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On Width Ideals of a Module

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The notion of width of a module was introduced by M.-P. Brameret and some properties of it were shown in the paper [1]. Moreover M. Wichman obtained some results on this subject in the case of modules over a commutative ring in [4]. On the other hand H. Fitting studied the determinantal ideals of a finitely generated module over a commutative ring for the first time in [2] and several authors used this notion for the study of modules. In particular it was shown by T. Matsuoka in [3] that some properties of the torsion submodule of a module have a close connection with Fitting's determinantal ideals.

The aim of this note is to show relations between these two notions. For this purpose we give the notion of weak width of a module over a commutative ring which is more fitting for us than that of width of a module, and elementary properties of it are shown. Next we define the width ideals of a module and show that these ideals are natural modifications of Fitting's determinantal ideals for a not necessarily finitely generated module. Moreover it is shown that the weak width of a module over an integral domain has a close connection with width ideals or Fitting's determinantal ideals of the module. Lastly we shall give a generalization of the results on the torsion submodule of a module in [3].

Throughout this paper all rings will be commutative with unit and all modules will be unitary.

§1. Weak width of a module

Let R be a commutative ring with unit and U the set of regular elements of R[1]. Let M be an R-module. Then we understand by the *weak width* $W'(R, M)$ *of* M *over* R the smallest integer n such that for any set $\{x_1, \cdots, x_{n+1}\}$ of $n+1$ elements of M, we have a soluution $ax_i=\sum_{j\neq i} a_j x_j$ for some i, a in U and a_j in R. In other words $W'(R, M)$ is the width $W(R_U, M_U)$ of M_U over R_U in the sence of [1]. If $W'(R, M)=n$, there exists a set $\{x_1, \cdots, x_n\}$ of n elements of M such that ax_i is not contained in $\sum_{j\neq i} Rx_j$ for any i and any a in U. We call a system with the above property *a set of*

1) An element of a ring R is called regular, if it is not a zero-divisor of R. If an ideal of R contains a regular element of R, it is called a regular ideal.

weak width determiners of M over R. The following proposition is a direct consequence of the definition of weak width and Proposition 1.1 in [4].

Proposition 1.1. *Let M be an R-module.*

(1) *$W'(R, M)=0$ if and only if M is a torison R-module.*

(2) *For any submodule N of M, $W'(R,N)\leq W'(R, M)$.*

(3) *If N is a homomorphic image of M,*

$$W'(R, M)\geq W'(R, N).$$

(4) *Assume that $W'(R, M)=n<\infty$ and let N be a finitely generated submodule of M with a system $\{x_1, \cdots, x_t\}$ of generators. Then there exists an element a in U such that a N is contained in a submodule of N generated by at most n elements among the elements $x_1, \cdots, x_{t-1}$* and *x_t.*

(5) *If $W'(R, M)<\infty$, there exists a finite R-submodule N of M such that $W'(R, M)=W'(R, N)$.*

(6) *Let N be a submodule of M. If $W'(R, M)=m$ and there exists a regular element a in R such that $aM\subset N$, then $W'(R, N)=m$.*

Proposition 1.2. *Let M be an R-module and N a submodvle of M. If $W'(R, N)=n$ and $W'(R, M/N)=l$, then $W'(R, M)\leq n+l$.*

Proof. Since $W'(R, M)$ (resp. $W(R, N)$ or $W'(R, N)$ is equal to the $W'(R, M/N)$) (resp. $W(R_U, M_U)$ or $W(R_U, M_U/N_U)$), this follows immediately from proposition 1.2 of [4]. q.e.d.

Corollary 1.3. *Let M be an R-module and M_t the torsion submodule of M. Then $W(R, M)=W'(R, M/M_t)$.*

Proof. By (3) of proposion 1.1, $W'(R, M/M_t)\leq W'(R, M.)$. Conversely, by (1) of proposition 1.1 and proposition 1.2, $W(R, M)\leq W'(R, M_t)+W'(R, M/N_t)=W'(R, M/M_t)$. q.e.d.

