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Energy Inequalities and the Cauchy Problem for a Pseudo-Differential System

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Let $\vec{A}(t)$ be an $N\times N$ matrix of pseudo-differential operators of order $\leqq 1$ which depend on a parameter t. Here the term "pseudo-differential operator" will be understood as described in the preceding paper [10], which has been designed to be the introductory part of the present paper. Certain pseudo-commutativity relations are assumed for $\vec{A}(t)$. Let us write $L=D_t+\vec{A}(t)$, $D_t=\frac{1}{i}\frac{\partial}{\partial t}$. Here we study the Cauchy problem which consists in finding a solution $\vec{u}=(u_1, u_2,\cdots, u_N)$, $u_j\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$, to the equation

$$L\vec{u}=\vec{f}\qquad \text{in } R_{n+1}^+=R_t^+\times(R_n)_x$$

with initial condition

$$\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}\vec{u}=\vec{\alpha},$$

when $\vec{f}=(f_1, f_2,\cdots, f_N)$, $f_j\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$ and $\vec{\alpha}=(\alpha_1, \alpha_2,\cdots, \alpha_N)$, $\alpha_j\in(\mathscr{D}'_{L^2})_x$ are arbitrarily given. It was shown in [10] that if a solution $\vec{u}$ exists, then $\vec{f}$ must admit the $\mathscr{D}'_{L^2}$-canonical extension. The energy inequalities of Friedrichs-Lewy type are assumed for L. Even if L is a system of differential operators, our treatments will give rise to some simplification and refinement to our related paper [9].

In Section 1 we shall show the approximation theorems, which are the analogues of the results [9] established for a system of differential operators. Sections 2 and 3 are devoted to the studies of the uniqueness and existence theorems for the Cauchy problem. In Section 4 we consider the pseudo-differential system with constant coefficients. The discussions are made here about the well-posedness in the L^2 norm and its connection with the energy inequalities. In Section 5 a characterization of regular hyperbolicity of a pseudo-differential operator is given. This is an analogue of our recent result established in [9] for a differential operator. Section 6 is concerned with generalization of S. Kaplan's result [11] about the Cauchy problem for parabolic equation. The method developed in Sections 2 and 3 will much simplify his treatments. In the final section the Cauchy problem for ordinary differential operators is considered. It is shown that the method developed in Sections 2, 3 and 4 also lead to generalization of basic theorem in [1].

1. Approximation theorem

Let $R_{n+1}=R\times R_n$ be an $(n+1)$-dimensional Euclidean space with generic points (t, x), $x=(x_1,\cdots, x_n)$ and $\Xi_{n+1}=\Xi\times\Xi_n$ be its dual with point (τ, ξ), $\xi=(\xi_1,\cdots, \xi_n)$. We write $|x|=(\sum_{j=1}^n x_j^2)^{1/2}$ and for an n-tuple $p=(p_1,\cdots, p_n)$ of non-negative integers we write $|p|=p_1+\cdots+p_n$, $x^p=x_1^{p_1}\cdots x_n^{p_n}$, $D_x^p=D_1^{p_1}\cdots D_n^{p_n}$ with $D_j=\frac{1}{i}\frac{\partial}{\partial x_j}$. By D_t we mean $\frac{1}{i}\frac{\partial}{\partial t}$. The Fourier transform, $\hat{\phi}$, $\phi\in\mathscr{S}(R_n)$, is defined by $\hat{\phi}(\xi)=\int\phi(x)e^{-i<x,\xi>}dx$, which is extended by continuity to a temperate distribution $u\in\mathscr{S}'(R_n)$ by the formula $<\hat{u}, \phi>=<u, \hat{\phi}>$, where $<x, \xi>=\sum_{j=1}^n x_j\xi_j$.

We shall continue to employ the notations in our preceding paper [10]. We have considered there the spaces $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)=\mathscr{D}'_t\varepsilon(\mathscr{D}'_{L^2})_x$, $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ and $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(\bar{R}^+_{n+1})$ and have shown that these spaces are reflexive, ultrabornological and Souslin. Let $A(t)$ be an OP_r-valued C^∞ function of $t\in R_t$, that is, $A(t)\in\mathfrak{C}^\infty_{(r)}$ in the notation used in [10]. For any $u\in\mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$ (resp. $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$), $A(t)u$ is well defined, belongs to the space $\mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$ (resp. $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$) and the map $u\to A(t)u$ is continuous. If $\vec{A}(t)$ is an $N\times N$ matrix of operators $A_{ij}(t)\in\mathfrak{C}^k_{(r)}$, then we shall also write $\vec{A}(t)\in\mathfrak{C}^k_{(r)}$. If, for a vector distribution $\vec{u}=(u_1,\cdots, u_N)$, each component u_j belongs to the same space $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$, then we shall write $\vec{u}\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$. When confusion appears impossible, we shall use a similar abbreviation.

Let Λ and S be operators with symbols $|\xi|$ and $(1+|\xi|^2)^{1/2}$ respectively. We shall denote by $\lambda(D_x)$ the operator with symbol $\lambda(\xi)$. Let us consider $\vec{A}(t)\in\mathfrak{C}^k_{(r)}$. In what follows, we assume that for any $T>0$ and any real λ there exists a constant $C_{\lambda,T}$ such that

$$(*)\quad \|(S^{-\lambda}\vec{A}(t)S^{\lambda}-\vec{A}(t))\vec{\chi}\|_{(0)}\leqq C_{\lambda,T}\|\vec{\chi}\|_{(r-1)},\ 0\leqq t\leqq T,\ \vec{\chi}\in C_0^\infty(R_n),$$

where we mean by $\|\vec{\chi}\|_{(r)}$ the norm defined by $\|\vec{\chi}\|^2_{(r)}=\sum_{j=1}^N\|\chi_j\|^2_{(r)}$ and $\|\chi_j\|_{(r)}=\left(\frac{1}{(2\pi)^n}\int|\hat{\chi}_j(\xi)|^2(1+|\xi|^2)^r d\xi\right)^{1/2}$. As shown in Section 4 in [10], a singular integral operator in the sense of $A.P.$ Calderón [3] has the pseudo-commutativity $(*)$. For a differential operator with constant coefficients, the commutativity in question is trivially satisfied. For any real s we can find a constant $C^{(s)}_{\lambda,T}$ such that $\|(S^{-\lambda}\vec{A}(t)S^{\lambda}-\vec{A}(t))\vec{\chi}\|_{(s)}\leqq C^{(s)}_{\lambda,T}\|\vec{\chi}\|_{(s+r-1)}$ and the adjoint operator $\vec{A}^*(t)$ has also the property $(*)$. In fact, for any $\vec{\chi}_1, \vec{\chi}_2\in C_0^\infty(R_n)$ the inequalities

$$|(\vec{\chi}_1, (S^{-\lambda}\vec{A}^*(t)S^{\lambda}-\vec{A}^*(t))\vec{\chi}_2)|=|((S^{\lambda}\vec{A}(t)S^{-\lambda}-\vec{A}(t))\vec{\chi}_1, \vec{\chi}_2)|$$
$$\leqq\|(S^{\lambda}\vec{A}(t)S^{-\lambda}-\vec{A}(t))\vec{\chi}_1\|_{(-r+1)}\|\vec{\chi}_2\|_{(r-1)}$$

$$\leqq C_{-\lambda,T}^{(-r+1)}\|\vec{\chi}_1\|_{(0)}\|\vec{\chi}_2\|_{(r-1)}$$

imply $\|(S^{-\lambda}\vec{A}^*(t)S^{\lambda}-A^*(t))\vec{\chi}_2\|_{(0)}\leqq C_{-\lambda,T}^{(-r+1)}\|\vec{\chi}_2\|_{(r-1)}$.

To prove the approximation theorem below (Theorem 1) we shall need the following two lemmas. For any $\varepsilon>0$ we put $S_\varepsilon=1+\varepsilon\Lambda$. Then we have

Lemma 1. *For any $\chi\in\mathscr{H}_{(s)}(R_n)$, $S_\varepsilon^{-1}\chi$ belongs to the space $\mathscr{H}_{(s+1)}(R_n)$ and it converges in $\mathscr{H}_{(s)}(R_n)$ to χ as $\varepsilon\downarrow 0$.*

Proof. For a fixed ε, $\dfrac{(1+|\xi|^2)^{1/2}}{1+\varepsilon|\xi|}$ is bounded and we can write

$$(1+|\xi|^2)^{(s+1)/2}(S_\varepsilon^{-1}\chi)^\wedge=\frac{(1+|\xi|^2)^{1/2}}{1+\varepsilon|\xi|}(1+|\xi|^2)^{s/2}\hat{\chi}(\xi),$$

and therefore $S_\varepsilon^{-1}\chi\in\mathscr{H}_{(s+1)}(R_n)$. If we write

$$(1+|\xi|^2)^{s/2}(S_\varepsilon^{-1}\chi-\chi)^\wedge=\Big(\frac{1}{1+\varepsilon|\xi|}-1\Big)(1+|\xi|^2)^{s/2}\hat{\chi}(\xi)$$

$$=-\frac{\varepsilon|\xi|}{1+\varepsilon|\xi|}(1+|\xi|^2)^{s/2}\hat{\chi}(\xi),$$

then $0\leqq\dfrac{\varepsilon|\xi|}{1+\varepsilon|\xi|}\leqq 1$ and $\dfrac{\varepsilon|\xi|}{1+\varepsilon|\xi|}$ converges to 0 for any fixed ξ as $\varepsilon\downarrow 0$. Thus we see that $\|S_\varepsilon^{-1}\chi-\chi\|_{(s)}$ converges to 0 as $\varepsilon\downarrow 0$.

Remark. Evidently $\|S_\varepsilon^{-1}\chi\|_{(s)}\leqq\|\chi\|_{(s)}$ and we see from the Banach-Steinhaus theorem that $S_\varepsilon^{-1}\chi$ converges to χ in $\mathscr{H}_{(s)}(R_n)$ uniformly when χ varies in a compact subset of $\mathscr{H}_{(s)}(R_n)$.

Lemma 2. *Let $A(t)\in\mathfrak{C}_{(r)}$. Then we have*

(i) *For any $T>0$ and ε with $0<\varepsilon\leqq 1$, there exists a constant $C_T^{(s)}$ such that*

$$\|(S_\varepsilon^{-1}\vec{A}(t)S_\varepsilon-\vec{A}(t))\vec{\chi}\|_{(s)}\leqq C_T^{(s)}\|\vec{\chi}\|_{(s+r-1)},\ 0\leqq t\leqq T,\ \vec{\chi}\in\mathscr{H}_{(s+r-1)}(R_n).$$

(ii) *For any $\vec{\chi}\in\mathscr{H}_{(s+r-1)}(R_n)$, $\|(S_\varepsilon^{-1}\vec{A}(t)S_\varepsilon-\vec{A}(t))\vec{\chi}\|_{(s)}$ converges to 0 as $\varepsilon\downarrow 0$.*

Proof. We may assume $\vec{\chi}\in C_0^\infty(R_n)$, for $C_0^\infty(R_n)$ is dense in $\mathscr{H}_{(s+r-1)}(R_n)$ and $\vec{\chi}\to\|(S_\varepsilon^{-1}\vec{A}(t)S_\varepsilon-\vec{A}(t))\vec{\chi}\|_{(s)}$ is semi-continuous from below.

(i) For each t, $0\leqq t\leqq T$, the operator $\vec{B}(t)=S\vec{A}(t)-\vec{A}(t)S$ is of order $\leqq r$. Putting $R=\Lambda-S$, we have $|\hat{R}(\xi)|\leqq\dfrac{1}{(1+|\xi|^2)^{1/2}}$ and therefore the operator $R\vec{A}(t)-\vec{A}(t)R$ is of order $\leqq r-1$. Thus the operator $\vec{B}_1(t)=\Lambda\vec{A}(t)-\vec{A}(t)\Lambda$ is of order $\leqq r$ and we can write $S_\varepsilon\vec{A}(t)-\vec{A}(t)S_\varepsilon=\varepsilon\vec{B}_1(t)$. Putting $\vec{\Gamma}_\varepsilon(t)=\vec{A}(t)-S_\varepsilon^{-1}\vec{A}(t)S_\varepsilon$, for any $\vec{\chi}\in\mathscr{H}_{(s)}(R_n)$ we have

$$\vec{\Gamma}_\varepsilon(t)\vec{\chi}=\frac{\varepsilon}{1+\varepsilon\Lambda}\vec{B}_1(t)\vec{\chi}=\frac{\varepsilon(1+\Lambda^2)^{1/2}}{1+\varepsilon\Lambda}S^{-1}\vec{B}_1(t)\vec{\chi},$$

where $\frac{\varepsilon(1+|\xi|^2)^{1/2}}{1+\varepsilon|\xi|}\leqq 1$ and $S^{-1}\vec{B}_1(t)$ is of order $\leqq r-1$. Thus we obtain

$$\|(\vec{A}(t)-S_\varepsilon^{-1}\vec{A}(t)S_\varepsilon)\vec{\chi}\|_{(s)}=\|\vec{\Gamma}_\varepsilon\vec{\chi}\|_{(s)}\leqq\|S^{-1}\vec{B}_1(t)\vec{\chi}\|_{(s)}\leqq C_T^{(s)}\|\vec{\chi}\|_{(s+r-1)},$$

where $C_T^{(s)}$ is a constant.

(ii) $\frac{\varepsilon(1+|\xi|^2)^{1/2}}{1+\varepsilon|\xi|}$ converges pointwise to 0 as $\varepsilon\downarrow 0$. If we let $\varepsilon\downarrow 0$ in (i), we see by the Banach-Steinhaus theorem that $\lim_{\varepsilon\downarrow 0}\|(S_\varepsilon^{-1}\vec{A}(t)S_\varepsilon-\vec{A}(t))\vec{\chi}\|_{(s)}=0$ for any $\vec{\chi}\in\mathscr{H}_{(s)}(R_n)$.

For any real numbers σ, s we shall denote by $\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ the space of $u\in\mathscr{D}'(R_{n+1}^+)$ such that ϕu belongs to the space $\mathscr{H}_{(\sigma,s)}(\bar{R}_{n+1}^+)$[5, p. 51] when ϕ is taken arbitrarily in $C_0^\infty(R)$. The topology in $\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ is defined by the semi-norms $\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)\ni u\to\|\phi u\|_{(\sigma,s)}$. By $\tilde{\mathscr{H}}^*_{(\sigma,s)}(\bar{R}_{n+1}^+)$ we mean the adjoint space of $\tilde{\mathscr{H}}_{(-\sigma,-s)}(\bar{R}_{n+1}^+)$, which consists of all $v\in\mathring{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ with support $\subset[0, T]\times R_n$ for some $T>0$. It is to be noticed that $\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ and $\mathring{\tilde{\mathscr{H}}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ may be identified for $|\sigma|<\frac{1}{2}$ (cf. Proposition 7 in [8, p. 416]) and that in the space $\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ the following conditions are equivalent (cf. Theorem 1 in [8, p. 410]):

(i) $\sigma>\frac{1}{2}$.

(ii) For any $u\in\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$, u has the $\mathscr{D}'_{L^2}$-boundary value $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}u$ [10, p. 375].

(iii) For any $u\in\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$, u has the distributional boundary value $\lim_{t\downarrow 0}u$ [7, p. 12].

and similarly the following conditions are equivalent (cf. Theorem 2 in [8, p. 413]):

(i)′ $\sigma>-\frac{1}{2}$.

(ii)′ For any $u\in\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$, u has the $\mathscr{D}'_{L^2}$-canonical extension over $t=0$ [10, p. 379].

(iii)′ For any $u\in\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$, u has the canonical extension over $t=0$ [7, p. 12].

Let $\vec{u}\in\mathscr{H}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ and assume that $\vec{A}(t)\in\mathfrak{C}_{(r)}^l$ with $l\geqq|\sigma|$. Then $\vec{A}(t)\vec{u}\in\tilde{\mathscr{H}}_{(\sigma,s-r)}(\bar{R}_{n+1}^+)$ and $\vec{u}\to\vec{A}(t)\vec{u}$ is a continuous map of $\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ into $\tilde{\mathscr{H}}_{(\sigma,s-r)}(\bar{R}_{n+1}^+)$ (cf. Proposition 14 in [10, p. 385]). From the equality $S^j\vec{\Gamma}_\varepsilon(t)\vec{\chi}=\frac{\varepsilon(1+\Lambda^2)^{1/2}}{1+\varepsilon\Lambda}S^{j-1}\vec{B}_1(t)\vec{\chi}$ for any $\vec{\chi}\in C_0^\infty(R_{n+1})$ we have $\|S^j\vec{\Gamma}_\varepsilon(t)\vec{\chi}\|_{(0,s)}\leqq$

$C_T^{(s)}\|\vec{\chi}\|_{(0,s+r-1+j)}$. In the same way as in the proof of Proposition 14 in [10, p. 385], we have immediately the following

COROLLARY 1. *Let* $\vec{A}(t) \in \mathfrak{C}^l_{(r)}$ *with* $l \geqq |\sigma|$. *Then*

(i) *For any* $T>0$ *and* ε *with* $0<\varepsilon\leqq 1$, *there exists a constant* $C_T^{(\sigma,s)}$ *such that*

$$\|(S_\varepsilon^{-1}\vec{A}(t)S_\varepsilon - \vec{A}(t))\vec{\chi}\|_{(\sigma,s)} \leqq C_T^{(\sigma,s)}\|\vec{\chi}\|_{(\sigma,s+r-1)},$$

$$0\leqq t\leqq T,\ \vec{\chi} \in \tilde{\mathscr{H}}_{(\sigma,s+r-1)}(\bar{R}^+_{n+1}).$$

(ii) *For any* $\vec{\chi} \in \tilde{\mathscr{H}}_{(\sigma,s+r-1)}(\bar{R}^+_{n+1})$, $\|(S_\varepsilon^{-1}\vec{A}(t)S_\varepsilon - \vec{A}(t))\vec{\chi}\|_{(\sigma,s)}$ *converges to* 0 *as* $\varepsilon \downarrow 0$.

