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Accretive Mappings in Banach Spaces

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Introduction

In the theory of semigroups of nonlinear contraction mappings, the notion of accretive mappings has appeared to be very practical (see [3], [6], [8]). In this paper, we study a multivalued accretive mapping A of a real Banach space X into itself. A is called *m-accretive* if the range of $I+A$ is the whole of X; thus it is useful in perturbation problems to know whether the given mapping is m-accretive.

It is known that if X is a Hilbert space, then an accretive mapping of X into itself is locally bounded at every point of the interior of its domain (see [10], [11]). We shall show that this is also true in case X is a reflexive Banach space provided that the duality mapping of X is bicontinuous (THEOREM 1), and use this fact to show that, under certain conditions, an accretive mapping is m-accretive if and only if it is maximal accretive (COROLLARY 1 of THEOREM 5).

In order to obtain the latter result, we consider the initial value problem of the evolution equation

$$\text{(E)} \qquad u'(t)+Au(t) \ni 0, \ u(0)=a.$$

This problem has a solution (in a certain sense) if A is m-accretive. However, it seems difficult to solve (E) without the m-accretiveness of A. It was shown in [7] that if X^* is uniformly convex and A is everywhere defined, singlevalued and hemicontinuous, then (E) has a global solution for any given $a \in X$ and A is m-accretive. We shall extend this result ot the case where A is multivalued, locally bounded, demiclosed and accretive (THEOREMS 4 and 5). As an application, we shall show that such a mapping A generates a nonlinear contraction semigroup on X (THEOREM 6).

§0. Definitions and notation

Throughout this paper let X be a real reflexive Banach space and X^* be the dual space. The natural pairing between $x \in X$ and $x^* \in X^*$ is denoted by $<x, x^*>$. The norms in X and X^* are denoted by $||\cdot||$. We denote by I the identity mapping of X onto X.

For a subset S of X, we denote by $\bar{S}$, $\mathring{S}$ and $co(S)$ the closure, the interior and the convex hull of S respectively. For $S, S' \subset X$ and a real λ, we denote by $S+S'$ the set $\{x+y; x \in S, y \in S'\}$ if $S \neq \emptyset$ and $S' \neq \emptyset$, and by λS the set $\{\lambda x; x \in S\}$. When S' consists of a single point y, we write $S+y$ for $S+S'$.

Let A be a multivalued mapping of X into X, that is, to each $x \in X$ a subset Ax of X be assigned. The sets $D(A)=\{x \in X; Ax \neq \emptyset\}$, $R(A)=\bigcup_{x \in X} Ax$ and $G(A)=\{(x, x') \in X \times X; x' \in Ax\}$ are called the domain, the range and the graph of A respectively. For a subset S of X, we denote by $A(S)$ the set $\bigcup_{x \in S} Ax$.

Let A and A' be two multivalued mappings of X into X and λ be a real. The mappings $A+A'$, AA' and λA are defined by $(A+A')x=Ax+A'x$, $(AA')x=A(A'x)$ and $(\lambda A)x=\lambda(Ax)$ respectively.

In what follows a mapping means a multivalued mapping unless otherwise stated.

The *duality mapping* F of X into X^* is defined by

$$Fx=\{x^* \in X^*; <x, x^*>=\|x\|^2=\|x^*\|^2\}.$$

The domain of F is all of X, the range of F is all of X^* and, in general, F is multivalued. The inverse F^{-1} is the duality mapping of X^* into X. We know that if X^* is strictly convex, then F is singlevalued and that if X^* is uniformly convex, then F is uniformly continuous on bounded subsets of X (see [6]).

A mapping A of X into X is called *accretive* if for any (x, x') aud (y, y') in $G(A)$ there exists an element $f \in F(x-y)$ such that $<x'-y', f> \geq 0$. An accretive mapping A of X into X is called *maximal accretive* if there is no proper accretive extention of A, and called *m-accretive* if $R(I+A)=X$.

A mapping A of X into X is called *locally bounded* at $x \in X$ if there is a neighborhood U of x such that $A(U)$ is bounded in X.

We denote by $B(x, r)$(resp. $B^*(x^*, r)$) the closed ball in X (resp. X^*) with center $x \in X$ (resp. $x^* \in X^*$) and radius r. We use the symbols "$\xrightarrow{s}$" (or "*s-lim*") and "$\xrightarrow{w}$" (or "*w-lim*") to denote the convergence in the strong and the weak topology respectively.

§ 1. Local boundedness

In this section we shall prove the following theorem.

THEOREM 1. *Let X and X^* be strictly convex and let A be an accretive mapping of X into X. Assume that the duality mapping F is bicontinuous. Then A is locally bounded at every point of $\overset{\circ}{\widehat{D(A)}}$.*

The method of proof is based on that in [11]. To prove THEOREM 1 we prepare three lemmas.

