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The Stable Homotopy Groups of Spheres I

Shichirô OKA

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Introduction and Summary

In this paper, p denotes always an odd prime number.

Let $\boldsymbol{K}_k=\{K_k(n)\}$ be the spectrum such that $K_k(n)$ is k-th element of the Postnikov system over S^n (see (1.1) of §1) and $\boldsymbol{S}=\{S^n\}$ be the sphere spectrum.

In [6: III], H. Toda has calculated $H^*(\boldsymbol{K}_k)$[1] for $k\leqq 2(p^2-1)(p-1)-2$ as the module over A^*, the Steenrod algebra mod p, by making use of several exact sequences of A^*-modules and by the induction on k using Lemmas 3.3 and 3.4 of [6: III], which are stated in Proposition 1.2 of §1.

Also in [6: III], for $k<2p^2(p-1)-3$, the p-primary component ${}_p\pi_k(\boldsymbol{S})$ of the k-th stable homotopy group $\pi_k(\boldsymbol{S})$ of spheres has been determined from the above results on $H^*(\boldsymbol{K}_k)$ by use of Lemma 3.1 of [6: III], which is quoted in (1.4) of §1. Furthermore in [6: IV], he has given the generators of ${}_p\pi_k(\boldsymbol{S})$ by means of the compositions and the secondary ones in the same range of k.

The purpose of this paper is to calculate $H^*(\boldsymbol{K}_k)$ and ${}_p\pi_k(\boldsymbol{S})$ for $k\leqq 2(p^2+p)(p-1)-3$ by use of the methods introduced by H. Toda [6] and some relations in ${}_p\pi_k(\boldsymbol{S})$. The results are summarized in Theorems 4.1, 4.4, 5.1, 6.2, 7.8 and 7.9. The exact sequences presented in [4] together with the ones in [6:I] are used in these calculations.

The beginning of our calculations is summarized in Theorem 4.1 of §4, which is due to H. Toda [6: III]. For further calculation, we use the following two relations:

$$\mathscr{P}^1 b_1^{p-1}=d_1,\ \Delta d_1=\mathscr{P}^{p^2}a_0 \text{ in } H^*(\boldsymbol{K}_{2(p^2-1)(p-1)-2}),$$

which are given in Lemma 4.2 of §4.

Using these results and the exact sequences in [4] and [6: I], $H^*(\boldsymbol{K}_{2(p^2-1)(p-1)-1})$ is determined (Theorem 4.4), and also $H^*(K_k)$, $2(p^2-1)(p-1)\leqq k\leqq 2(p^2+p-2)(p-1)-2$, in certain dimensional restriction (Theorem 5.1).

These results enable us to calculate the group ${}_p\pi_k(\boldsymbol{S})$ for $k\leqq 2(p^2+p-1)\cdot(p-1)-4$ (Theorem 6.2). In addition, theorems presented in §3 give the description of generators of ${}_p\pi_k(\boldsymbol{S})$ based on the compositions and the secondary compositions.

1) In this paper, the cohomology $H^*(\ )$ will be understood to have Z_p for coefficients.

For funrther calculation, it is necessary that the following two coefficients $x_3, x_4 \in Z_p$ are determined:

(7.1) $\mathscr{P}^1 e_{p-2}=x_3 b_p^0,\quad \mathscr{P}^1 \Delta e_{p-2}=x_4 \Delta b_p^0\quad in\quad H^*(\boldsymbol{K}_{2(p^2+p-2)(p-1)-2})$.

In [7], H. Toda has calculated the (unstable) homotopy groups ${}_p\pi_{2n+1+k}(S^{2n+1})$ for $k<2(p^2+p)(p-1)-5$ by the methods which differ from [6]. By use of the result ${}_p\pi_{2(p^2+p-1)(p-1)-3}(\boldsymbol{S})=0$ of [7], we obtain $x_3 \neq 0$ (Lemma 7.1). To determine x_4, in §8, we continue the calculations of [7] and obtain the partial results on ${}_p\pi_{2(p^2+p)(p-1)-3}(\boldsymbol{S})$. These imply $x_4=2x_3\neq 0$ (Proposition 7.7), and so $H^*(\boldsymbol{K}_k)$ is determined for $2(p^2+p-2)(p-1)-2\leqq k\leqq 2(p^2+p)(p-1)-3$ under certain dimensional restriction (Theorem 7.8).

In the forthcoming paper of the same title [5], we shall calculate the module $H^*(\boldsymbol{K}_k)$ for $k\leqq 2(p^2+3p)(p-1)-4$, $p>3$ and for $k\leqq 74$, $p=3$, and the group ${}_p\pi_k(\boldsymbol{S})$ for $k<2(p^2+3p+1)(p-1)-5$, $p>3$ and for $k\leqq 76$, $p=3$.

The contents of this paper are as follows: In §1, we review the method of H. Toda [6]. Section 2 is devoted to introducing some known facts on $H^*(\boldsymbol{K}_k)$ and ${}_p\pi_k(\boldsymbol{S})$, which are used in §§3–4. In §3, we discuss the relationships between some special relations in $H^*(\boldsymbol{K}_k)$ and the compositions in ${}_p\pi_k(\boldsymbol{S})$. The module $H^*(\boldsymbol{K}_k)$ is calculated for $k\leqq 2(p^2-1)(p-1)-1$ in §4, and for $2(p^2-1)(p-1)\leqq k\leqq 2(p^2+p-2)(p-1)-2$ under degree $<2(p^2+p+1)(p-1)-3$ in §5. Using the results in §§3–5, ${}_p\pi_k(\boldsymbol{S})$ is calculated for $k\leqq 2(p^2+p-1)(p-1)-4$ in §6. In §7, the non-triviality of the coefficients x_3 and x_4 is discussed by use of Propositions 7.5–6, and $H^*(\boldsymbol{K}_k)$ is calculated for $2(p^2+p-2)(p-1)-2\leqq k\leqq 2(p^2+p)(p-1)-3$. Also ${}_p\pi_k(\boldsymbol{S})$ for $2(p^2+p-2)(p-1)-3\leqq k\leqq 2(p^2+p)(p-1)-3$. In §8, by means of the methods established by H. Toda [7], the unstable group ${}_p\pi_{2n+1+k}(S^{2n+1})$ is calculated partially for $2(p^2+p)(p-1)-5\leqq k\leqq 2(p^2+p)(p-1)-2$, and in particular, ${}_p\pi_{2(p^2+p)(p-1)-3}(\boldsymbol{S})$ is determined for $p=3$ (Proposition 7.5). Moreover, by those methods together with the results of [3] [10], the non-triviality of the element $\alpha_1\varepsilon_{p-1}$ is proved for $p>3$ (Proposition 7.6).

§ 1. Postnikov system over spheres

In this section, we shall review the methods of H. Toda [6: III] (cf. [1]).

Let $\boldsymbol{K}_k=\{K_k(n)\}$ be the spectrum such that $K_k(n)$ is the k-th element of the Postnikov system over S^n. The indexing is given by

$$\text{(1.1)}\qquad \begin{aligned} &\pi_{j+n}(K_k(n))=0 && for\quad j\geqq k,\\ &i_*\colon \pi_{j+n}(S^n)\xrightarrow{\approx}\pi_{j+n}(K_k(n)) && for\quad j<k. \end{aligned}$$

Let $\boldsymbol{S}=\{S^n\}$ and $\boldsymbol{K}(G)=\{K(G,n)\}$ be the sphere spectrum and the

Eilenberg-MacLane spectrum respectively. The fibering i: $K_{k+1}(n)\rightarrow K_k(n)$ with the fiber $K(\pi_{n+k}(S^n),\ n+k)$ gives rise to an exact sequence of the cohomology of spectra, which is the sequence (3.1) of [6: III]:

$$\cdots\xrightarrow{j^*}H^i(\boldsymbol{K}_k)\xrightarrow{i^*}H^i(\boldsymbol{K}_{k+1})\xrightarrow{\delta^*}H^{i-k}({}_p\pi_k(\boldsymbol{S}))\xrightarrow{j^*}H^{i+1}(\boldsymbol{K}_k)\xrightarrow{i^*}\cdots, \tag{1.2}$$

where $H^n(G)=H^n(\boldsymbol{K}(G))$, ${}_pG$ denotes the p-component of G for any finitely generated abelian group G, and the cohomology $H^*(\ )$ is understood to have Z_p for coefficients.

By (1.1) and (1.2), it follows that

(1.3) ([6: III, (3.3)]) *$H^i(\boldsymbol{K}_k)=0$ for $0<i<k+1$ and j^*: $H^0({}_p\pi_k(\boldsymbol{S}))\rightarrow H^{k+1}(\boldsymbol{K}_k)$ is isomorphic. j^*: $H^1({}_p\pi_k(\boldsymbol{S}))\rightarrow H^{k+2}(\boldsymbol{K}_k)$ is monomorphic.*

Let Δ_r: $H^i(\ )\cap\mathrm{Ker}\ \Delta_{r-1}\rightarrow H^{i+1}(\ )/\mathrm{Im}\Delta_{r-1}$ ($\Delta_1=\Delta$) be the higher Bockstein operation of r-th kind. The following two statements are Lemmas 3.1 and 3.2 of [6: III] and are used to determine ${}_p\pi_k(\boldsymbol{S})$ in this paper.

(1.4) *The number of the direct summands of $\pi_k(\boldsymbol{S})$ which are isomorphic to Z_{p^r} is equal to the rank of the image of*

$$\Delta_r\colon H^{k+1}(\boldsymbol{K}_k)\cap\ \mathrm{Ker}\,\Delta_{r-1}\rightarrow H^{k+2}(\boldsymbol{K}_k)/\mathrm{Im}\Delta_{r-1}.$$

(1.5) *If $H^i(\boldsymbol{K}_k)=0$ for $0<i\leqq k+r$, $r>0$, then ${}_p\pi_j(\boldsymbol{S})=0$ for $k\leqq j<k+r$ and i^*: $H^*(\boldsymbol{K}_k)\rightarrow H^*(\boldsymbol{K}_j)$ is isomorphic for $k<j\leqq k+r$.*

The module $H^*({}_p\pi_k(\boldsymbol{S}))$ is the direct sum of the copies of A^* and $A^*/A^*\Delta$, where A^* denotes the Steenrod algebra mod p. Thus $H^i({}_p\pi_k(\boldsymbol{S}))=0$ for $2\leqq i\leqq 2p-3$ and $H^1({}_p\pi_k(\boldsymbol{S}))=0$ $(k>0)$ if and only if ${}_p\pi_k(\boldsymbol{S})=0$. This implies the following

LEMMA 1.1. (i) *The map i^*: $H^{k+1}(\boldsymbol{K}_{k-j})\rightarrow H^{k+1}(\boldsymbol{K}_k)$ is epimorphic for $0\leqq j\leqq 2p-4$.*

(ii) *Let $k>1$. The map i^*: $H^{k+1}(\boldsymbol{K}_{k-j})\rightarrow H^{k+1}(\boldsymbol{K}_{k-1})$ is monomorphic for $1\leqq j\leqq 2p-3$. i^*: $H^{k+1}(\boldsymbol{K}_{k-1})\rightarrow H^{k+1}(\boldsymbol{K}_k)$ is so if and only if ${}_p\pi_{k-1}(\boldsymbol{S})=0$.*

We can consider that the vector spaces $H^{k+1}(\boldsymbol{K}_k)$ and $H^{k+2}(\boldsymbol{K}_k)$ are given by

$$\begin{aligned} &H^{k+1}(\boldsymbol{K}_k)=Z_p\{a_i,\ b_j;\ 1\leqq i\leqq r,\ 1\leqq j\leqq s\},\\ &H^{k+2}(\boldsymbol{K}_k)=Z_p\{a_i',\ \Delta b_j,\ c_1,\ c_2,\cdots;\ 1\leqq i\leqq r,\ 1\leqq j\leqq s\},\end{aligned} \tag{1.6}$$

where $\Delta a_i=0$, $\Delta_{k_i}a_i=a_i'(k_i\geqq 2)$ and $Z_p\{d_1,\cdots,\ d_n\}$ denotes the vector space over Z_p spanned by the elements $d_1,\ \cdots,\ d_n$. Then we have

$$\begin{aligned} H^*({}_p\pi_k(\boldsymbol{S}))&=\sum_i A^*j^{*-1}a_i+\sum_i A^*j^{*-1}a_i'+\sum_j A^*j^{*-1}b_j\\ &\approx(\overset{r}{\oplus}A^*/A^*\Delta)\oplus(\overset{r}{\oplus}A^*/A^*\Delta)\oplus(\overset{s}{\oplus}A^*),\\ {}_p\pi_k(\boldsymbol{S})&\approx(\bigoplus_{i=1}^{r}Z_{t_i})\oplus(\overset{s}{\oplus}Z_p),\quad t_i=p^{k_i}.\end{aligned} \tag{1.7}$$

To determine the module structure of $H^*(\boldsymbol{K}_k)$ by the induction on k, we shall employ the following proposition, which is proved from (1.2) similarly to Lemmas 3.3 and 3.4 of [6 :III].

PROPOSITION 1.2. *In the notation* (1.6), *let*

$$\{\sum_i \alpha_{i,l}a_i + \sum_i \alpha'_{i,l}a'_i + \sum_j \beta_{j,l}b_j = 0; \quad l=1, 2,\cdots\}$$

be the system of relations in the submodule $\sum_i A^*a_i + \sum_i A^*a'_i + \sum_j A^*b_j$ *of* $H^*(\boldsymbol{K}_k)$ *and let*

$$\{\sum_l \gamma_{l,m}B_l = 0;\ m=1, 2,\cdots\},\quad B_l=(\alpha_{1,l},\cdots, \alpha_{r,l}, \alpha'_{1,l},\cdots, \alpha'_{r,l}, \beta_{1,l},\cdots, \beta_{s,l}),$$

be the system of relations in the submodule $\sum_l A^*B_l$ *of* $(\overset{r}{\oplus}A^*/A^*\varDelta)\oplus(\overset{r}{\oplus}A^*/A^*\varDelta)\oplus(\overset{s}{\oplus}A^*)$. *Then there exist elements*

$$d_l \in H^*(\boldsymbol{K}_{k+1}) \ \text{and}\ w_m \in H^*(\boldsymbol{K}_k)$$

such that

$$\delta^* d_l = \sum_i \alpha_{i,l}j^{*-1}a_i + \sum_i \alpha'_{i,l}j^{*-1}a'_i + \sum_j \beta_{j,l}j^{*-1}b_j, \quad \sum_l \gamma_{l,m}d_l = i^*w_m.$$

Let $\{e_n;\ n=1, 2,\cdots\}$ *and* $\{r_q=0;\ q=1, 2,\cdots\}$ *be the systems of generators and of relations of* $H^*(\boldsymbol{K}_k)$, *then* $H^*(\boldsymbol{K}_{k+1})$ *has the systems of generators* $\{i^*e_n, d_l\}$ *and of relations* $\{i^*r_q=0,\ i^*a_i=i^*a'_i=i^*b_j=0,\ \sum_l \gamma_{l,m}d_l - i^*w_m=0\}$.

§2. Some known results on $H^*(\boldsymbol{K}_k)$ and ${}_p\pi_k(\boldsymbol{S})$

Let a be an element of $H^i(\boldsymbol{K}_k)$. Then, following to H. Toda [6: III], we denote by a *in* $\boldsymbol{K}_l$ or simply a $(l\geqq k)$ the image of a under the map i^*: $H^*(\boldsymbol{K}_k)\to H^*(\boldsymbol{K}_l)$. Moreover, when $n>i-k$, by the stability $H^{i+n}(K_k(n))=H^i(\boldsymbol{K}_k)$, we use the same letter a for the corresponding element of $H^{i+n}(K_k(n))$. In particular, let $a_0 \in H^0(\boldsymbol{K}_k)=H^n(K_k(n))=Z_p$ denote a generator.

We shall define a map ϕ: ${}_p\pi_k(\boldsymbol{S})\to H^{k+1}(\boldsymbol{K}_k)$ as follows:

$$\begin{aligned}(2.1)\qquad \phi:\ {}_p\pi_k(\boldsymbol{S}) \underset{\approx}{\xleftarrow{\partial}} {}_p\pi_{k+1}(\boldsymbol{K}_k, \boldsymbol{S}) &\underset{\approx}{\xrightarrow{H}} H_{k+1}(\boldsymbol{K}_k, \boldsymbol{S}; Z)\\ &\xrightarrow{\rho} H_{k+1}(\boldsymbol{K}_k, \boldsymbol{S}) \underset{\approx}{\longleftarrow} H_{k+1}(\boldsymbol{K}_k) \underset{\approx}{\xrightarrow{D}} H^{k+1}(\boldsymbol{K}_k),\end{aligned}$$

where H, ρ and D denote the Hurewicz homomorphism, the reduction mod p and the duality map respectively.

Any element $a \in H^{k+1}(\boldsymbol{K}_k)$, $\varDelta_{r-1}a=0$, $\varDelta_r a\neq 0$, gives rise to a direct summand Z_{p^r} of ${}_p\pi_k(\boldsymbol{S})$ by (1.4). Then an element $\gamma \in {}_p\pi_k(\boldsymbol{S})$ generates this summand if $\phi(\gamma)=a$ (see [6: III, pp. 192–193]).

DEFINITION 2.1. Any element $a \in H^{k+1}(\boldsymbol{K}_k)$ together with the element $a_0 \in H^0(\boldsymbol{K}_k)$ forms a subcomplex of $K_k(n)$ (up to homotopy type mod p) by the first statement of (1.3). We denote this complex by $P_k^n(a)$. In more detail, there exist a complex

$$P_k^n(a)=S^n \cup e^{n+k+1}$$

and a map f_n: $P_k^n(a) \to K_k(n)$ such that $\tilde{H}^*(P_k^n(a))$ is spanned by the elements $f_n^*(a_0)$ and $f_n^*(a)$. Moreover we denote by $\boldsymbol{P}_k(a)=\{P_k^n(a)\}$ the spectrum of these subcomplexes. This spectrum is stable, since $SP_k^n(a)=P_k^{n+1}(a)$ for $n>k+1$.