Let $\vec{A}(t) \in \mathfrak{C}_{(r)}$ with any fixed real r. With the aid of Lemmas 1 and 2 we can show the following

THEOREM 1 (*Approximation theorem*). *Let* $\vec{u} \in \tilde{\mathscr{H}}_{(0,s+r-1)}(\bar{R}^+_{n+1})$ *and assume that*

$$\begin{cases} L\vec{u} \equiv D_t\vec{u} + \vec{A}(t)\vec{u} = \vec{f} \in \tilde{\mathscr{H}}_{(0,s)}(\bar{R}^+_{n+1}), \\ \mathscr{D}'_{L^2}\text{-}\lim\limits_{t\downarrow 0} \vec{u} = \vec{\alpha} \in \mathscr{H}_{(s)}(R_n). \end{cases}$$

Then there exists a sequence $\{\vec{\psi}_j\}$, $\vec{\psi}_j \in C_0^\infty(R_{n+1})$ *such that*

(i) $\vec{\psi}_j \to \vec{u}$ *in* $\tilde{\mathscr{H}}_{(0,s+r-1)}(\bar{R}^+_{n+1})$,

(ii) $L\vec{\psi}_j \to \vec{f}$ *in* $\tilde{\mathscr{H}}_{(0,s)}(\bar{R}^+_{n+1})$,

(iii) $\vec{\psi}_j(0,\cdot) \to \vec{\alpha}$ *in* $\mathscr{H}_{(s)}(R_n)$

as $j\to\infty$.

PROOF. Put $\vec{u}_\varepsilon = S_\varepsilon^{-1}\vec{u}$, $\vec{f}_\varepsilon = S_\varepsilon^{-1}\vec{f}$ and $\vec{\alpha}_\varepsilon = S_\varepsilon^{-1}\alpha$ for $\varepsilon>0$. Then $\vec{u}_\varepsilon \in \tilde{\mathscr{H}}_{(0,s+r)}(\bar{R}^+_{n+1})$, $\vec{f}_\varepsilon \in \tilde{\mathscr{H}}_{(0,s+1)}(\bar{R}^+_{n+1})$, $\vec{\alpha}_\varepsilon \in \mathscr{H}_{(s+1)}(\bar{R}_n)$ and we can write

$$L(\vec{u}_\varepsilon) = \vec{f}_\varepsilon + \vec{\Gamma}_\varepsilon(\vec{u}_\varepsilon),$$

where $\vec{\Gamma}_\varepsilon(\vec{u}_\varepsilon) = \vec{A}(t)\vec{u}_\varepsilon - S_\varepsilon^{-1}\vec{A}(t)S_\varepsilon\vec{u}_\varepsilon \in \tilde{\mathscr{H}}_{(0,s+1)}(R^+_{n+1})$ and $\lim\limits_{t\downarrow 0}\vec{u}_\varepsilon = \vec{\alpha}_\varepsilon$. Furthermore, we see from Lemmas 1 and 2 that

(1) $\vec{u}_\varepsilon \to \vec{u}$ in $\tilde{\mathscr{H}}_{(0,s+r-1)}(\bar{R}^+_{n+1})$,

(2) $\vec{f}_\varepsilon \to \vec{f}$ in $\tilde{\mathscr{H}}_{(0,s)}(\bar{R}^+_{n+1})$,

(3) $\vec{\Gamma}_\varepsilon(\vec{u}_\varepsilon) \to 0$ in $\tilde{\mathscr{H}}_{(0,s)}(\bar{R}^+_{n+1})$,

(4) $\vec{\alpha}_\varepsilon \to \vec{\alpha}$ in $\mathscr{H}_{(s)}(R_n)$

as $\varepsilon \downarrow 0$. We note here $D_t\vec{u}_\varepsilon=\vec{f}_\varepsilon+\vec{\Gamma}_\varepsilon(\vec{u}_\varepsilon)-\vec{A}(t)\vec{u}_\varepsilon \in \tilde{\mathscr{H}}_{(0,s)}(\bar{R}^+_{n+1})$.

For sufficiently small $\varepsilon_0>0$ if we put $\vec{v}^1=\vec{u}_{\varepsilon_0} \in \tilde{\mathscr{H}}_{(0,s+r)}(\bar{R}^+_{n+1})$, $\vec{f}^1=\vec{f}_{\varepsilon_0}+\vec{\Gamma}_{\varepsilon_0}(\vec{u}_{\varepsilon_0}) \in \tilde{\mathscr{H}}_{(0,s+1)}(\bar{R}^+_{n+1})$ and $\vec{\alpha}^1=\vec{\alpha}_{\varepsilon_0} \in \mathscr{H}_{(s+1)}(R_n)$, then $L\vec{v}^1=\vec{f}^1$, $\lim_{t\downarrow 0}\vec{v}^1=\vec{\alpha}^1$ and we have

$$\text{(5)} \qquad \vec{v}^1_\varepsilon \to \vec{v}^1 \quad \text{in} \quad \tilde{\mathscr{H}}_{(0,s+r)}(\bar{R}^+_{n+1}),$$

$$\text{(6)} \qquad \vec{f}^1_\varepsilon+\vec{\Gamma}_\varepsilon(\vec{v}^1_\varepsilon) \to \vec{f}^1 \quad \text{in} \quad \tilde{\mathscr{H}}_{(0,s+1)}(\bar{R}^+_{n+1}),$$

$$\text{(7)} \qquad \vec{\alpha}^1_\varepsilon \to \vec{\alpha}^1 \quad \text{in} \quad \mathscr{H}_{(s+1)}(R_n)$$

as $\varepsilon \downarrow 0$ and moreover $D_t\vec{v}^1_\varepsilon \in \tilde{\mathscr{H}}_{(0,s+1)}(\bar{R}^+_{n+1})$.

Determine $\vec{v}^k$, $k=2, 3, \dots$, successively, by $\vec{v}^k=\vec{v}^{k-1}_{\varepsilon_0}$, $\vec{f}^k=\vec{f}^{k-1}_{\varepsilon_0}+\vec{\Gamma}_{\varepsilon_0}(\vec{v}^{k-1}_{\varepsilon_0})$ and $\vec{\alpha}^k=\vec{\alpha}^{k-1}_{\varepsilon_0}$. Then $\vec{v}^k \in \tilde{\mathscr{H}}_{(0,s+r-1+k)}(\bar{R}^+_{n+1})$, $L\vec{v}^k=\vec{f}^k \in \tilde{\mathscr{H}}_{(0,s+k)}(\bar{R}^+_{n+1})$ and $\lim_{t\downarrow 0}\vec{v}^k=\vec{\alpha}^k \in \mathscr{H}_{(s+k)}(R_n)\subset \mathscr{H}_{(s)}(R_n)$ and we have $D_t\vec{v}^k \in \tilde{\mathscr{H}}_{(0,s+k-1)}(\bar{R}^+_{n+1})$. Thus $\vec{v}^k \in \tilde{\mathscr{H}}_{(1,s+r+k-2)}(\bar{R}^+_{n+1})$ for $r\leqq 1$ and $\vec{v}^k \in \tilde{\mathscr{H}}_{(1,s+k-1)}(\bar{R}^+_{n+1})$ for $r>1$.

Let us take k so that $k>2-r$ (resp. $k\geqq r$) in the case where $r\leqq 1$ (resp. $r>1$). There exists a sequence $\{\vec{\psi}_j\}$, $\vec{\psi}_j \in C_0^\infty(R_{n+1})$, such that $\vec{\psi}_j$ converges in $\tilde{\mathscr{H}}_{(1,s+r+k-2)}(\bar{R}^+_{n+1})$ (resp. in $\tilde{\mathscr{H}}_{(1,s+k-1)}(\bar{R}^+_{n+1})$) to $\vec{v}^k$ for $r\leqq 1$ (resp. for $r>1$). Then $\vec{\psi}_j$, $L\vec{\psi}_j$ and $\vec{\psi}_j(0,\cdot)$ converge in $\tilde{\mathscr{H}}_{(0,s+r-1)}(\bar{R}^+_{n+1})$, $\tilde{\mathscr{H}}_{(0,s)}(\bar{R}^+_{n+1})$ and $\tilde{\mathscr{H}}_{(s)}(R_n)$ to $\vec{v}^k$, $L\vec{v}^k$ and $\vec{v}^k(0,\cdot)$ respectively as $j\to\infty$.

Let $\sigma>-\dfrac{1}{2}$ and suppose $\vec{A}(t) \in \mathfrak{C}^l_{(r)}$ with $l\geqq|\sigma|$. In the same way as in the proof of the theorem we can prove the following

COROLLARY 2. *Let $\vec{u} \in \tilde{\mathscr{H}}_{(\sigma,s+r-1)}(\bar{R}^+_{n+1})$ and assume that*

$$\begin{cases} L\vec{u}=\vec{f} \in \tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1}), \\ \mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}\vec{u}=\vec{\alpha} \in \mathscr{H}_{(\nu)}(R_n) \end{cases}$$

for any real ν. Then there exists a sequence $\{\vec{\psi}_j\}$, $\vec{\psi}_j \in C_0^\infty(R_{n+1})$, such that

(i) $\vec{\psi}_j\to\vec{u}$ *in* $\tilde{\mathscr{H}}_{(\sigma,s+r-1)}(\bar{R}^+_{n+1})$,

(ii) $L\vec{\psi}_j\to\vec{f}$ *in* $\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$,

(iii) $\vec{\psi}_j(0,\cdot)\to\vec{\alpha}$ *in* $\mathscr{H}_{(\nu)}(R_n)$

as $j\to\infty$.

Let σ, s be any real numbers and r a fixed positive real number. According to S. Kaplan [11] we shall use the notation $\mathscr{K}^{(\sigma,s)}$ to denote the space $\mathscr{B}_{2,k}$ [5, p. 36], where $k=k_{\sigma,s}=(\tau^2+\lambda^{2r}(\xi))^{\sigma/2r}\lambda^s(\xi)$, $\lambda(\xi)=(1+|\xi|^2)^{1/2}$. $\mathscr{K}^{(\sigma,s)}(\bar{R}^+_{n+1})$, $\mathring{\mathscr{K}}^{(\sigma,s)}(\bar{R}^+_{n+1})$, $\tilde{\mathscr{K}}^{(\sigma,s)}(\bar{R}^+_{n+1})$ and the like will have obvious meanings. We shall denote the norm in $\mathscr{K}^{(\sigma,s)}$ by $\|\cdot\|_{\sigma,s}$. Then we see from Proposition 5 in [8,

p. 413] that the canonical extension $u_{\sim}$ exists for every $u \in \tilde{\mathscr{K}}^{(\sigma,s)}(\bar{R}^+_{n+1})$ if and only if $\sigma > -\frac{r}{2}$ and from Corollary 1 in [8, p. 412] that $\lim_{t\downarrow 0} u$ exists for every $u \in \tilde{\mathscr{K}}^{(\sigma,s)}(\bar{R}^+_{n+1})$ if and only if $\sigma > \frac{r}{2}$ and $\lim_{t\downarrow 0} u \in \mathscr{H}_{(\sigma+s-r/2)}(R_n)$. In this case the trace map $u \to u(0, \cdot)$ of $\tilde{\mathscr{K}}^{(\sigma,s)}$ into $\mathscr{H}_{(\sigma+s-r/2)}(R_n)$ is an epimorphism (cf. Theorem 1 in [6, p. 21]). It is also to be noticed that $\mathscr{K}^{(\sigma,s)}(\bar{R}^+_{n+1})$ and $\mathring{\mathscr{K}}^{(\sigma,s)}(\bar{R}^+_{n+1})$ may be identified for $|\sigma| < \frac{r}{2}$ (cf. Proposition 7 in [8, p. 416]). In the same way as in the proof of Proposition 14 in [10, p. 385] we can prove that $(A(t) \in \mathfrak{C}^l_{(r)}$, $lr \geqq |\sigma|$, is a continuous linear map of $\tilde{\mathscr{K}}^{(\sigma,s)}(\bar{R}^+_{n+1})$ into $\tilde{\mathscr{K}}^{(\sigma,s-r)}(\bar{R}^+_{n+1})$ for any real numbers σ, s. Similarly we have the following

COROLLARY 1′. *Let $A(t) \in \mathfrak{C}^l_{(r)}$ with $lr \geqq |\sigma|$. Then*

(i) *For any $T>0$ and ε with $0<\varepsilon\leqq 1$, there exists a constant $C_T^{(\sigma,s)}$ such that*

$$\|(S_\varepsilon^{-1}\vec{A}(t)S_\varepsilon - \vec{A}(t))\vec{\chi}\|_{\sigma,s} \leqq C_T^{(\sigma,s)}\|\vec{\chi}\|_{\sigma,s+r-1},$$
$$0 \leqq t \leqq T,\ \vec{\chi} \in \tilde{\mathscr{K}}^{(\sigma,s+r-1)}(\bar{R}^+_{n+1}).$$

(ii) *For any $\vec{\chi} \in \tilde{\mathscr{K}}^{(\sigma,s+r-1)}(\bar{R}^+_{n+1})$, $\|(S_\varepsilon^{-1}\vec{A}(t)S_\varepsilon - \vec{A}(t))\vec{\chi}\|_{\sigma,s}$ converges to 0 as $\varepsilon \downarrow 0$.*

COROLLARY 2′. *Let $\sigma > -\frac{r}{2}$ and $\vec{A}(t) \in \mathfrak{C}^l_{(r)}$ with $lr \geqq |\sigma|$. Let $\vec{u} \in \tilde{\mathscr{K}}^{(\sigma,s+r-1)}(\bar{R}^+_{n+1})$ and assume that*

$$\begin{cases} L\vec{u} = \vec{f} \in \tilde{\mathscr{K}}^{(\sigma,s)}(\bar{R}^+_{n+1}), \\ \mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} \vec{u} = \vec{\alpha} \in \mathscr{H}_{(\nu)}(R_n) \end{cases}$$

for any real ν. Then there exists a sequence $\{\vec{\psi}_j\}$, $\vec{\psi}_j \in C_0^\infty(R_{n+1})$, such that

(i) $\vec{\psi}_j \to \vec{u}$ *in* $\tilde{\mathscr{K}}^{(\sigma,s+r-1)}(\bar{R}^+_{n+1})$,

(ii) $L\vec{\psi}_j \to \vec{f}$ *in* $\tilde{\mathscr{K}}^{(\sigma,s)}(\bar{R}^+_{n+1})$,

(iii) $\vec{\psi}_j(0, \cdot) \to \vec{\alpha}$ *in* $\mathscr{H}_{(\nu)}(R_n)$

as $j \to \infty$.

2. Uniqueness and existence theorems for the Cauchy problem (I)

For the sake of simplicity we assume $A(t) \in \mathfrak{C}^\infty_{(1)}$ in this and next sections. Let H be a slab $[0, T] \times R_n$, $T>0$, and denote by $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$ the set of distributions $\in \mathscr{D}'(\mathring{H})$ which can be extended to distributions $\in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$.

The quotient topology is induced in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$. Similarly for $\mathscr{D}'(H)$ and $\mathscr{D}'((-\infty, T])$.

