the following integration with respect to V.

$$(15.19)\quad \int\mathrm{etr}\left[-\frac{1}{2}(V-\boldsymbol{B})'\Omega_{11\cdot2}^{-1}(V-B)A_{22}\right]|A_{22}|^{\frac{k}{2}}\mathrm{etr}\left[-\frac{l}{2}\Theta_0^{-1}VA_{22}V'\right]dV,$$

where the range of the integration is over a $k(p-k)$-dimensional Euclidean space. Put $Y=VA_{22}^{\frac{1}{2}}$. Then the range of Y is also a $k(p-k)$-dimensional space and the Jacobian is given by $|\partial Y/\partial V|=|A_{22}|^{\frac{k}{2}}$. So we can rewrite (15.19) as follows:

$$\text{(15.20)}\quad \operatorname{etr}\left[-\frac{1}{2}\boldsymbol{B}'\Omega_{11\cdot 2}^{-1}\boldsymbol{B}A_{22}\right]\int \operatorname{etr}\left[-\frac{1}{2}(\Omega_{11\cdot 2}^{-1}+l\Theta_0^{-1})YY'\right]\operatorname{etr}[\Omega_{11\cdot 2}^{-1}\boldsymbol{B}A_{22}^{\frac{1}{2}}Y']dY.$$

Using the following well-known result for $m\times n$ matrices S, X and $R>0$,

$$\text{(15.21)}\quad \int \operatorname{etr}[-RXX']\operatorname{etr}[SX']dX=\pi^{mn/2}\operatorname{etr}\left[\frac{1}{4}R^{-1}SS'\right]|R|^{-n/2},$$

where the range of the integral is a mn-dimensional space, we can express (15.20) as

$$\text{(15.22)}\quad (2\pi)^{k(p-k)/2}|\Omega_{11\cdot 2}^{-1}+l\Theta_0^{-1}|^{-(p-k)/2}\operatorname{etr}\left[-\frac{1}{2}\boldsymbol{B}'\Omega_{11\cdot 2}^{-1}\boldsymbol{B}A_{22}\right]$$

$$\cdot\operatorname{etr}\left[\frac{1}{2}\boldsymbol{B}'\Omega_{11\cdot 2}^{-1}(\Omega_{11\cdot 2}^{-1}+l\Theta_0^{-1})^{-1}\Omega_{11\cdot 2}^{-1}\boldsymbol{B}A_{22}\right].$$

We, finally, consider the following integration with respect to A_{22}.

$$\text{(15.23)}\quad (2\pi)^{k(p-k)/2}|\Omega_{11\cdot 2}^{-1}+l\Theta_0^{-1}|^{-\frac{1}{2}(p-k)}\int |A_{22}|^{\frac{1}{2}\{n-(p-k)-1\}}|\Omega_{22}|^{-\frac{n}{2}}$$

$$\cdot\operatorname{etr}\left[-\frac{1}{2}\Omega_{22}^{-1}A_{22}\right]\operatorname{etr}\left[-\frac{1}{2}\boldsymbol{B}'\Omega_{11\cdot 2}^{-1}\boldsymbol{B}A_{22}\right]\operatorname{etr}\left[\frac{1}{2}\boldsymbol{B}'\Omega_{11\cdot 2}^{-1}(\Omega_{11\cdot 2}^{-1}+l\Theta_0^{-1})^{-1}\right.$$

$$\left.\cdot\,\Omega_{11\cdot 2}^{-1}\boldsymbol{B}A_{22}\right]dA_{22}.$$

By noting that the matrix $\{\Omega_{22}^{-1}+\boldsymbol{B}'\Omega_{11\cdot 2}^{-1}\boldsymbol{B}-\boldsymbol{B}'\Omega_{11\cdot 2}^{-1}(\Omega_{11\cdot 2}^{-1}+l\Theta_0^{-1})^{-1}\Omega_{11\cdot 2}^{-1}\boldsymbol{B}\}$ is positive definite and that A_{22} is distributed according to the Wishart distribution, we have

$$\text{(15.24)}\quad (2\pi)^{\frac{1}{2}k(p-k)}|\Omega_{11\cdot 2}^{-1}+l\Theta_0^{-1}|^{-\frac{1}{2}(p-k)}c_{p-k,n}^{-1}|\Omega_{22}|^{-\frac{n}{2}}|\Omega_{22}^{-1}+\boldsymbol{B}'\Omega_{11\cdot 2}^{-1}\boldsymbol{B}$$

$$-\boldsymbol{B}'\Omega_{11\cdot 2}^{-1}(\Omega_{11\cdot 2}^{-1}+l\Theta_0^{-1})^{-1}\Omega_{11\cdot 2}^{-1}\boldsymbol{B}|^{-\frac{n}{2}}.$$

Hence the proof is completed.