Lemma 1.4. *Assume that $W'(R, R)=1$ and let $\mathfrak{a}$ be an ideal generated by n elements $a_1, \cdots, a_n$ of R. If $\mathfrak{a}$ is a regular ideal, one of them is contained in the set U of units of R.*

Proof. This is easily seen from (4) of proposition 1.1.

Lemma 1.5. *Let M be an R-module of the weak width $W'(R, M)=n$ and $\{x_1, \cdots, x_n\}$ a set of weak width determiners of M over R. Then the annihilator Ann (x_i) of x_i is zero for any i. Moreover if R is an integral domain, the submodule $\sum_{i=1}^{n} Rx_i$ is a free module with a free basis $\{x_1, \cdots, x_n\}$.*

This is easily seen from the definition of a set of weak width determiners.

§ 2. Width ideals of a module.

Let M be an R-module and S_n the set of the elements a in R such that aM is contained in the submodule $\sum_{i=1}^{n} Rx_i$ of M generated n by elements $x_1, \cdots, x_n$. Then we denote by $W_n(M)$ the ideal of R generated by S_n for a non-negative interger n and call it *the n-th width ideal of M over R.* The elements of S_n will be called *the generators of* $W_n(M)$. From the definition of $W_n(M)$, we see easily the following.

PROPOSITION 2.1. *Let M be an R-module.*
(1) $W_n(M) \subset W_{n+1}(M)$.
(2) *If N is a submodule of M,* $W_n(M) \subset W_n(M/N)$.

Let M be a finite R-module, and denote by $F_n(M)$ the n-th Fitting ideal of M over R[2]. Now we give some relations between Fitting ideals and width ideals.

PROPOSITION 2.2. *Let M be a finite R-module. Then, for any n,*

$$F_n(M) \subset W_n(M) \subset \sqrt{F_n(M)}.$$

PROOF. If $n=0$, since $(Ann(M))^s \subseteq F_0(M \subseteq Ann(M)$ for some m, the proof is easily seen. Now we assume $n \geqq 1$ and let $\{x\} = \{x_1, \cdots, x_m\}$ be a system of generators of M. Let $A=(a_{ij})(i=1, \cdots, m, j=1, \cdots, m-n)$ be a matrix such that $A\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}=0$, and let a be the minor det $(a_{ij})(i, j=1, 2, \cdots, m-n)$ of A. Then we can easily see that aM is contained in the submodule $N=\sum_{j=m+n-1}^{m} Rx_j$ of M. From this we see that $F_n(M)$ is contained in $W_n(M)$.

Conversely let a be a generator of $W_n(M)$. From the definition there exist n elements $x_1, \cdots, x_n$ in M such that aM is contained in the submodule $\sum_{i=1}^{n} Rx_i$ of M. Let $\{y_1, \cdots, y_m, x_1, \cdots, x_n\}$ be a system of generator of M. Then we have relations

$$a y_j + \sum_{j=1}^{n} a_{ij} x_i = 0 \qquad (i=1, \cdots, m).$$

Put $A = \begin{array}{c} \\ m\{ \\ n\{ \end{array}\left(\begin{array}{ccc:c} \overbrace{a \quad\quad}^{m} & & & \overbrace{\quad\quad}^{n} \\ & \ddots & & a_{ij} \\ & & a & \\ \hdashline & 0 & & 0 \end{array}\right)$. Since A annihilates $\{x\}$, a^m is a generator of

2) As to the definition and basic results of Fitting ideals of a module, see the papers [2] and [3].

$F_n(M)$. This implies $W_n(M) \subset \sqrt{F_n(M)}$ q.e.d.

We shall say that *an R-module M is of type* (W_n) if the $(n-1)$-th width ideal $W_{n-1}(M)$ of M is zero and the n-th width ideal $W_n(M)$ of M is regular.