Consider the Cauchy problem:

$$(8) \qquad \begin{cases} L\vec{u}=\vec{f} & \text{in} \quad \mathring{H}, \\ u_0 \equiv \mathscr{D}'_{L^2}\text{-}\lim\limits_{t\downarrow 0} \vec{u}=\vec{\alpha} \end{cases}$$

for given $\vec{f} \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$ and $\vec{\alpha} \in (\mathscr{D}'_{L^2})_x$. If a solution $\vec{u} \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$ exists, then $\vec{f}$ must have the $\mathscr{D}'_{L^2}$-canonical extension $\vec{f}_\smile$ over $t=0$ and $\vec{u}_\smile$ satisfies the equation

$$L(\vec{u}_\smile)=\vec{f}_\smile - i\delta_t \otimes \vec{\alpha}.$$

Conversely, if $\vec{v} \in \mathscr{D}'((-\infty, T])((\mathscr{D}'_{L^2})_x)$ vanishing for $t<0$ is a solution of $L\vec{v}=\vec{f}_\smile - i\delta_t\otimes\vec{\alpha}$, that is,

$$(9) \qquad ((\vec{v}, L^*\vec{w}))=((\vec{f}_\smile, \vec{w}))-i(\vec{\alpha}, \vec{w}_0), \qquad \vec{w} \in C_0^\infty((-\infty, T)\times R_n),$$

where ((,)) means the scalar product between $\mathscr{D}'((-\infty, T])((\mathscr{D}'_{L^2})_x)$ and $\mathscr{D}((-\infty, T))\hat{\otimes}_\iota(\mathscr{D}_{L^2})_x$, then the restriction $\vec{u}\,|\,\mathring{H} \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$ is a solution of the Cauchy problem (8) (cf. Corollary 3 in [10, p. 393]). The equation (9) implies Green's formula:

$$(((L\vec{u})_\smile, \vec{w}))-((\vec{u}, L^*\vec{w}))=i(\vec{u}_0, \vec{w}_0).$$

Let $\vec{f} \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$, $\vec{\alpha}$, $\vec{\beta} \in (\mathscr{D}'_{L^2})_x$ and assume that $\vec{f}$ has a two-sided $\mathscr{D}'_{L^2}$-canonical extension $\vec{f}^{\frown}_{\smile}$. The problem to find a solution $\vec{u} \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$ of the equation $L\vec{u}=\vec{f}$ in $\mathring{H}$ with the conditions $\vec{u}_0=\vec{\alpha}$, $\vec{u}_T \equiv \mathscr{D}'_{L^2}\text{-}\lim\limits_{t\uparrow T} \vec{u}=\vec{\beta}$ is reduced to the problem of finding $\vec{v} \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ with $\operatorname{supp}\vec{v}\subset H$ such that

$$(10) \qquad ((\vec{v}, L^*\vec{w}))=((\vec{f}^{\frown}_{\smile}, \vec{w}))-i(\vec{\alpha}, \vec{w}_0)+i(\vec{\beta}, \vec{w}_T), \ \vec{w} \in C_0^\infty(R_{n+1}),$$

where ((,)) means the scalar product between $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ and $\mathscr{D}_t\otimes(\mathscr{D}_{L^2})_x$. The equation (10) implies Green's formula:

$$(((L\vec{u})^{\frown}_{\smile}, \vec{w}))-((\vec{u}^{\frown}_{\smile}, L^*\vec{w}))=-i\{(\vec{u}_T, \vec{w}_T)-(\vec{u}_0, \vec{w}_0)\}.$$

In Sections 2 and 3, L will be assumed to admit the inequality:

$$(E^2_{(0)}\uparrow)_T:\ \|\vec{\phi}(t, \cdot)\|^2_{(0)} \leqq C_T(\|\vec{\phi}(0, \cdot)\|^2_{(0)}+\int_0^t \|L\vec{\phi}(t', \cdot)\|^2_{(0)}dt'),$$

$$0\leqq t\leqq T, \qquad \vec{\phi} \in C_0^\infty(R_{n+1}),$$

where C_T is a constant. We shall agree to write $(E^2_{(0)}\uparrow)$ if $(E^2_{(0)}\uparrow)_T$ holds true for every $T>0$.

We shall often need the following lemma (cf. Lemma 4 in [9, p. 78]).

LEMMA 3. *Let $r(t)$ and $\rho(t)$ be two real-valued functions defined in the interval $0 \leqq t \leqq T$ and suppose that r is continuous and ρ is non-decreasing. Then the inequality*

$$r(t) \leqq C(\rho(t) + \int_0^t r(t')dt')(C>0 \text{ is a constant})$$

implies

$$r(t) \leqq Ce^{Ct}\rho(t).$$

Let s be arbitrarily chosen. If we apply the inequality $(E_{(0)}^2 \uparrow)_T$ to $S^s\vec{\phi}$ instead of $\vec{\phi}$, then the pseudo-commutativity $(*)$ and Lemma 3 yield the following inequality:

$$(E_{(s)}^2 \uparrow)_T: \|\vec{\phi}(t, \cdot)\|_{(s)}^2 \leqq C_T^{(s)}(\|\vec{\phi}(0, \cdot)\|_{(s)}^2 + \int_0^t \|L\vec{\phi}(t', \cdot)\|_{(s)}^2 dt'),$$

$$0 \leqq t \leqq T, \qquad \vec{\phi} \in C_0^\infty(R_{n+1}),$$

where $C_T^{(s)}$ is a constant. We can also apply Lemma 3 to conclude that if $(E_{(0)}^2 \uparrow)_T$ holds for L and $\vec{B}(t) \in \mathfrak{C}_{(0)}$, then so does for $L^1 = L + \vec{B}(t)$.

Let us denote by $\mathscr{E}_t^0(\mathscr{H}_{(s)})$ the space of $\mathscr{H}_{(s)}(R_n)$-valued continuous functions of t defined on $[0, \infty)$. Then we have

PROPOSITION 1. *Suppose $(E_{(0)}^2 \uparrow)$ holds for L. If, for a given $\vec{u} \in \tilde{\mathscr{H}}_{(0,s)}(\bar{R}_{n+1}^+)$, $L\vec{u} = \vec{f} \in \tilde{\mathscr{H}}_{(0,s)}(\bar{R}_{n+1}^+)$ and $\lim_{t \downarrow 0} \vec{u} = \vec{\alpha} \in \mathscr{H}_{(s)}(R_n)$ hold, then $\vec{u} \in \mathscr{E}_t^0(\mathscr{H}_{(s)})$ and $\vec{u}$ satisfies the inequality $(E_{(s)}^2 \uparrow)$, that is,*

$$\|\vec{u}(t, \cdot)\|_{(s)}^2 \leqq C_T^{(s)}(\|\vec{\alpha}\|_{(s)}^2 + \int_0^t \|\vec{f}(t', \cdot)\|_{(s)} dt').$$

In particular, if $\vec{f} = 0$ and $\vec{\alpha} = 0$, then $\vec{u} = 0$.

PROOF. In virtue of Theorem 1 there exists a sequence $\{\vec{\phi}_k\}$, $\vec{\phi}_k \in C_0^\infty(R_{n+1})$, with properties mentioned there and we have

$$\|\vec{\phi}_k(t, \cdot) - \vec{\phi}_{k'}(t, \cdot)\|_{(s)}^2 \leqq C_T^{(s)}(\|\vec{\phi}_k(0, \cdot) - \vec{\phi}_{k'}, (0, \cdot)\|_{(s)}^2 +$$

$$+ \int_0^t \|L\vec{\phi}_k(t', \cdot) - L\vec{\phi}_{k'}(t', \cdot)\|_{(s)}^2 dt'),$$

which means that $\{\vec{\phi}_k(t, \cdot)\}$ is a Cauchy sequence in $\mathscr{E}_t^0(\mathscr{H}_{(s)})$. Let $\vec{v}$ be the limit of $\{\vec{\phi}_k\}$. Clearly $\vec{v}$ coincides with $\vec{u}$ as a distribution and $\vec{u}$ satisfies $(E_{(s)}^2 \uparrow)$ and the proposition is proved.

Let $u \in \tilde{\mathscr{H}}_{(0,s)}(\bar{R}_{n+1}^+)$. Then u may be considered as an $\mathscr{H}_{(s)}(R_n)$-valued measurable function $u(t, \cdot)$ defined for almost everywhere $t \in (0, \infty)$ and $\int_0^T \|u(t, \cdot)\|_{(s)}^2 dt < +\infty$ for any $T>0$. Thus almost all points $t_0 \in (0, \infty)$ are Lebesgue points of $u(t, \cdot)$:

$$\lim_{h\downarrow 0}\frac{1}{2h}\int_{t_0-h}^{t_0+h}\|u(t',\cdot)-u(t_0,\cdot)\|_{(s)}dt'=0.$$

Let t_0 be a Lebesgue point of $u(t,\cdot)$. For any $\phi\in C_0^\infty(R_t)$ such that $\phi\geqq 0$, $\int\phi(t)\,dt=1$ and $\operatorname{supp}\phi\subset[-1,1]$, we have for any small $\varepsilon>0$

$$\begin{aligned}&\left\|\frac{1}{\varepsilon}\int\phi\left(\frac{t-t_0}{\varepsilon}\right)u(t,\cdot)\,dt-u(t_0,\cdot)\right\|_{(s)}\\ &=\left\|\frac{1}{\varepsilon}\int\phi\left(\frac{t-t_0}{\varepsilon}\right)(u(t,\cdot)-u(t_0,\cdot))\,dt\right\|_{(s)}\\ &\leqq\frac{\sup\phi}{\varepsilon}\int_{t_0-\varepsilon}^{t_0+\varepsilon}\|u(t,\cdot)-u(t_0,\cdot)\|_{(s)}dt.\end{aligned}$$

Thus we see that $u(t_0,\cdot)$ is the section of u for $t=t_0$.

If $\vec{u}\in\widetilde{\mathscr{H}}_{(0,s)}(\bar{R}_{n+1}^+)$ and $L\vec{u}=\vec{f}\in\widetilde{\mathscr{H}}_{(0,s)}(\bar{R}_{n+1}^+)$, then $\vec{u}$ may be considered as an $\mathscr{H}_{(s)}$-valued continuous function of $t\in(0,\infty)$. In fact, let $t_0>0$ be a sufficiently small Lebesgue point of $\vec{u}(t,\cdot)$. Then $\lim_{t\downarrow t_0}\vec{u}$ exists in $\mathscr{H}_{(s)}(R_n)$ and therefore $\vec{u}$ is an $\mathscr{H}_{(s)}(R_n)$-valued continuous function of $t\in[t_0,\infty)$, where t_0 can be chosen arbitrarily small.

For any σ, s we denote by $\mathscr{H}_{(\sigma,s)}(H)$ the space of all distributions $u\in\mathscr{D}'(\mathring{H})$ such that there exists a distribution $U\in\mathscr{H}_{(\sigma,s)}(R_{n+1})$ with $U=u$ in $\mathring{H}$. The norm of u is defined by $\|u\|_{(\sigma,s)}=\inf\|U\|_{(\sigma,s)}$, the infimum being taken over all such U.

In the following Propositions 2 through 7 we assume that $(E_{(0)}^2\uparrow)_T$ holds for L.

PROPOSITION 2. *If $\vec{u}\in\mathscr{D}_t'((\mathscr{D}_{L^2}')_x)(H)$, $L\vec{u}=0$ in $\mathring{H}$ and $\mathscr{D}_{L^2}'$-$\lim_{t\downarrow 0}\vec{u}=0$, then $\vec{u}=0$ in $\mathring{H}$.*

PROOF. Since $\vec{u}\in\mathscr{D}_t'((\mathscr{D}_{L^2}')_x)(H)$ there exist integers k, l such that $\vec{u}\in\mathscr{H}_{(k,l)}(H)$. Suppose that $k<0$. From the relation $D_t\vec{u}=-\vec{A}(t)\vec{u}\in\mathscr{H}_{(k,l-1)}(H)$ it follows that $\vec{u}\in\mathscr{H}_{(k+1,l-1)}(H)$. Repeating this procedure, we see that $\vec{u}\in\mathscr{H}_{(0,l-k)}(H)$, and applying Proposition 1 we can conclude that $\vec{u}=0$ in $\mathring{H}$.

PROPOSITION 3. *For any given $\vec{g}\in\mathscr{H}_{(0,s)}(H)$ and $\vec{\beta}\in\mathscr{H}_{(s)}(R_n)$, s being a real number, the Cauchy problem:*

$$\left\{\begin{array}{l}L^*\vec{v}=\vec{g}\ \text{in}\ \mathring{H},\\ \lim_{t\downarrow T}\vec{v}=\vec{\beta}\end{array}\right.\tag{11}$$

has a solution $\vec{v}\in\mathscr{H}_{(0,s)}(H)$ such that $\vec{\gamma}=\lim_{t\downarrow 0}\vec{v}$ exists in $\mathscr{H}_{(s)}(R_n)$ and such that the inequality

$$\|\vec{\gamma}\|_{(s)}+\int_0^T \|\vec{v}(t,\cdot)\|_{(s)}dt \leqq C_T(\|\vec{\beta}\|_{(s)}+\int_0^T \|\vec{g}(t,\cdot)\|_{(s)}dt)$$

holds, where C_T is a constant.

PROOF. Consider the space $\boldsymbol{H}_{(-s)}=\mathscr{H}_{(-s)}(R_n)\times\mathscr{H}_{(0,-s)}(H)$ and its subspace $A=\{(\vec{u}(0,\cdot), L\vec{u}) : \vec{u}\in C_0^\infty(H)\}$. Then the map

$$l: A\ni(\vec{u}(0,\cdot), L\vec{u})\to\int_0^T(\vec{u}(t,\cdot), \vec{g}(t,\cdot))dt-i(\vec{u}_T, \vec{\beta})$$

is continuous. In fact, from the energy inequality $(E^2_{(-s)}\uparrow)_T$ for L we have

$$\left|\int_0^T(\vec{u}(t,\cdot), \vec{g}(t,\cdot))dt-i(\vec{u}_T, \vec{\beta})\right|$$

$$\leqq \max_{0\leqq t\leqq T}\|\vec{u}(t,\cdot)\|_{(-s)}\int_0^T\|\vec{g}(t,\cdot)\|_{(s)}dt+\|\vec{u}_T\|_{(-s)}\|\beta\|_{(s)}$$

$$\leqq\sqrt{C_T^{(-s)}}(\|\vec{u}(0,\cdot)\|^2_{(-s)}+\int_0^T\|L\vec{u}(t,\cdot)\|^2_{(-s)}dt)^{1/2}(\|\vec{\beta}\|_{(s)}+\int_0^T\|\vec{g}(t,\cdot)\|_{(s)}dt),$$

which implies the inequality

$$\|l\|\leqq\sqrt{C_T^{(-s)}}(\|\vec{\beta}\|_{(s)}+\int_0^T\|\vec{g}(t,\cdot)\|_{(s)}dt).$$

Thus there exists $(i\vec{\gamma}, \vec{v})\in\boldsymbol{H}_{(s)}$ such that

$$(12)\quad \int_0^T(L\vec{u}(t,\cdot), \vec{v}(t,\cdot))dt-i(\vec{u}_0, \vec{\gamma})=\int_0^T(\vec{u}(t,\cdot), \vec{g}(t,\cdot))dt-i(\vec{u}_T, \vec{\beta})$$

and

$$(\|\vec{\gamma}\|^2_{(s)}+\int_0^T\|\vec{v}(t,\cdot)\|^2_{(s)}dt)^{1/2}\leqq\sqrt{C_T^{(-s)}}(\|\vec{\beta}\|_{(s)}+\int_0^T\|g(t,\cdot)\|_{(s)}dt).$$

From Green's formula (12) we see that $\|\vec{v}\|$ is a solution of the Cauchy problem (11), which completes the proof.

We shall say that $(\mathrm{CP})_{(s)}$ holds for L if the Cauchy problem:

$$(13)\quad \begin{cases} L\vec{u}=\vec{f} & \text{in } \mathring{H}, \\ \lim_{t\downarrow 0}\vec{u}=\vec{\alpha} \end{cases}$$

has a solution $\vec{u}\in\mathscr{H}_{(0,s)}(H)$ for any given $\vec{f}\in\mathscr{H}_{(0,s)}(H)$ and $\vec{\alpha}\in\mathscr{H}_{(s)}(R_n)$. Then we have the following

PROPOSITION 4. *If $(\mathrm{CP})_{(s)}$ holds for L, then so does it for $L^1=L+\vec{B}(t)$, $\vec{B}(t)\in\mathfrak{C}_{(0)}$.*

PROOF. Let $(\mathrm{CP})_{(s)}$ hold for L and consider the Cauchy problem:

$$\begin{cases} L^1\vec{u}=\vec{h} & \text{in } \mathring{H}, \\ \lim_{t\downarrow 0} \vec{u}=\vec{\gamma} \end{cases}$$

for any given $\vec{h}\in\mathscr{H}_{(0,s)}(H)$ and $\vec{\gamma}\in\mathscr{H}_{(s)}(R_n)$.

Let $\vec{v}^0\in\mathscr{H}_{(0,s)}(H)$ be chosen so that

$$\begin{cases} L\vec{v}^0=\vec{h} & \text{in } \mathring{H}, \\ \lim_{t\downarrow 0} \vec{v}^0=\vec{\gamma}. \end{cases}$$

If there exists a $\vec{w}\in\mathscr{H}_{(0,s)}(H)$ such that

$$\begin{cases} L\vec{w}=-\vec{B}(t)\vec{w}-\vec{B}(t)\vec{v}^0, \\ \lim_{t\downarrow 0} \vec{w}=0, \end{cases}$$

then $\vec{u}=\vec{v}^0+\vec{w}$ will be the solution to be found. The method of successive approximation will be successful to this end.

Put $\vec{w}^0=0$ and determine $\vec{w}^l\in\mathscr{H}_{(0,s)}(H)$ successively by

$$\begin{cases} L\vec{w}^{l+1}=-\vec{B}(t)\vec{w}^l-\vec{B}(t)\vec{v}^0, \\ \lim_{t\downarrow 0} \vec{w}^{l+1}=0. \end{cases}$$

Then $L(\vec{w}^{l+1}-\vec{w}^l)=-\vec{B}(t)(\vec{w}^l-\vec{w}^{l-1})$ and we have from $(E^2_{(s)}\uparrow)_T$ for L

$$\begin{aligned} \|(\vec{w}^{l+1}-\vec{w}^l)(t,\cdot)\|^2_{(s)} &\leqq C_{s,T}\int_0^t \|(\vec{w}^l-\vec{w}^{l-1})(t',\cdot)\|^2_{(s)}\,dt' \\ &\leqq C^l_{s,T}\int_0^t \frac{(t-t')^{l-1}}{(l-1)!}\|\vec{w}^1(t,\cdot)\|^2_{(s)}\,dt \\ &\leqq \frac{(C_{s,T}T)^l}{l!}\sup_{0\leqq t'\leqq T}\|\vec{w}^1(t',\cdot)\|^2_{(s)},\ 0\leqq t\leqq T, \end{aligned}$$

and therefore $\|(\vec{w}^{l+l'}-\vec{w}^l)(t,\cdot)\|_{(s)}\leqq C'_{s,T}$, where $C'_{s,T}$ is a constant independent of l, l'. Thus $\{\vec{w}^l\}$ is a Cauchy sequence in $\mathscr{E}^0_t(\mathscr{H}_{(s)})$, $t\in[0,T)$. If we put $\vec{w}=\lim_{l\to\infty}\vec{w}^l$, then $\vec{w}$ will be the solution as desired.