Especially when $\Omega_{12}=0$, Mallows [15] gave the above theorem.

16. Asymptotic expansion under hypothesis. Now we can obtain the asymptotic expansion of the statistic $-2\log\Lambda$ under the hypothesis from Theorem 15.2. R.P. Gupta [8] showed that the limiting distribution $-2\log\Lambda$ has a chi-square distribution with $f=k(k+1)/2+k(p-k)$ degrees of freedom under the hypothesis. Under the hypothesis, Ω is reduced to

$$(16.1)\qquad \Omega=\begin{pmatrix}\Theta_0 & 0\\ 0 & \Omega_{22}\end{pmatrix}.$$

Thus the characteristic function of $-2\log\Lambda$ is given by

$$(16.2)\qquad C(t)=\left(\frac{n}{2e}\right)^{itnk}\prod_{j=1}^{k}\frac{\Gamma\left[\frac{1}{2}(n-p+k+1-j)-itn\right]}{\Gamma\left[\frac{1}{2}(n-p+k+1-j)\right]}(1-2it)^{-\frac{kn}{2}+itnk}.$$

From this expression (16.2), the distribution under the hypothesis is shown to be independent of the nuisance parameter Ω_{22}. We shall use the following formula for the gamma function, which is given in Anderson [1].

$$(16.3)\qquad \log\Gamma[x+h]=\frac{1}{2}\log2\pi+\left(x+h-\frac{1}{2}\right)\log x-x-\sum_{r=1}^{m}\frac{(-1)^r B_{r+1}(h)}{r(r+1)x^r}$$
$$+0(|x|^{-m-1}).$$

This formula holds for large value x with fixed h. $B_r(h)$ is the Bernoulli polynomial of degree r. Some of these are listed below;

$$(16.4)\qquad B_2(h)=h^2-h+\frac{1}{6},\qquad B_3(h)=h^3-\frac{3}{2}h^2+\frac{1}{2}h,$$
$$B_4(h)=h^4-2h^3+h^2-\frac{1}{30},\qquad B_5(h)=h^5-\frac{5}{2}h^4+\frac{5}{3}h^3-\frac{1}{6}h.$$

Applying the formula (16.3) to each gamma function in (16.2), we have

$$(16.5)\qquad \log C(t)=-\frac{f}{2}\log(1-2it)+\frac{B_2}{n}\{(1-2it)^{-1}-1\}-\frac{2B_3}{3n^2}\{(1-2it)^{-2}-1\}$$

$$+\frac{2B_4}{3n^3}\{(1-2it)^{-3}-1\}+O(n^{-4}),$$

where

$$B_2=\frac{1}{4}k(p-k-1)(p+2)+\frac{1}{24}k(2k^2+9k+11),$$

$$B_3=\frac{1}{16}k(k-p+1)\{2(k-p+1)^2-3(k+3)(k-p+1)+2k^2+9k+11\}$$

$$-\frac{1}{32}k(k+1)(k+2)(k+3), \tag{16.6}$$

$$B_4=\frac{k}{16}(k-p+1)\{(k-p+1)^3-2(k+3)(k-p+1)^2+(k-p+1)(2k^2+9k+11)$$

$$-(k+1)(k+2)(k+3)\}+\frac{k}{480}(6k^4+45k^3+110k^2+90k+3).$$

Therefore we have

$$C(t)=(1-2it)^{-\frac{f}{2}}\Big[1+\frac{1}{n}B_2\{(1-2it)^{-1}-1\}+\frac{1}{6n^2}\{(3B_2^2-4B_3)(1-2it)^{-2} \tag{16.7}$$

$$-6B_2^2(1-2it)^{-1}+(3B_2^2+4B_3)\}+\frac{1}{6n^3}\{(4B_4-4B_2B_3+B_2^3)(1-2it)^{-3}$$

$$+B_2(4B_3-2B_2^3)(1-2it)^{-2}+B_2(4B_3+3B_2^3)(1-2it)^{-1}$$

$$-(4B_4+4B_2B_3+B_2^3)\}\Big]+O(n^{-4}).$$

By inverting this characteristic function, we obtain the following theorem:

THEOREM 16.1. *Under the hypothesis H, the asymptotic expansion of the statistic* $-2\log\Lambda$ *defined in* (15.1) *is given by*

$$P(-2\log\Lambda\leq x)=P(\chi_f^2\leq x)+n^{-1}B_2\{P(\chi_{f+2}^2\leq x)-P(\chi_f^2\leq x)\} \tag{16.8}$$

$$+\frac{n^{-2}}{6}\{(3B_2^2-4B_3)P(\chi_{f+4}^2\leq x)-6B_2^2P(\chi_{f+2}^2\leq x)$$

$$+(3B_2^2+4B_3)P(\chi_f^2\leq x)\}+\frac{n^{-3}}{6}\{(4B_4-4B_2B_3+B_2^3)$$

$$\cdot P(\chi^2_{f+6}\leq x)+B_2(4B_3-3B_2^2)P(\chi^2_{f+4}\leq x)$$

$$+B_2(4B_3+3B_2^2)P(\chi^2_{f+2}\leq x)-(4B_4+4B_2B_3+B_2^3)$$

$$\cdot P(\chi^2_f\leq x)\}+O(n^{-4}),$$

where $f=\frac{1}{2}k(k+1)+k(p-k)$ *and the constants* B_r *are given by* (16.6).

This theorem is reduced to ones given by Sugiura [23] and Korin [12] in case $p=k$.

17. Asymptotic expansion under fixed alternative. Now we consider the characteristic function of $-2n^{-\frac{1}{2}}\log \varLambda$ under the alternative K. By Theorem 15.2, the characteristic function of $-2n^{-\frac{1}{2}}\log \varLambda$ is given by

$$C_K(t)=C_1(t)C_2(t)C_3(t), \tag{17.1}$$

where

$$C_1(t)=\left(\frac{n}{2e}\right)^{\sqrt{n}kit}\prod_{j=1}^{k}\frac{\Gamma\left[\frac{1}{2}(n-2it\sqrt{n}-p+k+1-j)\right]}{\Gamma\left[\frac{1}{2}(n-p+k+1-j)\right]}|\Theta_0|^{\sqrt{n}it}|\Omega|^{-\frac{n}{2}}, \tag{17.2}$$

$$C_2(t)=|\Omega_{11\cdot2}^{-1}-\frac{2it}{\sqrt{n}}\Theta_0^{-1}|^{-\frac{n}{2}+\sqrt{n}it}, \tag{17.3}$$

$$C_3(t)=|\Omega_{22}^{-1}+\boldsymbol{B}'\Omega_{11\cdot2}^{-1}\boldsymbol{B}-\boldsymbol{B}'\Omega_{11\cdot2}^{-1}\left(\Omega_{11\cdot2}^{-1}-\frac{2it}{\sqrt{n}}\Theta_0^{-1}\right)^{-1}\Omega_{11\cdot2}^{-1}\boldsymbol{B}|^{-\frac{n}{2}}. \tag{17.4}$$

Applying the asymptotic formula (16.3) to the first factor $C_1(t)$, we can expand asymptotically as follows:

$$\log C_1(t)=-\frac{n}{2}\log|\Omega|+\sqrt{n}it\log|\Theta_0|-itk\sqrt{n}+k(it)^2 \tag{17.5}$$

$$+\frac{1}{\sqrt{n}}\left\{\left(pk-\frac{k^2}{2}+\frac{k}{2}\right)it+\frac{2k}{3}(it)^3\right\}+\frac{1}{n}\left\{\left(pk-\frac{k^2}{2}+\frac{k}{2}\right)(it)^2\right.$$

$$+\frac{2k}{3}(it)^4\Big\}+O(n^{-\frac{3}{2}}).$$

Using the formula (7.17), we can express (17.3) as (17.6).

$$(17.6)\qquad \log C_2(t)=\left(\frac{n}{2}-\sqrt{n}\,it\right)\log|\Omega_{11\cdot 2}|+it\sqrt{n}\,\mathrm{tr}\Theta_0^{-1}\Omega_{11\cdot 2}$$

$$+(it)^2\{\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot 2})^2-2\mathrm{tr}\Theta_0^{-1}\Omega_{11\cdot 2}\}$$

$$+\frac{(it)^3}{\sqrt{n}}\left\{\frac{4}{3}\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot 2})^3-2\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot 2})^2\right\}$$

$$+\frac{(it)^4}{n}\left\{2\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot 2})^4-\frac{8}{3}\mathrm{tr}(\Theta_0^{-1}\Theta_{11\cdot 2})^3\right\}+O(n^{-\frac{3}{2}}).$$