PROPOSITION 2.3. *Let M be a finite R-module.*
(1) *If M is of type* (W_n), *the Mn is of type* (F_n).
(2) *If R is a reduced ring*[3] *and M is of type* (F_n), *then M is of type* (W_n).

PROOF. This is a direct consequence of the definitions of types (W_n) and (F_n) and of proposition 2.2. q.e.d.

Next we show that the weak width of a module has a close connection with width ideals of the module and the additivity of the weak widths of modules over an integral domain holds. For this purpose we give the following;

LEMMA 2.4. *Assume that the weak width* $W'(R, R)$ *of R is one, and let M be an R-module of type* (W_n) *for some* $n \geq 1$. *Then there exists an element x of M such that* M/Rx *is of type* (W_{n-1}) *and that* $Ann(x)=0$.

PROOF. By lemma 1.4, there exists a regular element g in R such that $gM \subset \sum_{i=1}^{n} Rx_i$ for some $x_1, \cdots, x_n$ in M. Since $W_{n-1}(M)$ is zero, $Ann(x_i)=(0)$ for any $i=1, \cdots, n$. Put $M'=M/Rx_i$. Then we can easily show that M' is of type (W_{n-1}). q.e.d.

Remark. If R is an integral domain, $W'(R, R)=1$.

PROPOSITION 2.5. *Assume that the weak width* $W'(R, R)$ *of R is one. If M is an R-module of type* (W_n), *then* $W'(R, M)=n$.

PROOF. We show our assertion by an induction on n. If $n=0$, $M=M_t$. Hence we have $W'(R, M)=0$ by (1) of proposition 1.1. Now we assume that $n>0$ and M is of type (W_n). By Lemma 1.4, there exists a regular generator a of $W_n(M)$. Therefore we may assume that aM is contained in the submodule $\sum_{j=1}^{n} Rx_j$ of $M (x_j \in M, j=1, \cdots, n)$. By lemma 2.4, there exists an element y in M such that M/Ry is of type (W_{n-1}) and $Ann(y)=0$. From the induction hypothesis, $W'(R, M/Ry)=n-1$. Since $R \cong Ry$, we have $W'(R, Ry)=W'(R, R)=1$. By Proposition 1.2, $W'(R, M) \leq W'(R, Ry)+W'(R, M/Ry)=n$. Since $W_{n-1}(M)=(0)$, the system $\{x_1, \cdots, x_n\}$ is a set of weak width determiners of M. Hence we have $W'(R, M) \geq n$ and have $W'(R, M)=n$. q.e.d.

3) A ring R is called reduced when R has no nilpotent elements except zero.

COROLLARY 2.6. *Let R be an integral domain and M an R-module. Then the following conditions are equivalent*:

(1) *M is of type (W_n).*

(2) *The weak width $W'(R, M)$ of M is n and aM is contained in a finitely generated submodule of M for an element a in U. Moreover if M is finitely generated, these conditions are equivalent to the following.*

(3) *M is of type (F_n).*

PROOF. In order to prove the first half, it is sufficient to show that (2) means (1). Since aM is contained in finitely generated submodule of M, there exists an integer s such that $W_s(M)$ is not zero and hence there exists an integer t such that $0=W_{t-1}(M)\subsetneqq W_t(M)$. Since R is an integral domain, $W_t(M)$ must be a regular ideal and hence M is of type (W_t). By Proposition 2.5, n is equal to t. The latter half is immediately seen from Proposition 2.3. q.e.d.

Example. Let K and L be two fields and R the direct product of K and L. Then we have $W'(R, R)=2$, but R is of type (W_1). This means that we cannot exclude the assumption $W'(R, R)=1$ in Proposition 2.5.

Let M be an R-module generated by m elements $x_1, \cdots, x_m$ of M and F a free R-module with a free basis $\{e_1, \cdots, e_m\}$. Denoting by ϕ the R-homomorphism of F onto M such that $\phi(e_i)=x_i$ for any i, let N be the kernel of ϕ.