LEMMA 1. *Let X, X^* and F be as in* THEOREM 1. *Let S be a subset of X such that $(-S)\cap \mathring{S}\neq\emptyset$. Then there are positive numbers ε and δ such that*

$$B^*(0, \varepsilon)\subset \bigcap_{x\in B(0,\delta)} co(F(S-x)). \tag{1.1}$$

PROOF. Let $x_0 \in (-S)\cap \mathring{S}$. First we shall show that for suitable numbers r and r'

$$B^*(Fx_0, r')\subset \bigcap_{x\in B(0,r)} F(S-x). \tag{1.2}$$

Indeed, since $x_0 \in \mathring{S}$, we have for some $r>0$

$$x_0+2B(0, r)\subset \mathring{S}.$$

Hence, $F(x_0+B(0, r))\subset F(S-x)$ for any $x \in B(0, r)$. By the bicontinuity of F, we have for some $r'>0$

$$B^*(Fx_0, r')\subset F(x_0+B(0, r))$$

Hence (1. 2) holds.

By the continuity of F at $-x_0$, for a number ε satisfying $\frac{r'}{4}>\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that

$$\|F(-x_0-x)-F(-x_0)\|\leq\varepsilon \qquad \textit{for all } x \in B(0, \delta(\varepsilon)). \tag{1.3}$$

Set $\delta=\min(r, \delta(\varepsilon))$. Then we have (1. 1). In fact, let y^* be any point of $B^*(0, \varepsilon)$ and x any point of $B(0, \delta)$. From (1. 3) it follows that

$$F(-x_0-x)=F(-x_0)+x^* \qquad \textit{for some } x^* \in B^*(0, \varepsilon). \tag{1.4}$$

We write

$$y^*=\frac{1}{2}(-F(-x_0)+2y^*-x^*)+\frac{1}{2}(F(-x_0)+x^*).$$

Since $F(-x_0)=-Fx_0$, $-F(-x_0)+2y^*-x^* \in B^*(Fx_0, r')$. From (1.2) and (1.4) it follows that $y^* \in co(F(S-x))$. q.e.d.

LEMMA 2. *Let A be a mapping of X into X, and assume that $0 \in \mathring{\widehat{D(A)}}$. Then there exist bounded subsets S, S' and Q such that*

$$(-S)\cap \mathring{S}\neq\emptyset, \tag{1.5}$$

$$\overline{S'}=S, \tag{1.6}$$

$$Ax\cap Q\neq\emptyset \qquad \textit{for all } x \in S'. \tag{1.7}$$

PROOF. We choose a positive number R such that $B(0, R)\subset \mathring{\widehat{D(A)}}$, and set for each positive integer n

$$S_n=\{x \in X;\ x \in B(0, n),\ Ax\cap B(0, n)\neq \emptyset\}.$$

Clearly, $D(A)=\bigcup_{n=1}^{\infty} S_n \subset \bigcup_{n=1}^{\infty} \bar{S}_n$. Therefore we have

$$B(0, R)=\bigcup_{n=1}^{\infty}(\bar{S}_n\cap B(0, R)).$$

From Baire's second category theorem it follows that there exists a positive integer n_0 such that the interior of $(\bar{S}_{n_0}\cap B(0, R))$ is non-empty. Let x_0 be a point in the interior of this set and y_0 be a point of $A(-x_0)$. We set $S'=(S_{n_0}\cap B(0, R))\cup\{-x_0\}$, $S=\bar{S}'$ and $Q=B(0, \|y_0\|+n_0)$. Such S, S' and Q satisfy (1.5), (1.6) and (1.7). q.e.d.

A mapping is called demiclosed if the following condition is satisfied: if $x_n \in D(A)$ for $n=1, 2, \cdots$, $x_n \xrightarrow{s} x$ and if there are $x_n' \in Ax_n$ such that $x_n' \xrightarrow{w} x'$, then $x \in D(A)$ and $x' \in Ax$.

LEMMA 3. *Let A be a maximal accretive mapping of X into X. Assume that X^* is strictly convex and F is continuous. Then A is demiclosed.*

PROOF. Let $\{x_n\}$ and $\{x_n'\}$ be sequences in X such that $(x_n, x_n') \in G(A)$, $x_n \xrightarrow{s} x_0$ and $x_n' \xrightarrow{w} x_0'$. From the accretiveness of A it follows that

$$< x_n'-x', F(x_n-x)> \geq 0 \qquad \textit{for all } (x, x') \in G(A).$$

Letting $n\to\infty$,

$$< x_0'-x', F(x_0-x)> \geq 0 \qquad \textit{for all } (x, x') \in G(A).$$

Since A is maximal accretive, this implies that $(x_0, x_0') \in G(A)$. q.e.d.

PROOF of THEOREM 1: Let x_0 be an arbitrary point of $\overset{\circ}{\widehat{D(A)}}$. We define A' by $A'x=A(x+x_0)$. It is easy to see that A' is accretive, $D(A')=D(A)-x_0$ and $0 \in \overset{\circ}{\widehat{D(A')}}$. The mapping A is locally bounded at x_0 if and only if A' is locally bounded at 0. Therefore it is sufficient to show that A is locally bounded at 0 in case $0 \in \overset{\circ}{\widehat{D(A)}}$.