LEMMA 2.2. *Let $a \in H^{k+1}(\boldsymbol{K}_k)$ and let $\gamma \in {}_p\pi_k(\boldsymbol{S})$ denote the attaching class of $(n+k+1)$-cell of $P_k^n(a)$. Then $\phi(\gamma)=a$.*

PROOF. Comparing the diagram (2.1) and the diagram which is obtained by the replacement of $\boldsymbol{K}_k$ by $\boldsymbol{P}_k(a)$ in (2.1), this lemma follows immediately.
Q.E.D.

Since $\boldsymbol{K}_1=\boldsymbol{K}(Z)$, we have

(2.2) *The module $H^*(\boldsymbol{K}_1)$ is generated by a_0 with the relation $\Delta a_0=0$.*

By use of (1.4) and (1.5), we have

(2.3) $H^*(\boldsymbol{K}_k) \approx H^*(\boldsymbol{K}_1)$ *for* $k \leqq 2p-3$, ${}_p\pi_k(\boldsymbol{S})=0$ *for* $1 \leqq k \leqq 2p-4$ *and* ${}_p\pi_{2p-3}(\boldsymbol{S}) \approx Z_p$.

Since $H^{2p-2}(\boldsymbol{K}_{2p-3})=Z_p\{\mathscr{P}^1 a_0\}$, we obtain a well-known fact: *the generator α_1 of ${}_p\pi_{2p-3}(\boldsymbol{S}) \approx Z_p$ is detected by $\mathscr{P}^1$ operation.*

The module $H^*(\boldsymbol{K}_k)$, $k \leqq 2p(p-2)-2$, is calculated by H. Toda in Theorems 3.6, 3.7 and Lemma 3.8 of [6: III].

THEOREM 2.3 (Toda). *Let $2(p-1) \leqq k \leqq 2p(p-1)-2$. Then $H^*(\boldsymbol{K}_k)$ has a minimal set of generators which is given by the following*

TABLE A1

Generator a	Degree of a	Range of k in which a exists	δ^*-image of a in $\boldsymbol{K}_{h(a)}$
a_0	0	$k \geqq 1$	
a_r $(2 \leqq r \leqq p)$	$2r(p-1)$	$k \geqq 2(r-1)(p-1)=h(a_r)$	$\delta^* a_2=R_1 j^{*-1}(\mathscr{P}^1 a_0)$ $\delta^* a_r=R_{r-1} j^{*-1} a_{r-1} (3 \leqq r \leqq p)$
a_p'	$2p(p-1)+1$	$k \geqq 2(p-1)(p-1)=h(a_p')$	$\delta^* a_p'=\Delta \mathscr{P}^1 \Delta j^{*-1} a_{p-1}$
b_1^0	$2p(p-1)-1$	$k \geqq 2(p-1)=h(b_1^0)$	$\delta^* b_1^0=\mathscr{P}^{p-1} j^{*-1}(\mathscr{P}^1 a_0)$

$(R_r=(r+1)\mathscr{P}^1\Delta-r\Delta\mathscr{P}^1)$.

The relations in $H^*(\boldsymbol{K}_k)$ are given by the relations in Table B1 *below.*

TABLE B1

(a-1) $\Delta a_0=\mathscr{P}^1 a_0=0$.	(a-2) $R_r a_r=\Delta\mathscr{P}^1\Delta a_{p-1}=\Delta a_p=\Delta a'_p=\Delta\mathscr{P}^1 a_p-\mathscr{P}^1 a'_p=0$.
(b-1) $\mathscr{P}^p a_0-\Delta b_1^0-\mathscr{P}^{p-2}a_2=0$.	(b-2) $\mathscr{P}^1 b_1^0=0$.
(l) $a=0$ *in* $\boldsymbol{K}_k$, $k\geqq\deg a$, *for the generator* $a\neq a_0$ *in Table* A1. *The relation* ($b-1$) *induces the following*	
(b-3) $(2\mathscr{P}^p\mathscr{P}^1-\mathscr{P}^{p+1})\Delta b_1^0=c(\mathscr{P}^{p(p-1)})\Delta b_1^0=0$ *in* $\boldsymbol{K}_k$, $k\geqq 4(p-1)$, *where* c: $A^*\to A^*$ *denotes the conjugation of* A^*.	

From this theorem and (1.4), ${}_p\pi_k(S)$ is calculated for $k\leqq 2p(p-1)-2$.

COROLLARY 2.4. *The group* ${}_p\pi_k(S)$ *is isomorphic to* Z_p *for* $k=2r(p-1)-1$, $1\leqq r\leqq p-1$, *and for* $k=2p(p-1)-2$, *and vanishes for other* $k\leqq 2p(p-1)-2$.

Now the element b_1^0 of Table A1 gives rise to a generator β_1 of ${}_p\pi_{2p(p-1)-2}(S)$. According to the relation ($b-1$) of Table B1, the generator β_1 can be determined uniquely by the following

(2.4) (see e.g. [3: Remark in p. 172], [7: III, p. 102]). *A map* f: $S^{n+2p(p-1)-2}\to S^n$ *of order* p *represents* β_1 *if and only if* $\mathscr{P}^p u=(-1)^n v$ *in* $H^*(L)$, *where* $L=S^n\cup e^{i-1}\cup e^i$, $i=n+2p(p-1)$, *is a complex such that the map* f *(resp. a map* $S^{i-1}\to S^{i-1}$ *of degree* p*) is the attaching map of* $(i-1)$*-cell (resp.* i*-cell) in* L *(resp.* L/S^n*), and* $u\in H^n(L)$ *and* $v\in H^i(L)$ *are the generators corresponding to the cells of* L.

From the results on ${}_p\pi_k(S)$ of Corollary 2.4, we have the following facts about the element β_1.

LEMMA 2.5 (see [6: IV, Lemma 4.10]). (i) *For* $0\leqq i<p$, *there exists a complex* $L_i^n=S^n\cup e^{n+2(p-1)}\cup\cdots\cup e^{n+2i(p-1)}$ *such that* $H^{n+2k(p-1)}(L_i^n)=Z_p$ *is spanned by* $\mathscr{P}^k u$ *for* $u\in H^n(L_i^n)$, $0\leqq k\leqq i$.

(ii) *There exist maps* A: $L_{p-2}^{n+2p-3}\to S^n$ *and* B: $S^{n+2p(p-1)-2}\to L_{p-2}^{n+2p-3}$ *such that the composition* AB *represents an element* $x\beta_1$, $x\not\equiv 0$ mod p, *and that both* Aj *and* πB *represent* α_1, *where* j: $S^n\to L_i^n$ *and* π: $L_i^n\to S^{n+2i(p-1)}$ *denote the inclusion and the projection respectively.*

(iii) *There exist maps* A': $L_{p-3}^{n+4p-5}\to L_1^n$ *and* B': $S^{n+2p(p-1)-2}\to L_{p-3}^{n+4p-5}$ *such that* $A'B'$ *represents* $j_*(y\beta_1)$, $y\not\equiv 0$ mod p, *and that both* $\pi A'j$ *and* $\pi B'$ *represent* α_1.

PROOF. (i) and (ii) are proved in [6: IV, Lemma 4.10]. The maps A' and B' are constructed by the following homotopy commutative diagram of cofiberings:

$$\begin{array}{ccccc}
\textit{point} & \longrightarrow & S^{n+2p(p-1)-2} & \xrightarrow{\mathrm{id}} & S^{n+2p(p-1)-2} \\
\downarrow & & \downarrow B & & \downarrow B' \\
S^{n+2p-3} & \xrightarrow{j} & L_{p-2}^{n+2p-3} & \longrightarrow & L_{p-3}^{n+4p-5}=L_{p-2}^{n+2p-3}/S^{n+2p-3} \\
\downarrow \mathrm{id} & & \downarrow A & & \downarrow A' \\
S^{n+2p-3} & \xrightarrow{\alpha_1} & S^n & \xrightarrow{j} & L_1^n,
\end{array}$$

which is obtained from (i) and (ii), and so (iii) follows from this diagram.

Q.E.D.

§3. Some relations on $H^*(\boldsymbol{K}_k)$ and the compositions in ${}_p\pi_k(S)$

Let $\boldsymbol{X}_k=\{X_k(n)\}$ be the spectrum such that $X_k(n)=\Omega(K_k(n), S^n)$, the space of paths in $K_k(n)$ starting from the base point and ending in S^n. Then p_k: $X_k(n)\to S^n$ is an $(n+k-1)$-connective fiber space over S^n with the fiber $\Omega K_k(n)$. The inclusion i_0: $K_k(n)\to(K_k(n), S^n)$ induces isomorphisms

$$(3.1)\ ([6: \mathrm{IV}, (4.7)])\quad \tau\colon H^i(\boldsymbol{X}_k)\underset{\approx}{\xrightarrow{(\Omega i_0)^*}} H^i(\Omega\boldsymbol{K}_k)\underset{\approx}{\longleftarrow} H^{i+1}(\boldsymbol{K}_k).$$

LEMMA 3.1. *The map ϕ of* (2.1) *coincides with the following composition.*

$$\phi\colon\ {}_p\pi_k(\boldsymbol{S})\underset{\approx}{\xleftarrow{(p_k)_*}}{}_p\pi_k(\boldsymbol{X}_k)\underset{\approx}{\xrightarrow{H}}H_k(\boldsymbol{X}_k;\ Z)\xrightarrow{\rho}H_k(\boldsymbol{X}_k)\underset{\approx}{\xrightarrow{D}}H^k(\boldsymbol{X}_k)\underset{\approx}{\xrightarrow{\tau}}H^{k+1}(\boldsymbol{K}_k).$$

PROOF. The following diagram is commutative:

$$\begin{array}{ccccccc}
{}_p\pi_k(\boldsymbol{S}) & \underset{\approx}{\xleftarrow{p_{k*}}} & {}_p\pi_k(\boldsymbol{X}_k) & \xrightarrow{\rho H} & H_k(\boldsymbol{X}_k) & \underset{\approx}{\xleftarrow{(\Omega i_0)_*}} & H_k(\Omega\boldsymbol{K}_k) \\
\| & & \approx\uparrow & & \approx\uparrow & & \uparrow\approx \\
{}_p\pi_k(\boldsymbol{S}) & \underset{\approx}{\xleftarrow{\partial}} & {}_p\pi_{k+1}(\boldsymbol{K}_k,\ \boldsymbol{S}) & \xrightarrow{\rho H} & H_{k+1}(\boldsymbol{K}_k,\ \boldsymbol{S}) & \underset{\approx}{\xleftarrow{i_{0*}}} & H_{k+1}(\boldsymbol{K}_k),
\end{array}$$

where all maps except ρH are isomorphic. Then this lemma is immediate.

Q.E.D.

PROPOSITION 3.2. *Let f: $S^{n+k}\to S^n$ be a representative of $\gamma\in{}_p\pi_k(\boldsymbol{S})$, and assume $\phi(\gamma)=a\neq 0$. Then there is a map F: $S^{n+k}\to X_k(n)$ such that $p_kF=f$, $F^*(\tau^{-1}a)\neq 0$ and $F^*(H^{n+k}(X_k(n))/Z_p\{\tau^{-1}a\})=0$.*

PROOF. By the covering homotopy property, there is a map F such that $p_kF=f$. Consider the map F_*: $H_{n+k}(S^{n+k})\to H_{n+k}(X_k(n))$. Let $\iota\in\pi_{n+k}(S^{n+k})$ be the class of the identity map and $\iota'=\rho H\iota\in H_{n+k}(S^{n+k})$ be the generator. Then $p_kF=f$ implies $(p_{k*})^{-1}\gamma=F_*\iota$. By Lemma 3.1, $D^{-1}\tau^{-1}a=\rho H(p_{k*})^{-1}\gamma$. Thus, $D^{-1}\tau^{-1}a=\rho HF_*\iota=F_*\iota'$. This implies the rest of the assertions.

Q.E.D.

The following theorems give the information about the compositions with the elements α_1 and β_1 in ${}_p\pi_k(\boldsymbol{S})$.

THEOREM 3.3. *Let $a \in H^{k+1}(\boldsymbol{K}_k)$ and $\gamma \in {}_p\pi_k(\boldsymbol{S})$, and assume that*

(1) $\phi(\gamma)=a \neq 0$,

(2) $\mathscr{P}^1 a=0$ *in* $\boldsymbol{K}_k$.

Then there is an element $b \in H^{k+2p-2}(\boldsymbol{K}_{k+1})$ such that $\delta^* b=\mathscr{P}^1 j^{*-1} a$ by Proposition 1.2. *Furthermore such b satisfies:*

$$b \neq 0 \text{ in } \boldsymbol{K}_{k+2p-3} \text{ and } \phi(\alpha_1\gamma)=\pm b.$$

THEOREM 3.4. *In the above theorem, assume also*

(3) $\mathscr{P}^{p-1} b=0$ *in* $\boldsymbol{K}_{k+2p-3}$.

Then $\beta_1\gamma \neq 0$ *in* ${}_p\pi_{k+2p(p-1)-2}(\boldsymbol{S})$.

Let $c \in H^{k+2p(p-1)-1}(\boldsymbol{K}_{k+2p-2})$ be an element such that $\delta^* c=\mathscr{P}^{p-1} j^{*-1} b$. Assume further

(4) $\mathscr{P}^{p-2} b \neq 0$ *in* $\boldsymbol{K}_{k+2p-3}$.

(5) $c \neq 0$ *in* $\boldsymbol{K}_{k+2p(p-1)-2}$.

Then $\phi(\beta_1\gamma)=xc$ *for some* $x \not\equiv 0$ mod p.

PROOF OF THEOREM 3.3. By Lemma 1.1, $b \neq 0$ in $\boldsymbol{K}_{k+2p-4}$. Assume that $b=0$ in $\boldsymbol{K}_{k+2p-3}$. Then by (1.3), $b=\Sigma x_i \Delta_{r_i} u_i$, $\Delta_{r_i-1} u_i=0$ (if $r_i \geqq 2$), $\Delta_{r_i} u_i \neq 0$, for some $x_i \in Z_p$ and $r_i \geqq 1$, where $H^{k+2p-3}(\boldsymbol{K}_{k+2p-4})=Z_p\{u_i\}$. By Lemma 1.1, u_i exists in $\boldsymbol{K}_k$ and $\Delta_{r_i-1} u_i=0$, $\Delta_{r_i} u_i \neq 0$ in $\boldsymbol{K}_k$. Thus b (in $\boldsymbol{K}_{k+1}$) is contained in $\operatorname{Im} i^*$. This contradicts to $\delta^* b \neq 0$. Thus $b \neq 0$ in $\boldsymbol{K}_{k+2p-3}$.

Now put $L=P^n_{k+2p-3}(b)$, $M=P^n_k(a)$, and let f': $L \to K_{k+2p-3}(n)$, g': $M \to K_k(n)$ be the inclusions (see Definition 2.1) and $f=if'$: $L \to K_{k+1}(n)$.

Consider the cofibering of spectra $\boldsymbol{K}_{k+1} \xrightarrow{i} \boldsymbol{K}_k \xrightarrow{j} \boldsymbol{Q}$, where $\boldsymbol{Q}=\{Q_n\}$ is the spectrum such that $Q_n=K({}_p\pi_{n+k}(S^n), n+k+1)$. Since the element γ generates a direct summand Z_{p^r} of ${}_p\pi_k(\boldsymbol{S})$ for some r, we have $Q_n=Q'_n \times Q''_n$, $Q'_n=K(Z_{p^r}, n+k+1)$, $Q''_n=K(G, n+k+1)$ for the decomposition: ${}_p\pi_k(\boldsymbol{S}) \approx Z_{p^r} \oplus G$, and so $H^*(\boldsymbol{Q}')$ is generated by the elements q and $q'=\Delta_r q$ which correspond to a and $\Delta_r a$, where $\boldsymbol{Q}'=\{Q'_n\}$. The element b corresponds to $\mathscr{P}^1 q$, since $\delta^* b=\mathscr{P}^1 j^{*-1} a$. Thus the cell of $\boldsymbol{Q}$ corresponding to b attaches only to the cell corresponding to q, since $H^*(\boldsymbol{Q}')$ is a direct summand (as A^*-module) of $H^*(\boldsymbol{Q})$. This implies that the map if: $L \to K_k(n)$ passes through the subcomplex M (up to homotopy). In other words, there is a map l: $L \to M$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} L & \xrightarrow{f} & K_{k+1}(n) \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle i} \\ M & \xrightarrow{g} & K_k(n). \end{array}$$

Let $\delta \in {}_p\pi_{k+2p-3}(\boldsymbol{S})$ and $\gamma_1 \in {}_p\pi_k(\boldsymbol{S})$ be the classes of the attaching maps of $(n+k+2p-2)$- and $(n+k+1)$-cells of L and M respectively. From the above discussions, we have $\delta=\pm\gamma_1\alpha_1=\pm\alpha_1\gamma_1$. By Lemma 2.2, $\phi(\delta)=b$, $\phi(\gamma_1)$

$=a=\phi(\gamma)$. The kernel of ϕ consists of all p-divisible elements. Hence $\phi(\alpha_1\gamma)=\pm b$. Q.E.D.

PROOF OF THEOREM 3.4. First we shall prove $\beta_1\gamma\neq 0$. Assume that $\beta_1\gamma=0$. Let $g\colon S^{n+k}\to S^n$ be a representative of γ. Then $(gA)B$ is null homotopic for the maps A, B of Lemma 2.5, so there is a map $f\colon L_{p-1}^{n+k+2p-3}\to S^n$ such that $f\mid S^{n+k+2p-3}$ represents $\pm\alpha_1\gamma$. Let $F\colon L_{p-1}^{n+k+2p-3}\to X_{k+2p-3}(n)$ be a lifting of f. By Proposition 3.2, $F^*(\tau^{-1}b)=u$ for a generator u of $H^{n+k+2p-3}(L_{p-1}^{n+k+2p-3})$. Hence, $F^*(\tau^{-1}(\mathscr{P}^{p-1}b))=\mathscr{P}^{p-1}u\neq 0$. This contradicts to $\mathscr{P}^{p-1}b=0$. Thus $\beta_1\gamma\neq 0$.

Let us put $L=P_{k+2p(p-1)-2}^n(c)$, $M_1=P_{k+2p-3}^n(b)$, $N=P_k^n(a)$, and denote the inclusions by $f'\colon L\to K_{k+2p(p-1)-2}(n)$, $g_1\colon M_1\to K_{k+2p-3}(n)$ and $h\colon N\to K_k(n)$ (see Definition 2.1).