Applying the asymptotic formulae (7.17) and (7.20) to the third factor $C_3(t)$, we obtain

$$(17.7)\qquad \log C_3(t)=\frac{n}{2}\log|\Omega_{22}|+it\sqrt{n}\,\mathrm{tr}G^{(1)}+(it)^2\{2\mathrm{tr}G^{(2)}+\mathrm{tr}G^{(1)^2}\}$$

$$+\frac{4(it)^3}{\sqrt{n}}\left\{\mathrm{tr}G^{(3)}+\mathrm{tr}G^{(1)}G^{(2)}+\frac{1}{3}\mathrm{tr}G^{(1)^3}\right\}+\frac{4(it)^4}{n}\Big\{2\mathrm{tr}\,G^{(4)}$$

$$+\mathrm{tr}G^{(2)^2}+2\mathrm{tr}G^{(1)}G^{(3)}+2\mathrm{tr}\,G^{(1)^2}G^{(2)}+\frac{1}{2}\mathrm{tr}\,G^{(1)^4}\Big\}+O(n^{-\frac{3}{2}}),$$

where matrices $G^{(j)}=F'C^jF(j=1, 2, 3,4)$ defined by $F=\Omega_{11\cdot 2}^{-\frac{1}{2}}\boldsymbol{B}\Omega_{22}^{\frac{1}{2}}$ and $C=\Omega_{11\cdot 2}^{\frac{1}{2}}\cdot\Theta_0^{-1}\Omega_{11\cdot 2}^{\frac{1}{2}}$. Thus we have (17.8) by adding up the expansions (17.5), (17.6) and (17.7).

$$(17.8)\qquad \log C(t)=\sqrt{n}E_1+E_2+\frac{1}{\sqrt{n}}E_3+\frac{1}{n}E_4+O(n^{-\frac{3}{2}}),$$

where the coefficients E_1, E_2, E_3 and E_4 of each term are given by

$$(17.9)\qquad E_1=it\{\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot 2}-I)-\log|\Omega_{11\cdot 2}\Theta_0^{-1}|+\mathrm{tr}G^{(1)}\},$$

$$(17.10)\quad E_2=(it)^2\{\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot 2}-I)^2+2\mathrm{tr}G^{(2)}+\mathrm{tr}G^{(1)2}\},$$

$$(17.11)\quad E_3=it\left(pk-\frac{k^2}{2}+\frac{k}{2}\right)+(it)^3\left\{\frac{2k}{3}+\frac{4}{3}\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^3-2\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^2\right.$$

$$\left.+4\mathrm{tr}G^{(3)}+4\mathrm{tr}G^{(1)}G^{(2)}+\frac{4}{3}\mathrm{tr}\,G^{(1)^3}\right\},$$

$$(17.12)\quad E_4=(it)^2\left(pk-\frac{k^2}{2}+\frac{k}{2}\right)+(it)^4\left\{\frac{2k}{3}+2\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^4-\frac{8}{3}\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^3\right.$$

$$+8\mathrm{tr}G^{(4)}+4\mathrm{tr}G^{(2)^2}+8\mathrm{tr}G^{(1)}G^{(3)}+8\mathrm{tr}G^{(1)^2}G^{(2)}$$

$$\left.+2\mathrm{tr}G^{(1)^4}\right\}.$$

This shows that the statistic $\Lambda^*=-2n^{-\frac{1}{2}}\log\ \Lambda-\sqrt{n}\{\mathrm{tr}\,(\Theta_0^{-1}\,\Omega_{11\cdot2}-I)-$ $\log\ |\Theta^{-1}_0\Omega_{11\cdot2}|+\mathrm{tr}G^{(1)}\}$ converges in law to the normal distribution with mean zero and variance $\tau^2=2\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2}-I)^2+4\mathrm{tr}G^{(2)}+2\mathrm{tr}G^{(1)^2}$and further it enable us to expand the characteristic function of Λ^*/τ up to order n^{-1}, that is,

$$(17.13)\quad C_{\Lambda^*/\tau}(t)=\exp\left[\frac{(it)^2}{2}\right]\left\{1+n^{-\frac{1}{2}}A_1+n^{-1}A_2\right\}+O(n^{-\frac{3}{2}}),$$

where A_1 and A_2 are given by

$$(17.14)\quad A_1=\frac{it}{2}\tau^{-1}(2pk-k^2+k)+\frac{(it)^3}{3}\tau^{-3}\{2k+4\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^3$$

$$-6\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^2\ +12\mathrm{tr}G^{(3)}+12\mathrm{tr}G^{(1)}G^{(2)}+4\mathrm{tr}G^{(1)^3}\},$$

$$(17.15)\ A_2=\frac{1}{8}(it)^2\tau^{-2}(2pk-k^2+k)(2pk-k^2+k+4)+\frac{1}{3}(it)^4\tau^{-4}[2k+k(2pk-k^2+k)$$

$$-3(2pk-k^2+k)\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^2+2(2pk-k^2+k-4)\mathrm{tr}(\Theta^{-1}_0\Omega_{11\cdot2})^3$$

$$+6\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^4+2(2pk-k^2+k)\{3\mathrm{tr}G^{(3)}+3\mathrm{tr}G^{(1)}G^{(2)}$$