Assume that $0 \in \overset{\circ}{\widehat{D(A)}}$. By LEMMA 2 there exist bounded sets S, S' and Q satisfying (1.5), (1.6) and (1.7). Set $\rho=\sup_{x\in S}\|x\|$ and $\rho'=\sup_{x\in Q}\|x\|$. By LEMMA 1, for suitable positive numbers ε and δ,

$$B^*(0, \varepsilon)\subset \bigcap_{x\in B(0,\delta)} co(F(S-x)). \tag{1.8}$$

We shall show that $A(B(0, \delta))$ is bounded. In fact, let x be any point of $B(0, \delta)\cap D(A)$ and let x' be any point of Ax. From the accretiveness of

A we infer that for $u \in S'$ and $u' \in Au \cap Q$

$$\begin{aligned}<x', F(u-x)> &\leq <u', F(u-x)> \\ &\leq (\|u\|+\|x\|)\|u'\| \\ &\leq (\rho+\delta)\rho'.\end{aligned}$$

Hence, $F(S'-x) \subset E \equiv \{x^* \in X^*; <x', x^*> \leq (\rho+\delta)\rho'\}$. By the continuity of F, $F(S-x) \subset \overline{(F(S'-x))}$. Since E is closed and convex, we have $co(F(S-x)) \subset E$. This relation and (1.8) imply that $B^*(0, \varepsilon) \subset E$. Hence, $\|x'\| \leq (\rho+\delta)\rho'/\varepsilon$. Thus $A(B(0, \delta)) \subset B(0, (\rho+\delta)\rho'/\varepsilon)$. q.e.d.

COROLLARY. *Let X, X^*, F and A be as in* THEOREM 1. *In addition assume that A is singlevalued and maximal accretive. Then A is demicontinuous at every point of $\overset{\circ}{\widehat{D(A)}}$ (i.e., if $x_n \xrightarrow{s} x$* and *$x \in \overset{\circ}{\widehat{D(A)}}$, then $Ax_n \xrightarrow{w} Ax$).*

PROOF Let $x \in \overset{\circ}{\widehat{D(A)}}$, $x_n \in \overset{\circ}{\widehat{D(A)}}$ and $x_n \xrightarrow{s} x$. By the local boundedness of A at x, $\{Ax_n\}$ is weakly relatively compact. Let x' be any weak cluster point of $\{Ax_n\}$. Then there is a subsequence $\{x_{n_k}\}$ such that $Ax_{n_k} \xrightarrow{w} x'$. By LEMMA 3, $Ax = x'$. It follows that $Ax_n \xrightarrow{w} Ax$. q.e.d.

§ 2. Maximal accretive mappings

There is another notion of maximality for singlevalued mappings. A singlevalued accretive mapping A is called *f-maximal accretive* ([4]) if there is no proper singlevalued accretive extention of A.

THEOREM 2. *Let X and X^* be strictly convex and F be bicontinuous and let A be a singlevalued accretive mapping of X into X with open domain. Then A is maximal accretive if and only if A is f-maximal accretive and demicontinuous.*

PROOF. Assume that A is maximal accretive. Then it is clear that A is f-maximal accretive. By the COROLLARY of THEOREM 1, A is demiconti nuous in $D(A)$.

Conversely, assume that A is f-maximal accretive and demicontinuous. Let A' be any accretive extention of A and (x_0, x_0') be any element of $G(A')$. Since A is f-maximal accretive, $D(A) = D(A')$. For small $t > 0$, $x_t = x_0 + t(x_0' - Ax_0) \in D(A)$ and

$$0 \leq <Ax_t - x_0', F(x_t - x_0)> = t <Ax_t - x_0', F(x_0' - Ax_0)>.$$

Hence, $<Ax_t - x_0', F(Ax_0 - x_0')> \leq 0$ for small $t > 0$. Letting $t \searrow 0$, $\|Ax_0 - x_0'\|^2 \leq 0$ by the demicontinuity of A. Hence, $x_0' = Ax_0$. Thus $A' = A$. This implies that A is maximal accretive. q.e.d.

REMARK. THEOREM 2 is a generalization of THEOREM 2.5 in [5].

THEOREM 3. *Let X and X^* be strictly convex and F be bicontinuous, and let A be an accretive mapping of X into X. Suppose that $D(A)$ is dense in X and there exists a point at which A is locally bounded. Then A is locally bounded at every point of X. In additidn if A is maximal accretive, then $D(A)=X$.*

PROOF. It is sufficient to show that A is locally bounded at 0. By the assumptions of the theorem there exist a point x_0 and a bounded neighborhood U of x_0 such that $-x_0 \in D(A)$ and $A(U)$ is bounded. Let y_0 be a point of $A(-x_0)$. Choose a positive number r such that $y_0 \in B(0, r)$ and $A(U) \subset B(0, r)$, and set $S'=(U \cap D(A)) \cup \{-x_0\}$, $S=\overline{S'}$ and $Q=B(0, r)$. Then $x_0 \in \mathring{S}$. From LEMMA 1, for suitable positive numbers ε and δ

$$B^*(0, \varepsilon) \subset \bigcap_{x \in B(0,\delta)} co(F(S-x)).$$

Just as in the proof of THEOREM 1, we see that $A(B(0, \delta))$ is bounded. Under the additional assumption we infer from the local boundedness of A and the reflexivity of X that for each $x \in X$ there exist a sequence $(x_n, x_n') \in G(A)$ and a point $x' \in X$ such that $x_n \xrightarrow{s} x$ and $x_n' \xrightarrow{w} x'$. By LEMMA 3, $x \in D(A)$. Thus $D(A)=X$. q.e.d.