LEMMA 2.7. *Let M, F and N be as above, If R is an integral domain, then M is of type (W_n) if and only if N is of type (W_{m-n}).*

PROOF. We assume that M is of type (W_n). By (4) of proposition 1.1, there exists a non-zero element a in R such that, $aM\subset\sum_{i=1}^{n}Rx_i$ by exchanging the order of $x_1, \cdots, x_m$ if necessary. Since $W_{n-1}(M)=(0)$, the system $\{x_1, \cdots, x_n\}$ is a set of weak width determiners of M. Put $ax_{n+j}=\sum_{i=1}^{n}a_{j.i}x_i$ ($j=1, \cdots, m-n$, $a_{ji}\in R$) and put $\alpha_j=ae_{n+j}-\sum_{i=1}^{n}a_{j,i}e_i$. Then $\alpha_j\in N$ for $j=1, \cdots, m-n$. If $\gamma\in aN$, there exists $\gamma'=\sum_{i=1}^{m}b_ie_i$ in N such that $\gamma=a\gamma'$. Since $\gamma'\in N$, $\sum_{i=1}^{m}b_ix_i=0$. Then we have the following relation

$$\sum_{i=1}^{n}(b_ia+\sum_{1\leq j\leq m-n}b_{n+j}a_{j.i})x_i=0.$$

Since $\{x_1, \cdots, x_n\}$ is lineary independent over R, $b_ia+\sum_{1\leq j\leq m-n}b_{n+j}a_{ji}=0$ for $i=1, \cdots, n$. Thus, $\gamma=a\gamma'=\sum_i(b_ia+\sum_j b_{n+j}a_{ji})e_i+\sum_j b_{n+j}\alpha_j=\sum_{1\leq j\leq m-n}b_{n+j}\alpha_j$. This implies $aN \subset \sum_j R\alpha_j$. Since $\{\alpha_1, \cdots, \alpha_{m-n}\}$ is linearly independent over R. $N' = \sum_j R\alpha_j$ is of type (W_{m-n}). Then N is also of type (W_{m-n}). q.e.d.

THEOREM 2.8. *Let R be an integral domain and let the sequence*

$$0 \longrightarrow L \longrightarrow M \xrightarrow{\phi} N \longrightarrow 0$$

of R-modules be exact. Then the weak width of M is the sum of those of N and L.

PROOF. Put $W'(R, N) = n$, $W'(R, M) = m$ and $W'(R, L) = l$, Let $\{\phi(z_1), \cdots, \phi(z_n)\}$ (resp. $\{y_1, \cdots, y_m\}$) be a set of weak width determiners of N (resp. M), where z_i is in M. By (5) of proposition 1.1, there exists a finite R-submodule L_0 of L such that $W'(R, L_0) = W'(R, L)$. Put $M_1 = L_0 + Ry_1 + \cdots + Ry_m + Rz_1 + \cdots + Rz_n$, $N_1 = \phi(M_1)$ and $L_1 = L \cap M_1$. Then we have the next exact sequence

$$0 \longrightarrow L_1 \longrightarrow M_1 \xrightarrow{\phi} N_1 \longrightarrow 0.$$

By Corollary 2.6, M_1 (resp. N_1) is of type (W_m) (resp. of type (W_n)). Since M_1 is a finite R-module of type (W_m), there exists a non-zero element a in R such that aM_1 is contained in the submodule $\sum_{j=1}^{m} Ru_j$ of M_1 genarated dy m elements u_j in M_1. As $W_{m-1}(M_1) = (0)$, $\{u_1, \cdots, u_m\}$ is a set of weak width determiners. By Lemma 1.5 $M' = \sum_j Ru_j$ is a free module with free basis $\{u_1, \cdots, u_m\}$. Now put $N' = \phi(M')$ and $L' = L \cap M'$. Then we have the following exact sequence

$$0 \longrightarrow L' \longrightarrow M' \xrightarrow{\phi} N' \longrightarrow 0.$$

Since aN_1 (resp. aL_1) is contained in N' (resp. L'), N') (resp. L') is of type (W_n) (resp. of type (W_1)) by (6) of Proposition 1.1 and Corollary 2.6, and hence $m = l + n$ by Lemma 2.7. q.e.d.