PROPOSITION 5. *If* $(\mathrm{CP})_{(s)}$ *holds for some* s, *then it does also for any* s.

PROOF. Let $(\mathrm{CP})_{(s)}$ hold for L. This means that the set $A=\{(\vec{\phi}(0,\cdot), L\vec{\phi}):\vec{\phi}\in C_0^\infty(H)\}$ is dense in $\boldsymbol{H}_{(s)}=\mathscr{H}_{(s)}\times\mathscr{H}_{(0,s)}(H)$. Let s' be any real number. Then the map $[S^{s'-s}]$: $(\vec{\alpha},\vec{f})\to(S^{s'-s}\vec{\alpha}, S^{s'-s}\vec{f})$ is an isomorphism of $\boldsymbol{H}_{(s)}$ onto

$\boldsymbol{H}_{(s')}$ and $[S^{s'-s}](A)$ is also dense in $\boldsymbol{H}_{(s')}$. If we put $\vec{\psi}=S^{s'-s}\vec{\phi}$, $\vec{\phi}\in C_0^\infty(H)$, then we have

$$(S^{s'-s}(\vec{\phi}(0,\cdot)),\ S^{s'-s}L\vec{\phi})=(\vec{\psi}(0,\cdot),\ S^{s'-s}LS^{s-s'}\vec{\psi})$$
$$=(\vec{\psi}(0,\cdot),\ L\vec{\psi}+\vec{B}(t)\vec{\psi}),$$

where $\vec{B}(t)=S^{s'-s}LS^{s-s'}-L=S^{s'-s}\vec{A}(t)S^{s-s'}-\vec{A}(t)$ is of order $\leqq 0$. Thus $(\mathrm{CP})_{(s')}$ holds for $L^1=L+\vec{B}(t)$ and therefore so does it for L.

PROPOSITION 6. *If for any $\vec{f}\in\mathscr{H}_{(0,s)}(H)$ and $\vec{\alpha}\in\mathscr{H}_{(s)}(R_n)$ the Cauchy problem* (8) *has a solution $\vec{u}\in\mathscr{D}_t'((\mathscr{D}_{L^2}')_x)(H)$, then $\vec{u}\in\mathscr{H}_{(0,s)}(H)$*

PROOF. There exist integers k, l such that $\vec{u}\in\mathscr{H}_{(k,l)}(H)$. Suppose that $k<0$. Then from the equation $D_t\vec{u}=\vec{f}-\vec{A}(t)\vec{u}$ we see that $D_t\vec{u}\in\mathscr{H}_{(k,s_1)}(H)$, $s_1=\min(s-k,\ l-1)$ and therefore $\vec{u}\in\mathscr{H}_{(k+1,s_2)}(H)$, $s_2=\min(l-1,\ s_1)$. Repeating this procedure, we can find s' such that $\vec{u}\in\mathscr{H}_{(0,s')}(H)$. For any $(\vec{\alpha},\vec{f})\in\boldsymbol{H}_{(s)}=\mathscr{H}_{(s)}(R_n)\times\mathscr{H}_{(0,s)}(H)$ a solution of the Cauchy problem (13) belongs to the space $\bigcup_{m=-\infty}^{\infty}\mathscr{H}_{(0,m)}(H)$. Since a solution is unique in $\mathscr{D}_t'((\mathscr{D}_{L^2}')_x)(H)$ for $\vec{f}$ and $\vec{\alpha}$, we see from the closed graph theorem that $(\vec{\alpha},\vec{f})\to\vec{u}$ is a continuous map of $\boldsymbol{H}_{(s)}$ into $\mathscr{D}_t'((\mathscr{D}_{L^2}')_x)(H)$. The space $\mathscr{H}_{(s)}(R_n)$ and $\mathscr{H}_{(0,s)}(H)$ are both of type $(\boldsymbol{F})$. Thus by Theorem A in A. Grothendieck [4, p. 16] there exists a fixed m such that the corresponding solution $\vec{u}$ belonging to the space $\mathscr{H}_{(0,m)}(H)$ for every $(\vec{\alpha},\vec{f})\in\boldsymbol{H}_{(s)}$.

Suppose that $m<s$. For any $\vec{g}\in\mathscr{H}_{(0,m)}(H)$ and $\vec{\beta}\in\mathscr{H}_{(m)}(R_n)$ there exist sequences $\{\vec{g}_j\}$, $\{\vec{\beta}_j\}$, $\vec{g}_j\in C_0^\infty(H)$, $\vec{\beta}_j\in C_0^\infty(R_n)$ such that $\vec{g}_j$, $\vec{\beta}_j$ converge to $\vec{g}$, $\vec{\beta}$ in $\mathscr{H}_{(0,m)}(H)$, $\mathscr{H}_{(m)}(R_n)$ as $j=\infty$ respectively. Denote by $\vec{v}_j$ a unique solution $\in\mathscr{H}_{(0,m)}(H)$ of the Cauchy problem (13) associated with $\vec{g}_j$ and $\vec{\beta}_j$. Owing to $(E_{(m)}^2\uparrow)_T$, $\{\vec{v}_j\}$ is a Cauchy sequence in $\mathscr{E}_t^0(\mathscr{H}_{(m)})$, $t\in[0,\ T)$, and therefore $\vec{v}_j$ has the limit $\vec{v}\in\mathscr{H}_{(0,m)}(H)$ and $\vec{v}$ is a solution of the Cauchy problem (13) associated with $\vec{g}$ and $\vec{\beta}$. In virtue of Proposition 5, it follows that $\vec{u}\in\mathscr{H}_{(0,-s)}(H)$, which was to be proved.

PROPOSITION 7. $(\mathrm{CP})_{(s)}$ *holds for L if and only if the conditions that $\vec{w}\in\mathscr{H}_{(0,s)}(H)$, $L^*\vec{w}=0$ in $\mathring{H}$ and $\lim_{t\uparrow T}\vec{w}=0$ imply $\vec{w}=0$ in $\mathring{H}$.*

PROOF. Let $(\mathrm{CP})_{(s)}$ hold for L and $\vec{w}\in\mathscr{H}_{(0,-s)}(H)$ and assume that $L^*\vec{w}=0$ in $\mathring{H}$ with $\lim_{t\uparrow T}\vec{w}=0$. For any $\vec{f}\in C_0^\infty(\mathring{H})$ let $\vec{u}\in\mathscr{H}_{(0,s)}(H)$ be a solution of $L\vec{u}=\vec{f}$. Since $(E_{(s)}^2\uparrow)_T$ holds for L, there exists a sequence $\{\vec{\phi}_j\}$, $\vec{\phi}_j\in C_0^\infty(H)$, vanishing near $t=0$ and we have $\int_0^T(L\vec{\phi}_j(t,\cdot),\ \vec{w}(t,\cdot))\,dt=0$. Thus $\vec{w}=0$ in $\mathring{H}$.

To prove the converse, it suffices to show that $A=\{(\vec{\phi}(0,\cdot),\ L\vec{\phi}):\vec{\phi}\in C_0^\infty(H)\}$ is dense in $\boldsymbol{H}_{(s)}=\mathscr{H}_{(s)}(R_n)\times\mathscr{H}_{(0,s)}(H)$. Let $(i\vec{\beta},\ \vec{w})\in\boldsymbol{H}_{(-s)}$ such that

$$\int_0^T (L\vec{\phi}(t,\cdot), \vec{w}(t,\cdot))dt - i(\vec{\phi}_0, \vec{\beta}) = 0, \qquad \vec{\phi} \in C_0^\infty(H),$$

which implies $L^*\vec{w}=0$ in $\mathring{H}$ and $\lim_{t\uparrow T}\vec{w}=0$. Thus we see that $\vec{w}=0$ in $\mathring{H}$ and $\vec{\beta}=0$, completing the proof.

PROPOSITION 8. *Let* $(\mathrm{CP})_{(s)}$ *hold for* L. *Then the energy inequality* $(E_{(0)}^2\uparrow)_T$ *implies the following*:

$$(E_{(0)}^1\uparrow)_T:\ \|\vec{\phi}(t,\cdot)\|_{(0)} \leqq C_T'(\|\vec{\phi}(0,\cdot)\|_{(0)} + \int_0^t \|L\vec{\phi}(t',\cdot)\|_{(0)}dt')$$

$$0\leqq t\leqq T, \qquad \vec{\phi}\in C_0^\infty(R_{n+1}).$$

PROOF. From the fact that $(\mathrm{CP})_{(s)}$ holds for L in any slab $H_1=[0, T_1]\times R_n$, $0<T_1\leqq T$, we see by the preceding proposition that the conditions $\vec{w}\in\mathscr{H}_{(0,-s)}(H_1)$, $L^*\vec{w}=0$ in $\mathring{H}_1$ and $\lim_{t\uparrow T_1}\vec{w}=0$ imply $\vec{w}=0.$ in $\mathring{H}_1$, and therefore by Proposition 3 we can conclude that $(E_{(s)}^1\uparrow)_T$ holds for L. In virtue of the pseudo-commutativity (*) and Lemma 3 we see that $(E_{(s)}^1\uparrow)_T$ implies $(E_{(s')}^1\uparrow)_T$ for any s', completing the proof.

We shall say that $(\mathrm{CP})_{(s)}^{\sim}$ holds for L if the Cauchy problem:

$$(14)\qquad \begin{cases} L\vec{u}=\vec{f} & \text{in } R_{n+1}^+,\\ \lim_{t\downarrow 0}\vec{u}=\vec{\alpha} \end{cases}$$

has a solution $\vec{u}\in\mathscr{H}_{(0,s)}(\bar{R}_{n+1}^+)$ for any given $\vec{g}\in\mathscr{H}_{(0,s)}(\bar{R}_{n+1}^+)$ and $\vec{\alpha}\in\mathscr{H}_{(s)}(R_n)$.

Consider the Cauchy problem:

$$(15)\qquad \begin{cases} L\vec{u}=\vec{f} & \text{in } R_{n+1}^+,\\ \mathscr{D}_{L^2}'\text{-}\lim_{t\downarrow 0}\vec{u}=\vec{\alpha} \end{cases}$$

for given $\vec{\alpha}\in(\mathscr{D}_{L^2}')_x$ and $\vec{f}\in\mathscr{D}'(R_t^+)((\mathscr{D}_{L^2}')_x)$, which has the $\mathscr{D}_{L^2}'$-canonical extension $\vec{f}_{\sim}$. For the Cauchy problem (15) we can prove with necessary modifications the analogues of Propositions 2 through 8, which were obtained for the slab H.

THEOREM 2. *Suppose* $(E_{(0)}^2\uparrow)$ *holds for* L. *Then*

(1) *A solution of the Cauchy problem* (15) *is unique in* $\mathscr{D}'(R_t^+)((\mathscr{D}_{L^2}')_x)$.

(2) *For any given* $\vec{g}\in\tilde{\mathscr{H}}_{(0,s)}^*(\bar{R}_{n+1}^+)$, *the equation* $L^*\vec{w}=\vec{g}$ *in* R_{n+1}^+ *has a solution* $\vec{w}\in\tilde{\mathscr{H}}_{(0,s)}^*(\bar{R}_{n+1}^+)$ *such that* $\vec{\gamma}=\lim_{t\downarrow 0}\vec{w}$ *exists in* $\mathscr{H}_{(s)}(R_n)$ *and*

$$\|\vec{\gamma}\|_{(s)} + \int_0^\infty \|\vec{w}(t,\cdot)\|_{(s)}dt \leqq C_s\int_0^\infty \|\vec{g}(t,\cdot)\|_{(s)}dt$$

with a constant C_s.

(3) *The following conditions are equivalent*:

(i) $(\mathrm{CP})_{(s)}^{\sim}$ *holds for some real* s.

(ii) $(\mathrm{CP})_{(s)}^{\sim}$ *holds for every real* s.

(iii) $(\mathrm{CP})_{(s)}^{\sim}$ *holds for* $L^1=L+\vec{B}(t)$ *with* $\vec{B}(t)\in\mathfrak{C}_{(0)}$.

(iv) *If* $\vec{w}\in\tilde{\mathscr{H}}^*_{(0,s)}(\bar{R}^+_{n+1})$ *and* $L\vec{w}=0$ *in* R^+_{n+1}, *then* $\vec{w}=0$.

If each of these conditions is satisfied, then the energy inequality $(E^1_{(s)}\uparrow)$ *holds true for any* s.

Let k be a non-negative integer and s a real number. Along the same line as in the proofs of Proposition 5 and Corollary 3 in [9, p. 89, p. 90] we can obtain

Proposition 9. *Suppose* $(E^2_{(0)}\uparrow)$ *and* $(\mathrm{CP})_{(0)}^{\sim}$ *hold for* L. *Then for any* $\vec{f}\in\tilde{\mathscr{H}}_{(k,s)}(\bar{R}^+_{n+1})$ *and* $\vec{\alpha}\in\mathscr{H}_{(k+s)}(R_n)$ *the Cauchy problem* (14) *has a unique solution* $\vec{u}\in\mathscr{H}_{(k+1,s-1)}(\bar{R}^+_{n+1})$ *and* $\vec{u}$ *has the following properties*:

(i) $(\vec{u},\cdots,D_t^k\vec{u})\in\mathscr{E}_t^0(\mathscr{H}_{(k+s)})\times\cdots\times\mathscr{E}_t^0(\mathscr{H}_{(s)})$,

(ii) $$\sum_{j=0}^{k}\|D_t^j\vec{u}(t,\cdot)\|^2_{(k+s-j)}\leqq C_T(\|\vec{\alpha}\|^2_{(k+s)}+\sum_{j=0}^{k-1}\|D_t^j\vec{f}(0,\cdot)\|^2_{(k+s-1-j)}+$$
$$+\sum_{j=0}^{k}\int_0^t\|D_t^j\vec{f}(t',\cdot)\|^2_{(k+s-j)}dt'),\qquad 0\leqq t\leqq T$$

for any $T>0$.

Applying the interpolation theorem for the Hilbert scales and proceeding along the same lines as in the proof of Corollary 4 in [9, p. 96] we can obtain

Proposition 10. *Suppose* $(E^2_{(0)}\uparrow)$ *and* $(\mathrm{CP})_{(0)}^{\sim}$ *hold for* L. *Then for any* $\vec{f}\in\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$ *and* $\vec{\alpha}\in\mathscr{H}_{(\sigma+s)}(R_n)$, σ *being a non-negative number, the Cauchy problem* (14) *has a unique solution* $\vec{u}\in\tilde{\mathscr{H}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$ *and* $(\vec{\alpha},\vec{f})\to\vec{u}$ *is a continuous map of* $\mathscr{H}_{(\sigma+s)}(R_n)\times\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$ *into* $\tilde{\mathscr{H}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$.

Next we show the following

Theorem 3. *Suppose* $(E^2_{(0)}\uparrow)$ *and* $(\mathrm{CP})_{(0)}^{\sim}$ *hold for* L. *Let* $\sigma=k+\sigma'$ *with non-negative integer* k *and* $-\dfrac{1}{2}<\sigma'\leqq\dfrac{1}{2}$. *Then for any* $\vec{f}\in\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$ *and* $\vec{\alpha}\in\mathscr{H}_{(\sigma+s)}(R_n)$ *the Cauchy problem* (15) *has a unique solution* $\vec{u}\in\tilde{\mathscr{H}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$ *and* $\vec{u}$ *has the following properties*:

(i) $(\vec{u},\cdots,D_t^k\vec{u})\in\mathscr{E}_t^0(\mathscr{H}_{(\sigma+s)})\times\cdots\times\mathscr{E}_t^0(\mathscr{H}_{(\sigma'+s)})$,

(ii) $(\vec{\alpha},\vec{f})\to\vec{u}$ *is a continuous map of* $\mathscr{H}_{(\sigma+s)}(R_n)\times\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$ *into* $\tilde{\mathscr{H}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$.

PROOF. As shown in Theorem 2 a solution of the Cauchy problem (15) is unique in $\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$. We shall first consider the case $\sigma\geqq 0$. Owing to Proposition 9 and Corollary 3, there exists a solution $\vec{u}\in\tilde{\mathscr{H}}_{(\sigma+1,s-1)}(\bar{R}_{n+1}^+)$ and $\vec{u}$ has the property (ii). We have only to show that $(\vec{u},\cdots, D_t^k\vec{u})\in\mathscr{E}_t^0(\mathscr{H}_{(\sigma+s)})\times\cdots\times\mathscr{E}_t^0(\mathscr{H}_{(s)})$. Clearly $\vec{u}\in\mathscr{E}_t^0(\mathscr{H}_{(\sigma+s)})$ and $\vec{f}\in\mathscr{E}_t^0(\mathscr{H}_{(\sigma+s-1/2)})$ for $\sigma>\frac{1}{2}$ and therefore $D_t\vec{u}=\vec{f}-\vec{A}(t)\vec{u}\in\mathscr{E}_t^0(\mathscr{H}_{(\sigma+s-1)})$. Repeating this process, we see that (i) holds true.