COROLLARY 1. *Let X, X^* and F be as in* THEOREM 3, *and let A be an accretive mapping with dense domain in X. If $\overset{\circ}{\widehat{D(A)}}$ is non-empty, then A is locally bounded at every point of X. In addition, if A is maximal accretive, then $D(A)=X$.*

COROLLARY 2. *Let X be finite dimensional and strictly convex and X^* be strictly convex, and let A be a maximal accretive mapping with dense domain in X. Then $D(A)=X$.*

PROOF. It is sufficient to show that A is locally bounded at 0. If otherwise, there exists a sequence $(x_n, x_n') \in G(A)$ such that $x_n \to 0$ and $\|x_n'\| \to \infty$ as $n \to \infty$. Set $y_n' = x_n'/\|x_n'\|$ and choose a subsepuence $\{y_{n_k}'\}$ such that $y_{n_k}' \to y'$. Then $\|y'\|=1$. From the accretiveness of A it follows that

$$< \frac{x'}{\|x_{n_k}'\|} - y_{n_k}', F(x-x_{n_k}) > \geq 0 \qquad \textit{for all } (x, x') \in G(A).$$

Letting $k \to \infty$, we have $<y', Fx> \leq 0$ for all $x \in D(A)$. Since F is topological and $D(A)$ is dense in X, $F(D(A))$ is dense in X^*, and hence, $y'=0$. This is a contradiction.

§ 3. Nonlinear evolution equations

In this section let A be a mapping of X into X. We consider the differential equation

$$u'(t)+Au(t)\ni 0 \tag{3.1}$$

where t is a real variable and $u'(t)=du(t)/dt$.

An X-valued function $u(t)$ is called a *strong solution* of (3.1) on a real interval Ω if $u(t)$ is strongly absolutely continuous on any finite closed interval contained in Ω and $u'(t)+Au(t)\ni 0$ for a.e. $t\in\Omega$.

THEOREM 4. *Let X^* be uniformly convex and let A be an accretive mapping with open domain. Suppose that Ax is closed and convex for every $x\in D(A)$ and A is demiclosed and locally bounded at every point of $D(A)$. Then for each $a\in D(A)$ the equation* (3.1) *has a unique strong solution on $[0,\infty)$ with $u(0)=a$.*

To prove this theorem we prepare several lemmas.

LEMMA 4. *Let u_n, $n=1,2,\cdots$, be strongly measurable functions on $(0,r)$ into X such that $\|u_n(t)\|\leq K$ for a.e. $t\in(0,r)$ and $u_n\xrightarrow{w}u$ in $L^p(0,r;X)$, $1<p<\infty$. Let $V(t)$ be the set of all weak cluster points of the sequence $\{u_n(t)\}$. Then*

$$u(t)\in\overline{co(V(t))}\qquad \textit{for a.e. } t\in(0,r).$$

For a proof of LEMMA 4 see [7]. Using LEMMA 4, we obtain the following lemma.

LEMMA 5. *Let X^* and A be as in* THEOREM 4, *and let $u(t)$ be an X-valued continuous function on the finite closed interval $[0,r]$ such that $u(t)\in D(A)$ for all $t\in[0,r]$ and $Au(t)\subset B(0,K)$ for all $t\in[0,r]$. Then there exists an X-valued strongly measurable function $U(t)$ on $(0,r)$ such that $U(t)\in Au(t)$ and $\|U(t)\|\leq K$ for a.e. $t\in(0,r)$.*

PROOF. Set $Z=\{u(t);0\leq t\leq r\}$ and for each positive integer n, define a step function $u_n(t)$ on $[0,r)$ into Z by

$$u_n(t)=\sum_{k=0}^{n-1}\chi_{[t_k,t_{k+1})}(t)u(t_k)$$

where $t_k=\frac{r}{n}k$, $k=0,1,\cdots,n-1$ and $\chi_{[t_k,t_{k+1})}$ are the characteristic functions. Then $\|u_n(t)-u(t)\|\to 0$ uniformly on $[0,r)$ as $n\to\infty$. Define $U_n(t)$ by

$$U_n(t)=\sum_{k=1}^{n-1}\chi_{[t_k,t_{k+1})}(t)a_k^{(n)}$$

where $a_k^{(n)} \in Au_n(t_k) \subset B(0, K)$. Then $\|U_n(t)\| \leq K$ for all $t \in (0, r)$ and for all n. Hence there exists a subsequence $\{U_{n_j}\}$ of $\{U_n\}$ such that $U_{n_j} \xrightarrow{w} U$ in $L^p(0, r; X)$, $1 < p < \infty$. Let $V(t)$ be the set of all weak cluster points of $\{U_{n_j}(t)\}$ for $t \in (0, r)$. Clearly, $V(t) \subset B(0, K)$ for all $t \in (0, r)$. Since A is demiclosed, $V(t) \subset Au(t)$. From LEMMA 4, $U(t) \in \overline{co(V(t))}$ for a.e. $t \in (0, r)$. Since $Au(t)$ is closed and convex in X, we have

$$\overline{co(V(t))} \subset Au(t) \cap B(0, K).$$

Thus $U(t) \in Au(t)$ and $\|U(t)\| \leq K$ for a.e. $t \in (0, r)$. a.e.d.

The following lemma gives a local solution of (3.1).