From the discussion of Theorem 3.3, the attaching map of $(n+k+2p-2)$-cell of M_1 represents $\pm\alpha_1\gamma$ by the replacement of this map if necessary. Then there is a homotopy commutative diagram of cofiberings:

$$\begin{tikzcd} S^{n+k} \arrow[r, "j"] \arrow[d] & L_1^{n+k} \arrow[r, "\pi"] \arrow[d, "q_1"] & S^{n+k+2p-2} \arrow[d, equal] \\ S^n \arrow[r, "j'"] & M_1 \arrow[r, "\pi'"] & S^{n+k+2p-2}, \end{tikzcd}$$

where the left vertical arrow represents $\pm\gamma$ and L_i^n is in Lemma 2.5. Let $\lambda_i\in {}_p\pi_{n+k+2(i+1)(p-1)-1}(L_i^{n+k})$ be the attaching class of the top cell of L_{i+1}^{n+k}. Then for $1\leqq i\leqq p-1$, we can construct inductively a complex $M_i=S^n\cup e^{n+k+2p-2}\cup\cdots\cup e^{n+k+2i(p-1)}$ and a map $q_i\colon L_i^{n+k}\to M_i$ such that for $i<p-1$ the following is a homotopy commutative diagram of cofiberings:

$$\begin{tikzcd} S^{n+k+2(i+1)(p-1)-1} \arrow[r, "\lambda_i"] \arrow[d, equal] & L_i^{n+k} \arrow[r, "j"] \arrow[d, "q_i"] & L_{i+1}^{n+k} \arrow[r, "\pi"] \arrow[d, "q_{i+1}"] & S^{n+k+2(i+1)(p-1)} \arrow[d, equal] \\ S^{n+k+2(i+1)(p-1)-1} \arrow[r, "q_i\lambda_i"] & M_i \arrow[r, "j'"] & M_{i+1} \arrow[r, "\pi'"] & S^{n+k+2(i+1)(p-1)}. \end{tikzcd}$$

By Lemma 2.5 (ii), $\pi'_*(q_i\lambda_i)=\pi_*(\lambda_i)$ generates ${}_p\pi_{2p-3}(S)$, hence $\tilde{H}^*(M_i)$ is spanned by the elements u, v, $\mathscr{P}^1v,\cdots,\mathscr{P}^{i-1}v$, where $\deg u=n$, $\deg v=n+k+2p-2$.

Set $M=M_{p-1}$. The map g_1 has an extension $g'\colon M\to K_{k+2p-3}(n)$ by (1.1). Then $\tilde{H}^*(M)$ is spanned by the g'^*-images of the elements a_0, b, $\mathscr{P}^1b,\cdots,\mathscr{P}^{p-2}b$. Put $f=if'\colon L\to K_{k+2p-2}(n)$ and $g=ig'\colon M\to K_{k+1}(n)$.

By the similar discussions in the above proof of Theorem 3.3, we obtain the following

(3.2) *There exist maps $l\colon L\to M$ and $m\colon M\to N$ such that the following diagrams are homotopy commutative for some x_1, $x_2\not\equiv 0 \bmod p$:*

$$\begin{tikzcd}
S^{n+k+2p(p-1)-1}=L/S^n \arrow[d, "x_1B"] & L \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "l"] & K_{k+2p-2}(n) \arrow[d, "i"] \\
L^{n+k+2p-2}_{p-2}=M/S^n & M \arrow[l, "\pi"'] \arrow[r, "g'"] & K_{k+2p-3}(n),
\end{tikzcd}$$

$$\begin{tikzcd}
L^{n+k+2p-2}_{p-2}=M/S^n \arrow[d, "x_2A"] & M \arrow[l, "\pi"'] \arrow[r, "g"] \arrow[d, "m"] & K_{k+1}(n) \arrow[d, "i"] \\
S^{n+k+1}=N/S^n & N \arrow[l, "\pi"'] \arrow[r, "h"] & K_k(n),
\end{tikzcd}$$

where A, B and L^n_i are in Lemma 2.5, and π denotes the projection.

By this, we have a homotopy commutative diagram:

$$\begin{tikzcd}
S^{n+k+2p(p-1)-1} \arrow[d, "C"] & L \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "ml"] & K_{k+2p-2}(n) \arrow[d, "i"] \\
S^{n+k+1} & N \arrow[l, "\pi"'] \arrow[r, "h"] & K_k(n),
\end{tikzcd}$$

where C represents $y\beta_1$ for some $y \not\equiv 0 \bmod p$.

Let γ_1 be the class of the attaching map of $(n+k+1)$-cell of N. Then by this diagram, $\phi(\beta_1\gamma_1)=xc$ for some $x\not\equiv 0 \bmod p$. Since $\gamma_1=\gamma+p\lambda$ for some λ, we obtain the equality $\phi(\beta_1\gamma)=xc$. Q.E.D.

Similarly to the above theorems, we obtain the following

THEOREM 3.5. *Let $a' \in H^{k+1}(\boldsymbol{K}_k)$ and $\gamma' \in {}_p\pi_k(\boldsymbol{S})$ with*

(1) $\phi(\gamma')=a'$, $\alpha_1\gamma'=0$, $\mathscr{P}^2a'=0$ *in* $\boldsymbol{K}_k$.

Then the secondary composition $\{\gamma', \alpha_1, \alpha_1\}$ does not contain zero.

Let $b' \in H^{k+4p-4}(\boldsymbol{K}_{k+1})$ be an element such that $\delta^*b'=\mathscr{P}^2j^{*-1}a'$. Assume also

(2) $b'\neq 0$ *in* $\boldsymbol{K}_{k+4p-5}$.

Then there is an element $\varepsilon \in \{\gamma', \alpha_1, \alpha_1\}$ such that $\phi(\varepsilon)=\pm b'$.

Assume further

(3) $\mathscr{P}^{p-2}b'=0$ *in* $\boldsymbol{K}_{k+4p-5}$.

Then $\beta_1\gamma'\neq 0$.

Let $c' \in H^{k+2p(p-1)-1}(\boldsymbol{K}_{k+4p-4})$ be an element satisfing $\delta^*c'=\mathscr{P}^{p-2}j^{*-1}b'$. In addition, assume

(4) $\mathscr{P}^{p-3}b'\neq 0$ *in* $\boldsymbol{K}_{k+4p-5}$, $c'\neq 0$ *in* $\boldsymbol{K}_{k+2p(p-1)-2}$.

Then $\phi(\beta_1\gamma')=x'c'$ for some $x'\not\equiv 0 \bmod p$.

PROOF. Assume that $\{\gamma', \alpha_1, \alpha_1\} \ni 0$. Then there is a map $F: L^{n+k}_2 \to X_k(n)$ such that $p_kF|S^{n+k}$ represents γ'. By Proposition 3.2, $F^*(\tau^{-1}a')=u$ for a generator u of $H^{n+k}(L^{n+k}_2)$. Then $F^*(\tau^{-1}(\mathscr{P}^2a'))=\mathscr{P}^2u\neq 0$. This contradicts to $\mathscr{P}^2a'=0$. Thus $\{\gamma', \alpha_1, \alpha_1\} \not\ni 0$.

The assumption $\alpha_1\gamma'=0$ implies $\mathscr{P}^1a'\neq 0$ in $\boldsymbol{K}_k$. From the discussion of Theorem 3.4, there exist a complex $M=P^n_k(a')\cup e^{n+k+2p-1}$ satisfying $M/S^n=L^{n+k+1}_1$ and a map $g: M\to K_k(n)$ such that $\tilde{H}^*(M)$ is spanned by g^*a_0, g^*a', $\mathscr{P}^1g^*a'$. Put $L=P^n_{k+4p-5}(b')$ and let $f: L\to K_{k+4p-5}(n)$ be a map such that $\tilde{H}^*(L)$ is spanned by f^*a_0 and f^*b'.

By the similar argument of the above theorems, we have a map $l: L \to M$ and a homotopy commutative diagram:

$$\begin{tikzcd} S^{n+k+4p-4} \arrow[d] & L \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "l"] & K_{k+4p-5}(n) \arrow[d, "i"] \\ S^{n+k+2p-1} & M \arrow[l, "\pi"'] \arrow[r, "g"] & K_k(n), \end{tikzcd}$$

where the left vertical arrow represents $\pm\alpha_1$. This implies $\phi(\varepsilon) = \pm b'$ for some $\varepsilon \in \{\gamma', \alpha_1, \alpha_1\}$.

The rest of the assertions is proved similarly to Theorem 3.4, by use of Lemma 2.5 (iii) instead of (ii). Q.E.D.

The following theorem is obtained similarly to the previous theorems and the proof is omitted.

THEOREM 3.6. *Let $a'' \in H^{k+1}(\boldsymbol{K}_k)$ and $\gamma'' \in {}_p\pi_k(S)$ with $\phi(\gamma'') = a''$, and assume that $\Delta a'' \neq 0$, $\mathscr{P}^1\Delta a'' = 0$. Then $\{\gamma'', p\iota, \alpha_1\}$ is defined and does not contain zero (ι denotes the class of the identity map). Let $b'' \in H^{k+2p-1}(\boldsymbol{K}_{k+1})$ with $\delta^* b'' = \mathscr{P}^1\Delta j^{*-1}a''$, and assume also $b'' \neq 0$ in $\boldsymbol{K}_{k+2p-2}$. Then there exists an element $\lambda \in \{\gamma'', p\iota, \alpha_1\}$ such that $\phi(\lambda) = \pm b''$.*

§4. $H^*(\boldsymbol{K}_k)$ for $k \leqq 2(p^2-1)(p-1)-1$

In this section, we shall compute $H^*(\boldsymbol{K}_k)$, $k \leqq 2(p^2-1)(p-1)-1$, continued from Theorem 2.3 of §2, for our further calculation.

For any non-zero element $a \in H^i(\boldsymbol{K}_k)$, $i > 0$, we define

$$h(a) = \min \{l : \textit{there is } a' \in H^i(\boldsymbol{K}_l) \textit{ such that } a' = a \textit{ in } \boldsymbol{K}_k\}.$$

We put $q = 2(p-1)$ in the rest of this paper.

Almost all of the following theorem is occupied in Theorem 3.10 of [6: III].

THEOREM 4.1. *Let $pq-1 \leqq k \leqq (p^2-1)q-2$. In degree $< (2p^2+p)q-2$, $H^*(\boldsymbol{K}_k)$ has a minimal set of generators given by the following table:*

TABLE A2

Generator a	Degree of a	$h(a)$	δ^*-image of a in $\boldsymbol{K}_{h(a)}$
a_0	0		
a_r $(p \leqq r \leqq p^2-1)$	rq	$(r-1)q$	$R_{r-1}j^{*-1}a_{r-1}$ *for* $r \not\equiv 1 \bmod p$ $\Delta\mathscr{P}^1 j^{*-1}a_{sp} - \mathscr{P}^1 j^{*-1}a'_{sp}$ *for* $r = sp+1$
a'_{sp} $(1 \leqq s < p)$	$spq+1$	$(sp-1)q$	$\Delta\mathscr{P}^1\Delta j^{*-1}a_{sp-1}$

b_s^r $(r \geqq 0,\ s \geqq 1,\ r+s \leqq p)$	$((r+s)p+s-1)q$ $-2r-1$	q	$\mathscr{P}^{p-1}j^{*-1}(\mathscr{P}^1 a_0)$ *for* $r=0,\ s=1$
		$((s-1)p+s-2)q-1$	$W_{s-1}j^{*-1}b_{s-1}^0$ *for* $r=0,\ s \geqq 2$
		$((r+s-1)p+s)q-2r$	$\mathscr{P}^{p-1}j^{*-1}c_s^{r-1}$ *for* $r \geqq 1$
c_s^r $(r \geqq 0,\ s \geqq 1,\ r+s<p)$	$((r+s)p+s)q$ $-2r-2$	$((r+s)p+s-1)q$ $-2r-1$	$\mathscr{P}^{1*}j^{-1}b_s^r$
d_1	p^2q-1	$pq-1$	$c(\mathscr{P}^{p(p-1)})\Delta j^{*-1}b_1^0$
d_2	$(p^2+p)q+1$	$(p^2-2)q-1$	$\Delta \mathscr{P}^{p+1}\Delta \mathscr{P}^1 \Delta j^{*-1}b_{p-1}^0$

Here $R_t=(t+1)\mathscr{P}^1\Delta - t\Delta\mathscr{P}^1$, $W_t=(t+1)\mathscr{P}^p\mathscr{P}^1\Delta - t\mathscr{P}^{p+1}\Delta+(t-1)\Delta\mathscr{P}^{p+1}$, *and* c: $A^* \to A^*$ *denotes the conjugation of* A^*.

The relations in $H^*(\boldsymbol{K}_k)$, *degree* $<(2p^2+p)q-2$, *are given by the following*

TABLE B2

(a-1) $\Delta a_0=\mathscr{P}^1 a_0=\mathscr{P}^p a_0-\Delta b_1^0=0$.

(a-2) $R_r a_r=\Delta a_{sp}=\Delta a'_{sp}=\Delta\mathscr{P}^1 a_{sp}-\mathscr{P}^1 a'_{sp}=\Delta\mathscr{P}^1\Delta a_{sp-1}=0$.

(b-1) $\mathscr{P}^1 b_1^0=\mathscr{P}^1 b_s^r=0$ $(r \geqq 1,\ (r,s) \neq (p-1,1))$, $\mathscr{P}^1 b_s^0-\bar{W}_s c_{s-1}^0=0$ $(2 \leqq s \leqq p-1)$,
$\mathscr{P}^1 b_p^0-\bar{W}_p c_{p-1}^0 \equiv 0 \bmod A^* b_1^{p-1}$.

(b-1)′ $\mathscr{P}^1 b_1^{p-1}-x_1 d_1=0$ *for some* $x_1 \in Z_p$.

(b-2) $W_1 b_1^0=0$, $W_s b_s^0-A_s c_{s-1}^0 \equiv 0 \bmod A^* a_{sp+s-1}$ $(2 \leqq s \leqq p-1)$,
$W_p b_p^0-A_p c_{p-1}^0 \equiv 0 \bmod A^* a_{p^2-2}+A^* b_1^{p-1}$.

(b-3) $c(\mathscr{P}^{p(p-1)})\Delta b_1^0=0$.

(b-4) $\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta b_{p-1}^0-Ac_{p-2}^0-\lambda_1 a_p-\lambda_2 a'_p=0$, $\lambda_i=0$ *if* $p>3$.

(b-5) $\Delta\mathscr{P}^1\Delta b_p^0 \equiv 0 \bmod A^* a_{p^2-2}+A^* b_1^{p-1}$.

(c) $\mathscr{P}^{p-1} c_s^r=0$.

(d-1) $\mathscr{P}^1 d_1-B_3 c_1^0-B_4 b_2^0 \equiv 0 \bmod A^* a_p+A^* a'_p$,
$\mathscr{P}^{p+1} d_1-B_5 c_1^0-B_6 b_2^0 \equiv 0 \bmod A^* a_p+A^* a'_p$,
$\mathscr{P}^{2p} d_1-B_7 c_1^0-B_8 b_2^0 \equiv 0 \bmod A^* a_p+A^* a'_p$.

(d-1)′ $\Delta d_1-x_2\mathscr{P}^{p^2}a_0-B_1 c_1^0-B_2 b_2^0 \equiv 0 \bmod A^* a_p+A^* a'_p$ *for some* $x_2 \in Z_p$.

(d-2) $\Delta d_2 \equiv \mathscr{P}^1 d_2-Cc_{p-1}^0 \equiv \mathscr{P}^p d_2-Db_p^0 \equiv 0 \bmod A^* a_{p^2-2}+A^* b_1^{p-1}$.

(l) $a=0$ *in* $\boldsymbol{K}_k$, $k \geqq \deg a-1$, *for* $a=a'_{sp}$,
$a=0$ *in* $\boldsymbol{K}_k$, $k \geqq \deg a$, *for other* a *in Table* A2 *with* $0<\deg a \leqq (p^2-1)q-2$.

Here $\bar{W}_s$, A_s, A, B_i, C *and* D *are elements of* A^* *such that*

$\mathscr{P}^1 W_{s-1}=\bar{W}_s\mathscr{P}^1$, $W_s W_{s-1}=A_s\mathscr{P}^1$, $\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta W_{p-2}=A\mathscr{P}^1$,
$\Delta c(\mathscr{P}^{p(p-1)})\Delta=B_1\mathscr{P}^1+B_2 W_1$, $\mathscr{P}^1 c(\mathscr{P}^{p(p-1)})\Delta=B_3\mathscr{P}^1+B_4 W_1$,
$\mathscr{P}^{p+1}c(\mathscr{P}^{p(p-1)})\Delta=B_5\mathscr{P}^1+B_6 W_1$, $\mathscr{P}^{2p}c(\mathscr{P}^{p(p-1)})\Delta=B_7\mathscr{P}^1+B_8 W_1$,
$\mathscr{P}^1\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta=C\mathscr{P}^1$, $\mathscr{P}^p\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta=DW_{p-1}$.

REMARK. The element b_s^r (resp. c_s^r) corresponds to $b_{r+s}^{(s-1)}$ (resp. $c_{r+s}^{(s-1)}$) of [6: III, pp. 201–202] so that our b_s^r (resp. c_s^r) corresponds to the element $\beta_1^r\beta_s$

(resp. $\alpha_1\beta_1^r\beta_s$) of ${}_p\pi_k(\boldsymbol{S})$ by the map ϕ of (2.1). The element d_1 corresponds to d of [6: III]. The elements b_p^0 and d_2 do not appear in [6: III] by the dimensional reason.

Proof of Theorem 4.1. Except the relations about the elements b_p^0, d_1 and d_2, this theorem is proved in Theorem 3.10 of [6: III].

Let $r=pq-2$. By Theorem 2.3, $H^{r+1}(\boldsymbol{K}_r)=Z_p\{b_1^0\}$, and the submodule $A^*b_1^0$ has the relations:

$$\text{B1 } (b\text{-}2) \quad \mathscr{P}^1 b_1^0=0, \qquad (b\text{-}3) \quad W_1 b_1^0=c(\mathscr{P}^{p(p-1)})\Delta b_1^0=0.$$

By Proposition 1.2, these relations give the elements c_1^0, b_2^0 and d_1 of $H^*(\boldsymbol{K}_{r+1})$.