$$+\mathrm{tr}G^{(1)^3}\}+24\mathrm{tr}G^{(4)}+12\mathrm{tr}G^{(2)^2}+24\mathrm{tr}G^{(1)}G^{(3)}$$

$$+24\mathrm{tr}G^{(1)^2}G^{(2)}+6\mathrm{tr}G^{(1)^4}]+2(it)^6\tau^{-6}\left\{\frac{k}{3}+\frac{2}{3}\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^3\right.$$

$$-\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^2+2\mathrm{tr}G^{(3)}+2\mathrm{tr}G^{(1)}G^{(2)}+\frac{2}{3}\mathrm{tr}G^{(1)^3}\Big\}^2.$$

Thus, inverting this characteristic function (17.13), we have the following theorem:

THEOREM 17.1. *The distribution of the statistic* $-2\log\Lambda$ *defined by* (15.1) *is expanded asymptotically by, under the fixed alternative,*

$$\begin{aligned}(17.16)\quad & P[n^{-\frac{1}{2}}\tau^{-1}[-2\log\Lambda-\sqrt{n}\{\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2}-I)-\log|\Theta_0^{-1}\Omega_{11\cdot2}| \\ & +\mathrm{tr}G^{(1)}\}]\leq x]=\Phi(x)-n^{-\frac{1}{2}}\Big[\frac{1}{2}\tau^{-1}(2pk-k^2+k)\Phi^{(1)}(x) \\ & +\frac{1}{3}\tau^{-3}\{2k+4\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^3-6\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^2+12\mathrm{tr}G^{(3)} \\ & +12\mathrm{tr}G^{(1)}G^{(2)}+4\mathrm{tr}G^{(1)^3}\}\Phi^{(3)}(x)\Big]+n^{-1}\Sigma_{\alpha=1}^3 g_{2\alpha}\tau^{-2\alpha}\Phi^{(2\alpha)}(x) \\ & +O(n^{-\frac{3}{2}}),\end{aligned}$$

where $\tau^2=2\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2}-I)^2+4\mathrm{tr}G^{(2)}+2\mathrm{tr}G^{(1)^2}$ *with* $G^{(j)}=F'C^jF$, $F=\Omega_{11\cdot2}^{-\frac{1}{2}}\boldsymbol{B}\Omega_{22}^{-\frac{1}{2}}$, $C=\Omega_{11\cdot2}^{\frac{1}{2}}\Theta_0^{-1}\Omega_{11\cdot2}^{\frac{1}{2}}$. *The coefficients* g_2, g_4, g_6 *are given by*

$$\begin{aligned} & g_2=\frac{1}{2}(2pk-k^2+k)+\frac{1}{8}(2pk-k^2+k)^2, \\ & g_4=\frac{1}{6}(2pk-k^2+k)\{2k+4\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^3-6\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^2+12\mathrm{tr}G^{(3)} \\ & \quad +12\mathrm{tr}G^{(1)}G^{(2)}+4\mathrm{tr}G^{(1)^3}\}+\frac{2}{3}k+2\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^4 \\ (17.17)\quad & \quad -\frac{8}{3}\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^3+8\mathrm{tr}G^{(4)}+4\mathrm{tr}G^{(2)^2}+8\mathrm{tr}G^{(1)}G^{(3)} \\ & \quad +8\mathrm{tr}G^{(1)^2}G^{(2)}+2\mathrm{tr}G^{(1)^4}, \\ & g_6=2\Big\{\frac{k}{3}+\frac{2}{3}\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^3-\mathrm{tr}(\Theta_0^{-1}\Omega_{11\cdot2})^2\end{aligned}$$

$$+2\mathrm{tr}G^{(3)}+2\mathrm{tr}G^{(1)}G^{(2)}+\frac{2}{3}\mathrm{tr}G^{(1)^3}\}^2.$$

For the special case $p=k$, Sugiura[25] gave the above theorem.

18. Asymptotic expansions under local alternatives. Since the asymptotic variance of $-2\log\Lambda$ given in Theorem 17.1 vanishes at the null hypothesis, we shall consider the asymptotic expansion under the sequence of alternatives $K_\delta : \Theta_0^{-\frac{1}{2}}\Omega_{11\cdot2}\Theta_0^{-\frac{1}{2}}=I+n^{-\delta}\Theta_1$ and $\boldsymbol{B}\Omega_{22}^{\frac{1}{2}}=n^{-\delta}\Theta_2\left(\delta\geqq\frac{1}{2}\right)$. By Theorem 15.2, we get the characteristic function of $-2\log\Lambda$ under K_δ.