LEMMA 6. *Let X^* and A be as in* THEOREM 4. *Then for any given $a \in D(A)$ there are a positive number r and a strong solution $u(t)$ of* (3.1) *on $[0, r)$ with $u(0) = a$.*

PROOF. Since $D(A)$ is open and A is locally bounded at a, we can choose positive numbers R and K such that $B(a, R) \subset D(A)$ and $A(B(a, R)) \subset B(0, K)$. Set $r = \frac{R}{K}$ and $\varepsilon_n = \frac{r}{n}$, $n = 1, 2, \dots$. By induction, define a function $u_n(t)$ on $[0, r)$ for each n as follows. Let $u_n(t) = a$ for $t \in [0, \varepsilon_n]$. For a positive integer k, $1 \leq k < n$, assume that $u_n(t)$ is already defined on $[0, k\varepsilon_n]$ in such a way that $u_n(t)$ is strongly absolutely continuons on $[0, k\varepsilon_n]$ and $\|u_n(t) - a\| \leq (k-1)\varepsilon_n K$ for all $t \in [0, k\varepsilon_n]$. Then $u_n(t) \in D(A)$ and $Au_n(t) \subset B(0, K)$ for all $t \in [(k-1)\varepsilon_n, k\varepsilon_n]$. Hence, by LEMMA 5, there exists a strongly integrable function $U_n^{(k)}(s)$ on $[(k-1)\varepsilon_n, k\varepsilon_n]$ such that $U_n^{(k)}(s) \in Au_n(s)$ and $\|U_n^{(k)}(s)\| \leq K$ a.e. on $[(k-1)\varepsilon_n, k\varepsilon_n]$. Let us define $u_n(t)$ on $[k\varepsilon_n, (k+1)\varepsilon_n]$ by

$$u_n(t) = u_n(k\varepsilon_n) - \int_{k\varepsilon_n}^t U_n^{(k)}(s - \varepsilon_n)\, ds.$$

Then, $u_n(t)$ is strongly absolutely continuous on $[0, (k+1)\varepsilon_n]$ and $\|u_n(t) - a\| \leq k\varepsilon_n K$ for all $t \in [0, (k+1)\varepsilon_n]$. Thus, for each positive integer n, a function u_n on $[0, r)$ into $B(a, R)$ is defined. Set

$$U_n(s) = \begin{cases} U_n^{(1)}(s) & \text{on } (0, \varepsilon_n) \\ U_n^{(2)}(s) & \text{on } (\varepsilon_n, 2\varepsilon_n) \\ \vdots & \\ U_n^{(n-1)}(s) & \text{on } ((n-2)\varepsilon_n, (n-1)\varepsilon_n). \end{cases}$$

Then

$$u_n(t) = \begin{cases} a & \text{on } [0, \varepsilon_n) \\ a - \displaystyle\int_{\varepsilon_n}^t U_n(s - \varepsilon_n)\, ds & \text{on } [\varepsilon_n, r), \end{cases}$$

and satisfies

(3.2) $$u_n'(t) = -U_n(t-\varepsilon_n) \in Au_n(t-\varepsilon_n) \qquad a.e.\ on\ (\varepsilon_n, r).$$

To show that $\{u_n(t)\}$ converges uniformly on $[0, r)$ as $n\to\infty$, we observe for any n and m with $n \geq m$

(3.3)
$$\begin{aligned}
&\frac{d}{dt}(\|u_n(t)-u_m(t)\|^2)\\
&=2\|u_n(t)-u_m(t)\|\frac{d}{dt}(\|u_n(t)-u_m(t)\|)\\
&=2<u_n'(t)-u_m'(t), F(u_n(t)-u_m(t))>\\
&=-2<U_n(t-\varepsilon_n)-U_m(t-\varepsilon_m), F(u_n(t)-u_m(t))>\\
&\leq -2<U_n(t-\varepsilon_n)-U_m(t-\varepsilon_m), F(u_n(t)-u_m(t))\\
&\quad -F(u_n(t-\varepsilon_n)-u_m(t-\varepsilon_m))>
\end{aligned}$$

for a.e. $t \in (\varepsilon_m, r)$. The above relations follow from LEMMA 1.3 in [6], (3.2) and the accretiveness of A. Integrating both sides of (3.3) on (ε_m, t), we obtain

(3.4)
$$\begin{aligned}
&\|u_n(t)-u_m(t)\|^2\\
&\leq \|u_n(\varepsilon_m)-a\|^2+4K\int_{\varepsilon_m}^{r}\|F(u_n(s)-u_m(s))\\
&\quad -F(u_n(s-\varepsilon_n)-u_m(s-\varepsilon_m))\|ds \qquad for\ all\ t\in[\varepsilon_m, r).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\|(u_n(t)-u_m(t))-u_n(t-\varepsilon_n)-u_m(t-\varepsilon_m))\|\\
&\leq \|u_n(t)-u_n(t-\varepsilon_n)\|+\|u_m(t)-u_m(t-\varepsilon_m)\|\\
&\leq K(\varepsilon_n+\varepsilon_m) \qquad for\ all\ t\in[\varepsilon_m, r),
\end{aligned}$$

and $\|u_n(\varepsilon_m)-a\| \leq \int_0^{\varepsilon_m}\|u_n'(s)\|ds \leq \varepsilon_m K$. Hence, $\|u_n(t)-u_m(t)-(u_n(t-\varepsilon_n)-u_m(t-\varepsilon_m))\| \to 0$ uniformly on $(0, r)$ as $n, m\to\infty$. Since F is uniformly continuous on $B(a, R)$, $\|F(u_n(s)-u_m(s))-F(u_n(s-\varepsilon_n)-u_n(s-\varepsilon_m))\| \to 0$ uniformly on $(0, r)$ as $n, m\to\infty$. Therefore, it follows from (3.4) that

$$\|u_n(t)-u_m(t)\|\to 0\ uniformly\ on\ [0, r),$$

and $\{u_n\}$ converges uniformly on $[0, r)$ to a continuous function $u(t)$ with $u(0)=a$.