Now we consider the relation $\alpha\mathscr{P}^1+\beta W_1+\gamma c(\mathscr{P}^{p(p-1)})\Delta=0$ in A^*. The exact sequence (4.11) of [4] implies $\gamma=\gamma_1\Delta+\gamma_2\mathscr{P}^1+\gamma_3\mathscr{P}^{p+1}+\gamma_4\mathscr{P}^{2p}$ for some $\gamma_i \in A^*$, and the following relations in A^*:

$$\Delta c(\mathscr{P}^{p(p-1)})\Delta-B_1\mathscr{P}^1-B_2W_1=0, \quad \mathscr{P}^1 c(\mathscr{P}^{p(p-1)})\Delta-B_3\mathscr{P}^1-B_4W_1=0,$$
$$\mathscr{P}^{p+1} c(\mathscr{P}^{p(p-1)})\Delta-B_5\mathscr{P}^1-B_6W_1=0, \quad \mathscr{P}^{2p} c(\mathscr{P}^{p(p-1)})\Delta-B_7\mathscr{P}^1-B_8W_1=0,$$

for some $B_i \in A^*$. We can check that these four relations generate the relations in $A^*\mathscr{P}^1+A^*W_1+A^*c(\mathscr{P}^{p(p-1)})\Delta\subset A^*$ which contain the element $c(\mathscr{P}^{p(p-1)})\Delta$. Hence, by Proposition 1.2, we obtain the relations about the element d_1:

$$\Delta d_1-B_1c_1^0-B_2b_2^0=i^*w_1, \quad \mathscr{P}^1 d_1-B_3c_1^0-B_4b_2^0=i^*w_2,$$
$$\mathscr{P}^{p+1} d_1-B_5c_1^0-B_6b_2^0=i^*w_3, \quad \mathscr{P}^{2p} d_1-B_7c_1^0-B_8b_2^0=i^*w_4,$$

for some $w_i \in H^*(\boldsymbol{K}_r)$.

By Theorem 2.3, $i^*H^*(\boldsymbol{K}_r)$, $i: \boldsymbol{K}_{r+1}\to\boldsymbol{K}_r$, is generated by a_0, a_p, a_p' with the relations $\Delta a_0=\mathscr{P}^1a_0=\mathscr{P}^pa_0=\Delta a_p=\Delta a_p'=\Delta\mathscr{P}^1a_p-\mathscr{P}^1a_p'=0$. Thus the relations $(d-1)$ and $(d-1)'$ are obtained.

The relations about b_p^0 and d_2 are obtained similarly by making use of the exact sequences (4.8) and (4.14) of [4]. Q.E.D.

Let $t=(p^2-1)q-2$ in the rest of this section.

To compute $H^*(\boldsymbol{K}_{t+1})$, it is necessary that the coefficients x_1 and x_2 in the relations B2$(b-1)'$ and $(d-1)'$ are determined.

H. Gershenson has proved the non-triviality of x_2 [1: Lemma 4.2], from the triviality of the mod Hopf invariant.

H. Toda has proved the relation $\alpha_1\beta_1^p=0$ in ${}_p\pi_{t+q}(\boldsymbol{S})$ ([8] and [9: Theorem 3]). The element b_1^{p-1} gives rise to the generator $\beta_1^p=\beta_1\circ\cdots\circ\beta_1$ (p-fold composition) of ${}_p\pi_t(\boldsymbol{S})\approx Z_p$, and so the relation $\alpha_1\beta_1^p=0$ implies $\mathscr{P}^1b_1^{p-1}\neq 0$ by Theorem 3.3.

Since $a_p=a_p'=c_1^0=b_2^0=0$ in $\boldsymbol{K}_t$ by B2(l), by the suitable replacement of the generators, we have

LEMMA 4.2. *Let* $t=(p^2-1)q-2$, *then*

$$(b-1)' \quad \mathscr{P}^1 b_1^{p-1}=d_1 \quad in\ \boldsymbol{K}_t, \qquad (d-1)' \quad \Delta d_1=\mathscr{P}^{p^2} a_0 \quad in\ \boldsymbol{K}_t.$$

Now let $\gamma b_1^{p-1}=0$, $\gamma \in A^*$, be any relation of $A^* b_1^{p-1}$ in $H^*(\boldsymbol{K}_t)$. Then $\gamma=\gamma_1\mathscr{P}^1$, $\gamma_1 d_1=0$, by B2$(b-1)'$, and $\gamma_1=\gamma_2\Delta+\gamma_3\mathscr{P}^1+\gamma_4\mathscr{P}^{p+1}+\gamma_5\mathscr{P}^{2p}$, $\gamma_2\mathscr{P}^{p^2}a_0=0$, by B2$(d-1)$ and $(d-1)'$. The element γ_2 is contained in the kernel of the right translation:

$$(\mathscr{P}^{p^2})^*:\ A^* \to A^*/(A^*\Delta+A^*\mathscr{P}^1+A^*\mathscr{P}^p),$$

hence $\gamma_2=\gamma_6\Delta+\gamma_7\mathscr{P}^1+\gamma_8\mathscr{P}^{2p}$ in degree $<(p^2+p)q$ by Proposition 1.7 of [6: I]. Using the Adem relations, we have $\gamma=\delta_1\mathscr{P}^2+\delta_2\mathscr{P}^2\Delta+\delta_3\mathscr{P}^{p+1}\mathscr{P}^1+\delta_4\mathscr{P}^{2p}\mathscr{P}^1$, in degree $<(p^2+p+1)q+1$, for some $\delta_i \in A^*$. Conversely $\mathscr{P}^2 b_1^{p-1}=\mathscr{P}^2\Delta b_1^{p-1}=\mathscr{P}^{p+1}\mathscr{P}^1 b_1^{p-1}=\mathscr{P}^{2p}\mathscr{P}^1 b_1^{p-1}=0$ in $\boldsymbol{K}_t$. Therefore the following lemma is obtained.

LEMMA 4.3. *Let* $t=(p^2-1)q-2$. *In degree* $<(2p^2+p)q-1$, *the submodule* $A^*b_1^{p-1}$ *of* $H^*(\boldsymbol{K}_t)$ *has the relations given by*

$$\mathscr{P}^2 b_1^{p-1}=\mathscr{P}^2\Delta b_1^{p-1}=\mathscr{P}^{p+1}\mathscr{P}^1 b_1^{p-1}=\mathscr{P}^{2p}\mathscr{P}^1 b_1^{p-1}=0.$$

From this lemma, we calculate $H^*(\boldsymbol{K}_{t+1})$.

THEOREM 4.4. *In degree* $<(2p^2+p)q-3$, $H^*(\boldsymbol{K}_{(p^2-1)q-1})$ *has a minimal set of generators*:

$$\{a_0,\ a_{p^2-1},\ b_s^{p-s}(2\leqq s\leqq p),\ e_1',\ e_1,\ d_2,\ g_0,\ d_3\},$$

where the new generators are given by

TABLE A3

Generator a	Degree of a	$h(a)$	$\delta^*(a)$
e_1'	$(p^2+1)q-2$	$(p^2-1)q-1$	$\mathscr{P}^2 j^{*-1} b_1^{p-1}$
e_1	$(p^2+1)q-1$	$(p^2-1)q-1$	$\mathscr{P}^2\Delta j^{*-1} b_1^{p-1}$
g_0	$(p^2+p+1)q-2$	$(p^2-1)q-1$	$\mathscr{P}^{p+1}\mathscr{P}^1 j^{*-1} b_1^{p-1}$
d_3	$(p^2+2p)q-2$	$(p^2-1)q-1$	$\mathscr{P}^{2p}\mathscr{P}^1 j^{*-1} b_1^{p-1}$

The relations of $H^*(\boldsymbol{K}_{(p^2-1)q-1})$ *are given by*

TABLE B3

$(a\text{-}1)$	$\Delta a_0=\mathscr{P}^1 a_0=\mathscr{P}^p a_0=\mathscr{P}^{p^2} a_0=0.$	$(a\text{-}2)$	$R_{p^2-1} a_{p^2-1}=\Delta\mathscr{P}^1\Delta a_{p^2-1}=0.$
(b)	$\mathscr{P}^1 b_s^{p-s}=\Delta\mathscr{P}^1\Delta b_p^0=W_p b_p^0=0.$	$(d\text{-}1)$	$\Delta d_2=\mathscr{P}^1 d_2=\mathscr{P}^p d_2-D b_p^0=0.$

(*e*-1) $-R_2e'_1+\mathscr{P}^1e_1=0$ *if* $p>3$,

$\mathscr{P}^1e_1-x_3b_p^0\equiv 0 \bmod A^*b_{p-1}^1$ *for some* $x_3\in Z_p$ *if* $p=3$.

(*e*-2) $-2\Delta\mathscr{P}^1\Delta e'_1+\mathscr{P}^1\Delta e_1=0$ *if* $p>3$,

$\Delta\mathscr{P}^1\Delta e'_1+\mathscr{P}^1\Delta e_1-x_4\Delta b_p^0=0$ *for some* $x_4\in Z_p$ *if* $p=3$.

(*e*-3) $\mathscr{P}^{p-2}e'_1\equiv 0 \bmod A^*b_{p-1}^1$ *if* $p>3$, $\mathscr{P}^1e'_1=0$ *if* $p=3$.

(*g*) $\mathscr{P}^1g_0-A'_1e'_1-\lambda_1a_8-\lambda_2b_2^1=\mathscr{P}^1\Delta g_0-A'_2e'_1-A_2e_1-\lambda_3b_2^1-\lambda_4b_3^0=0$,

$\lambda_i\in A^*$, $\lambda_i=0$ *if* $p>3$.

(*d*-2) *If* $p>3$, $\mathscr{P}^1d_3-A'_3e'_1-A''_3g_0\equiv 0 \bmod A^*b_{p-1}^1$,

$\mathscr{P}^{p+1}d_3-A'_4e'_1-A''_4g_0=0$,

$W_2d_3-A'_5e'_1-A_5e_1-A''_5g_0=0$,

$\mathscr{P}^{p(p-2)}d_3-A'_6e'_1\equiv 0 \bmod A^*b_{p-1}^1$.

If $p=3$, $\mathscr{P}^1d_3-A'_3e'_1-A''_3g_0\equiv 0 \bmod A^*a_8+A^*b_2^1$,

$\mathscr{P}^3d_3-A'_6e'_1\equiv 0 \bmod A^*a_8+A^*b_2^1$,

$\mathscr{P}^4\Delta\mathscr{P}^1\Delta d_3-A'_7e'_1-A_7e_1-A''_7g_0\equiv 0 \bmod A^*a_8+A^*b_2^1+A^*b_3^0$.

Here D, A'_i, A_i *and* A''_i *satisfy the following.*

$\mathscr{P}^p\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta=DW_{p-1}$, $\mathscr{P}^1\mathscr{P}^{p+1}\mathscr{P}^1=A'_1\mathscr{P}^2$, $\mathscr{P}^1\Delta\mathscr{P}^{p+1}\mathscr{P}^1=A'_2\mathscr{P}^2+A_2\mathscr{P}^2\Delta$,

$\mathscr{P}^1\mathscr{P}^{2p}\mathscr{P}^1=A'_3\mathscr{P}^2+A''_3\mathscr{P}^{p+1}\mathscr{P}^1$, $\mathscr{P}^{p+1}\mathscr{P}^{2p}\mathscr{P}^1=A'_4\mathscr{P}^2+A''_4\mathscr{P}^{p+1}\mathscr{P}^1$,

$W_2\mathscr{P}^{2p}\mathscr{P}^1=A'_5\mathscr{P}^2+A_5\mathscr{P}^2\Delta+A''_5\mathscr{P}^{p+1}\mathscr{P}^1$, $\mathscr{P}^{p(p-2)}\mathscr{P}^{2p}\mathscr{P}^1=A'_6\mathscr{P}^2$,

$\mathscr{P}^4\Delta\mathscr{P}^1\Delta\mathscr{P}^6\mathscr{P}^1=A'_7\mathscr{P}^2+A_7\mathscr{P}^2\Delta+A''_7\mathscr{P}^4\mathscr{P}^1 (p=3)$.

PROOF. The existence of the elements e'_1, e_1, g_0 and d_3 follows from Lemma 4.3. The relations B3(a–1), (a–2), (b) and (d–1) follow from Theorem 4.1 and Lemma 4.2. To investigate the relations related with new generators, we consider the relations in the submodule

$$A^*\mathscr{P}^2+A^*\mathscr{P}^2\Delta+A^*\mathscr{P}^{p+1}\mathscr{P}^1+A^*\mathscr{P}^{2p}\mathscr{P}^1 \text{ of } A^*.$$

By (4.13) of [4], we have the following relations in A^*:

$$(*)\quad\begin{cases}\mathscr{P}^1\mathscr{P}^{2p}\mathscr{P}^1=A'_3\mathscr{P}^2+A_3\mathscr{P}^2\Delta+A''_3\mathscr{P}^{p+1}\mathscr{P}^1, & \\ \mathscr{P}^{p+1}\mathscr{P}^{2p}\mathscr{P}^1=A'_4\mathscr{P}^2+A_4\mathscr{P}^2\Delta+A''_4\mathscr{P}^{p+1}\mathscr{P}^1 & \textit{for } p>3,\\ W_2\mathscr{P}^{2p}\mathscr{P}^1=A'_5\mathscr{P}^2+A_5\mathscr{P}^2\Delta+A''_5\mathscr{P}^{p+1}\mathscr{P}^1 & \textit{for } p>3,\\ \mathscr{P}^{p(p-2)}\mathscr{P}^{2p}\mathscr{P}^1=A'_6\mathscr{P}^2+A_6\mathscr{P}^2\Delta+A''_6\mathscr{P}^{p+1}\mathscr{P}^1, & \\ \mathscr{P}^4\Delta\mathscr{P}^1\Delta\mathscr{P}^6\mathscr{P}^1=A'_7\mathscr{P}^2+A_7\mathscr{P}^2\Delta+A''_7\mathscr{P}^4\mathscr{P}^1 & \textit{for } p=3.\end{cases}$$

By the Adem relations, we can put

$$A'_3=-2\mathscr{P}^{2p},\ A_3=0,\ A''_3=\mathscr{P}^p,\ A'_4=-12\mathscr{P}^{3p},\ A_4=0,\ A''_4=3\mathscr{P}^{2p},$$

$$A'_5=-(\mathscr{P}^{3p}\Delta+8\Delta\mathscr{P}^{3p}),\ A_5=3\mathscr{P}^{3p},\ A''_5=3\Delta\mathscr{P}^{2p},\ A'_6=2\sum_{i=1}^{p-2}(-1)^{i-1}\frac{1}{i!}\mathscr{P}^{p^2-i}\mathscr{P}^{i-1},$$

$$A_6=A''_6=0,\ A'_7=\Delta\mathscr{P}^{10}\Delta,\ A_7=\mathscr{P}^{10}\Delta,\ A''_7=-\Delta\mathscr{P}^7\Delta.$$

Let $\alpha\mathscr{P}^2+\beta\mathscr{P}^2\Delta+\gamma\mathscr{P}^{p+1}\mathscr{P}^1+\delta\mathscr{P}^{2p}\mathscr{P}^1=0$ be any relation in $A^*\mathscr{P}^2+A^*\mathscr{P}^2\Delta+A^*\mathscr{P}^{p+1}\mathscr{P}^1+A^*\mathscr{P}^{2p}\mathscr{P}^1$. Then, by (4.13) of [4],

$$\delta=\delta_1\mathscr{P}^1+\delta_2\mathscr{P}^{p+1}+\delta_3 W_2+\delta_4\mathscr{P}^{p(p-2)}+\delta_5\mathscr{P}^4\Delta\mathscr{P}^1\Delta,\quad \delta_i\in A^*,$$

where $\delta_2=\delta_3=0$ for $p=3$, $\delta_5=0$ for $p>3$[1].

Hence we have

$$\begin{aligned}&\alpha'\mathscr{P}^2+\beta'\mathscr{P}^2\Delta+\gamma'\mathscr{P}^{p+1}\mathscr{P}^1=0,\\ &\qquad\alpha'=\alpha+\delta_1A_3'+\delta_2A_4'+\delta_3A_5'+\delta_4A_6'+\delta_5A_7'\\ &\qquad\beta'=\beta+\delta_3A_5+\delta_5A_7\\ &\qquad\gamma'=\gamma+\delta_1A_3''+\delta_2A_4''+\delta_3A_5''+\delta_5A_7''.\end{aligned}$$

By (4.12) of [4] and the Adem relations, we have the following relations:

$$(**)\quad\begin{cases}\mathscr{P}^1\mathscr{P}^{p+1}\mathscr{P}^1=A_1'\mathscr{P}^2,\ \ A_1'=\mathscr{P}^{p+1}-\dfrac{1}{2}\mathscr{P}^p\mathscr{P}^1,\\ \mathscr{P}^1\Delta\mathscr{P}^{p+1}\mathscr{P}^1=A_2'\mathscr{P}^2+A_2\mathscr{P}^2\Delta,\ A_2'=\mathscr{P}^{p+1}\Delta-2\mathscr{P}^p\mathscr{P}^1\Delta+2\Delta\mathscr{P}^{p+1},\ A_2=\mathscr{P}^{p+1}.\end{cases}$$

Furthermore $\gamma'=\gamma_1\mathscr{P}^1+\gamma_2\mathscr{P}^1\Delta$ in degree $<(p^2-1)q$[2] for some $\gamma_i\in A^*$. Hence $(\alpha'+\gamma_1A_1'+\gamma_2A_2')\mathscr{P}^2+(\beta'+\gamma_2A_2)\mathscr{P}^2\Delta=0$.