COROLLARY 2.9. *Let R, M, N and L be the same as in proposition* 2.5.
(1) *If M is of type* (W_m), *then N is of type* (W_n) *if and only if L is of type* (W_{m-n}).
(2) *If N (resp. L) is of type* (W_n) *(resp.* (W_1)*), then M is of type* (W_{n+1}).
This is a direct cousequence of proposition 2.8.

COROLLARY 2.10. *Let a sequence*

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow 0$$

be an exact sequence of R-modules. If M_i *is of type* (W_{m_i}), $i = 0, 1, \cdots, n$, *then* $\sum_{i=0}^{n} (-1)^i m_i = 0$.

§ 3. Torsion submodules

THEOREM 3.1. *Assume that the weak width $W'(R, R)$ of R is one. and let M be an R-module of type (W_1). Then there exist a regular ideal $\mathfrak{a}$ of R and an R-homomorphism ϕ on M into $\mathfrak{a}$ can be defined such that the next sequence*

$$0 \longrightarrow M_t \longrightarrow M \xrightarrow{\phi} \mathfrak{a} \longrightarrow 0$$

is exact.

PROOF. By Lemma 1.4, there exists an element a in U such that aM is contained in the submodule Rx_0 of M for some x_0 in M.

Since $Ann(M)=0$, $Ann(x_0)=0$. Therefore there exists an isomorphism f on Rx_0 onto R such that $f(bx_0)=b$ for any b in R. Now we put $\phi=f\cdot\phi_a$ where ϕ_a is an R-homomorphism on M into Rx such that $\phi_a(y)=ay$ for any y in M. Then ϕ is an R-homomorphism on M into R. Let $\{x_i\}_{i\in I}$ be a system of generators of M and c_i the elements of R such that $ax_i=c_ix_0(i \in I$, $i_i \in R)$. If we write $x_0=\sum_{i\in I} a_ic_i$, we have $a=\sum_{i\in I} a_ic_i$ since $ax_0=\sum_{i\in I}(a_ic_i)x_0$. On the other hand, $\phi(x_0)=a$ and hence $\mathfrak{a}=\phi(M)$ is a regular ideal of R.

Now if x is contained in M_t, there exists a regular element c in R such that $cx=0$. Since $ax=\phi(x)x_0$, we have $c\phi(x)x_0=cax=0$. This means $c\cdot\phi(x)=0$ and hence $\phi(x)=0$. Therefore x is contained in the kernel of ϕ. Conversely if x is in the kernel of ϕ, $ax=0\cdot x_0=0$. Since a is a regular element, x is in M_t. Therefore M_t is the kernel of ϕ. q.e.d.

PROPOSITION 3.2. *Let R be a noetherian ring such that Krull dimension of R is one and that the weak width $W'(R, R)$ is one. If M is an R-module of type (W_1), then the following conditions are equivalent:*

(1) *The module M is the direct sum of its torsion submodule and a free module of rank one (resp. a projective module).*

(2) *The module* $\mathrm{Hom}_R(M, R)$ *is a free module of rank one (resp. a proiective module).*

This is a direct consequence of prop. 3.1. and prop. 2 and 3 in [3]

Remark If R is an integral domain and M is a finite R-module then Theorem 3 in [3] is obtained from prop. 3.2.

LEMMA 3.3. *Let S be a multiplicatively closed subset of R and M a finite R-module. Then the n-th Fitting ideal $F_n(M_S)$ of the R_S-module M_S is $F_n(M)_S$. In particular if M is of type (F_n), so is M_S.*

This is easily seen by a routine calculation and hence we omit the proof.