Next, consider the case $-\frac{1}{2}<\sigma<0$. The canonical extension $\vec{f}_\sim$ belongs to the space $\mathring{\tilde{\mathscr{H}}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$. If we put

$$\vec{g}=\frac{\vec{f}_\sim}{D_t-i\lambda(D_x)},\quad \text{i.e.}\quad D_t\vec{g}-i\lambda(D_x)\vec{g}=\vec{f}_\sim,$$

where $\lambda(\xi)=(1+|\xi|^2)^{1/2}$ and $\lambda(D_x)\in\mathfrak{C}_{(1)}^\infty$, then $\vec{g}\in\mathring{\tilde{\mathscr{H}}}_{(\sigma+1,s)}(\bar{R}_{n+1}^+)$, $\frac{1}{2}<\sigma+1<1$. From Corollary 3 in [8, p. 419] we see that $\lim\limits_{t\downarrow 0} g$ exists and equals 0. The Cauchy problem (15) can be written in the form

$$\begin{cases} D_t(\vec{u}-\vec{g})+\vec{A}(t)(\vec{u}-\vec{g})=-i\lambda(D_x)\vec{g}-\vec{A}(t)\vec{g} & \text{in } R_{n+1}^+,\\ \mathscr{D}'_{L^2}\text{-}\lim\limits_{t\downarrow 0}(\vec{u}-\vec{g})=\vec{\alpha}, \end{cases}$$

where $-i\lambda(D_x)\vec{g}-\vec{A}(t)\vec{g}\in\mathring{\tilde{\mathscr{H}}}_{(\sigma+1,s-1)}(\bar{R}_{n+1}^+)$, $\sigma+1>\frac{1}{2}$. Thus there exists a unique solution $\vec{v}=\vec{u}-\vec{g}\in\tilde{\mathscr{H}}_{(\sigma+2,s-2)}(\bar{R}_{n+1}^+)\cap\mathscr{E}_t^0(\mathscr{H}_{(\sigma+s)})$ and therefore $\vec{u}=\vec{v}+\vec{g}\in\tilde{\mathscr{H}}_{(\sigma+1,s-1)}(\bar{R}_{n+1}^+)\cap\mathscr{E}_t^0(\mathscr{H}_{(\sigma+s)})$. In view of the closed graph theorem it follows that $(\vec{\alpha},\vec{f})\rightarrow\vec{u}$ is a continuous map of $\mathscr{H}_{(\sigma+s)}(R_n)\times\tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}_{n+1}^+)$ into $\tilde{\mathscr{H}}_{(\sigma+1,s-1)}(\bar{R}_{n+1}^+)$. This completes the proof of the theorem.

We shall close this section with some remarks on energy inequalities.

PROPOSITION 11. *If the following inequality for L:*

$$(\tilde{E}_{(0)}^2\uparrow)_T: \|\vec{\phi}(t_1,\cdot)\|_{(0)}^2\leqq C_T(\|\vec{\phi}(t_0,\cdot)\|_{(0)}^2+\int_{t_0}^{t_1}\|L\vec{\phi}(t,\cdot)\|_{(0)}^2dt),\ \vec{\phi}\in C_0^\infty(R_{n+1})$$

holds for any t_0, t_1, $0\leqq t_0\leqq t_1\leqq T$ with a constant C_T, then the condition that $\vec{w}\in\mathscr{H}_{(0,s)}(H)$, $L^*\vec{w}=0$ in $\mathring{H}$ and $\lim\limits_{t\uparrow T}\vec{w}=0$ imply $\vec{w}=0$ in $\mathring{H}$ is equivalent to saying that the inequality for L^*:

$$(E_{(0)}^1\downarrow)_T: \|\vec{\phi}(t_0,\cdot)\|_{(0)}\leqq C_t'(\|\vec{\phi}(t_1,\cdot)\|_{(0)}+\int_{t_0}^{t_1}\|L^*\vec{\phi}(t,\cdot)\|_{(0)}dt),\ \vec{\phi}\in C_0^\infty(R_{n+1})$$

holds for any t_0, t_1, $0\leqq t_0\leqq t_1\leqq T$, where C_T' is a constant.

If this is the case, then $(\tilde{E}_{(0)}^1\uparrow)$ holds true for L.

PROOF. We may take $s=0$. Suppose a solution of the Cauchy problem for L^* is unique in $\mathscr{H}_{(0,0)}(H)$. Then it is unique in $\mathscr{H}_{(0,0)}(H_1)$, $H_1=[0, t_1]\times R_n$. We shall first show that it is also unique in $\mathscr{H}_{(0,0)}(H')$, $H'=[t_0, t_1]\times R_n$. Let $\vec{w}\in\mathscr{H}_{(0,0)}(H')$, $L^*\vec{w}=0$ in $\mathring{H}'$ and $\lim_{t\uparrow t_1}\vec{w}=0$. Let t_0' be a Lebesgue point of the $\mathscr{H}_{(0)}(R_n)$-valued function $\vec{w}(t, \cdot)$ defined on (t_0, t_1). Then $\vec{w}$ has the section $\vec{w}(t_0', \cdot)=\vec{\beta}\in\mathscr{H}_{(0)}(R_n)$ for $t=t_0'$. The Cauchy problem $L^*\vec{w}_1=0$ in $(0, t_0')\times R_n$ with initial condition $\lim_{t\uparrow t_0'}\vec{w}_1=\vec{\beta}$ has a unique solution $\vec{w}_1\in\mathscr{H}_{(0,0)}([0, t_0']\times R_n)$. If we put $\vec{W}=\vec{w}$ in $[t_0', t_1)\times R_n$ and $\vec{W}=\vec{w}_1$ in $(0, t_0']\times R_n$, then $L^*\vec{W}=0$ in $(0, t_1)\times R_n$ and $\lim_{t\uparrow t_1}\vec{W}=0$. Our assumption implies $\vec{W}=0$ and therefore $\vec{w}=0$.

Thus, replacing 0, T by t_0, t_1 in the proof of Proposition 3, and repeating the same procedure as given there, we see that for given $\vec{g}\in\mathscr{H}_{(0,0)}(H')$ and $\vec{\beta}\in\mathscr{H}_{(0)}(R_n)$ the Cauchy problem $L^*\vec{v}=\vec{g}$ in $\mathring{H}'$ with initial condition $\lim_{t\uparrow t_1}\vec{v}=\vec{\beta}$ has a unique solution $\vec{v}\in\mathscr{H}_{(0,0)}(H')$ and $\vec{v}$ satisfies the following:

$$\|\vec{\gamma}\|_{(0)}+\int_{t_0}^{t_1}\|\vec{v}(t, \cdot)\|_{(0)}dt\leqq C_T'(\|\vec{\beta}\|_{(0)}+\int_{t_0}^{t_1}\|\vec{g}(t, \cdot)\|_{(0)}dt),$$

where $\vec{\gamma}=\lim_{t\downarrow t_0}\vec{v}$ and C_T' is a constant. As a result, we can conclude that $(E_{(0)}^1\downarrow)_T$ holds true for any $\vec{\phi}\in C_0^\infty(R_{n+1})$.

The converse is trivial, since the approximation theorem holds for L^*.

PROPOSITION 12. *Suppose $(E_{(0)}^2\uparrow)_T$ holds for L and L^*. Then*

(i) *$(\tilde{E}_{(0)}^1\uparrow)_T$ holds for L and L^*.*

(ii) *$(\mathrm{CP})_{(0)}$ holds for L if and only if $(\tilde{E}_{(0)}^1\downarrow)_T$ holds for L^*.*

PROOF. (i) Let t_0, t_1 be any two points such that $0\leqq t_0\leqq t_1\leqq T$. Then Proposition 3 implies that for any given $\vec{\beta}\in\mathscr{H}_{(0)}(R_n)$ the Cauchy problem

$$\begin{cases} L^*\vec{v}=0 & \text{in } \mathring{H}_1, \\ \lim_{t\uparrow t_1}\vec{v}=\vec{\beta}, \end{cases}$$

where $H_1=[0, t_1]\times R_n$, has a solution $\vec{v}\in\mathscr{H}_{(0,0)}(H_1)$ such that $\|\vec{v}(0, \cdot)\|_{(0)}\leqq C_1\|\vec{\beta}\|_{(0)}$ with a constant C_1 independent of t_1. From the fact that $(E_{(0)}^2\uparrow)_T$ holds for L^* it follows that

$$\|\vec{v}(t, \cdot)\|_{(0)}\leqq C\|\vec{v}(0, \cdot)\|_{(0)}\leqq C_2\|\vec{\beta}\|_{(0)}, \qquad 0\leqq t\leqq t_1.$$

From Green's formula

$$\int_{t_0}^{t_1}(L\vec{u}(t', \cdot), \vec{v}(t', \cdot))dt'=-i\{(\vec{u}(t_1, \cdot), \vec{\beta})-(\vec{u}(t_0, \cdot), \vec{v}(t_0, \cdot))\}$$

for any $\vec{u} \in C_0^\infty(R_{n+1})$, we have

$$|(\vec{u}(t_1, \cdot), \vec{\beta})| \leqq C_3\{\|\vec{u}(t_0, \cdot)\|_{(0)}\|\vec{v}(t_0, \cdot)\|_{(0)}+ + \int_{t_0}^{t_1}\|L\vec{u}(t', \cdot)\|_{(0)}\|\vec{v}(t', \cdot)\|_{(0)}dt'\}$$

$$\leqq C_4\|\vec{\beta}\|_{(0)}\{\|\vec{u}(t_0, \cdot)\|_{(0)}+\int_{t_0}^{t_1}\|L\vec{u}(t', \cdot)\|_{(0)}dt'\},$$

where C_2, C_3 and C_4 are constants independent of t_0 and t_1. This implies that we have with a constant C_T

$$\|\vec{u}(t_1, \cdot)\|_{(0)} \leqq C_T(\|\vec{u}(t_0 \cdot,)\|_{(0)}+\int_{t_0}^{t_1}\|L\vec{u}(t', \cdot)\|_{(0)}dt').$$

Combining (i) with Proposition 11 leads to (ii), which completes the proof.

3. Uniqueness and existence theorems for the Cauchy problem (II)

Let σ, s be any real numbers and write $\sigma=k+\sigma'$ with integer k and $-\frac{1}{2}<\sigma'\leqq\frac{1}{2}$. Then we have the following

PROPOSITION 13. *Suppose $(E^2_{(0)}\uparrow)$ and $(\mathrm{CP})^{\sim}_{(0)}$ hold for L. Then*

(i) *For any $\vec{\alpha} \in \mathscr{H}_{(\sigma+s)}(R_n)$ and $\vec{f} \in \tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$, where $\vec{f}$ is assumed to have the $\mathscr{D}'_{L^2}$-canonical extension $\vec{f}_\sim \in \mathring{\tilde{\mathscr{H}}}_{(\sigma,s)}(\bar{R}^+_{n+1})$, the Cauchy problem (15) has a unique solution $\vec{u} \in \tilde{\mathscr{H}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$.*

(ii) *Let $\vec{u} \in \mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$ and assume that $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}\vec{u}$ exists, $L\vec{u}=\vec{f} \in \mathscr{H}_{(\sigma,s)}(\bar{R}^+_{n+1})$ and the $\mathscr{D}'_{L^2}$-canonical extension $\vec{f}_\sim$ exists in $\mathring{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$ for some real σ, s. Then $\vec{u} \in \tilde{\mathscr{H}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$. In particular, if $\vec{\alpha}=0$ then $\vec{u} \in \mathring{\tilde{\mathscr{H}}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$.*

PROOF. Consider the case $k\geqq 0$. In Theorem 2 we have shown that there exists a solution $\vec{u} \in \tilde{\mathscr{H}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$ for the Cauchy problem (I5). Since a solution of the Cauchy problem (15) is unique in $\mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$ we have only to show that if $\alpha=0$ then $\vec{u}_\sim \in \mathring{\tilde{\mathscr{H}}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$. Suppose $\vec{\alpha}=0$. Then $\lim_{t\downarrow 0}(\vec{u},\cdots, D^k_t\vec{u})=0$. In fact, if $k=0$ then $\lim_{t\downarrow 0}\vec{u}=\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}\vec{u}=0$. Let $k>0$. Then the condition $\vec{f}_\sim \in \mathring{\tilde{\mathscr{H}}}_{(k+\sigma',s)}(\bar{R}^+_{n+1})$ implies $\lim_{t\downarrow 0}(\vec{f},\cdots, D^{k-1}_t\vec{f})= \mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(\vec{f},\cdots, D^{k-1}_t\vec{f})=0$ (cf. Theorem 3 in [8, p. 419]). Since $\lim_{t\downarrow 0}\vec{A}(t)\vec{u}= \lim_{t\downarrow 0}\vec{A}'(t)\vec{u}=\cdots=0$, it follows from the equation $D_t\vec{u}=\vec{f}-\vec{A}(t)\vec{u}$ that $\lim_{t\downarrow 0}D_t\vec{u}=0$. Then from the equation $D^2_t\vec{u}=D_t\vec{f}+i\vec{A}'(t)\vec{u}+\vec{A}(t)D_t\vec{u}$ we obtain $\lim_{t\downarrow 0}D^2_t\vec{u}=0$.

Repeating this procedure, we see that $\lim_{t\downarrow 0}(\vec{u},\cdots, D_t^k\vec{u})=0$. In the case where $\sigma'<\frac{1}{2}$, by Theorem 3 in [8, p. 419] we have $\vec{u}_\sim \in \mathring{\tilde{\mathscr{H}}}_{(\sigma+1,s-1)}\bar{R}^+_{n+1})$. Let $\sigma'=\frac{1}{2}$. Then $\vec{u}_\sim \in \mathring{\tilde{\mathscr{H}}}_{(\sigma+1-\varepsilon,s-1+\varepsilon)}(\bar{R}^+_{n+1})\subset \mathring{\tilde{\mathscr{H}}}_{(\sigma,s)}(\bar{R}^+_{n+1})$, $0<\varepsilon<1$, and therefore $(D_t-i\lambda(D_x))\vec{u}_\sim=\vec{f}_\sim-\vec{A}(t)\vec{u}_\sim-i\lambda(D_x)\vec{u}_\sim \in \mathring{\tilde{\mathscr{H}}}_{(\sigma,s-1)}(\bar{R}^+_{n+1})$. Consequently $\vec{u}_\sim \in \mathring{\tilde{\mathscr{H}}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$.

Consider the case where $k\leqq 0$. We shall reason by descending induction over k. Assume that the results are valid for any $k+1$. Let $\vec{f}\in \tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$ with $\vec{f}_\sim \in \mathring{\tilde{\mathscr{H}}}_{(\sigma,s)}(\bar{R}^+_{n+1})$, $\sigma=k+\sigma'$ and $\vec{\alpha}\in \mathscr{H}_{(\sigma+s)}(R_n)$. Let $\vec{g}\in \mathring{\tilde{\mathscr{H}}}_{(\sigma+1,s)}(\bar{R}^+_{n+1})$ be such that

$$D_t\vec{g}-i\lambda(D_x)\vec{g}=\vec{f}_\sim.$$

Then it follows from Corollary 3 in [10, p. 393] that $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}\vec{g}=0$. The Cauchy problem (15) can be written in the form

$$\begin{cases} D_t(\vec{u}-\vec{g})+\vec{A}(t)(\vec{u}-\vec{g})=-i\lambda(D_x)\vec{g}-\vec{A}(t)\vec{g} & \text{in } R^+_{n+1},\\ \mathscr{D}'_{L^2}\text{-}\lim\limits_{t\downarrow 0}(\vec{u}-\vec{g})=\vec{\alpha}, \end{cases}$$

where $-i\lambda(D_x)\vec{g}-\vec{A}(t)\vec{g}\in \mathring{\tilde{\mathscr{H}}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$. Then there exists a solution $\vec{v}=\vec{u}-\vec{g}\in \tilde{\mathscr{H}}_{(\sigma+2,s-2)}(\bar{R}^+_{n+1})$ and therefore $\vec{u}=\vec{v}+\vec{g}\in \tilde{\mathscr{H}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$. In particular, if $\vec{\alpha}=0$ then $\vec{v}\in \mathring{\tilde{\mathscr{H}}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$. Thus the proof is complete.