By (1.1) and (3.4) of [4] and the Adem relations, the relations in $A^*\mathscr{P}^2+A^*\mathscr{P}^2\Delta$ are generated by the following ones:

$$(***)\quad\begin{cases}\mathscr{P}^{p-2}\mathscr{P}^2=0,\\ \mathscr{P}^1\mathscr{P}^2\Delta-\varepsilon R_2\mathscr{P}^2=0,\quad \varepsilon=0\ \text{ for } p=3,\quad \varepsilon=1\ \text{ for } p>3,\\ \mathscr{P}^1\Delta\mathscr{P}^2\Delta-2\Delta\mathscr{P}^1\Delta\mathscr{P}^2=0.\end{cases}$$

Thus we have

$$\begin{aligned}&\alpha'+\gamma_1A_1'+\gamma_2A_2'+\beta_1R_2+2\beta_2\Delta\mathscr{P}^1\Delta=\alpha_1\mathscr{P}^{p-2},\\ &\beta'+\gamma_2A_2=\beta_1\mathscr{P}^1+\beta_2\mathscr{P}^1\Delta,\quad\text{for some }\alpha_1,\ \beta_i\in A^*,\quad \beta_1=0\quad\text{for } p=3.\end{aligned}$$

From the above calculations, α, β, γ and δ are determined:

$$\begin{aligned}&\alpha=\alpha_1\mathscr{P}^{p-2}-\beta_1R_2-2\beta_2\Delta\mathscr{P}^1\Delta-\gamma_1A_1'-\gamma_2A_2'-\sum_{i=1}^{5}\delta_iA_{i+2}'\\ &\beta=\beta_1\mathscr{P}^1+\beta_2\mathscr{P}^1\Delta-\gamma_2A_2-\delta_3A_5-\delta_5A_7\end{aligned}$$

1) In the case $p=3$ of (4.13) of [4], we can omit the term $(\mathscr{P}^4\Delta)^*$, since $\mathscr{P}^4\Delta=\mathscr{P}^1\Delta\mathscr{P}^3\in A^*\mathscr{P}^3$.

2) For degree $\geqq(p^2-1)q$, this relation is understood up to modulo $A^*\mathscr{P}^{p^2-1}$, and so there is a new relation in A^*g_0 of degree $(2p^2+p)q-2$. This gives no effect for the calculation of $H^*(K_k)$ under degree $<(2p^2+p)q-3$.

$$\gamma=\gamma_1\mathscr{P}^1+\gamma_2\mathscr{P}^1\varDelta-\delta_1A_3''-\delta_2A_4''-\delta_3A_5''-\delta_5A_7''$$
$$\delta=\delta_1\mathscr{P}^1+\delta_2\mathscr{P}^{p+1}+\delta_3W_2+\delta_4\mathscr{P}^{p(p-2)}+\delta_5\mathscr{P}^4\varDelta\mathscr{P}^1\varDelta,$$

where $\beta_1=\delta_2=\delta_3=0$ for $p=3$, $\delta_5=0$ for $p>3$.

Thus, it follows that the relations $(*)$, $(**)$ and $(***)$ generate the relations in $A^*\mathscr{P}^2+A^*\mathscr{P}^2\varDelta+A^*\mathscr{P}^{p+1}\mathscr{P}^1+A^*\mathscr{P}^{2p}\mathscr{P}^1$.

By Proposition 1.2, there are elements $w_i\in H^*(\boldsymbol{K}_{(p^2-1)q-2})$ such that

$$\mathscr{P}^{p-2}e_1'=i^*w_1,\qquad \mathscr{P}^1e_1-\varepsilon R_2e_1'=i^*w_2,$$
$$\mathscr{P}^1\varDelta e_1-2\varDelta\mathscr{P}^1\varDelta e_1=i^*w_3,\qquad \mathscr{P}^1g_0-A_1'e_1'=i^*w_4,$$
$$\mathscr{P}^1\varDelta g_0-A_2'e_1'-A_2e_1=i^*w_5,\qquad \mathscr{P}^1d_3-A_3''g_0-A_3'e_1'=i^*w_6,$$
$$\mathscr{P}^{p+1}d_3-A_4''g_0-A_4'e_1'=i^*w_7(p>3),\quad W_2d_3-A_5''g_0-A_5e_1-A_5'e_1'=i^*w_8(p>3),$$
$$\mathscr{P}^{p(p-2)}d_3-A_6'e_1'=i^*w_9,\quad \mathscr{P}^4\varDelta\mathscr{P}^1\varDelta d_3-A_7''g_0-A_7e_1-A_7'e_1'=i^*w_{10}\ (p=3).$$

By Theorem 4.1, $i^*H^*(\boldsymbol{K}_{(p^2-1)q-2})$ is generated by a_0, a_{p^2-1}, $b_s^{p-s}(2\leqq s\leqq p)$ and d_2 with the relations B3(a-1), (a-2), (b) and (d-1). By the dimensional reason and the relation $\mathscr{P}^1b_{p-1}^1=0$, $w_1=x\mathscr{P}^1\varDelta b_{p-1}^1$ for some $x\in Z_p$. For $p=3$, replacing e_1' by $e_1'-x\varDelta b_{p-1}^1$, we obtain the relation (e-3)[1]. By dimensional reason, (e-1) is obtained. For $p>3$, $\mathscr{P}^1\varDelta i^*w_3=\mathscr{P}^1\varDelta(\mathscr{P}^1\varDelta e_1-2\varDelta\mathscr{P}^1\varDelta e_1')=\varDelta\mathscr{P}^1\varDelta\mathscr{P}^1e_1=\varDelta\mathscr{P}^1\varDelta R_2e_1'=0$. For $p=3$, $R_1i^*w_3=R_1(\mathscr{P}^1\varDelta e_1+\varDelta\mathscr{P}^1\varDelta e_1')=0$. Set $w_3=y\mathscr{P}^3a_{p^2-1}+x_4\varDelta b_p^0$, $x_4=0$ for $p>3$. Then $\mathscr{P}^1\varDelta\mathscr{P}^3a_{p^2-1}\neq 0(p>3)$ and $R_1\mathscr{P}^3a_{p^2-1}\neq 0$ $(p=3)$ imply $y=0$, and (e-2) is obtained.

The relations (g) and (d-2) are obtained similarly. Q.E.D.

§5. $H^*(\boldsymbol{K}_k)$ for $k\leqq 2(p^2+p-2)(p-1)-2$

In this section, we shall continue the calculations of $H^*(\boldsymbol{K}_k)$ in certain dimensional restriction. Results are stated as follows.

THEOREM 5.1. *Let $(p^2-1)q\leqq k\leqq(p^2+p-2)q-2$. In degree $<(p^2+p+1)q-3$, $H^*(\boldsymbol{K}_k)$ has a minimal set of generators*:

$$\{a_0,\ \ a_r(p^2\leqq r\leqq p^2+p-2),\ \ a_{p^2}',\ \ b_s^{p-s}(2\leqq s\leqq p),\ \ b_1^p,\ \ b_2^{p-1}(\textit{if } p>3),$$
$$c_s^{p-s}(2\leqq s\leqq p-1),\ \ e_i'(1\leqq i\leqq p-2),\ \ e_i(1\leqq i\leqq p-2),\ \ d_2(\textit{if } p>3)\}$$

---

1) If we use a result of §6: $\phi(\beta_1\beta_{p-1})=b_{p-1}^1$, then $x\not\equiv 0 \bmod p$ implies $\{\beta_1\beta_{p-1},\ p\iota,\ \alpha_1\}\not\ni 0$ by Theorem 3.6. But $0=\beta_{p-1}\{\beta_1,\ p\iota,\ \alpha_1\}\subset\{\beta_1\beta_{p-1},\ p\iota,\ \alpha_1\}$, hence $x\equiv 0 \bmod p$.

The new generators are given by

TABLE A4

Generator a	Degree of a	$h(a)$	$\delta^*(a)$
a_r $(p^2 \leqq r \leqq p^2+p-2)$	rq	$(r-1)q$	$R_{r-1} j^{*-1} a_{r-1} (r \neq p^2+1)$ $\Delta \mathscr{P}^1 j^{*-1} a_{p^2} - \mathscr{P}^1 j^{*-1} a'_{p^2} (r = p^2+1)$
a'_{p^2}	$p^2 q+1$	$(p^2-1)q$	$\Delta \mathscr{P}^1 \Delta j^{*-1} a_{p^2-1}$
b_1^p	$(p^2+p-1)q-3$	$(p^2+1)q-2$	$\mathscr{P}^{p-2} j^{*-1} e'_1$
b_2^{p-1}	$(p^2+p)q-1$	$(p^2+1)q$	$\mathscr{P}^{p-1} j^* c_2^{p-2}$
$c_s^{p-s} (2 \leqq s < p)$	$(p^2+s-1)q+2s-4$	$(p^2+s-2)q+2s-3$	$\mathscr{P}^1 j^{*-1} b_s^{p-s}$
$e'_i (2 \leqq i \leqq p-2)$	$(p^2+i)q-2$	$(p^2+i-1)q-1$	$\mathscr{P}^1 j^{*-1} e_{i-1}$
$e_i (2 \leqq i \leqq p-2)$	$(p^2+i)q-1$	$(p^2+i-1)q-1$	$\mathscr{P}^1 \Delta j^{*-1} e_{i-1}$

The relations in the submodule of $H^*(\boldsymbol{K}_k)$ generated by the above elements are given by Table B3 *and the following*

TABLE B4

(*a*) $R_r a_r = \Delta a_{p^2} = \Delta a'_{p^2} = \Delta \mathscr{P}^1 a_{p^2} - \mathscr{P}^1 a'_{p^2} = 0,\ p^2+1 \leqq r \leqq p^2+p-2.$

(*b*-1) $\mathscr{P}^2 b_1^p \equiv 0 \bmod \operatorname{Im} i^*.$ (*b*-2) $\mathscr{P}^1 b_2^{p-1} \equiv 0 \bmod \operatorname{Im} i^*.$

(*c*) *If* $p>3$, $\mathscr{P}^{p-1} c_2^{p-2} = 0$, $\mathscr{P}^{p-1} c_s^{p-s} \equiv 0 \bmod \operatorname{Im} i^*$ $(3 \leqq s \leqq p-1)$. *If* $p=3$, $\mathscr{P}^2 c_2^1 = 0$.

(*e*-1) $(p>3)$ $-R_1 e'_i + \mathscr{P}^1 e_i = 0,\ 2 \leqq i \leqq p-3,$

$-R_1 e'_{p-2} + \mathscr{P}^1 e_{p-2} - x_3 b_p^0 = 0$ *for some* $x_3 \in Z_p$.

(*e*-2) $(p>3)$ $-\Delta \mathscr{P}^1 \Delta e'_i + \mathscr{P}^1 \Delta e_i = 0,\ 2 \leqq i \leqq p-3,$

$-\Delta \mathscr{P}^1 \Delta e'_{p-2} + \mathscr{P}^1 \Delta e_{p-2} - x_4 \Delta b_p^0 = 0$ *for some* $x_4 \in Z_p$.

(*e*-3) $(p>3)$ $\mathscr{P}^{p-1} e'_i \equiv 0 \bmod \operatorname{Im} i^*,\ 2 \leqq i \leqq p-2.$

(*l*) $a'_{p^2} = 0$ *in* $\boldsymbol{K}_k$, $k \geqq p^2 q$,

$a = 0$ *in* $\boldsymbol{K}_k$, $k \geqq \deg a$, *for the above generator* $a \neq a'_{p^2}$ *with* $\deg a \leqq (p^2+p-1)q-4$.

PROOF. The proof is done by the induction on k. The following cases are considered.

(i) $H^{k+1}(\boldsymbol{K}_k) = Z_p\{a_r\}$ for $k = \deg a_r - 1,\ p^2-1 \leqq r \leqq p^2+p-3,\ r \neq p^2+1.$

(ii) $H^{k+1}(\boldsymbol{K}_k) = Z_p\{a_{p^2+1},\ c_2^{p-2}\}$ for $k = (p^2+1)q-1.$

(iii) $H^{k+1}(\boldsymbol{K}_k) = Z_p\{b_s^{p-s}\}$ for $k = \deg b_s^{p-s} - 1,$ $2 \leqq s \leqq p-1.$

(iv) $H^{k+1}(\boldsymbol{K}_k) = Z_p\{c_s^{p-s}\}$ for $k = \deg c_s^{p-s} - 1,$ $2 \leqq s \leqq p-2.$

(v) $H^{k+1}(\boldsymbol{K}_k) = Z_p\{e'_i\}$ for $k = \deg e'_i - 1,$ $1 \leqq i \leqq p-3.$

(vi) $H^{k+1}(\boldsymbol{K}_k) = Z_p\{e_i\}$ for $k = \deg e_i - 1,$ $1 \leqq i \leqq p-3.$

(vii) $H^{k+1}(\boldsymbol{K}_k) = 0$ for other k.

Assume that the theorem is true for $\boldsymbol{K}_k$ in each case of the above. In the case (vii), the theorem is true for $\boldsymbol{K}_{k+1}$ obviously. In the case (i), by Proposition 1.2, new generators of $H^*(\boldsymbol{K}_{k+1})$ are a_{r+1} and a'_{p^2} (if $r = p^2-1$), and by Proposition 1.5 of [6: I], new relations are the following:

$$R_{r+1}a_{r+1}=i^*w_1(r>p^2+1),\quad \Delta a_{p^2}=i^*w_2,\quad \Delta a'_{p^2}=i^*w_3,\quad \Delta\mathscr{P}^1a_{p^2}-\mathscr{P}^1a'_{p^2}=i^*w_4.$$

Since w_1 and w_4 belong to $H^{(t+1)q+1}(\boldsymbol{K}_{(t-1)q-1})$, $t\geqq p^2$, and $i^*H^{(t+1)q+1}(\boldsymbol{K}_{(t-1)q-1})=0$ from the assumption of the induction, we see $i^*w_1=i^*w_4=0$. Since $i^*w_2\in i^*H^{p^2q+1}(\boldsymbol{K}_{(p^2-1)q-1})=Z_p\{b_2^{p-2}\}$ and $i^*w_3\in i^*H^{p^2q+2}(\boldsymbol{K}_{(p^2-1)q-1})=Z_p\{\Delta b_2^{p-2}\}$ ($+Z_p\{e'_1\}$ if $p=3$), the possibility of $i^*w_2\neq 0$ or $i^*w_3\neq 0$ is the following.

$$\Delta a_{p^2}=i^*w_2=xb_2^{p-2},\quad \Delta a'_{p^2}=i^*w_3=y\Delta b_2^{p-2}+ze'_1.$$

Hence $x\Delta b_2^{p-2}=\Delta\Delta a_{p^2}=0$, $z\Delta e'_1=\Delta\Delta a'_{p^2}-y\Delta\Delta b_2^{p-2}=0$, and it follows from $\Delta b_2^{p-2}\neq 0$ and $\Delta e'_1\neq 0$ that $x=z=0$. Thus $i^*w_2=0$. By the replacement of a'_{p^2} by $a'_{p^2}-yb_2^{p-2}$, we have $i^*w_3=0$.

Consequently, by a suitable choice of a'_{p^2}, the relations (a) are established. Thus, the theorem is true for $\boldsymbol{K}_{k+1}$ in the case (i).

Next we consider the case (iii). Since $A^*b_s^{p-s}(2\leqq s\leqq p-1)$ has the relations generated by $\mathscr{P}^1b_s^{p-s}=0$, new generator of $H^*(\boldsymbol{K}_{k+1})$ is $c_s^{p-s}(2\leqq s\leqq p-1)$, and by (1.1) of [4], new relation is given by the form $\mathscr{P}^{p-1}c_s^{p-s}=i^*w$. But for $s\geqq 3(p>3)$ the degree of this relation exceeds the range of degree in this theorem[1]. For $s=2$, the possibility of $i^*w\neq 0$ is $i^*w=x\mathscr{P}^{p-1}a_{p^2+1}$. Replacing c_2^{p-2} by $c_2^{p-2}-xa_{p^2+1}$, we obtain $i^*w=0$. Thus the relations (c) are obtained, and the theorem is true for $\boldsymbol{K}_{k+1}$.

The cases (ii) and (iv) are similar to (i) and (iii).

Next we consider the case (vi). By Theorem 4.4 and the assumption of the induction, A^*e_i (in $H^*(\boldsymbol{K}_k)$) has the relations $\mathscr{P}^1e_i=\mathscr{P}^1\Delta e_i=0$ and new generators e'_{i+1} and e_{i+1} of $H^*(\boldsymbol{K}_{k+1})$ are obtained. By (1.1) and (3.3) of [4] and the Adem relations, new relations are

$$-R_1e'_{i+1}+\mathscr{P}^1e_{i+1}=i^*w_1,\quad -\Delta\mathscr{P}^1\Delta e'_{i+1}+\mathscr{P}^1\Delta e_{i+1}=i^*w_2,\quad \mathscr{P}^{p-1}e'_{i+1}=i^*w_3.$$

The possibility of $i^*w_1\neq 0$ or $i^*w_2\neq 0$ is as follows:

$$i^*w_1=x\Delta\mathscr{P}^1\Delta b_{p-1}^1+x_3b_p^0\quad \text{for } i+1=p-2\quad (i^*w_1=0\ \text{ for } i+1<p-2),$$

$$i^*w_2=y\mathscr{P}^2a_{p^2+i}(+x_4\Delta b_p^0\quad \text{if } i+1=p-2).$$

Since $R_1\Delta+\Delta\mathscr{P}^1\Delta=0$, $i^*w_1=x_3b_p^0$ for $i+1=p-2$ by a suitable choice of e'_{p-2}. Since $\mathscr{P}^1\Delta(i^*w_2)=\mathscr{P}^1\Delta(-\Delta\mathscr{P}^1\Delta e'_{i+1}+\mathscr{P}^1\Delta e_{i+1})=\Delta\mathscr{P}^1\Delta\mathscr{P}^1e_{i+1}=\Delta\mathscr{P}^1\Delta(R_1e_{i+1}+i^*w_1)=0$ and $\mathscr{P}^1\Delta\mathscr{P}^2a_{p^2+i}\neq 0$, we have $y=0$. Thus (e-1) and (e-2) are obtained and the theorem is true for $\boldsymbol{K}_{k+1}$.

Finally, we consider the case (v). $A^*e'_1$ has the relation $\mathscr{P}^{p-2}e'_1=0$ in $\boldsymbol{K}_k$ ($k=$ deg e'_1-1), and the new generator of $H^*(\boldsymbol{K}_{k+1})$ is b_1^p. By (1.1) of [4],

1) In this case, the proof of $i^*w=0$ will be given in [5: Theorem 13.1]. Also in [5: Theorems 10.1 and 13.1], we shall discuss the relations B4 (b-1), (b-2) and (e-3).

the new relation is $\mathscr{P}^2 b_1^p = i^* w$. For $i>1$, $A^* e_i'$ has the relation of the form $\mathscr{P}^{p-1} e_i' = i^* w'$ and this gives no new generators in degree $<(p^2+p+1)q-3$. Thus, the theorem is true for $\boldsymbol{K}_{k+1}$. Q.E.D.