We are to show that u is a strong solution of (3.1). Since $\|u_n'(t)\| \leq K$ for a.e. $t \in (0, r)$ by (3.2), there exist a subsequence $\{u_{n_j}'\}$ and $v \in L^p(0, r; X)$, $1<p<\infty$, such that $u_{n_j}' \xrightarrow{w} v$ in $L^p(0, r; X)$. Then $v=u'$ in the distribution sense. It follows that u is strongly absolutely continuous and $u'=v$ a.e. on $(0, r)$. Just as in the proof of LEMMA 5, we see that $u'(t)=v(t) \in -Au(t)$ for a.e. $t \in (0, r)$. Thus $u(t)$ is a strong solution of (3.1) on $[0, r)$.

q.e.d.

LEMMA 7. *Let A be an accretive mapping and $u(t)$ be a strong solution of* (3.1) *on* $[0, r)$ *with* $u(0)=a \in D(A)$. *Then*

$$\|u'(t)\| = \inf_{y \in Au(t)} \|y\| \leq \inf_{y \in Aa} \|y\| \qquad \textit{for a.e. } t \in (0, r).$$

For a proof of LEMMA 7 see [2].

PROOF OF THEOREM 4: By LEMMA 6 there exists a strong solution of $u(t)$ (3.1) on $[0, r)$ with $u(0)=a$ for some $r>0$. Let $[0, r^+)$ be the maximal interval of existence. We shall show that $r^+=\infty$.

Assume that $r^+<\infty$. From LEMMA 7, $\|u'(t)\| \leq \inf_{y \in Aa} \|y\| < \infty$ for a.e. $t \in (0, r^+)$. Hence, the limit $s\text{-}\lim_{t \nearrow r^+} u(t)$ exists. We can choose a sequence $\{t_k\}$ such that $t_k \nearrow r^+$ and $\{u'(t_k)\}$ is weakly convergent. Since A is demiclosed, it follows that $b = s\text{-}\lim_{t \nearrow r^+} u(t)$ belongs to $D(A)$. If we apply LEMMA 6 with the initial time r^+ and initial condition $u(r^+)=b$, we obtain a continuation of u beyond r^+. This contradicts the definition of r^+.

Next we prove the uniqueness of the solution. Let $u_1(t)$ and $u_2(t)$ be strong solutions of (3.1) on $[0, \infty)$ such that $u_1(0)=u_2(0)=a$. Then, for any $t \in [0, \infty)$

$$\begin{aligned}\|u_1(t)-u_2(t)\|^2 &= \int_0^t \frac{d}{ds}(\|u_1(s)-u_2(s)\|^2)ds \\ &= 2\int_0^t < u_1'(s)-u_2'(s), F(u_1(s)-u_2(s)) > ds \\ &\leq 0,\end{aligned}$$

since $-u_i'(s) \in Au_i(s)$ for a.e. $s \in (0, \infty)$, $i=1, 2$. Hence, $u_1(t)=u_2(t)$ on $[0, \infty)$.

§4. m-accretiveness

The purpose of this section is to prove

THEOREM 5. *Let X^* and A be as in* THEOREM 4. *Then A is m-accretive, that is,* $R(I+A)=X$.

To prove this theorem we consider the differential equation

$$(4.1) \qquad u'(t)+(I+A)u(t) \ni 0.$$

If A is as in THEOREM 4, then $I+A$ is accretive, demiclosed and locally bounded at every point of $D(I+A)=D(A)$ and $(I+A)x=x+Ax$ is closed and convex in X for each $x \in D(I+A)$. Hence, by THEOREM 4, for each $a \in D(I+A)$ the equation (4.1) has a unique strong solution $u(t)$ on $[0, \infty)$ with $u(0)=a$.

LEMMA 8. *Let $u(t)$ be a strong solution of* (4.1) *on* $[0, \infty)$. *Suppose that $u(t)$ is differentiable and satisfies* (4.1) *at $t=s$ and s', $0<s<s'<\infty$. Then*

$$\|u'(s')\| \leq e^{s-s'}\|u'(s)\|.$$

For a proof of LEMMA 8 see [7].

PROOF of THEOREM 5: Let $u(t)$ be the strong solution of (4.1) on $[0, \infty)$ with $u(0)=a \in D(I+A)$. Choose $r>0$ such that $u(t)$ is differentiable and satisfies (4.1) at $t=r$. From LEMMA 8, for t and t', $t \geq t' > r$

$$\begin{aligned}\|u(t)-u(t')\| &\leq \int_{t'}^{t} \|u'(s)\| ds \\ &\leq \|u'(r)\| e^{r} \int_{t'}^{t} e^{-s} ds \\ &= \|u'(r)\| e^{r}(-e^{-t}+e^{-t'}).\end{aligned}$$

Hence, $\|u(t)-u(t')\| \to 0$ as $t, t' \to \infty$, that is, the limit $s\text{-}\lim_{t\to\infty} u(t)=u_0$ exists. By LEMMA 8 again there is a sequence $\{t_n\}$ such that $t_n \to \infty$ and $u'(t_n) \xrightarrow{s} 0$ as $n \to \infty$. Since $I+A$ is demiclosed, we obtain $(I+A)u_0 \ni 0$. Thus $0 \in R(I+A)$.