PROPOSITION 14. *Suppose $(E^2_{(0)}\uparrow)$ and $(\mathrm{CP})^\sim_{(0)}$ hold for L. For any given $\vec{h}\in \mathring{\tilde{\mathscr{H}}}_{(\sigma,s)}(\bar{R}^+_{n+1})$ there exists a unique solution $\vec{v}\in \mathring{\tilde{\mathscr{H}}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$ of $L\vec{v}=\vec{h}$.*

PROOF. First we let $\sigma>-\frac{1}{2}$. The problem to find a solution $\vec{v}\in(\mathscr{D}'_t)_+\varepsilon((\mathscr{D}'_{L^2})_x)$ for $L\vec{v}=\vec{h}$ is equivalent to the one to find a solution $\vec{u}\in\mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$ of the Cauchy problem $L\vec{u}=\vec{f}$, $\vec{f}=\vec{h}\,|\,R^+_{n+1}\in \tilde{\mathscr{H}}_{(\sigma,s)}(\bar{R}^+_{n+1})$, with initial condition $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}\vec{u}=0$. Thus we see that there exists a solution $\vec{u}\in \tilde{\mathscr{H}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$ and $\vec{u}_\sim \in \mathring{\tilde{\mathscr{H}}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$. Moreover we can conclude that $\vec{v}$ is unique in $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$.

Let $\sigma\leqq-\frac{1}{2}$. We can then show the existence of a solution $\vec{v}\in \mathring{\tilde{\mathscr{H}}}_{(\sigma+1,s-1)}(\bar{R}^+_{n+1})$ by proceeding along the same line as in the proof of Proposition 13. And the proof is now complete.

THEOREM 4. *Suppose $(E^2_{(0)}\uparrow)^\sim$ and $(\mathrm{CP})_{(0)}$ hold for L. Then for any $\vec{h}\in(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ there exists a unique solution $\vec{v}\in(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ of $L\vec{v}=\vec{h}$ and $\vec{h}\to\vec{v}$ is a continuous map of $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ onto itself.*

PROOF. Let $\{t_j\}$ be a sequence of real numbers such that $t_0<0<t_1<t_2<\cdots$ and $\lim_{j\to\infty}t_j=\infty$ and put $U_j=(t_j, t_{j+2})$. Then $\{U_j\}_{j=0,1,\dots}$ is an open covering of (t_0, ∞). We can choose a partition of unity $\{\phi_j\}$ subordinate to the covering. Then $\vec{h}=\sum_{j=0}^{\infty}\phi_j\vec{h}$. Consider the equations

$$L\vec{v}_j=\phi_j\vec{h}, \qquad j=0, 1,\cdots,$$

where $\phi_j\vec{h} \in \mathring{\vec{\mathscr{H}}}_{(\sigma_j, s_j)}(\bar{R}^+_{n+1})$ for some real numbers σ_j, s_j. In virtue of Proposition 14 it follows that there exists a solution $\vec{v}_j \in (\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$. From our assumption that $(E^2_{(0)}\uparrow)$ holds for L we see that $\vec{v}_j$ vanishes for $t<t_j$. Thus $\vec{v}=\sum_{j=0}^{\infty}\vec{v}_j$ is well defined in the space $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ and $\vec{v}$ is unique in $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$.

Let us consider the map

$$l: (\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x) \ni \vec{v} \rightarrow L\vec{v} \in (\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x),$$

which is linear, continuous and onto. Since the space $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ is ultrabornological and Souslin (Corollary 1 in [10, p. 374]), we see from Corollary in [16, p. 604] that l is an epimorphism. Thus the proof is complete.

Now we can state the following theorem which is an immediate consequence of Theorem 4 and the discussions given just before Lemma 3.

THEOREM 5. *Suppose* $(E^2_{(0)}\uparrow)$ *and* $(\mathrm{CP})^{\sim}_{(0)}$ *hold for* L. *Then for any* $\vec{\alpha} \in (\mathscr{D}'_{L^2})_x$ *and* $\vec{f} \in \mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$, *where* $\vec{f}$ *is assumed to have the* $\mathscr{D}'_{L_2}$*-canonical extension* $\vec{f}_\sim \in (\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$, *the Cauchy problem* (15) *has a unique solution* $\vec{u} \in \mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$ *and* $(\vec{\alpha}, \vec{f}_\sim)\rightarrow\vec{u}$ *is a continuous map under the topology of* $(\mathscr{D}'_{L^2})_x\times(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ *and the topology of* $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$.

4. Pseudo-differential operators with constant coefficients

Let $A_{ij} \in \mathrm{OP}_r$, $i, j=1, 2,\cdots, N$, such that $\frac{\partial}{\partial x_k}(A_{ij}\phi)=A_{ij}\left(\frac{\partial}{\partial x_k}\phi\right)$, $k=1, 2,\cdots, N$, hold for any $\phi \in C_0^\infty(R_n)$. Then there exist distributions $T_{A_{ij}} \in \mathscr{D}'(R_n)$, with which we can write $A_{ij}u=T_{A_{ij}}*'u$, $u \in (\mathscr{D}'_{L^2})_x$, where by $*'$ we mean the partial convolution with respect to the variable x. By taking δ as u we see that $T_{A_{ij}} \in (\mathscr{D}'_{L^2})_x$. We shall denote by $\vec{A}$ (resp. $\vec{T}_A$) the $N\times N$ matrix with entries A_{ij} (resp. $T_{A_{ij}}$). Then we can write $\vec{A}\vec{u}=\vec{T}_A*\vec{u}$. The map $l: \hat{\vec{u}}\rightarrow(1+|\xi|^2)^{-\frac{r}{2}}\times\hat{\vec{T}}_A(\xi)\hat{\vec{u}}$ is a bounded operator of $L^2(\Xi_n)$ into itself and its norm is given by the formula

$$\|l\|=\text{ess. sup } |(1+|\xi|^2)^{-\frac{r}{2}}\hat{\vec{T}}_A(\xi)|,$$

where we mean by $|\vec{X}|$ the operator norm of a matrix $\vec{X}$. Thus $\hat{\vec{T}}_{A_{ij}}(\xi)$ is a locally summable function for $i, j=1, 2,\cdots, N$ and

$$|\hat{\vec{T}}_A(\xi)| \leqq C(1+|\xi|^2)^{\frac{r}{2}}$$

with a constant $C=\|l\|$.

In this section we shall deal with the operator $L=D_t+\vec{A}$, where $\vec{A} \in \mathrm{OP}_r$ is a convolution operator given above.

PROPOSITION 15. *If* $\vec{u} \in \mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$ *satisfies* $L\vec{u}=0$ *in* R_{n+1}^+ *and* $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} \vec{u}=0$, *then* $\vec{u}=0$ *in* R_{n+1}^+.

PROOF. By Proposition 9 we see that $\vec{u}$ may be considered as a $\mathscr{D}'_{L^2}$-valued C^∞ function of t. If we write $\vec{A}\vec{u}=\vec{T}_A *' \vec{u}$ with a $\vec{T}_A \in (\mathscr{D}'_{L^2})_x$ such that $|\hat{\vec{T}}_A(\xi)| \leqq C(1+|\xi|^2)^{r/2}$, then the Fourier transformation of $D_t\vec{u}+\vec{A}(t)\vec{u}$ with respect to x is written in the form

$$D_t\hat{\vec{u}}(t, \xi)+\hat{\vec{T}}_A(\xi)\hat{\vec{u}}(t, \xi)=0.$$

Since $e^{i\hat{\vec{T}}_A(\xi)t}$ is a locally summable function of ξ, $e^{i\hat{\vec{T}}_A(\xi)t}\hat{\vec{u}}$ is well defined as $\mathscr{D}'(\Xi_n)$-valued C^∞ function of t and $D_t(e^{i\hat{\vec{T}}_A(\xi)t}\hat{\vec{u}})=0$ and therefore $\vec{U}(\xi)=e^{i\hat{\vec{T}}_A(\xi)t}\hat{\vec{u}}(t, \xi) \in \mathscr{D}'(\Xi_n)$. On the other hand, from $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} u=0$ we see that $\lim_{t\downarrow 0} \vec{u}=\lim_{t\downarrow 0} \hat{\vec{u}}=0$. Thus $\vec{U}(\xi)=0$ as a distribution. Thus we can conclude that $\vec{u}=0$ as a distribution.

We shall say the Cauchy problem for L is well posed in the L^2 norm if for any $\vec{\alpha} \in C_0^\infty(R_n)$ the Cauchy problem:

$$\begin{cases} L\vec{u}=0 \quad \text{in} \quad (0, T)\times R_n, \\ \lim_{t\downarrow 0} \vec{u}=\vec{\alpha} \end{cases}$$

has a unique solution $\vec{u} \in \mathscr{E}_t^0(\mathscr{H}_{(0)})$, $0\leqq t\leqq T$, and

$$\|\vec{u}(t, \cdot)\|_{(0)} \leqq C_T\|\vec{u}(0, \cdot)\|_{(0)}, \qquad 0\leqq t\leqq T,$$

where $T>0$ is arbitrary.

Then the Cauchy problem for L is well posed in the L^2 norm if and only if

$$|e^{-i\hat{\vec{T}}_A(\xi)t}| \leqq C_T, \qquad 0\leqq t\leqq T.$$

If we put $k=(\log C_T)/T$, then

$$|e^{-i\hat{\vec{T}}_A(\xi)t}| \leqq C_Te^{kt}, \qquad 0\leqq t<\infty,$$

and therefore

$$|e^{-(i\hat{\vec{T}}_A(\xi)+kI)t}| \leqq C_T, \qquad 0\leqq t<\infty.$$

In [17, p. 411] G. Strang gave a necessary and sufficient condition in order that a Kowalewski system may be strongly hyperbolic. In connection with his studies we shall show the following

PROPOSITION 16. *The following conditions are equivalent*:
(1) *The Cauchy problem for* L *is well posed in the* L^2 *norm.*
(2) $(E^1_{(0)}\uparrow)$ *holds for* L *and* $(E^1_{(0)}\downarrow)$ *holds for* L^*.
(3) $(E^2_{(0)}\uparrow)$ *holds for* L *and* $(E^2_{(0)}\downarrow)$ *holds for* L^*.
(4) $(E^2_{(0)}\uparrow)$ *holds for* L.

PROOF Since the implications (2)⇒(3), (3)⇒(4) are trivial, we have only to show the implications (1)⇒(2) and (4)⇒(1).

(1)⇒(2). For any $\vec{u}\in C_0^\infty(R_{n+1})$ if we put $\vec{f}=L\vec{u}$, then we have

$$D_t\hat{\vec{u}}(t,\xi)+\hat{\vec{T}}_A(\xi)\hat{\vec{u}}(t,\xi)=\hat{\vec{f}}(t,\xi),$$

and therefore

$$\hat{\vec{u}}(t,\xi)=e^{-i\hat{\vec{T}}_A(\xi)t}\hat{\vec{u}}(0,\xi)+i\int_0^t e^{-i\hat{\vec{T}}_A(\xi)(t-t')}\hat{\vec{f}}(t',\xi)dt',$$

which implies that $(E^1_{(0)}\uparrow)$ holds for L and similarly $(E^1_{(0)}\downarrow)$ holds for L^*.

(4)⇒(1). Consider the set $A=\{(\vec{\phi}(0,\cdot),L\vec{\phi}):\vec{\phi}\in C_0^\infty(_{n+1})\}$. Then the set A is dense in $\mathscr{H}_{(0)}(R_n)\times\tilde{\mathscr{H}}_{(0,0)}(\bar{R}^+_{n+1})$. In fact, let $(-i\vec{\beta},\vec{w})$ be any element of $\mathscr{H}_{(0)}(R_n)\times\tilde{\mathscr{H}}^*_{(0,0)}(\bar{R}^+_{n+1})$ such that

$$\int_0^\infty(L\vec{\phi}(t,\cdot),\vec{w}(t,\cdot))dt-i(\vec{\phi}(0,\cdot),\vec{\beta})=0.$$

This means that $L^*\vec{w}=0$ in R^+_{n+1}, and therefore the preceding proposition implies $\vec{w}=0$ and $\vec{\beta}=0$. Thus there exists a sequence $\{\vec{\phi}_j\}$, $\vec{\phi}_j\in C_0^\infty(R_{n+1})$, such that $L\vec{\phi}_j\to 0$ in $\tilde{\mathscr{H}}_{(0,0)}(\bar{R}^+_{n+1})$ and $\vec{\phi}_j(0,\cdot)\to\vec{\alpha}$. In virtue of $(E^2_{(0)}\uparrow)$ we see that $\{\vec{\phi}_j(t,\cdot)\}$ is a Cauchy sequence in $\mathscr{E}^0_t(\mathscr{H}_{(0)})$. If we put $\vec{u}=\lim_{j\to\infty}\vec{\phi}_j$, then $\vec{u}\in\mathscr{E}^0_t(\mathscr{H}_{(0)})$ and for any $T>0$ we have $\|\vec{u}(t,\cdot)\|_{(0)}\leqq C_T\|\vec{\alpha}\|_{(0)}$, $0\leqq t\leqq T$.

Let $\mathfrak{F}$ be a family of $N\times N$ matrices $\vec{M}(\xi)$ of measurable functions of ξ. As an analogue of a result of H.-O. Kreiss [12, p. 71; 13, p. 113], we can show the following

PROPOSITION 17. *The following conditions are equivalent*:
(1) $|e^{\vec{M}(\xi)t}|\leqq C$ *for all* $t\geqq 0$ *and* $\vec{M}\in\mathfrak{F}$ a.e. *on* Ξ_n.
(1)′ *For any complex number* s *with* Re $s>0$ *there exists a constant* C *such that for all* $\vec{M}\in\mathfrak{F}$

$$(\vec{M}(\xi)-sI)^{-1}\leqq C/\text{Re } s \text{ a.e. on } \Xi_n.$$

(2) *There exist a constant* C *and a matrix* $\vec{S}$, *whose entries are measurable functions of* ξ, *such that for all* $\vec{M}\in\mathfrak{F}$

$$|\vec{S}(\xi)|,\ |\vec{S}^{-1}(\xi)|\leqq C$$

and

$$SMS^{-1}=\begin{pmatrix} \varkappa_1 & b_{12} & \cdots & b_{1N} \\ 0 & \varkappa_2 & b_{23} \cdots & b_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \varkappa_N \end{pmatrix} \quad \text{a.e. } on\ \Xi_n,$$

where $0 \geqq \operatorname{Re}\varkappa_1 \geqq \operatorname{Re}\varkappa_2 \geqq \cdots \geqq \operatorname{Re}\varkappa_N$ *and* $|b_{ij}| \leqq C|\operatorname{Re}\varkappa_i|$.

(3) *There exist a constant* C *and a positive definite Hermitian matrix* $\vec{H}(\xi)$ *such that for all* $\vec{M} \in \mathfrak{F}$

$$|\vec{H}(\xi)|,\ |\vec{H}^{-1}(\xi)| \leqq C \text{ and } \vec{H}\vec{M} + \vec{M}^*\vec{H} \leqq 0 \qquad \text{a.e. } on\ \Xi_n.$$

The proposition with $\vec{M}(\xi)$ replaced by $-i\hat{\vec{T}}_A(\xi) - kI$ yields the following

Corollary 3. *The following conditions are equivalent*:

(1) $|e^{-(i\hat{\vec{T}}_A(\xi)+kI)t}| \leqq C$ *for* $t \geqq 0$ *and* a.e. *on* Ξ_n.

(2) *There exist a constant* C *and a matrix* $\vec{S}$ *such that*

$$|S(\xi)|,\ |S^{-1}(\xi)| \leqq C$$

and

$$\vec{S}\hat{\vec{T}}_A\vec{S}^{-1}=\begin{pmatrix} \varkappa_1 & b_{12} & \cdots & b_{1N} \\ 0 & \varkappa_2 & b_{23} \cdots & b_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \varkappa_N \end{pmatrix} \quad \text{a.e. } on\ \Xi_n,$$

where $k \geqq \operatorname{Im}\varkappa_1 \geqq \cdots \geqq \operatorname{Im}\varkappa_N$ *and* $|b_{ij}| \leqq C(|\operatorname{Im}\varkappa_i - k|)$.

(3) *There exist a constant* C *and a positive definite Hermitian matrix* $\vec{H}(\xi)$ *such that*

$$|\vec{H}(\xi)|,\ |\vec{H}^{-1}(\xi)| \leqq C \text{ and } -i(\vec{H}(\xi)\hat{\vec{T}}_A(\xi) - \hat{\vec{T}}_A^*(\xi)\vec{H}(\xi)) \leqq kC \quad \text{a.e. } on\ \Xi_n.$$

Proposition 18. *Suppose* $\hat{\vec{T}}_A(\xi)$ *is positive homogeneous of degree* $r > 0$, *that is*, $\hat{\vec{T}}_A(\lambda\xi) = \lambda^r \hat{\vec{T}}_A(\xi)$ *for* $\lambda > 0$. *For the operator* $L = D_t + \vec{A}(t)$, *the energy inequality* $(E^2_{(0)} \uparrow \downarrow)$ *holds if and only if the eigenvalues* $\varkappa_j$ *of the matrix* $\hat{\vec{T}}_A(\xi)$ *are real and* $\hat{\vec{T}}_A(\xi)$ *is symmetrizable.*

Proof. Suppose $(E^2_{(0)} \uparrow)$ holds for L. Let $\varkappa_j(\xi)$, $j=1, 2,\cdots, N$, be the eigenvalues of $\hat{\vec{T}}_A(\xi)$. From that $(E^2_{(0)} \uparrow)$ holds for L it follows by Corollary 3 (2) that $k \geqq \operatorname{Im}\varkappa_j(\xi)$ and $\varkappa_j(\lambda\xi) = \lambda^r\varkappa_j(\xi)$, and therefore $\frac{k}{\lambda^r} \geqq \operatorname{Im}\varkappa_j(\xi)$. On the other hand, that $(E^2_{(0)} \downarrow)$ holds for L means that $(E^2_{(0)} \downarrow)$ holds for $L^1 = D_t - \vec{A}$. In the same way as above, we can conclude that $0 \geqq -\operatorname{Im}\varkappa_j(\xi)$, and therefore $\operatorname{Im}\varkappa_j(\xi) = 0$ for $j=1, 2,\cdots, N$.