§6. ${}_p\pi_k(S)$ for $k<2(p^2+p-1)(p-1)-3$

In this section, we shall compute the group ${}_p\pi_k(S)$ for $k<(p^2+p-1)q-3$, using the results on $H^*(\boldsymbol{K}_k)$ in previous sections.

In [7], H. Toda has calculated the unstable group ${}_p\pi_{2n+1+k}(S^{2n+1})$, hence the stable group ${}_p\pi_k(S)$, for $k<(p^2+p)q-5$. Our results in this section are independent of his results.

PROPOSITION 6.1 (cf. [6: III, Proposition 3.11]). *The vector space $H^{k+1}(\boldsymbol{K}_k)$, $k<(p^2+p-1)q-3$, is as follows:*

$Z_p\{\mathscr{P}^1 a_0\}$, $\Delta\mathscr{P}^1 a_0 \neq 0$, *for* $k=q-1$.

$Z_p\{a_r\}$, $\Delta a_r \neq 0 (r \not\equiv 0 \bmod p)$, $\Delta_2 a_{sp} \neq 0 (r=sp \not\equiv 0 \bmod p^2)$, $\Delta_3 a_{p^2} \neq 0$ $(r=p^2)$,
for $k=rq-1$, $2 \leqq r \leqq p^2+p-2$, $r \neq p^2-p$, p^2+1.

$Z_p\{a_{p^2-p}, c_1^{p-2}\}$, $\Delta_2 a_{p^2-p} \neq 0$, $\Delta c_1^{p-2} \neq 0$, *for* $k=(p^2-p)q-1$.

$Z_p\{a_{p^2+1}, c_2^{p-2}\}$, $\Delta a_{p^2+1} \neq 0$, $\Delta c_2^{p-2} \neq 0$, *for* $k=(p^2+1)q-1$.

$Z_p\{b_s^r\}$, $\Delta b_s^r \neq 0$, *for* $k=((r+s)p+s-1)q-2r-2$, $r \geqq 0$, $1 \leqq s \leqq p-1$,
$r+s \leqq p$ *and* $(r, s)=(p, 1)$.

$Z_p\{c_s^r\}$, $\Delta c_s^r \neq 0$, *for* $k=((r+s)p+s)q-2r-3$, $r \geqq 0$, $1 \leqq s \leqq p-1$,
$r+s \leqq p$, $(r, s) \neq (p-2, 1)$, $(p-1, 1)$, $(p-2, 2)$.

$Z_p\{e_i'\}$, $\Delta e_i' \neq 0$, *for* $k=(p^2+i)q-3$, $1 \leqq i \leqq p-2$. .

$Z_p\{e_i\}$, $\Delta e_i \neq 0$, *for* $k=(p^2+i)q-2$, $1 \leqq i \leqq p-2$.

0 *for other value of* k.

PROOF. For $k \leqq (p^2+p-2)q-2$, $H^{k+1}(\boldsymbol{K}_k)$ is computed directly from (2.2), (2.3), Theorems 2.3, 4.1, 4.4 and 5.1. For $(p^2+p-2)q-1 \leqq k < (p^2+p-1)q-3$, it is computed easily.

The assertions on the Bockstein operations Δ_2 and Δ_3 follow quite similarly to Lemma 3.12 of [6: III]. Q.E.D.

By (1.4) and this proposition, the group ${}_p\pi_k(S)$ is calculated.

THEOREM 6.2. *Let $k<(p^2+p-1)q-3$. The group ${}_p\pi_k(S)$ is the direct sum of the cyclic groups generated by the following elements of degree k:*

Generator γ	Degree of γ $(=k)$	Order of γ	$\phi(\gamma)$
α_r $(r\not\equiv 0 \bmod p)$	$rq-1$	p	$\mathscr{P}^1 a_0 (r=1)$ a_r $(r>1)$
α'_{sp} $(s\not\equiv 0 \bmod p)$	$spq-1$	p^2	a_{sp}
α''_{p^2}	p^2q-1	p^3	a_{p^2}
$\beta_1^r\beta_s$ $(r\geqq 0,\ 1\leqq s<p)$	$((r+s)p+s-1)q-2r-2$	p	b_s^r
$\alpha_1\beta_1^r\beta_s$ $(r\geqq 0,\ 1\leqq s<p,\ (r,s)\neq(p-1,1))$	$((r+s)p+s)q-2r-3$	p	c_s^r
ε'	$(p^2+1)q-3$	p	e'_1
ε_i $(1\leqq i\leqq p-2)$	$(p^2+i)q-2$	p	e_i
$\alpha_1\varepsilon_i$ $(1\leqq i\leqq p-3)$	$(p^2+i+1)q-3$	p	e'_{i+1}

The elements α_r, α'_{sp}, α''_{p^2}, ε' and ε_i $(i>1)$ are given by the following formulas:

(6.1) $\qquad \alpha_r \in \{\alpha_{r-1}, p\iota, \alpha_1\},\quad \alpha_{sp}=p\alpha'_{sp},\quad \alpha'_{p^2}=p\alpha''_{p^2}.$

(6.2) $\qquad \varepsilon'=\{\beta_1^p, \alpha_1, \alpha_1\}.$

(6.3) $\qquad \varepsilon_i=\{\varepsilon_{i-1}, p\iota, \alpha_1\},\quad 2\leqq i\leqq p-2.$

For $p=3$ the following relation is satisfied:

(6.4) ([7: III, Lemma 15.5]) $(p=3)\quad \alpha_1\varepsilon'=\pm\beta_1^4.$

REMARK. We shall prove in [5: Corollary 12.4] that the element ε_1 is chosen so that it satisfies the following

(6.5) $\qquad \varepsilon_1=\{\alpha_1, p\iota, \beta_1^p, \alpha_1\},$

where the right side is a tertiary composition (for the definition see e.g. [2]).

REMARK. In our situation, the element α_r is determined up to the indeterminacy of the secondary composition and the element β_s of ${}_p\pi_{(sp+s-1)q-2}(S)\approx Z_p$, $1<s<p$, is determined up to a multiple of the non-zero element of Z_p. In [10] these elements are determined uniquely.

REMARK. The generator ε'_i in [7: III] correspond to ε' (for $i=1$) and $\alpha_1\varepsilon_{i-1}$ (for $i>1$). The non-triviality of $\alpha_1\varepsilon_i$ and the relations (6.2) (for $p>3$)

and (6.3) do not appear in [7], and appear in [8: Proposition 1] without proof.

PROOF OF THEOREM 6.2. By Proposition 6.1, it follows directly that the group ${}_p\pi_k(S)$ is the direct sum of the cyclic groups generated by $\phi^{-1}(\mathscr{P}^1 a_0)$, $\phi^{-1}(a_r)$, $\phi^{-1}(b_s^r)$, $\phi^{-1}(c_s^r)$, $\phi^{-1}(e_i')$ and $\phi^{-1}(e_i)$.

By Theorem 3.3 and the relation $\mathscr{P}^1 b_s^r=0$, $\phi(\beta_1^r\beta_s)=b_s^r$ implies $\phi(\alpha_1\beta_1^r\beta_s)=\pm c_s^r$ for $(r, s)\neq(p-1, 1)$. By Theorem 3.4 and the relation $\mathscr{P}^{p-1}c_s^r=0$, $\phi(\alpha_1\beta_1^r\beta_s)=c_s^r$ implies $\phi(\beta_1^{r+1}\beta_s)=xb_s^{r+1}$, $x\not\equiv 0 \bmod p$. By Theorem 3.5 and the relations $\mathscr{P}^2 b_1^{p-1}=\mathscr{P}^{p-2}e_1'=0$, $\phi(\beta_1^p)=b_1^{p-1}$ implies $\phi(\varepsilon')=\pm e_1'$ and $\phi(\beta_1^{p+1})=xb_1^p$, $x\not\equiv 0 \bmod p$, where the indeterminacy of $\{\beta_1^p, \alpha_1, \alpha_1\}$ is trivial and ε' satisfies (6.2). By Theorem 3.3 and $\mathscr{P}^1 e_i=0$, $\phi(\varepsilon_i)=e_i$ implies $\phi(\alpha_1\varepsilon_i)=\pm e_{i+1}'$ for $1\leqq i\leqq p-3$. By Theorem 3.6 and $\mathscr{P}^1\Delta e_i=0$, $\phi(\varepsilon_i)=e_i$ implies $\phi(\varepsilon_{i+1})=\pm e_{i+1}$ for $1\leqq i\leqq p-3$, where ε_{i+1} satisfies (6.3).

The relation (6.1) is quite similar to (4.11–12) of [6: IV], and (6.4) follows from Theorem 3.3 and $\mathscr{P}^1 e_1'=0$. Q.E.D.

§7. $H^*(\boldsymbol{K}_k)$ and ${}_p\pi_k(S)$ for $k\leqq 2(p^2+p)(p-1)-3$

We shall start from the discussion on the following coefficients x_3, $x_4\in Z_p$ in the relations B3 (e–1), B3 (e–2), B4 (e–1) and B4 (e–2):

$$\mathscr{P}^1 e_{p-2}=x_3 b_p^0,\quad \mathscr{P}^1\Delta e_{p-2}=x_4\Delta b_p^0 \quad in\ H^*(\boldsymbol{K}_{(p^2+p-2)q-2}). \tag{7.1}$$

Set $t=(p^2+p-2)q-2$ throughout this section.

H. Toda has proved ${}_p\pi_{2n+t+q}(S^{2n+1})=0$ for $n>p^2-1$ hence ${}_p\pi_{t+q-1}(S)=0$ [7: III]. By Theorem 3.3, $x_3=0$ implies $\alpha_1\varepsilon_{p-2}\neq 0$ in ${}_p\pi_{t+q-1}(S)$. Thus $x_3\neq 0$. Replacing b_p^0 by $(1/x_3)b_p^0$, we have the following

LEMMA 7.1. *$\mathscr{P}^1 e_{p-2}=b_p^0$ in $\boldsymbol{K}_t$, $t=(p^2+p-2)q-2$.*

Let $R=x_4\Delta\mathscr{P}^1-\mathscr{P}^1\Delta$. Since the submodule $A^*b_p^0$ of $H^*(\boldsymbol{K}_t)$ has the relations $\mathscr{P}^1 b_p^0=\Delta\mathscr{P}^1\Delta b_p^0=W_p b_p^0=0$ in degree $<(2p^2+p)q$ by B2 (b–1), (b–2) and (b–5), we obtain the following

LEMMA 7.2. *In degree $<(2p^2+p)q$, the submodule A^*e_{p-2} of $H^*(\boldsymbol{K}_t)$, $t=(p^2+p-2)q-2$, has the relations $\mathscr{P}^2 e_{p-2}=Re_{p-2}=W_p\mathscr{P}^1 e_{p-2}=0$.*

PROOF. Let $\gamma e_{p-2}=0$. Then $\gamma=\gamma_1\mathscr{P}^1+\gamma_2\mathscr{P}^1\Delta$, and γ_1 and γ_2 satisfy $(\gamma_1+x_4\gamma_2\Delta)b_p^0=0$, $\gamma_1+x_4\gamma_2\Delta=\gamma_3\mathscr{P}^1+\gamma_4\Delta\mathscr{P}^1\Delta+\gamma_5 W_p$. Then

$$\begin{aligned}\gamma&=(-x_4\gamma_2\Delta+\gamma_3\mathscr{P}^1+\gamma_4\Delta\mathscr{P}^1\Delta+\gamma_5 W_p)\mathscr{P}^1+\gamma_2\mathscr{P}^1\Delta\\&=(-\gamma_2+(1/2)\gamma_4(x_4\mathscr{P}^1\Delta-\Delta\mathscr{P}^1))R+2\gamma_3\mathscr{P}^2+\gamma_5 W_p\mathscr{P}^1.\end{aligned}$$

Thus the lemma follows from this relation. Q.E.D.

Let e_{p-1} and f be elements of $H^*(\boldsymbol{K}_{t+1})$ such that

$$\delta^* e_{p-1} = Rj^{*-1} e_{p-2}, \quad \delta^* f = \mathscr{P}^2 j^{*-1} e_{p-2}. \tag{7.2}$$

The following lemma is used to determine the relations related with e_{p-1} and f, and follows from routine calculations by making use of the methods employed in [4].

LEMMA 7.3. *The kernel of $R^*: A^* \to A^*$, the right translation by R, is equal to*

$$A^*((x_4-2)\mathscr{P}^1\varDelta + \varDelta\mathscr{P}^1) \qquad \textit{if } x_4 \not\equiv 1, 2 \bmod p,$$

$$A^*(\varDelta\mathscr{P}^1 - \mathscr{P}^1\varDelta) + A^*\varDelta\mathscr{P}^1\varDelta \qquad \textit{if } x_4 \equiv 1 \bmod p,$$

$$A^*\varDelta\mathscr{P}^1 + A^*\varDelta\mathscr{P}^1\varDelta \qquad \textit{if } x_4 \equiv 2 \bmod p.$$

The kernel of $(\mathscr{P}^2)^*: A^* \to A^*/A^*R$ is equal to

$$A^*((2x_4-3)\mathscr{P}^1\varDelta - (x_4-2)\varDelta\mathscr{P}^1) + A^*\mathscr{P}^{p-2} \quad \textit{if } x_4 \not\equiv 0, 2, 3/2 \bmod p,$$

$$A^*\varDelta + A^*\mathscr{P}^{p-2} \qquad \textit{if } x_4 \equiv 2 \bmod p,$$

$$A^*R_2 + A^*\mathscr{P}^{p-2} \qquad \textit{if } x_4 \equiv 0 \bmod p, \not\equiv 3/2 \bmod p, p>3,$$

$$A^*\varDelta\mathscr{P}^1 + A^*\varDelta\mathscr{P}^1\varDelta + A^*\mathscr{P}^{p-2} \qquad \textit{if } x_4 \equiv 3/2 \bmod p.$$

PROPOSITION 7.4. *Let $t+1 \leqq k \leqq t+2q-1$, $t=(p^2+p-2)q-2$. In degree $<t+2q+2$, $H(\boldsymbol{K}_k)$ has minimal sets of generators:*

$$\{a_0, \quad a_{p^2+p-2}, \quad a_{p^2+p-1}, \quad c^1_{p-1}, \quad b^p_1, \quad b^{p-1}_2, \quad e_{p-1}, \quad f, \quad f'\}$$

and of relations:

$$\{\varDelta a_0 = \mathscr{P}^1 a_0 = \mathscr{P}^p a_0 = \mathscr{P}^{p^2} a_0 = 0, \quad R_{p^2+p-2} a_{p^2+p-2} = 0, \quad \mathscr{P}^2 c^1_2 = 0 \ (\textit{if } p=3),$$

$$(\varDelta\mathscr{P}^1 + (x_4-2)\mathscr{P}^1\varDelta) e_{p-1} = 0, \quad \varDelta f - (1/2)\mathscr{P}^1 e_{p-1} = 0 \ \textit{if } x_4 \equiv 2 \bmod p,$$

$$a = 0 \textit{ in } \boldsymbol{K}_k, \ k \geqq \deg a, \textit{ for } a = a_{p^2+p-2}, a_{p^2+p-1}, c^1_{p-1}, b^p_1, e_{p-1}\}.$$

Here a_{p^2+p-1} and f' are given by

$$\delta^* a_{p^2+p-1} = R_{p^2+p-2} j^{*-1} a_{p^2+p-2},$$

$$\delta^* f' = (\varDelta\mathscr{P}^1 + (x_4-2)\mathscr{P}^1\varDelta) j^{*-1} e_{p-1}.$$

PROOF. The new generators of $H^*(\boldsymbol{K}_{t+1})$ are e_{p-1} and f of (7.2). From the above lemma and the Adem relations, the relations in the submodule $A^*R + A^*\mathscr{P}^2$ of A^*, degree $<3q$, are given by

$$(\Delta\mathscr{P}^1+(x_4-2)\mathscr{P}^1\Delta)R=0,$$

$$\Delta\mathscr{P}^2-(1/2)\mathscr{P}^1R=0 \quad \text{if } x_4\equiv 2 \bmod p,$$

$$\Delta\mathscr{P}^1\Delta R=0 \quad \text{if } x_4\equiv 1,\ 2 \bmod p.$$

Hence, the new relations are the following:

$$\Delta f-(1/2)\mathscr{P}^1 e_{p-1}=i^*w_1 \quad \text{if } x_4\equiv 2 \bmod p,$$

$$(\Delta\mathscr{P}^1+(x_4-2)\mathscr{P}^1\Delta)e_{p-1}=i^*w_2,$$

$$\Delta\mathscr{P}^1\Delta e_{p-1}=i^*w_3 \quad \text{if } x_4\equiv 1,\ 2 \bmod p.$$

The degree of the last relation exceeds our restriction of the degree. The possibility of $i^*w_1\neq 0$ or $i^*w_2\neq 0$ is the following:

$$i^*w_1=x\Delta\mathscr{P}^1\Delta b_1^p+yb_2^{p-1}\ (y=0 \quad \text{if } p=3),$$

$$i^*w_2=z\Delta b_2^{p-1}\ (z=0 \quad \text{if } p=3).$$

By the replacement of f by $f+x\mathscr{P}^1\Delta b_1^p$, we have $i^*w_1=yb_2^{p-1}$.

Before proving $y=z=0$ for $p>3$, we shall prove the proposition for $k>t+1$. Since $H^{t+2}(\boldsymbol{K}_{t+1})=Z_p\{a_{p^2+p-2}\}(+Z_p\{c_2^1\}$ if $p=3$), the new generators of $H^*(\boldsymbol{K}_{t+2})$ are a_{p^2+p-1} and b_2^2 in addition for $p=3$, and new relations are $R_{p^2+p-1}a_{p^2+p-1}\equiv 0 \bmod \mathrm{Im}\,i^*$, $\Delta\mathscr{P}^1\Delta a_{p^2+p-1}\equiv 0 \bmod \mathrm{Im}\,i^*$ and $\mathscr{P}^1b_2^2\equiv 0 \bmod \mathrm{Im}\,i^*$. These are of degree $\geqq t+2q+2$. Thus the proposition is true for $k=t+2$. For $k>t+2$, the proposition is proved rather easily.