PROPOSITION 3.4. *Let M be a finite R-module. Then the following conditions are equivalent:*

(a) *M is a projective module of rank n.*
(b) *M is of type (F_n) and the n-th Fitting ideal $F_n(M)$ of M is the unit ideal.*

PROOF. This is easily seen by Lemma 3.3 and Theorem 1 in [3].

PROPOSITION 3.5. *Let M be an R-module. If $W_n(M)=R$ and $W_{n-1}(M)=(0)$, then M is a finite projective module of rank n. Moreover if R is a reduced ring and if M is a finite projective modvle of rank n, then $W_n(M)=R$ and $W_{n-1}(M)=(0)$.*

PROOF. Assume that $W_n(M)=R$ and $W_{n-1}(M)=(0)$. Then there exists a set $\{\alpha_1, \cdots, \alpha_t\}$ of generators of $W_n(M)$ such that $a_1\alpha_1+\cdots+a_t\alpha_t=1$ for some a_i in R. Therefore there exist $t\cdot n$ elements $x_j^{(i)}$ in $M(i=1, \cdots, t, j=1, 2, \cdots, n)$ such that $\alpha_i M\subset \sum_{j=1}^{n} Rx_j^{(i)}$ and hence we see that $M=\sum_{i=1}^{t}\sum_{j=1}^{n} Rx_j^{(i)}$. This means that M is a finite R-module. Then, by Proposition 2.3 and Proposition 3.4, M is a projective module of rank n. The converse is also a direct consequence of Prop. 2.3 and Proposition 3.4. q.e.d.

LEMMA 3.6. *Let M be a finite R-module. Then the following conditions are equivalent:*
(1) *M is of type (F_n).*
(2) *M/M_t is of type (F_n).*

PROOF. First we note that $F_s(M)\subset F_s(M/M_t)$ and $aF_s(M/M_t)\subset F_s(M)$ for some regular element a in R. In fact if $\{x_{1u}, \cdots, x_u\}$ is a system of generators of M, $\{\bar{x}_1, \cdots, \bar{x}_u\}$ is that of M/M_t, where $\bar{x}_i$ is the class of x_i modulo M_t. If M is of type (F_n), $F_{n-1}(M)=0$ and $F_n(M)$ is a regular ideal. Therefore $F_n(M/M)$ is also regular and $aF_{n-1}(M/M_t)=0$ for some regular element a of R from the above assertion. This means $F_{n-1}(M/M_t)=0$ and hence M/M_t is of type (F_n). For the converse we can give a proof similarly but we omit the detail. q.e.d.

PROPOSITION 3.7. *Let M be a finite R-module. Then M is of type (F_n) and $F_n(M/M_t)=R$, if and only if M is a direct sum of the torsion submodule M_t and a finite projective module of rank n.*

This is easily seen from Lemma 3.6 and Proposition 3.4.

Remark 1. If R is a reduced ring in Proposition 3.7, we may replace F_n by W_n by Proposition 2.3.

Remark 2. It is well known that if R is a semi-local ring or a principal ideal domain, a projective R-module of rank n is a free module. Therefore we may replace "*projective*" by "*free*" in Propositions 3.4 and 3.7.

Example. Let K pe a field and $\mathfrak{a}$ the ideal of the polynomial ring $K[X, Y]$

of two variables X and Y generated by X^2 and Y^2. Put $R=K[X, Y]/\mathfrak{a}$ and let x and y be the classes of X and Y in R respectively. Let F be a free R-module with a free basis $\{e_1, e_2\}$. Then we see that $(xy)F$ is contained in $R(xe_1+ye_2)$, and hence the first width ideal $W_1(R)$ of R contains a non-zero element xy. This means that R is not of type (W_2). Therefore the condition that R is reduced is necessary in (2) of Proposition 2.3 and in Proposition 3.5.

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