$$(18.1)\qquad C_{K_\delta}(t)=\left(\frac{n}{2e}\right)^{nkit}\prod_{j=1}^{k}\frac{\Gamma\left[\frac{1}{2}(n-2itn-p+k+1-j)\right]}{\Gamma\left[\frac{1}{2}(n-p+k+1-j)\right]}(1-2it)^{-\frac{nk}{2}(1-2it)}$$

$$\cdot|I+n^{-\delta}\Theta_1|^{-nit}\left|I-\frac{2it}{1-2it}n^{-\delta}\Theta_1\right|^{-\frac{n}{2}(1-2it)}$$

$$\cdot|I+n^{-2\delta}\Theta_2'\Theta_0^{-\frac{1}{2}}(I+n^{-\delta}\Theta_1)^{-1}\Theta_0^{-\frac{1}{2}}\Theta_2-n^{-2\delta}\Theta_2'\Theta_0^{-\frac{1}{2}}(I-2itI$$

$$-2itn^{-\delta}\Theta_1)^{-1}(I+n^{-\delta}\Theta_1)^{-1}\Theta_0^{-\frac{1}{2}}\Theta_2|^{-\frac{n}{2}}.$$

The first factor of $C_{K_\delta}(t)$ in (18.1) is the characteristic function of $-2\log\Lambda$ under the null hypothesis, which is asymptotically expanded by (16.7). The second factor, using (7.17) and (7.20), is given by

$$(18.2)\ \exp\left[n^{1-2\delta}\frac{it}{(1-2it)}\left(\frac{1}{2}t_2+r_1\right)+n^{1-3\delta}\left\{\left(\frac{1}{6}t_3+\frac{1}{2}s_1\right)(1-2it)^{-2}-\left(\frac{1}{2}t_3+s_1\right)\right.\right.$$

$$\left.\cdot(1-2it)^{-1}+\left(\frac{1}{3}t_3+\frac{1}{2}s_1\right)\right\}+n^{1-4\delta}\left\{\left(\frac{1}{8}t_4+\frac{1}{2}s_2\right)(1-2it)^{-3}-\left(\frac{1}{2}t_4+\frac{3}{2}s_2\right.\right.$$

$$\left.\left.-\frac{1}{4}r_2\right)(1-2it)^{-2}+\left(\frac{3}{4}t_4+\frac{3}{2}s_2-\frac{1}{2}r_2\right)(1-2it)^{-1}-\left(\frac{3}{8}t_4+\frac{1}{2}s_2-\frac{1}{4}r_2\right)\right\}$$

$$\left.+O(n^{1-5\delta})\right],$$

where $t_j=\mathrm{tr}\Theta_1^j$, $r_j=\mathrm{tr}(\Theta_2'\Theta_0^{-1}\Theta_2)^j$ and $s_j=\mathrm{tr}\Theta_2'\Theta_0^{-\frac{1}{2}}\Theta_1^j\Theta_0^{-\frac{1}{2}}\Theta_2$ for abbreviation.

Multiplying (16.7) to (18.2), we can see that

$$(18.3)\qquad C_{K_\delta}(t)=(1-2it)^{-\frac{f}{2}}+o(1)\quad \text{when } \delta>\frac{1}{2}$$

$$=(1-2it)^{-\frac{f}{2}}\exp\left[\frac{it}{(1-2it)}\left\{\frac{1}{2}\mathrm{tr}\Theta_1^2+\mathrm{tr}\,\Theta_2'\Theta_0^{-1}\Theta_2\right\}\right]+o(1)$$

$$\text{when } \delta=\frac{1}{2},$$

which implies that the limiting distribution of $-2\log\Lambda$ is χ^2 with $f=\frac{1}{2}k(k+1)+k(p-k)$ degrees of freedom when $\delta>\frac{1}{2}$ and non-central χ^2 with f degrees of freedom and noncentrality parameter $\Delta^2=\frac{1}{4}\mathrm{tr}\,\Theta_1^2+\frac{1}{2}\mathrm{tr}\Theta_2'\Theta_0^{-1}\Theta_2$ when $\delta=\frac{1}{2}$. Further we can get the asymptotic expansion of the characteristic function of $-2\log\Lambda$ when $\delta=\frac{1}{2}$ as follows:

$$(18.4)\ C_{K_{\frac{1}{2}}}(t)=(1-2it)^{-\frac{f}{2}}\exp\left[\frac{2it}{(1-2it)}\Delta^2\right]\Big[1+n^{-\frac{1}{2}}\Big\{\Big(\frac{1}{6}t_3+\frac{1}{2}s_1\Big)(1-2it)^{-2}$$