Let b be an arbitrary point of X. We define the mapping A_b by $A_b x = Ax-b$. Applying the same argument for A_b, we conclude that $b \in R(I+A)$. Thus $R(I+A)=X$. q.e.d.

COROLLARY 1. *Let X be strictly convex, X^* be uniformly convex, F be bicontinuous, and A be an accretive mapping of X into X with open domain. Then A is m-accretive if and only if A is maximal accretive.*

PROOF. Assume that A is maximal accretive. Then A satisfies conditions in THEOREM 4. In fact, the maximal accretiveness of A implies that Ax is closed and convex for all $x \in D(A)$. The demiclosedness and the local boundedness of A follow from LEMMA 3 and THEOREM 1 respectively. Hence, by THEOREM 5, A is m-accretive.

The converse is true in general (see [7; LEMMA 5.3]) q.e.d.

REMARK. In Banach spaces m-accretiveness always implies maximal accretiveness, but the converse is not true (for a counter-example see [4]). In Hilbert spaces both notions coincide.

COROLLARY 2. *Let X, X^* and F be as in* COROLLARY 1 *and let A be an f-maximal accretive mapping of X into X with open domain. Then A has an m-accretive extension.*

PROOF. By Zorn's lemma, A has a maximal accretive extension $\tilde{A}$ The f-maximal accretiveness of A implies $D(A)=D(\tilde{A})$ By COROLLARY 1, $\tilde{A}$ is m-accretive. q.e.d.

§ 5. A certain class of nonlinear contraction semigroups

Let $T=\{T(t);t\geq 0\}$ be a family of nonlinear singlevalued mappings of X into X with $D(T(t))=X$ for all $t\geq 0$. We say that T is a contraction semigroup on X if the following conditions (5.1), (5.2) and (5.3) are satisfied;

$$T(t+t')x=T(t)T(t')x \qquad \text{for } t,\ t'\geq 0 \text{ and } x\in X, \tag{5.1}$$

$$T(0)x=x \qquad \text{for } x\in X, \tag{5.2}$$

$$\|T(t)x-T(t)y\|\leq\|x-y\| \quad \text{for } t\geq 0 \text{ and } x,\ y\in X. \tag{5.3}$$

We define the strong infinitesimal generator G_s of T by

$$G_s x=s\text{-}\lim_{h\searrow 0}\frac{T(h)x-x}{h},$$

and the weak infinitesimal generator G_w of T dy

$$G_w x=w\text{-}\lim_{h\searrow 0}\frac{T(h)x-x}{h}$$

whenever the right sides exist.

It is easy to see that $-G_s$ and $-G_w$ are accretive and G_w is an extension of G_s.

For mappings A and B, by $B\subset A$ we mean that A is an extension of B, that is, $G(B)\subset G(A)$.

THEOREM 6. *Let X^* be uniformly convex.*
(a) *Let A be an m-accretive mapping of X into X with $D(A)=X$ and suppose that A is locally bounded at every point of X. Then there exists a unique contraction semigroup T on X satisfying $-G_s\subset A$ and the following condition:*
(5.4) *There exists a real-valued and locally bounded function $K(x)$ on X such that*

$$\|T(t)x-T(t')x\|\leq K(x)|t-t'| \qquad \text{for } t,\ t'\geq 0 \text{ and } x\in X.$$

(b) *Let T be a contraction semigroup on X satisfying* (5.4). *Then there exists a unique m-accretive mapping A with $-G_s\subset A$. Furthermore $D(A)=X$ and A is locally bounded at every point of X.*

PROOF of (a): First we prove the existence of such a semigroup on X. By THEOREM 4, for each $x\in X$ there exists a unique strong solution $u(x;t)$ of (3.1) on $[0,\infty)$ with $u(x;0)=x$. Put $T(t)x=u(x;t)$. Then $T=\{T(t);t\geq 0\}$

clearly satisfies conditions (5.1) and (5.2). For $x,\ y \in X$ and $t \geq 0$,

$$(5.5)\qquad \|T(t)x - T(t)y\|^2 - \|x-y\|^2$$
$$= \int_0^t \frac{d}{ds}(\|T(s)x - T(s)y\|^2)ds$$
$$= 2\int_0^t < \frac{d}{ds}(T(s)x - T(s)y),\ F(T(s)x - T(s)y) > ds.$$

Since $-\frac{d}{ds}T(s)x \in A(T(s)x)$ and $-\frac{d}{ds}T(s)y \in A(T(s)y)$ for a.e. $s \in (0, t)$, the last integral of (5.5) is non-positive. Hence, $\|T(t)x - T(t)y\| \leq \|x-y\|$. Thus T satisfies (5.3). Further (5.4) follows from Lemma 7 and the local boundedness of A. The inclusion $-G_s \subset A$ follows from Corollary 1 in [9]. In fact, the corollary say that $-G_w \subset A^0$, where A^0 is defined by $A^0x = \{x' \in Ax;\ \|x'\| = \inf_{y \in Ax} \|y\|\}$. Since $G_s \subset G_w$ and $A^0 \subset A$, we conclude that $-G_s \subset A$.