From the relation $|e^{-i\hat{\vec{T}}_A(\xi)t}|\leqq C_T$, $0\leqq t\leqq T$, together with the fact that $\hat{\vec{T}}_A(\xi)$ is positive homogeneous of degree r, we may take $k=0$, and by Corollary 3 (2) we can conclude that $b_{ij}=0$ for any i, j. Taking $\vec{H}(\xi)=\vec{S}^*(\xi)\vec{S}(\xi)$, we see that $\vec{H}(\xi)$ is a positive definite Hermitian matrix, $\vec{H}(\xi)$, $\vec{H}^{-1}(\xi)$ are bounded and $\vec{H}(\xi)\hat{\vec{T}}_A(\xi)$ is Hermitian.

The converse is well known [3, p. 111].

Let $\vec{A}\in \mathrm{OP}_r$, $r>0$ and $\varkappa_j(\xi)$, $j=1, 2,\cdots, N$, be the characteristic roots of the matrix $\hat{\vec{T}}_A(\xi)$. If there exist constants $C>0$ and C_0 such that

$$\mathrm{Im}\,\varkappa_j(\xi)\leqq -C|\xi|^r+C_0,$$

then the Cauchy problem for the operator $L=D_t+\vec{A}$ is well posed in the L^2 norm. In fact, we have for any $T>0$

$$\begin{aligned}|e^{-i\hat{\vec{T}}_A(\xi)t}| &\leqq (1+2t\,|\hat{\vec{T}}_A(\xi)|+\cdots+(2t)^{N-1}|\hat{\vec{T}}_A(\xi)|^{N-1})e^{t\max_j \mathrm{Im}\,\varkappa_j(\xi)}\\ &\leqq C_1(1+t|\xi|^r+\cdots+(t|\xi|^r)^{N-1})e^{-Ct|\xi|^r}\\ &\leqq C_2, \qquad 0\leqq t\leqq T.\end{aligned}$$

For example, the Cauchy problem for the operator $D_t-i\lambda^r(D_x)$, $\lambda(\xi)=(1+|\xi|^2)^{1/2}$ is well posed in the L^2 norm.

Let us consider a pseudo-differential operator with the form:

$$P(D)=D_t^m+\sum_{j=1}^{m} A_jD_t^{m-j},$$

where A_j are convolution operators such that $A_j\in \mathrm{OP}_{r_j}$. For any given $\vec{\alpha}=(\alpha_0,\cdots,\alpha_{m-1})$, $\alpha_j\in(\mathscr{D}'_{L^2})_x$ and $f\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$, where f is assumed to have the $\mathscr{D}'_{L^2}$-canonical extension $f_\sim\in(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$, the problem to find a solution $u\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$ of the Cauchy problem

$$(16)\qquad \begin{cases} P(D)u=f & \text{in } R^+_{n+1},\\ \mathscr{D}'_{L^2}\text{-}\lim\limits_{t\downarrow 0}(u, D_tu,\cdots, D_t^{m-1}u)=\vec{\alpha}\end{cases}$$

is reduced to the problem to find $w\in(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ such that

$$Pw=f_\sim+\sum_{k=0}^{m-1}D_t^k\delta\otimes\gamma_k,$$

where $\gamma_k=-i(\alpha_{m-k-1}+\sum_{\nu=1}^{m-k-1}A_{m-\nu-k}\alpha_{\nu-1})$ ([9, p. 82]) and $u=(w|R^+_{n+1})_\sim$. We shall use the notation $\Gamma(\vec{\alpha})=(\gamma_0,\cdots,\gamma_{m-1})$.

On the other hand, by the Calderón transformation

$$v_j=S^{m-j}D_t^{j-1}u,\ j=1, 2,\cdots, m,$$

the Cauchy problem (16) can be written in the form

$$(17)\qquad \begin{cases} L\vec{v}\equiv D_t\vec{v}+\vec{A}\vec{v}=\vec{f} & \text{in } R_{n+1}^{+}, \\ \mathscr{D}'_{L^2}\text{-}\lim\limits_{t\downarrow 0}\vec{v}=\vec{\beta}, \end{cases}$$

where $\vec{f}=(0,\cdots,0,f)$, $\vec{\beta}=(S^{m-1}\alpha_0,\cdots,\alpha_{m-1})$,

$$\vec{A}=\begin{pmatrix} 0 & -S & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & & 0 & -S \\ A_mS^{-m+1} & A_{m-1}S^{-m+2} & \cdots & & A_1 \end{pmatrix}$$

and $\vec{A}\in \mathrm{OP}_r$, $r=\max(1, r_j+(1-j))$.

Let us denote by $[E^2_{(0)}\uparrow]$ the following energy inequality:

$$[E^2_{(0)}\uparrow]:\ \sum_{j=0}^{m-1}\|D_t^j\phi(t,\cdot)\|^2_{(m-1-j)}\leqq C_T\Big(\sum_{j=0}^{m-1}\|D_t^j\phi(0,\cdot)\|^2_{(m-1-j)}+$$
$$+\int_0^t\|P\phi(t',\cdot)\|^2_{(0)}dt'\Big),\ 0\leqq t\leqq T,\ \phi\in C_0^\infty(R_{n+1})$$

with a constant C_T. We shall use the notations $[E^2_{(0)}\downarrow]$, $[E^1_{(0)}\uparrow]$, $[E^1_{(0)}\uparrow]$ and the like with obvious meanings. If $[E^2_{(0)}\uparrow]$ holds for P, then $[E^2_{(0)}\downarrow]$ holds for $P^*=D_t^m+\sum_{j=0}^{m-1}\bar{A}_jD_t^{m-j}$. In fact, if we put $\psi(t)=\bar{\phi}(-t)$ for any $\phi\in C_0^\infty(R_{n+1})$, then $\overline{P^*(D)\phi}=P(D)\psi$ and

$$\sum_{j=0}^{m-1}\|D_t^j\psi(t,\cdot)\|^2_{(m-1-j)}\leqq C_T\Big(\sum_{j=0}^{m-1}\|D_t^j\psi(0,\cdot)\|^2_{(m-1-j)}+$$
$$+\int_0^t\|P(D)\psi(t',\cdot)\|^2_{(0)}dt'\Big),\qquad 0\leqq t\leqq T,$$

which implies

$$\sum_{j=0}^{m-1}\|D_t^j\phi(-t,\cdot)\|^2_{(m-1-j)}\leqq C_T\Big(\sum_{j=0}^{m-1}\|D_t^j\phi(0,\cdot)\|^2_{(m-1-j)}+$$
$$+\int_{-t}^0\|P^*(D)\phi(t',\cdot)\|^2_{(0)}dt'\Big).$$

Similarly if $[E^1_{(0)}\uparrow]$ holds for P, then $[E^1_{(0)}\downarrow]$ holds for P^*.

PROPOSITION 19. *Suppose* $[E^2_{(0)}\uparrow]$ *holds true for* P. *For any* $f\in \tilde{\mathscr{H}}_{(0,s)}(\bar{R}_{n+1}^{+})$ *and* $\vec{\alpha}\in\mathscr{H}_{(m-1)}(R_n)\times\cdots\times\mathscr{H}_{(0)}(R_n)$ *the Cauchy problem* (16) *has a unique solution* $u\in\mathscr{D}'(R_t^{+})((\mathscr{D}'_{L^2})_x)$ *and* u *has the properties*

(i) $D_t^ju\in\mathscr{E}_t^0(\mathscr{H}_{(s+m-1-j)})$, $j=0,1,\cdots,m-1$,

(ii) $\sum_{j=0}^{m-1}\|D_t^j u(t,\cdot)\|_{(m+s-1-j)}^2 \leqq C_T^{(s)}(\sum_{j=0}^{m-1}\|\alpha_j\|_{(m+s-1-j)}^2+$

$$+\int_0^t\|f(t',\cdot)\|_{(s)}^2 dt')$$

with a constant $C_T^{(s)}$.

PROOF. Uniqueness of a solution is trivial by Proposition 15. It is sufficient to show that the set $A=\{(\Gamma(\vec{\phi}_0), P\vec{\phi}): \phi\in C_0^\infty(R_{n+1})\}$, $\vec{\phi}_0=(\phi(0,\cdot),\cdots, D_t^{m-1}\phi(0,\cdot))$, is dense in $(\mathscr{H}_{(s+m-1)}(R_n)\times\cdots\times\mathscr{H}_{(s)}(R_n))\times\tilde{\mathscr{H}}_{(0,s)}(\bar{R}_{n+1}^+)$. Let $\vec{\beta}\in\mathscr{H}_{(0)}(R_n)\times\cdots\times\mathscr{H}_{(-m+1)}(R_n)$ and $w\in\tilde{\mathscr{H}}^*_{(0,-s)}(\bar{R}_{n+1}^+)$ such that

$$\int(P\phi(t,\cdot), w(t,\cdot))dt+(\Gamma(\vec{\phi}_0), \vec{\beta})=0, \qquad \phi\in A.$$

Then $P^*w=0$ in R_{n+1}^+, and therefore $w=0$ and $\beta=0$.

If for any $\vec{\alpha}\in C_0^\infty(R_{n+1})$ the Cauchy problem:

$$\begin{cases} P(D)u=0 & \text{in } (0,T)\times R_n, \\ \lim_{t\downarrow 0}(u, D_t u,\cdots, D_t^{m-1}u)=\alpha, \end{cases}$$

where $T>0$ is arbitrary, has a unique solution $u\in\mathscr{E}_t^0(\mathscr{H}_{(m-1)})$ such that $D_t^j u\in\mathscr{E}_t^0(\mathscr{H}_{(m-1-j)})$, $j=0,1,\cdots,m-1$, and

$$\sum_{j=0}^{m-1}\|D_t^j u(t,\cdot)\|_{(m-1-j)}^2\leqq C_T(\sum_{j=0}^{m-1}\|D_t^j u(0,\cdot)\|_{(m-1-j)}^2), \qquad 0\leqq t\leqq T$$

with a constant C_T, then we shall say that P satisfies the property (W).

Let L be the operator associated with P by the Calderón transformation. Then we have the following

THEOREM 6. *The following conditions are equivalent*:

(1) *P satisfies the property* (W).
(2) $[E_{(0)}^1\uparrow]$ *holds for P and* $[E_{(0)}^1\downarrow]$ *holds for* P^*.
(3) $[E_{(0)}^2\uparrow]$ *holds for* P.
(4) *The Cauchy problem for L is well posed in the L^2 norm.*
(5) $(E_{(0)}^1\uparrow)$ *holds for L and* $(E_{(0)}^1\downarrow)$ *holds for* L^*.
(6) $(E_{(0)}^2\uparrow)$ *holds for* L.

PROOF. The conditions (4), (5) and (6) are equivalent by Proposition 16 and the equivalences of (2) and (5), (3) and (6) are trivial by the definition. The implication $(3)\Rightarrow(1)$ is an immediate consequence of the preceding proposition. We have only to show the implication $(1)\Rightarrow(4)$. Let $\vec{\alpha}=(\alpha_0,\cdots, \alpha_{m-1})\in C_0^\infty(R_n)$. Then $(S^{-m+1}\alpha_0, S^{-m+2}\alpha_1,\cdots,\alpha_{m-1})\in(\mathscr{D}_{L^2})_x$, where $C_0^\infty(R_n)$ is dense in $(\mathscr{D}_{L^2})_x$. Since the property (W) holds also true of the initial data

$(S^{-m+1}\alpha_0,\cdots,\alpha_{m-1})\in(\mathscr{D}_{L^2})x$, the Cauchy problem for L is well posed in the L^2 norm.

Now let us consider the case where A_j can be written in the form

$$A_j=A_j^0\Lambda^j+B_j,\qquad j=1,2,\cdots,m,$$

where $\hat{T}_{A_j^0}(\xi)$ are positive homogeneous of degree 0, $\hat{T}_{A_j^0}(\xi)$ continuous on $|\xi|=1$ and B_j of order $\leqq j-1$. We put

$$P_0(D)=D_t^m+\sum_{j=1}^m A_j^0\Lambda^jD_t^{m-j}.$$

Then we have the following

PROPOSITION 20. *$[E^2_{(0)}\uparrow\downarrow]$ holds for $P(D)$ if and only if the roots of the polynomial $P_0(\tau,\xi)$ of τ are real and distinct.*

PROOF. Suppose $[E^2_{(0)}\uparrow\downarrow]$ holds for $P(D)$. Then $(E^2_{(0)}\downarrow\uparrow)$ holds for L. L can be written in the form $L=\vec{A}^0\Lambda+\vec{B}$, where $\hat{\vec{T}}_{A^0}(\xi)$ is positive homogeneous of degree 0, $\hat{\vec{T}}_{A^0}(\xi)$ continuous on $|\xi|=1$ and $\vec{B}$ of order$\leqq 0$. Thus we see by Lemma 3 that $(E^2_{(0)}\uparrow\downarrow)$ holds also for $L_0=\vec{A}^0\Lambda$. Observe that $P_0(\tau,\xi)$ is the minimal polynomial of the matrix $\hat{\vec{T}}_{A^0}(\xi)$. In virtue of Proposition 18 we see that the roots are real and distinct.

Conversely, suppose the roots of $P_0(\tau,\xi)=0$ are real and distinct. Then by necessary modifications of the proofs of Theorems 24 and 25 in A.P. Calderón [3, p. 109, p. 110] there exists a positive definite Hermitian matrix $\vec{H}(\xi)$ such that $\vec{H}(\xi)\hat{\vec{T}}_{A^0}(\xi)=\hat{\vec{T}}^*_{A^0}(\xi)\vec{H}(\xi)$. Applying Corollary 3 and Lemma 3 we see that $[E^2_{(0)}\downarrow\uparrow]$ holds for P.

COROLLARY 4. *In the case where the coefficients of the operators A_j^0, $j=1,2,\cdots,m$, are real, $[E^2_{(0)}\uparrow]$ holds for P if and only if the roots of $P_0(\tau,\xi)=0$ are real and distinct.*

5. A characterization of regular hyperbolicity

In our previous paper [9, p. 101] it is shown that a differential operator $P(D)=D_t^m+\sum_{j=0}^{m-1}\sum_{j+|p|\leqq m}a_{j,p}(t,x)D_t^jD_x^p$, $a_{j,p}\in\mathscr{B}$, is regularly hyperbolic if and only if, for any fixed $T>0$, $P(D)$ satisfies the energy inequality:

$$[\tilde{E}^1_{(0)}\uparrow]:\ \sum_{j=0}^{m-1}\|D_t^j\phi(t_1,\cdot)\|_{(m-1-j)}\leqq C_T\Big(\sum_{j=0}^{m-1}\|D_t^j\phi(t_0,\cdot)\|_{(m-1-j)}+$$

$$+\int_{t_0}^{t_1}\|P(D)\phi(t,\cdot)\|_{(0)}dt,\qquad 0\leqq t_0\leqq t_1\leqq T,\qquad \phi\in C_0^\infty(R_{n+1}),$$

where C_T is a constant.

The aim of this section is to generalize this result to a pseudo-differential operator.

T. Bałabon [1] has investigated the Cauchy problem for a pseudo-differential operators with the form:

$$P(D)=D_t^m+\sum_{j=1}^m A_j(t)D_t^{m-j},$$

where $A_j(t)=A_j(t, x, D_x)$ is a pseudo-diffenertial operator of order j in the sense of J.J. Kohn and L. Nirenberg, which depends smoothly on t and its asymptotic expansion has only operators of integral order.

Let $\mathfrak{B}$ be the space of all B_∞ singular integral operators in the sense of A.P. Calderón [3, 14] with semi-norms $\{p_m\}_m=0, 1,\cdots$:

$$p_m: \qquad K\to\|K\|_m=\max_{0\leqq|\alpha|\leqq 2n}\left\{\sup_{|\varsigma|=1}\left\|\left(\frac{\partial}{\partial\varsigma}\right)^\alpha\sigma(K)(x, \varsigma)\right\|_m\right\},$$

where $\|K\|_m$ is the norm of B_m singular integral operator K.

Consider a pseudo-differential operator with the form

$$P(D)=D_t^m+\sum_{j=1}^m A_j(t)D_t^{m-j},\ A_j(t)=A_j^0(t)\Lambda^j+B_j(t),$$

where $A_j^0(t)=A_j^0(t, x, D_x)$ are $\mathfrak{B}$-valued continuous functions of $t\in R_t$ and $B_j(t)\in\mathfrak{C}_{(j-1)}$. We shall give a characterization of the regular hyperbolicity of $P(D)$ by making use of the energy inequalities.

We shall denote by $P^0(D)$ the principal part of $P(D)$:

$$P^0(D)=D_t^m+\sum_{j=1}^m A_j^0(t)\Lambda^j D_t^{m-j}$$

and we put

$$P_{(t_0,x_0)}^0(D)=D_t^m+\sum_{j=1}^m A_j^0(t_0, x_0, D_x)\Lambda^j D_t^{m-j},$$

where the point (t_0, x_0) is fixed. Let T be any fixed positive number.