Consequently, if $x_4\not\equiv 2 \bmod p$, $p>3$, $H^{t+2q+1}(\boldsymbol{K}_{t+2q})=Z_p\{b_2^{p-1}\}$, $\Delta b_2^{p-1}=0$ or $=Z_p\{b_2^{p-1},f'\}$, $\Delta b_2^{p-1}\neq 0$ according as $z\neq 0$ or $z=0$. Since $\phi(\beta_1^{p-1}\beta_2)=b_2^{p-1}$ and $p\beta_1^{p-1}\beta_2=0$, it follows from $\Delta b_2^{p-1}\neq 0$ that $z=0$. If $x_4\equiv 2 \bmod p$, $p>3$, the triviality of y is equivalent to $z=0$, by comparing Δi^*w_1 and i^*w_2, and so $H^{t+2q+1}(\boldsymbol{K}_{t+2q})=0$ for the case $y\neq 0$. By Theorem 3.4, $\beta_1^{p-1}\beta_2\neq 0$ in ${}_p\pi_{t+2q}(\boldsymbol{S})$, hence $y=z=0$. Q.E.D.

By this proposition, $H^{t+q+1}(\boldsymbol{K}_{t+q})=Z_p\{e_{p-1}\}$, $H^{t+2q}(\boldsymbol{K}_{t+2q-1})=Z_p\{f\}$. Hence ${}_p\pi_{t+q}(\boldsymbol{S})$ and ${}_p\pi_{t+2q-1}(\boldsymbol{S})$ are the cyclic groups generated by the elements ε_{p-1} and φ such that $\phi(\varepsilon_{p-1})=e_{p-1}$ and $\phi(\varphi)=f$.

The following two propositions are proved in the next section.

PROPOSITION 7.5. *For $p=3$, ${}_3\pi_{45}(\boldsymbol{S})$ is isomorphic to Z_9.*

PROPOSITION 7.6. *For $p>3$, $\alpha_1\varepsilon_{p-1}\neq 0$.*

Using these propositions, we determine the coefficient x_4 in (7.1).

PROPOSITION 7.7. $x_4\equiv 2 \bmod p$.

PROOF. By Proposition 7.4, it follows that

$$\Delta f=0 \quad \textit{if and only if} \quad x_4\equiv 2 \bmod p.$$

For $p=3$, by use of (1.4) and Proposition 7.5, $x_4\equiv 2 \bmod 3$ holds.

For the case $p>3$, since the indeterminacy of $\{\varepsilon_{p-2}, \alpha_1, \alpha_1\}$ is the subgroup generated by $\alpha_1\varepsilon_{p-1}$, which is isomorphic to Z_p by Proposition 7.6 and $p\alpha_1=0$, and since $\{\varepsilon_{p-2}, \alpha_1, \alpha_1\}$ does not contain zero by Theorem 3.5, the group ${}_p\pi_{t+2q-1}(\boldsymbol{S})$ consists of more than p elements. Thus $x_4\equiv 2 \bmod p$.

Q.E.D.

By a little calculation of $H^*(\boldsymbol{K}_k)$ and the above propositions. we have the following

THEOREM 7.8. *Let $(p^2+p-2)q-1\leqq k\leqq (p^2+p)q-3$. In degree $<(p^2+p+1)q-4$, $H^*(\boldsymbol{K}_k)$ has a system of generators:*

$$a_0, \quad a_{p^2+p-2}, \quad c^1_{p-1}, \quad b^p_1, \quad d_2(\textit{if } p>3),$$

and the following elements

TABLE A5

Generator a	Degree of a	$h(a)$	$\delta^*(a)$
a_{p^2+p-1}	$(p^2+p-1)q$	$(p^2+p-2)q$	$R_{p^2+p-2}j^{*-1}a_{p^2+p-2}$
b^{p-1}_2	$(p^2+p)q-1$	$(p^2+1)q$	$\mathscr{P}^{p-1}j^{*-1}c^{p-2}_2$
e_{p-1}	$(p^2+p-1)q-1$	$(p^2+p-2)q-1$	$R_{p-2}j^{*-1}e_{p-2}$
f	$(p^2+p)q-2$	$(p^2+p-2)q-1$	$\mathscr{P}^2j^{*-1}e_{p-2}$
f'	$(p^2+p)q-1$	$(p^2+p-1)q-1$	$\Delta\mathscr{P}^1j^{*-1}e_{p-1}$
a_{p^2+p}	$(p^2+p)q$	$(p^2+p-1)q$	$R_{p^2+p-1}j^{*-1}a_{p^2+p-1}$
a'_{p^2+p}	$(p^2+p)q+1$	$(p^2+p-1)q$	$\Delta\mathscr{P}^1\Delta j^{*-1}a_{p^2+p-1}$

where by the dimensional reason we take off the last two elements if $p=3$, and we add the element d'_2 such that $\delta^*d'_2=\Delta\mathscr{P}^1\Delta j^{*-1}e_{p-1}$, $h(d'_2)=(p^2+p-1)q-1$ and deg *$d'_2=(p^2+p)q$ if $p>3$ and $x_5=0$ in the relation* (*e*-2) *in Table* B5 *below.*

The relations in the submodule of $H^*(\boldsymbol{K}_k)$ generated by the above elements (except d'_2) are given by

TABLE B5

(*a*-1) $\Delta a_0=\mathscr{P}^1a_0=\mathscr{P}^pa_0=\mathscr{P}^{p^2}a_0=0.$

(*a*-2) $R_{p^2+p-2}a_{p^2+p-2}=0,$

$R_{p^2+p-1}a_{p^2+p-1}=\Delta\mathscr{P}^1\Delta a_{p^2+p-1}=\Delta a_{p^2+p}=\Delta a'_{p^2+p}=0$ *if* $p>3$,

$\Delta\mathscr{P}^1a_{p^2+p}-\mathscr{P}^1a'_{p^2+p}\equiv 0 \bmod \operatorname{Im} i^*$ *if* $p>3$,

$R_{11}a_{11}\equiv\Delta\mathscr{P}^1\Delta a_{11}\equiv 0 \bmod \operatorname{Im} i^*$ *if* $p=3$.

(*b*) $\mathscr{P}^2b^p_1\equiv 0 \bmod \operatorname{Im} i^*$, $\mathscr{P}^1b^{p-1}_2\equiv 0 \bmod \operatorname{Im} i^*$.

(*c*) $\mathscr{P}^{p-1}c^1_{p-1}\equiv 0 \bmod \operatorname{Im} i^*$ *if* $p>3$, $\mathscr{P}^2c^1_2=0$ *if* $p=3$.

(*d*) $\Delta d_2=\mathscr{P}^1 d_2=\mathscr{P}^p d_2=0.$

(*e*-1) $\Delta\mathscr{P}^1 e_{p-1}=0.$

(*e*-2) $\Delta\mathscr{P}^1\Delta e_{p-1}-x_5 d_2\equiv 0 \bmod A^* a_{p^2+p-2}$ *for some* $x_5\in Z_p$ *if* $p>3$,
$\Delta\mathscr{P}^1\Delta e_2\equiv 0 \bmod \mathrm{Im}\ i^*$ *if* $p=3$.

(*f*-1) $\Delta f-(1/2)\mathscr{P}^1 e_{p-1}=0,\quad \mathscr{P}^{p-2}f\equiv 0 \bmod \mathrm{Im}\ i^*.$

(*f*-2) $\Delta f'=0,\quad \mathscr{P}^1\Delta\mathscr{P}^{p-1}f'\equiv 0 \bmod \mathrm{Im}\ i^*.$

(*l*) $a=0$ *in* $\boldsymbol{K}_k$, $k\geqq \deg a$, *for* $a=a_{p^2+p-2}, a_{p^2+p-1}, b_1^p, e_{p-1}, f$,
$a=0$ *in* $\boldsymbol{K}_k$, $k\geqq \deg a-1$, *for* $a=f'$,
$d_2=0$ *in* $\boldsymbol{K}_k$, $k\geqq(p^2+p-1)q-1$, *if* $p>3$ *and* $x_5=0$.
Furthermore the following relation holds.

(*f*-3) $\Delta_2 f=(1/2)f'$ *in* $\boldsymbol{K}_k$, $k\geqq(p^2+p-1)q-1$.

PROOF. By use of Lemma 3.5. i) of [6: III], the relation (*f*-3) follows from the first relation of (*f*-1). The first relation of (*f*-2) follows from (*f*-3). The second relation of (*f*-2) follows from (3.10) of [4].

Others are proved from Proposition 7.4 and Theorem 5.1 by a little calculation. Q.E.D.

THEOREM 7.9. *The group* ${}_p\pi_k(\boldsymbol{S})$, $(p^2+p-1)-3\leqq k\leqq(p^2+p)q-3$, *is as follows*:

$$\begin{aligned} {}_p\pi_k(\boldsymbol{S})&\approx Z_p \quad \text{generated by } \varepsilon_{p-1} && \text{for } k=(p^2+p-1)q-2,\\ &\approx Z_p \quad \text{generated by } \alpha_{p^2+p-1} && \text{for } k=(p^2+p-1)q-1,\\ &\approx Z_{p^2} \quad \text{generated by } \varphi && \text{for } k=(p^2+p)q-3,\\ &=0 \quad \text{for other } k. \end{aligned}$$

The generators are given by

(7.3) $$\varepsilon_{p-1}=\{\varepsilon_{p-2}, p\iota, \alpha_1\}.$$

(7.4) $$\alpha_{p^2+p-1}=\{\alpha_{p^2+p-2}, p\iota, \alpha_1\}.$$

(7.5) $$\varphi\in\{\varepsilon_{p-2}, \alpha_1, \alpha_1\}.$$

And the following relations are satisfied:

(7.6) $$p\varphi=\alpha_1\varepsilon_{p-1},\quad 2\varepsilon_{p-1}=\{\varepsilon_{p-2}, \alpha_1, p\iota\}.$$

PROOF. By Theorem 7.8, we have easily

$$\begin{aligned} H^{k+1}(\boldsymbol{K}_k)\ &=Z_p\{e_{p-1}\},\ \Delta e_{p-1}\neq 0, \quad \text{for } k=(p^2+p-1)q-2,\\ &=Z_p\{f\},\ \Delta f=0,\ \Delta_2 f\neq 0, \quad \text{for } k=(p^2+p)q-3,\\ &=Z_p\{a_{p^2+p-1}\},\ \Delta a_{p^2+p-1}\neq 0, \quad \text{for } k=(p^2+p-1)q-1,\\ &=0 \quad \text{for other } k. \end{aligned}$$

Then, using (1.4), the group ${}_p\pi_k(S)$ is determined immediately.

The secondary composition $\{\varepsilon_{p-2}, p\iota, \alpha_1\}$ consists of single element, since it has zero indeterminacy. Assume that $\{\varepsilon_{p-2}, p\iota, \alpha_1\}=0$. Then any representative $h: S^{n+t}\to S^n$, $t=(p^2+p-2)q-2$, of ε_{p-2} is extended to $K=S^{n+t}\cup e^{n+t+1}\cup e^{n+t+q+1}$ with $\mathscr{P}^1\Delta H^{n+t}(K)=H^{n+t+q+1}(K)$. The extension $H: K\to S^n$ is liftable to $X_t(n)$ of §3. By Proposition 3.2, the lifting $\bar{H}$ satisfies $\bar{H}^*(\tau^{-1}e_{p-2})=u$ for a generator u of $H^{n+t}(K)=Z_p$. Then $\bar{H}^*(\tau^{-1}(R_{p-2}e_{p-2}))=R_{p-2}u=-\mathscr{P}^1\Delta u\neq 0$. This contradicts to $R_{p-2}e_{p-2}=0$. Thus $\{\varepsilon_{p-2}, p\iota, \alpha_1\}\neq 0$, and we can choose ε_{p-1} so that it satisfies (7.3).

(7.4) is similar to (7.3). (7.5) is an easy consequence of Theorem 3.5. The secondary composition $\{\varepsilon_{p-2}, \alpha_1, p\iota\}$ consists of single element, hence $a\varepsilon_{p-1}=\{\varepsilon_{p-2}, \alpha_1, p\iota\}$ for some $a\in Z_p$. Similarly to Theorem 4.14. ii) of [6: IV], the relation $R_{p-2}e_{p-2}=0$ implies $a=2$. By Theorem 4.4 of [6: IV], $\{\alpha_1, p\iota, \varepsilon_{p-2}\}=-\varepsilon_{p-1}$ and $\{p\iota, \varepsilon_{p-2}, \alpha_1\}=-\varepsilon_{p-1}$. By (4.4). i) of [6: IV], $p\varphi=p\iota\{\varepsilon_{p-2},\alpha_1, \alpha_1\}=-\{p\iota, \varepsilon_{p-2}, \alpha_1\}\alpha_1=\varepsilon_{p-1}\alpha_1=\alpha_1\varepsilon_{p-1}$. Thus (7.6) is established. Q.E.D.

§8. Some relations in ${}_p\pi_k(S)$

In this section, we shall prove Propositions 7.5–6 in §7, by making use of the methods of [7].

The inclusion $S^{2n-1}\to\Omega^2S^{2n+1}$ induces the homomorphism of homotopy groups, which is equivalent to the double suspension $S^2: \pi_i(S^{2n-1})\to\pi_{i+2}(S^{2n+1})$, and the fibering $p: Q_2^{2n-1}\to S^{2n-1}$ with the fiber Ω^3S^{2n+1} gives rise to an exact sequence

(8.1) ((1.7) of [7: I])

$$\cdots\longrightarrow\pi_i(Q_2^{2n-1})\xrightarrow{p_*}\pi_i(S^{2n-1})\xrightarrow{S^2}\pi_{i+2}(S^{2n+1})\xrightarrow{H^{(2)}}\pi_{i-1}(Q_2^{2n-1})\longrightarrow\cdots,$$

where $Q_2^{2n-1}=\Omega(\Omega^2S^{2n+1}, S^{2n-1})$.

For the k-fold suspension S^k, this sequence is generalized as follows:

(8.2) ((1.7) of [7: I])

$$\cdots\longrightarrow\pi_i(Q_k^n)\xrightarrow{p_*}\pi_i(S^n)\xrightarrow{S^k}\pi_{i+k}(S^{n+k})\xrightarrow{H^{(k)}}\pi_{i-1}(Q_k^n)\longrightarrow\cdots,$$

where $Q_k^n=\Omega(\Omega^kS^{n+k}, S^n)$.

The main tools of [7] are these sequences and the following exact sequence:

(8.3) ((2.5) of [7: I])

$$\cdots\longrightarrow{}_p\pi_{i+4}(S^{2mp+1})\xrightarrow{\triangle}{}_p\pi_{i+2}(S^{2mp-1})\xrightarrow{I'}{}_p\pi_i(Q_2^{2m-1})\xrightarrow{I}{}_p\pi_{i+3}(S^{2mp+1})\longrightarrow\cdots.$$

Following to H. Toda, we use the notation: *for* $\gamma\in{}_p\pi_i(S)$, $\gamma(n_0)\in{}_p\pi_{i+n_0}$

(S^{n_0}) *denotes an element such that* $S^\infty\gamma(n_0)=\gamma$ *and* $\gamma(n_0)\notin \operatorname{Im} S$, *and* $\gamma(n)$ *denotes* $(n-n_0)$*-fold suspension of* $\gamma(n_0)$ *for* $n\geqq n_0$. Also we use the notations $Q^m(\gamma)$ and $\bar{Q}^m(\delta)$ to the elements of ${}_p\pi_i(Q_2^{2m-1})$ for some elements $\gamma\in {}_p\pi_{i-2mp+3}(S)$ and $\delta\in {}_p\pi_{i-2mp+2}(S)$ such that $Q^m(\gamma)=I'(\gamma(2mp-1))$ and $I(\bar{Q}^m(\delta))=\delta(2mp+1)$ (see (6.3) of [7: I]).

The map Δ of (8.3) satisfies (2.7) of [7: I] and Corollaries 9.4–5 of [7: II], which are important properties for the determination of Δ. By use of these properties, the sequence (8.3) and the results for ${}_p\pi_{2m+1+j}(S^{2m+1})$, $j<(p^2+p-1)q$, of [7], we have the following

(8.4) *The group* ${}_p\pi_{2n-1+k}(Q_2^{2n-1})$, $l-6\leqq k\leqq l-2$, $l=(p^2+p)q$, *is the direct sum of cyclic groups of order p generated by the following elements*:

$(k=l-6)$ $Q^1(\beta_1^{p+1})$, $\bar{Q}^{(p-s-1)(p+1)+1}(\alpha_1\beta_1\beta_s)$ $(1\leqq s<p)$,
$Q^{(p-s-1)(p+1)+2}(\beta_1\beta_s)$ $(1\leqq s<p)$,

and $\bar{Q}^i(\alpha_{11-i})(1\leqq i\leqq 10)$, $Q^{11}(\iota)$ *in addition for* $p=3$.

$(k=l-5)$ $Q^i(\alpha_1\varepsilon_{p-1-i})$ $(2\leqq i\leqq p-2)$, $Q^{p-1}(\varepsilon')$, $\bar{Q}^{(p-s-1)(p+1)+2}(\beta_1\beta_s)$ $(1\leqq s<p)$,
$Q^{(p-s)(p+1)}(\alpha_1\beta_s)$ $(1\leqq s<p)$, $\bar{Q}^1(\beta_1^{p+1})$.

$(k=l-4)$ $\bar{Q}^i(\alpha_1\varepsilon_{p-1-i})$ $(2\leqq i\leqq p-2)$, $\bar{Q}^{p-1}(\varepsilon')$, $u=I'p_*Q^{p^2}(\iota)$, $Q^{p+1}(\beta_1^p)$,
$Q^{(p-s)(p+1)+1}(\beta_s)$ $(1\leqq s<p)$, $\bar{Q}^{(p-s)(p+1)}(\alpha_1\beta_s)$ $(1\leqq s<p)$,

$(k=l-3)$ $Q^i(\alpha_{p^2+p-i})$ $(i\not\equiv 0 \bmod p)$, $Q^{jp}(\alpha'_{p^2+p-jp})$ $(2\leqq j\leqq p)$, $Q^p(\alpha''_{p^2})$,
$Q^{p-1}(\alpha_1\beta_1^{p-2}\beta_2)$, $\bar{Q}^{p+1}(\beta_1^p)$, $Q^{2p}(\alpha_1\beta_1^{p-1})$, $\bar{Q}^{(p-s)(p+1)+1}(\beta_s)$ $(1\leqq s<p)$.