Now we prove the uniqueness of such a semigroup. Let $\tilde{T} = \{\tilde{T}(t);\ t \geq 0\}$ be another contraction semigroup on X satisfying $-\tilde{G}_s \subset A$ and (5.4) for $\tilde{T}$, where $\tilde{G}_s$ is the strong infinitesimal generator of $\tilde{T}$. By (5.4) the function $t \rightarrow \tilde{T}(t)x$ is Lipschitz continuous on $[0, \infty)$ for each $x \in X$. Therefre $\tilde{T}(t)x$ is differentiable a.e. on $[0, \infty)$, and hence $\frac{d}{dt}\tilde{T}(t)x = \tilde{G}_s(\tilde{T}(t)x) \in -A(\tilde{T}(t)x)$ for a.e. $t \in [0, \infty)$, that is, $\tilde{T}(t)x$ is the strong solution of (3.1) on $[0, \infty)$ with the initial value x. By the uniqueness of a strong solution we have $\tilde{T}(t)x = T(t)x$ for $t \geq 0$ and $x \in X$. Thus $T = \tilde{T}$. q.e.d.

To prove (b) we use the following lemma.

Lemma 9. *Let X^* be uniformly convex and let A be an accretive mapping of X into X with open domain. Suppose that for each $x \in D(A)$ there exist a neighborhood U_x of x and a bounded subset V_x of X such that $Ay \cap V_x \neq \emptyset$ for all $y \in U_x$. Then A is locally bounded at every point of $D(A)$.*

Proof. For each $x \in D(A)$, choose a closed ball $B(x, r)$ such that $B(x, r) \subset U_x$. Put $K = \sup_{z \in V_x} \|z\|$. Let $y \in B\left(x, \frac{r}{2}\right)$ and $y' \in Ay$. For any $z \in B\left(0, \frac{r}{2}\right)$, $y + z \in U_x$. Let $z' \in A(y+z) \cap V_x$. Then we have by the accretiveness of A

$$<y', Fz> = <y', F(y+z-y)> \leq <z', F(y+z-y)>$$
$$\leq K\|z\|$$
$$\leq \frac{Kr}{2}.$$

Hence, $B\left(0, \frac{r}{2}\right) \subset \left\{z;\ <y', Fz> \leq \frac{Kr}{2}\right\}$. Since $B^*\left(0, \frac{r}{2}\right) = F\left(B\left(0, \frac{r}{2}\right)\right)$, we have $B^*\left(0, \frac{r}{2}\right) \subset \left\{z^*; <y', z^*> \leq \frac{Kr}{2}\right\}$. This implies that $\|y'\| \leq K$. Thus $A\left(B\left(x, \frac{r}{2}\right)\right) \subset B(0, K)$. q.e.d.

PROOF of (b): We shall prove the existence of such an m-accretive mapp ing. Let A be a maximal accretive extension of $-G_s$. Then, $D(A)=X$ and A is locally bounded at every point of X and m-accretive. In fact, let x be any point of X. Since the function $T(t)x$ is Lipschtiz continuous on $[0, \infty)$ by (5.4), there is a sequence $\{t_n\}$ such that $T(t)x$ is differentiable at $t=t_n$, $t_n \searrow 0$ and $\frac{d}{dt}T(t)x\Big|_{t=t_n} = G_s(T(t_n)x) \xrightarrow{w} x'$ for some $x' \in X$ as $n \to \infty$. Since $T(t_n)x \xrightarrow{s} x$ as $n \to \infty$ and A is demiclosed by LEMMA 3, $x \in D(A)$ and $-x' \in Ax$. Thus $D(A)=X$. Furtheremore, we easily obtain $\|-x'\| \leq K(x)$. Since $K(x)$ is locally bounded, there is a neighborhood U_x of x such that $\rho = \sup_{y \in U_x} K(y) < \infty$. By taking $V_x = B(0, \rho)$ in LEMMA 9, we see that A is locally bounded at every point of X, and hence, A is m-accretive by THEOREM 5.

Next we prove the uniqueness of such an m-accretive mapping. Let $\tilde{A}$ be any m-accretive extension of $-G_s$. Then, by (a) of THEOREM 6, there is a semigroup $\tilde{T}=\{\tilde{T}(t);\ t \geq 0\}$ such that $\tilde{A} \supset -\tilde{G}_s$, where $\tilde{G}_s$ is the strong infinitesimal generator of $\tilde{T}$. For each $x \in X$, $T(t)x$ satisfies $\frac{d}{dt}T(t)x = G_s(T(t)x) \in -\tilde{A}(T(t)x)$ a.e. on $[0, \infty)$, and $\tilde{T}(t)x$ satisfies $\frac{d}{dt}\tilde{T}(t)x = \tilde{G}_s(\tilde{T}(t)x) \in -\tilde{A}(\tilde{T}(t)x)$ a.e. on $[0, \infty)$. Therefore, $T(t)x = \tilde{T}(t)x$ for all $t \geq 0$. Thus $T = \tilde{T}$. From COROLLARY 2 in [1] we obtain $A = \tilde{A}$. q.e.d.

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