$$-\Big(\frac{1}{2}t_3+s_1\Big)(1-2it)^{-1}+\Big(\frac{1}{3}t_3+\frac{1}{2}s_1\Big)\Big\}+n^{-1}\textstyle\sum_{\alpha=0}^{4}g_{2\alpha}(1-2it)^{-\alpha}$$

$$+n^{-1}B_2\{(1-2it)^{-1}-1\}+O(n^{-\frac{3}{2}})\Big],$$

where the coefficients $g_{2\alpha}$ are given by

$$g_0=-\Big(\frac{3}{8}t_4+\frac{1}{2}s_2-\frac{1}{4}r_2\Big)+\frac{1}{2}\Big(\frac{1}{3}t_3+\frac{1}{2}s_1\Big)^2,$$

$$g_2=\Big(\frac{3}{4}t_4+\frac{3}{2}s_2-\frac{1}{2}r_2\Big)-\Big(\frac{1}{2}t_3+s_1\Big)\Big(\frac{1}{3}t_3+\frac{1}{2}s_1\Big),$$

$$(18.5)\qquad g_4=-\Big(\frac{1}{2}t_4+\frac{3}{2}s_2-\frac{1}{4}r_2\Big)+\frac{1}{2}\Big(\frac{1}{2}t_3+s_1\Big)^2+\Big(\frac{1}{6}t_3+\frac{1}{2}s_1\Big)\Big(\frac{1}{3}t_3+\frac{1}{2}s_1\Big),$$

$$g_6=\Big(\frac{1}{8}t_4+\frac{1}{2}s_2\Big)-\Big(\frac{1}{6}t_3+\frac{1}{2}s_1\Big)\Big(\frac{1}{2}t_3+s_1\Big),$$

$$g_8=\frac{1}{2}\left(\frac{1}{6}t_3+\frac{1}{2}s_1\right)^2.$$

When $\delta=1$, we can get another asymptotic formula from (16.7) and (18.2) as (18.6).

$$\text{(18.6)}\quad C_{K_1}(t)=(1-2it)^{-\frac{f}{2}}\Big[1+n^{-1}\Big\{\Big(\frac{1}{4}t_2+\frac{1}{2}r_1+B_2\Big)(1-2it)^{-1}-\Big(\frac{1}{4}t_2+\frac{1}{2}r_1+B_2\Big)\Big\}$$

$$+\frac{n^{-2}}{6}\{(3B_2^2-4B_3)(1-2it)^{-2}-6B_2^2(1-2it)^{-1}+(3B_2^2+4B_3)\}$$

$$+n^{-2}\sum_{\alpha=0}^{2}g'_{2\alpha}(1-2it)^{-\alpha}+O(n^{-3})\Big],$$

where coefficients are given by

$$g'_0=\frac{B_2}{4}(t_2+2r_1)+\frac{1}{3}t_3+\frac{1}{2}s_1+\frac{1}{8}\left(\frac{1}{2}t_2+r_1\right)^2,$$

$$\text{(18.7)}\qquad g'_2=-\frac{B_2}{2}(t_2+2r_1)-\frac{1}{2}(t_3+2s_1)-\frac{1}{4}\left(\frac{1}{2}t_2+r_1\right)^2,$$

$$g'_4=\frac{B_2}{4}(t_2+2r_1)+\frac{1}{6}t_3+\frac{1}{2}s_1+\frac{1}{8}\left(\frac{1}{2}t_2+r_1\right)^2.$$

Hence, inverting the characteristic functions (18.4) and (18.6), we have the following theorem:

THEOREM 18.1. *Under the sequence of alternatives* K_δ: $\Theta_0^{-\frac{1}{2}}\Omega_{11\cdot 2}\Theta_0^{-\frac{1}{2}}=I+n^{-\delta}\theta_1$ *and* $\boldsymbol{B}\Omega_{22}^{-\frac{1}{2}}=n^{-\delta}\theta_2$, *when* $\delta=\frac{1}{2}$,

$$\text{(18.8)}\qquad P(-2\log\Lambda\leq x)=P(\chi_f^2(\Delta^2)\leq x)+n^{-\frac{1}{2}}\Big\{\Big(\frac{1}{6}t_3+\frac{1}{2}s_1\Big)P(\chi_{f+4}^2(\Delta^2)\leq x)$$

$$-\Big(\frac{1}{2}t_3+s_1\Big)P(\chi_{f+2}^2(\Delta^2)\leq x)+\Big(\frac{1}{3}t_3+\frac{1}{2}s_1\Big)P(\chi_f^2(\Delta^2)\leq x)\Big\}$$