$(k=l-2)$ $\bar{Q}^i(\alpha_{p^2+p-i})$, $\bar{Q}^{p-1}(\alpha_1\beta_1^{p-2}\beta_2)$, $Q^p(\beta_1^{p-2}\beta_2)$, $\bar{Q}^{2p}(\alpha_1\beta_1^{p-1})$, $Q^{2p+1}(\beta_1^{p-1})$,
$Q^{p^2+p}(\iota)$, *and* $Q^2(\beta_1^4)$ *in addition for* $p=3$.

In the above, we use the notations ε' and $\alpha_1\varepsilon_i$ instead of ε'_i of [7: III] (see the third remark after Theorem 6.2).

By Lemma 6.1 of [7: I], the p_*-images of the following elements are trivial:

$\bar{Q}^{(p-s-1)(p+1)+1}(\alpha_1\beta_1\beta_s)$, $Q^{(p-s)(p+1)}(\alpha_1\beta_s)$ $(s>1)$, $\bar{Q}^{(p-s)(p+1)}(\alpha_1\beta_s)$,
$Q^{2p}(\alpha_1\beta_1^{p-1})$, $\bar{Q}^{p-1}(\alpha_1\beta_1^{p-2}\beta_2)$,

and also the p_*-images of the following elements are the unstable elements of first type [1]:

1) The classification of the unstable elements (S^∞-kernel) of ${}_p\pi_{2n+1+k}(S^{2n+1})$ is due to H. Toda ([7: I, p. 88]).

$\bar{Q}^{(p-s-1)(p+1)+2}(\beta_1\beta_s)(s>1)$, $\bar{Q}^i(\alpha_{11-i})$, $Q^{11}(\iota)$, $Q^{(p-s)(p+1)+1}(\beta_s)(s>1)$, $\bar{Q}^{(p-s)(p+1)+1}(\beta_s)$, $Q^{2p+1}(\beta_1^{p-1})$.

By Proposition 8.8 of [7: II], the following elements give the unstable elements of second type[1] of ${}_p\pi_{2n+1+l-2}(S^{2n+1})$:

$$Q^i(\alpha_{p^2+p-i})(i \leqq p^2+p-2),\ \ Q^{jp}(\alpha'_{p^2+p-jp}),\ \ Q^p(\alpha''_{p^2}),\ \ \bar{Q}^i(\alpha_{p^2+p-i})(i \geqq 3),\ \ Q^{p^2+p}(\iota).$$

By Theorems 10.3 and 10.6 of [7: II], the following elements give the unstable elements of third type[1] of ${}_p\pi_{2n+1+k}(S^{2n+1})$:

$$Q^{p^2-p}(\beta_1^2),\ \ Q^{p^2-1}(\alpha_1\beta_1),\ \ Q^{p^2}(\beta_1),\ \ Q^{p^2+p-1}(\alpha_1),\ \ \bar{Q}^{p+1}(\beta_1^p),\ \ \bar{Q}^{2p}(\alpha_1\beta_1^{p-1}),$$
$$Q^p(\beta_1^{p-2}\beta_2).$$

Consequently we obtain the following

Lemma 8.1. *Let $t=(p^2+p)q-3$ and $\gamma'=\bar{Q}^i(\alpha_1\varepsilon_{p-1-i}), \bar{Q}^{p-1}(\varepsilon')$, u or $Q^{p+1}(\beta_1^p)$. If $p_*\gamma'=0$, then there is an element $\gamma \in {}_p\pi_{2n+1+t}(S^{2n+1})$ such that $H^{(2)}\gamma=\gamma'$ and $S^\infty\gamma\neq 0$ in ${}_p\pi_t(S)$. Furthermore ${}_p\pi_t(S)$ is generated by such elements $S^\infty\gamma$.*

Now we shall prove Proposition 7.5 of §7.

Proof of Proposition 7.5. By the relation (6.4) of §6, Lemma 6.1 (i) (ii) of [7: I] also holds for ε' and β_1^4 instead of $\beta_1^r\beta_s$ and $\alpha_1\beta_1^r\beta_s$, and so $p_*\bar{Q}^2(\varepsilon')\neq 0$. By the discussion previous to Lemma 8.1, there is no element $\gamma \in {}_3\pi_{48}(S^5)$ such that $S^\infty\gamma\neq 0$. By (2.11) and (2.13) of [7 :1], $\alpha_1\beta_1^4(3)=\pm GQ^1(\beta_1^4)\neq 0$. Since $S^\infty\alpha_1\beta_1^4(3)=(\alpha_1\beta_1^3)\beta_1=0$, $p_*Q^2(\varepsilon')=\pm\alpha_1\beta_1^4(3)\neq 0$. Thus, by (8.1), ${}_3\pi_{49}(S^5)={}_3\pi_{51}(S^7)=0$, and $p_*u=0$, $p_*Q^4(\beta_1^3)=0$. Then ${}_3\pi_{45}(S)$ is of order 9 by Lemma 8.1. It is also cyclic as is seen after Proposition 7.4, and the proposition follows. Q.E.D.

For any map f: $S^{m+n}\to S^n$, the map $\Omega^k S^k f$: $\Omega^k S^{m+n+k}\to\Omega^k S^{n+k}$ induces a map $Q_k(f): Q_k^{m+n}\to Q_k^n$, and for the class $\alpha \in \pi_{m+n}(S^n)$ of f, we denote by $Q_k(\alpha)$ the class of $Q_k(f)$. Then we have

(8.5) $$H^{(k)}(S^k\alpha\circ\beta)=Q_k(\alpha)H^{(k)}(\beta) \quad \text{for } \alpha \in \pi_j(S^n),\ \beta \in \pi_i(S^{j+k}),$$

where $H^{(k)}$: $\pi_{i+k}(S^{n+k})\approx\pi_i(\Omega^k S^{n+k})\to\pi_i(\Omega^k S^{n+k}, S^n)\approx\pi_{i-1}(Q_k^n)$ is the homomorphism in (8.2).

Furthermore, for the inclusion j: $Q_k^h\to Q_{k+l}^h$, the following is verified easily.

1) The classification of the unstable elements (S^∞-kernel) of ${}_p\pi_{2n+1+k}(S^{2n+1})$ is due to H. Toda ([7: I, p. 88]).

(8.6) (i) $H^{(k+l)}(S^l\alpha)=j_*H^{(k)}(\alpha)$ *for* $\alpha \in \pi_{i+k}(S^{n+k})$,

(ii) $j_*Q_k(\beta)\gamma=Q_{k+l}(\beta)j_*\gamma$ *for* $\beta \in \pi_{m+n}(S^n)$, $\gamma \in \pi_i(Q_k^{m+n})$.

Let $V_{2m+1,2}=O(2m+1)/O(2m-1)$ denote the Stiefel manifold. This is an S^{2m-1} bundle over S^{2m} with the characteristic class $2\iota_{2m-1} \in \pi_{2m-1}(S^{2m-1})$. Let ρ: $\Omega S^{2m}\to S^{2m-1}$ be a map such that ρ_* is equivalent to the boundary homomorphism ∂ of the homotopy sequence of the fibering $V_{2m+1,2}\to S^{2m}$, i.e., the following diagram commutes:

$$\begin{tikzcd} \pi_i(\Omega S^{2m}) \arrow[r, "\rho_*"] & \pi_i(S^{2m-1}) \\ \pi_{i+1}(S^{2m}) \arrow[u, "\Omega_0", "\approx"'] \arrow[ur, "\partial"'] & \end{tikzcd}$$

Then the following proposition is established.

PROPOSITION 8.2. *There exists a map* $Q_{2k}(\rho)$: $\Omega Q_{2k}^{2m}\to Q_{2k}^{2m-1}$ *such that the following are satisfied*:
(i) *The following diagram is homotopy commutative*:

$$\begin{tikzcd} \Omega^{2k+2}S^{2m+2k} \arrow[r] \arrow[d, "\Omega^{2k+1}\rho"] & \Omega Q_{2k}^{2m} \arrow[r] \arrow[d, "Q_{2k}(\rho)"] & \Omega S^{2m} \arrow[r] \arrow[d, "\rho"] & \Omega^{2k+1}S^{2m+2k} \arrow[d, "\Omega^{2k}\rho"] \\ \Omega^{2k+1}S^{2m+2k-1} \arrow[r] & Q_{2k}^{2m-1} \arrow[r] & S^{2m-1} \arrow[r] & \Omega^{2k}S^{2m+2k-1}, \end{tikzcd}$$

where the horizontal lines are sequences of fiberings giving the exact sequence (8.2).
(ii) *The following diagram is commutative*:

$$\begin{tikzcd} \pi_{i+1}(Q_{2k}^{2m}) \arrow[r, "\Omega_0", "\approx"'] \arrow[d, "j_*"] & \pi_i(\Omega Q_{2k}^{2m}) \arrow[r, "Q_{2k}(\rho)_*"] \arrow[d, "(\Omega j)_*"] & \pi_i(Q_{2k}^{2m-1}) \arrow[d, "j_*"] \\ \pi_{i+1}(Q_{2k+2l}^{2m}) \arrow[r, "\Omega_0", "\approx"'] & \pi_i(\Omega Q_{2k+2l}^{2m}) \arrow[r, "Q_{2k+2l}(\rho)_*"] & \pi_i(Q_{2k+2l}^{2m-1}). \end{tikzcd}$$

(iii) *Let* σ: $\pi_i(Q_{2k}^{2n-1})\to\pi_{i+1}(Q_{2k}^{2n})$ *be the homomorphism induced by the inclusion* $Q_{2k}^{2n-1}\to\Omega Q_{2k}^{2n}$. *Then*

$$Q_{2k}(\rho)_*\Omega_0(\sigma\alpha)=2\alpha \qquad \textit{for } \alpha \in \pi_i(Q_{2k}^{2n-1})\cap \mathrm{Im}\, H^{(2k)}.$$

PROOF. Using the map $S^2V_{2n+1,2}\to V_{2n+3,2}$ of Proposition 2.1 of [3], we have a map f: $V_{2m+1,2}\to\Omega^{2k}V_{2m+2k+1,2}$ and a homotopy commutative diagram of fiberings:

$$\begin{tikzcd} S^{2m-1} \arrow[r] \arrow[d, "i_1"] & V_{2m+1,2} \arrow[r] \arrow[d, "f"] & S^{2m} \arrow[d, "i_1"] \\ \Omega^{2k}S^{2m+2k-1} \arrow[r] & \Omega^{2k}V_{2m+2k+1,2} \arrow[r] & \Omega^{2k}S^{2m+2k}, \end{tikzcd}$$

where $i_1\colon S^l \to \Omega^n S^{n+l}$ denotes the inclusion. By this diagram, we have a homotopy commutative diagram:

$$\begin{array}{ccc} \Omega S^{2m} & \xrightarrow{\rho} & S^{2m-1} \\ {\scriptstyle \Omega i_1}\downarrow & & \downarrow{\scriptstyle i_1} \\ \Omega^{2k+1}S^{2m+2k} & \xrightarrow{\Omega^{2k}\rho} & \Omega^{2k}S^{2m+2k-1}. \end{array}$$

Then $\Omega^{2k}\rho$ defines

$$Q_{2k}(\rho)\colon \Omega Q_{2k}^{2m} = \Omega(\Omega^{2k+1}S^{2m+2k}, \Omega S^{2m}) \to \Omega(\Omega^{2k}S^{2m+2k-1}, S^{2m-1}) = Q_{2k}^{2m-1}.$$

Then we see easily that this map $Q_{2k}(\rho)$ satisfies (i) and (ii). Since the characteristic class of the sphere bundle $V_{2m+1,2}$ is $2\iota_{2m-1} \in \pi_{2m-1}(S^{2m-1}) \approx Z$,

$$\rho_* \Omega_0 S\alpha = 2\alpha \quad \text{for any } \alpha \in \pi_i(S^{2m-1}).$$

Then (iii) follows from this and (i). Q.E.D.

Now we shall give the proof of Proposition 7.6 of §7.

Proof of Proposition 7.6. Set $r=(p^2+p-1)q-2$. Theorem 15.2 of [7: III] states the following

(8.7) *There exists an element* $\varepsilon=\varepsilon_{p-1}(2p+3) \in {}_p\pi_{2p+3+r}(S^{2p+3})$ *of order* p *such that* $S^\infty\varepsilon=\varepsilon_{p-1}$ *and* $H^{(2)}\varepsilon=a_1\bar{Q}^{p+1}(\beta_{p-1})$, $a_1 \not\equiv 0 \bmod p$.

Since the suspension $S\colon {}_p\pi_{2p+2+r}(S^5) \to {}_p\pi_{2p+3+r}(S^6)$ is monomorphic, there is an element $\mu \in {}_p\pi_{2p+2+r}(S^5)$ such that $S\mu=S^3A\circ\varepsilon$ and $S^\infty\mu=\alpha_1\varepsilon_{p-1}$, where $A \in {}_p\pi_{2p}(S^3)$ is an element such that $S^\infty A=\alpha_1$.

By (8.5) and (8.6) (i), $j_*H^{(2)}\mu=Q_4(A)H^{(4)}(S\varepsilon)$, where $j\colon Q_2^3 \to Q_4^3$. By Proposition 8.2 (iii),

$$2j_*H^{(2)}\mu=Q_4(\rho)_*\Omega_0\sigma(Q_4(A)H^{(4)}(S\varepsilon)).$$

Since $\sigma(Q_4(A)H^{(4)}(S\varepsilon))=Q_4(SA)H^{(4)}(S^2\varepsilon)$, it follows from (8.6) that $\sigma(Q_4(A)H^{(4)}(S\varepsilon))=j_*Q_2(SA)H^{(2)}\varepsilon$. Thus, from Proposition 8.2 (ii),

$$2j_*H^{(2)}\mu=j_*Q_2(\rho)_*(\Omega Q_2(SA))_*\Omega_0H^{(2)}(\varepsilon).$$

Since ${}_p\pi_{2p+r+2}(Q_2^5)$ is generated by $Q^3(\alpha_{p^2+p-3})$ and $p_*Q^3(\alpha_{p^2+p-3})=0$, the map j_* is monomorphic by (3.3), (3.4) and (5.2) of [7: I]. Thus we obtain the following

$$2H^{(2)}\mu=Q_2(\rho)_*(\Omega Q_2(SA))_*\Omega_0H^{(2)}\varepsilon. \tag{8.8}$$

Now let $M^n=S^n\cup e^{n+1}$ be the Moore space of type (Z_p, n), i.e., the mapping cone of a map $S^n \to S^n$ of degree p. Then we have a cofibering:

$$S^n \xrightarrow{i} M^n \xrightarrow{\pi} S^{n+1}.$$

Also let $\mathscr{A}_t(M)=\varinjlim_n [M^{n+t}, M^n]$. N. Yamamoto [10] has proved that there exist uniquely the elements $\alpha \in \mathscr{A}_q(\boldsymbol{M})$ and $\beta_{(s)} \in \mathscr{A}_{(sp+s-1)q-1}(\boldsymbol{M})(1 \leqq s<p)$ such that $\pi\alpha i=\alpha_1$, $\pi\beta_{(s)}i=\beta_s$, $\alpha\beta_{(s)}=\beta_{(s)}\alpha=0$.

By the definition of $\bar{Q}^m(\ )$, Lemma 2.5 of [7: I] and (8.7), we have

$$H^{(2)}\varepsilon=a_2G_*(\beta_{(p-1)}i), \quad a_2 \not\equiv 0 \bmod p,$$

where $G: M^{2mp-h-3} \to \Omega^h Q_2^{2m-1}$ is the map of Lemma 2.5 of [7: I].

The map $Q_2(\rho)(\Omega Q_2(SA))$ coincides with the map $Q_2(\rho_1)$ of [3: (3.8)] up to a multiple by non-zero element of Z_p, by Definition 2.2 of [3] and the definition of $Q_2(\rho_n)$ of [3: pp. 171-172]. Therefore by (3.8) and (4.22) of [3],

$$(8.9) \quad Q_2(\rho)_*(\Omega Q_2(SA))_*\Omega_0 H^{(2)}\varepsilon=a_3G_*(\lambda\beta_{(p-1)}i) \quad \textit{for some } a_3 \not\equiv 0 \bmod p,$$

$$\lambda=\beta_{(1)}+\gamma\alpha \quad \textit{for some } \gamma \in \mathscr{A}_{(p-1)q-1}(\boldsymbol{M}).$$

By Lemma 4.2 of [10], we have

$$\pi\lambda\beta_{(p-1)}i=\pi\beta_{(1)}\beta_{(p-1)}i=\{\beta_1, p\iota, \beta_{p-1}\}.$$

By (15.6) of [7: III], $\{\beta_1, p\iota, \beta_{p-1}\}=a_4\alpha_1\varepsilon_{p-3}$ for some $a_4 \not\equiv 0 \bmod p$. Using Lemma 2.5 of [7: I], we get $G_*(\lambda\beta_{(p-1)}i)=a_5\bar{Q}^2(\alpha_1\varepsilon_{p-3})$ for some $a_5 \not\equiv 0 \bmod p$. Thus, from (8.8) and (8.9), we obtain

$$H^{(2)}\mu=x\bar{Q}^2(\alpha_1\varepsilon_{p-3}) \quad \text{for some } x \not\equiv 0 \bmod p.$$

By use of Lemma 8.1, it follows that $\alpha_1\varepsilon_{p-1}=S^\infty\mu \neq 0$. Q.E.D.

REMARK. For $p=3$, the above proof is not valid, since the element $\varepsilon_{p-1}(2p+3)$ of (8.7) does not exist. In this case, the element $\varepsilon_2(11)$ exists. This satisfies $H^{(2)}\varepsilon_2(11)=\pm Q^5(\alpha_1\beta_1^2)$ and $S^\infty\varepsilon_2(11)=\varepsilon_2$ (see [7: III, Proposition 15.6]).

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Department of Mathematics,
Faculty of Science,
Hiroshima University