$$+n^{-1}B_2\{P(\chi_{f+2}^2(\Delta^2)\leq x)-P\chi_f^2(\Delta^2)\leq x)\}$$

$$+n^{-1}\sum_{\alpha=0}^{4}g_{2\alpha}P(\chi_{f+2\alpha}^2(\Delta^2)\leq x)+O(n^{-\frac{3}{2}}),$$

where $f=\frac{1}{2}k(k+1)+k(p-k)$ *and* $\Delta^2=\frac{1}{4}\mathrm{tr}\,\Theta_1^2+\frac{1}{2}\mathrm{tr}\Theta_2'\Theta_0^{-1}\Theta_2$, $t_j=\mathrm{tr}\,\Theta_1^j$, $r_j=\mathrm{tr}\,(\Theta_2'\Theta_0^{-1}\Theta_2)^j$, $s_j=\mathrm{tr}\Theta_2'\Theta_0^{-\frac{1}{2}}\Theta_1^j\Theta_0^{-\frac{1}{2}}\Theta_2$. *The coefficients* $g_{2\alpha}$ *and* B_2 *are given by* (18.5) *and* (16.6) *respectively. When* $\delta=1$, *we have*

$$(18.9)\qquad P(-2\log\Lambda\leq x)=P(\chi_f^2\leq x)+n^{-1}\Big(\frac{1}{4}t_2+\frac{1}{2}r_1+B_2\Big)\{P(\chi_{f+2}^2\leq x)$$

$$-P(\chi_f^2\leq x)\}+\frac{n^{-2}}{6}\{(3B_2^2-4B_3)P(\chi_{f+4}^2\leq x)$$

$$-6B_2^2P(\chi_{f+2}^2\leq x)+(3B_2^2+4B_3)P(\chi_f^2\leq x)\}$$

$$+n^{-2}\textstyle\sum_{\alpha=0}^{2}g_{2\alpha}'P(\chi_{f+2\alpha}^2\leq x)+O(n^{-3}),$$

where $B_\alpha(\alpha=2,\ 3)$ *and the coefficients* $g_{2\alpha}'$ *are given by* (16.6) *and* (18.7) *respectively.*

19. Numerical examples. Applying the general inverse expansion formula of Hill and Davis [11] to the asymptotic null distribution of $-2\log\Lambda$ given by Theorem 16.1, we can get the asymptotic formula of $100\alpha\%$ point of $-2\log\Lambda$ in terms of the $100\alpha\%$ point of the χ^2 distribution with $f=\frac{1}{2}k(k+1)+k(p-k)$ degrees of freedom as follows:

$$(19.1)\qquad u+\frac{2B_2}{nf}u+\frac{1}{3n^2}\Big\{\frac{u^2}{f^2(f+2)}(-6B_2^2-4fB_3)+\frac{u}{f^2}(6B_2^2-4fB_3)\Big\}$$

$$+\frac{1}{n^3}(u^3g_3+u^2g_2+ug_1)+O(n^{-4}),$$

where u is chosen such that $P(\chi_f^2>u)=\alpha$ and

$$(19.2)\qquad g_1=\frac{4}{3f^3}(f^2B_4-2fB_2B_3+B_2^3),$$

$$g_2=\frac{4}{3f^3(f+2)}(f^2B_4-2fB_2B_3-5B_2^3),$$

$$g_3 = \frac{4}{3f^3(f+2)(f+4)}(f^2B_4+4fB_2B_3+4B_2^3),$$

with B_2, B_3, B_4 in (16.6). This formula (19.1) is the same as that given by Sugiura [24] in the case of $p=k$.

EXAMPLE 19.1. When $p=2$, $k=1$ and $n=50$,the approximate 5% point of $-2\log\Lambda$ can be obtained from (19.1).

first term	5.99147
term of order n^{-1}	0.10984
term of order n^{-2}	0.00249
term of order n^{-3}	0.00005
approx. value	6.1039

Since the distribution under the null hypothesis is independent of θ_0, Γ_{10} and Γ_2, the above 5% point is of the null hypothesis $\Omega=\text{diag }(a, b)$ with known constant $a>0$ and unknown constant $b>0$. For the specified hypothesis $H:\Omega=\text{diag }(1,b)$ against the alternative K:

$$\Omega = \begin{pmatrix} 1 & \Delta \\ \Delta & 2 \end{pmatrix}, \tag{19.3}$$

the following approximate powers are computed by the different formulae according to the value of Δ.

	(17.6)	(18.8)	(18.9)
	normal approximation;	noncentral χ^2approximation;	χ^2 approximation
	$\Delta = 1$	$\Delta = 0.2$	$\Delta = 0.01$
first term	0.4521	0.1283	0.0473
second term	0.0499	-0.0004	0.0035
third term	0.0001	0.0034	0.0001
approx. power	0.502	0.131	0.051

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