PROPOSITION 21. *Suppose the following energy inequality* $[\tilde{E}_{(0)}^1\uparrow]$ *holds for* $P(D)$:

$$[\tilde{E}_{(0)}^1\uparrow]:\ \sum_{j=0}^{m-1}\|D_t^j\phi(t_1, \cdot)\|_{(m-1-j)}\leqq C_T\Big(\sum_{j=0}^{m-1}\|D_t^j\phi(t_0, \cdot)\|_{(m-1-j)}+$$

$$+\int_{t_0}^{t_1}\|(P\phi)(t', \cdot)\|_{(0)}dt', \qquad 0\leqq t_0\leqq t_1\leqq T, \qquad \phi\in C_0^\infty(R_{n+1}),$$

where C_T is a constant. Then $[E^1_{(0)}\uparrow]$ holds for $P^0_{(t_0,x_0)}(D)$ with a constant independent of $(t_0, x_0)\in[0, T]\times R_n$.

PROOF. Let $(t_0, x_0)\in[0, T)\times R_n$. For any fixed $\tilde{t}\in(0, T]$ we take λ so large that $t_0<t_1=t_0+\dfrac{\tilde{t}}{\lambda}\leqq T$. Let $\phi_\lambda(t, x)=u(\lambda(t-t_0), \lambda(x-x_0))$ for any $u\in C_0^\infty(R_{n+1})$. Then we have

$$(18)\qquad \sum_{j=0}^{m-1}\|D_t^j\phi_\lambda(t_1, \cdot)\|_{(m-1-j)}\leqq C_T\Big(\sum_{j=0}^{m-1}\|D_t^j\phi_\lambda(t_0, \cdot)\|_{(m-1-j)}+ +\int_{t_0}^{t_1}\|(P(D)\phi_\lambda)(t', \cdot)\|_{(0)}dt'\Big),$$

where

$$\|D_t^j\phi_\lambda(t_1, \cdot)\|^2_{(m-1-j)}=\sum_{|p|\leqq m-1-j}\lambda^{2(j+|p|-n/2)}\binom{m-1-j}{p}\|D_t^jD_p^xu(\tilde{t}, \cdot)\|^2_{(0)}$$

$$\|D_j^t\phi_\lambda(t_0, \cdot)\|^2_{(m-1-j)}=\sum_{|p|\leqq m-1-j}\lambda^{2(j+|p|-n/2)}\binom{m-1-j}{p}\|D_t^jD_x^pu(0, \cdot)\|^2_{(0)}.$$

Moreover, we can write

$$P(D)\phi_\lambda=D_t^m\phi_\lambda+\sum_{j=1}^m A_j^0(t)\Lambda^jD_t^{m-j}\phi_\lambda+\sum_{j=1}^m B_j(t)D_t^{m-j}\phi_\lambda,$$

where

$$A_j^0(t)\Lambda^jD_t^{m-j}\phi_\lambda=\frac{1}{(2\pi)^n}\lambda^{m-n}\int\sigma(A_j^0)(t, x, \xi)\left|\frac{\xi}{\lambda}\right|^jD_t^{m-j}\hat{u}\Big(\lambda(t-t_0), \frac{\xi}{\lambda}\Big)e^{i<x-x_0,\xi>}d\xi$$

$$=\frac{\lambda^m}{(2\pi)^n}\int\sigma(A_j^0)(t, x, \xi)|\xi|^jD_t^{m-j}\hat{u}(\lambda(t-t_0), \xi)e^{i<\lambda(x-x_0),\xi>}d\xi$$

$$=\lambda^m\Big(A_j^0\Big(\frac{\cdot}{\lambda}+t_0, \frac{\cdot}{\lambda}+x_0, D_x\Big)\Lambda^jD_t^{m-j}u\Big)(\lambda(t-t_0), \lambda(x-x_0)),$$

and therefore

$$\|(P(D)\phi_\lambda)(t, \cdot)\|_{(0)}\leqq\lambda^{m-(n/2)}\|(P_{0(\frac{t}{\lambda}+t, \frac{x}{\lambda}+x_0)}(D)u)(\lambda(t-t_0), \cdot)\|_{(0)}+ +C_1\lambda^{m-1-\frac{n}{2}}\sum_{j=1}^m\|(\lambda^{-2}+|\xi|^2)^{\frac{j-1}{2}}D_t^{m-j}\hat{u}(\lambda(t-t_0), \cdot)\|_{(0)},$$

where C_1 is a constant independent of u but depends on B_j. Thus we have

$$\int_{t_0}^{t_1}\|(P(D)\phi_\lambda)(t', \cdot)\|_{(0)}dt\leqq\lambda^{m-1-n/2}\int_0^{\tilde{t}}\|(P^0{}_{(\frac{t}{\lambda}+t_0, \frac{x'}{\lambda}+x_0)}(D)u)(t', \cdot)\|_{(0)}dt'+ +C_1\lambda^{m-2-n/2}\sum_{j=1}^m\int_0^{\tilde{t}}\|(\lambda^{-2}+|\xi|^2)^{\frac{j-1}{2}}D_t^{m-j}\hat{u}(t', \cdot)\|_{(0)}dt'.$$

In the case where $n\geqq 2$ we can expand the operator $A_j^0\left(\frac{t'}{\lambda}+t_0, \frac{x'}{\lambda}+x_0, D_x\right)f$ in the form

$$A_0^j\left(\frac{t'}{\lambda}+t_0, \frac{x'}{\lambda}+x_0, D_x\right)f=a_0^j\left(\frac{t'}{\lambda}+t_0, \frac{x'}{\lambda}+x_0\right)f$$

$$+\sum_{m=1}^{\infty}\sum_{l=1}^{d(m)} a_{lm}^j\left(\frac{t'}{\lambda}+t_0, \frac{x'}{\lambda}+x_0\right)G_{lm}f,$$

where $\{G_{lm}\}$ be a system of Giraud operators associated with a complete orthonormal system of spherical harmonics of degree m and $d(m)=g(m)-g(m-2)$, $g(m)=\binom{m+n-1}{n-1}$, and we set $g(-1)=g(-2)=0$. Since $\sup\left|a_0^j\left(\frac{t}{\lambda}+t_0, \frac{x}{\lambda}+x_0\right)\right|\leqq C'$, $\sup\left|a_{lm}^j\left(\frac{t}{\lambda}+t_0, \frac{x}{\lambda}+x_0\right)\right|\leqq C'm^{-\frac{3}{2}n}\|A_j^0\|_0$, $\|G_{lm}f\|_{(0)}\leqq C'm^{\frac{n-2}{2}}\gamma_m\|f\|_{(0)}$ with $\gamma_m=-i^m(2\sqrt{\pi})^{-n}\Gamma(m)\left(\Gamma\left(\frac{m+n}{2}\right)\right)^{-1}$ and $d(m)\leqq C'm^{n-2}$, where C' is a constant independent of (t_0, x_0) [3, 14], we obtain with a constant C'' independent of (t_0, x_0)

$$\left\|A_j^0\left(\frac{t}{\lambda}+t_0, \frac{x}{\lambda}+x_0, D_x\right)A_jD_t^{m-j}u(t', \cdot)\right\|_{(0)} \tag{19}$$

$$\leqq C''\left(1+\sum_{m=1}^{\infty} m^{-3}\right)\|A_j^0\|_0\|D_t^{m-j}u(t', \cdot)\|_{(j)}.$$

In the case where $n=1$ we can write

$$A_j^0\left(\frac{t'}{\lambda}+t_0, \frac{x'}{\lambda}+x_0, D_x\right)f=a_0^j\left(\frac{t'}{\lambda}+t_0, \frac{x'}{\lambda}+x_0\right)f+$$

$$+a_1^j\left(\frac{t'}{\lambda}+t_0, \frac{x'}{\lambda}+x_0\right)\lim_{\varepsilon\downarrow 0}\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y}dy,$$

where $a_0^j(t, \cdot)$ and $a_1^j(t, \cdot)$ are B_∞-valued continuous function of t. Thus the estimate (19) remains valid.

Dividing both sides of (18) by $\lambda^{m-1-(n/2)}$, letting $\lambda\to\infty$ and applying Lebesgue's convergence theorem we obtain the estimate

$$\sum_{j=0}^{m-1}\|(\Lambda^{m-1-j}D_t^ju)(\bar{t}, \cdot)\|_{(0)}\leqq C_T\left(\sum_{j=0}^{m-1}(\Lambda^{m-1-j}D_t^ju)(0, \cdot)\|_{(0)}+\right. \tag{20}$$

$$\left.+\int_0^{\bar{t}}\|P_{(t_0,x_0)}^0(D)u(t', \cdot)\|_{(0)}dt'\right),$$

where C_T is a constant independent of (t_0, x_0) and u.

If we take $u(t, x)e^{i\langle x,\xi_0\rangle}$, $\xi_0=(1, 0,\cdots, 0)$, instead of $u(t, x)$, then we get

$\hat{u}(t, \xi-\xi_0)$ as the partial Fourier transform and we have

$$(21)\quad \sum_{j=0}^{m-1} \|(\Lambda(D_x+\xi_0))^{m-1-j} D_t^j u(\tilde{t}, \cdot)\|_{(0)} \leqq C_T\Big(\sum_{j=0}^{m-1} (\Lambda(D_x+\xi_0))^{m-1-j} D_t^j u(\tilde{t}, \cdot)\|_{(0)} + $$
$$+\int_0^{\tilde{t}} \|(P^0_{(t_0,x_0)}(D) u e^{i<x,\xi_0>})(t', \cdot)\|_{(0)} dt'\Big),$$

where $\Lambda(D_x+\xi_0)$ is defined by $(\Lambda(D_x+\xi_0)u)^\wedge = |\xi+\xi_0|\hat{u}$.

It is evident that

$$(P^0_{(t_0,x_0)}(D) u e^{i<x,\xi_0>})^\wedge$$
$$= D_t^m \hat{u}(t, \xi-\xi_0) + \sum_{j=1}^{m} \sigma(A_j^0)(t_0, x_0, \xi)|\xi|^j D_t^{m-j} \hat{u}(t, \xi-\xi_0)$$
$$= D_t^m \hat{u}(t, \xi-\xi_0) + \sum_{j=1}^{m} \sigma(A_j^0)(t_0, x_0, \xi-\xi_0)|\xi-\xi_0|^j D_t^{m-j} \hat{u}(t, \xi-\xi_0) +$$
$$+ \sum_{j=1}^{m} \sigma(A_j^0)(t_0, x_0, \xi)(|\xi|^j - |\xi-\xi_0|^j) D_t^{m-j} \hat{u}(t, \xi-\xi_0) +$$
$$+ \sum_{j=1}^{m} (\sigma(A_j^0)(t_0, x_0, \xi) - \sigma(A_j^0)(t_0, x_0, \xi-\xi_0))|\xi-\xi_0|^j D_t^{m-j} \hat{u}(t, \xi-\xi_0).$$

From the following estimates:

$$|\sigma(A_j^0)(t_0, x_0, \xi)(|\xi|^j - |\xi-\xi_0|^j)| \leqq C_2(1+|\xi-\xi_0|^2)^{(j-1)/2}$$

and

$$|\sigma(A_j^0)(t_0, x_0, \xi) - \sigma(A_j^0)(t_0, x_0, \xi-\xi_0)|\,|\xi-\xi_0|^j$$
$$= \left|\sigma(A_j^0)\Big(t_0, x_0, \frac{\xi}{|\xi|}\Big) - \sigma(A_j^0)\Big(t_0, x_0, \frac{\xi-\xi_0}{|\xi-\xi_0|}\Big)\right| |\xi-\xi_0|^j$$
$$\leqq C_3\left|\frac{\xi}{|\xi|} - \frac{\xi-\xi_0}{|\xi-\xi_0|}\right| |\xi-\xi_0|^j$$
$$\leqq 2C_3 \frac{|\xi-\xi_0|^j}{|\xi|+|\xi-\xi_0|} \leqq C_4(1+|\xi-\xi_0|^2)^{(j-1)/2},$$

we obtain

$$|P^0_{(t_0,x_0)}(D) u e^{i<x,\xi_0>})^\wedge|$$
$$\leqq |D_t^m u(t, \xi-\xi_0) + \sum_{j=1}^{m} \sigma(A_j^0)(t_0, x_0, \xi)|\xi|^j D_t^{m-j} \hat{u}(t, \xi-\xi_0)|$$
$$+(C_2+C_4) \sum_{j=1}^{m} (1+|\xi-\xi_0|^2)^{(j-1)/2} |D_t^{m-j} \hat{u}(t, \xi-\xi_0)|,$$

and therefore

$$(22)\qquad \int_0^{\tilde{t}} \|(P^0_{(t_0,x_0)}(D)ue^{i<x,\xi_0>})(t',\cdot)\|_{(0)}dt'$$

$$\leqq \int_0^{\tilde{t}} \|(P^0_{(t_0,x_0)}(D)u)(t',\cdot)\|_{(0)}dt' + C^0_T \sum_{j=1}^{m-1}\int_0^{\tilde{t}} \|D_t^j u(t',\cdot)\|_{(m-1-j)}dt',$$

where C_2, C_3, C_4 and C^0_T are constants independent of (t_0, x_0).

There exists a constant C_5 such that

$$\frac{1}{C_5^2}(1+|\xi|^2)^{m-1-j} \leqq |\xi|^{2(m-1-j)} + |\xi+\xi_0|^{2(m-1-j)}$$

$$\leqq C_5^2(1+|\xi|^2)^{m-1-j}.$$

Thus we obtain the following estimate from (20), (21) and (22)

$$\frac{1}{C_5}\sum_{j=0}^{m-1}\|S^{m-1-j}D_t^j u(\tilde{t},\cdot)\|_{(0)} \leqq C_T\{C_2\sum_{j=0}^{m-1}\|S^{m-1-j}D_t^j u(0,\cdot)\|_{(0)}$$

$$+2\int_0^{\tilde{t}}\|(P^0_{(t_0,x_0)}(D)u)(t',\cdot)\|_{(0)}dt' + C^0_T \sum_{j=0}^{m-1}\int_0^{\tilde{t}}\|D_t^j u(t',\cdot)\|_{(m-1-j)}dt'.$$

By Lemma 3 we see that $[\tilde{E}^1_{(0)}\uparrow]$ holds for $P^0_{(t_0\ x_0)}(D)$ with a constant independent of $(t_0, x_0)\in[0, T)\times R_n$ and u. By letting $t_0\uparrow T$, we see that $[E^1_{(0)}\uparrow]$ holds for $P^0_{(T,x_0)}(D)$ with the same constant. Thus the proof is complete.

Let $\lambda_j(t, x, \xi)$, $j=1, 2,\cdots, m$, be the roots of the algebraic equation $P^0(t, x, \tau, \xi)=0$ with respect to τ. If (i) $\lambda_j(t, x, \xi)$ are real for $j=1, 2,\cdots, m$ and (ii) there exists a positive constant d_T depending on T such that $|\lambda_j(t, x, \xi)-\lambda_k(t, x, \xi)|\geqq d_T$, $j\neq k$, hold for $t\in[0, T]$, $x\in R_n$ and $\xi\in\Xi_n$ with $|\xi|=1$, then P is said to be regularly hyperbolic.

THEOREM 7. *P is regularly hyperbolic if $[\tilde{E}^1_{(0)}\uparrow\downarrow]$ holds for $P(D)$. The converse holds true when there exists a constant C_T such that*

$$\|A^0_j(t)-A^0_j(t')\|_0 \leqq C_T|t-t'|, \qquad 0\leqq t, t'\leqq T.$$

PROOF. Suppose $[\tilde{E}^1_{(0)}\uparrow\downarrow]$ hold for $P(D)$. Then, it follows from the preceding proposition that $[E^1_{(0)}\uparrow\downarrow]$ holds for $P^0_{(t_0,x_0)}(D)$, $(t_0, x_0)\in[0, T]\times R_n$, with a constant independent of (t_0, x_0). Owing to Theorem 6, $(E^1_{(0)}\uparrow\downarrow)$ holds for $L^0_{(t_0,x_0)}(D)$, the system of linear operators corresponding to $P^0_{(t_0,x_0)}(D)$ under the Calderón transformation, and therefore from Proposition 18 we see that for any fixed (t_0, x_0) the roots $\lambda_j(t_0, x_0, \xi)$, $j=1, 2,\cdots, m$, are real and distinct.

For any ξ', $\xi''\in\Xi_n$ on $|\xi|=1$ we denote by l the spherical distance between ξ', ξ'' and $\xi(s)$, $0\leqq s\leqq l\leqq\pi$, a point on an arc of a great circle with end points ξ', ξ''. Writing $\sigma(A_j)=\hat{A}_j(t, x, \xi)$, we have

$$|\hat{A}_j(t, x, \xi'')-\hat{A}_j(t, x, \xi')|=\left|\int_0^l \sum_{k=1}^n \frac{\partial}{\partial \xi_k}\hat{A}_j(t, x, \xi(s))\frac{\partial \xi_k}{\partial s}ds\right|$$
$$\leqq \int_0^l \left(\sum_{k=1}^n \left\|\frac{\partial}{\partial \xi_k}\hat{A}_j(t, x, \xi(s))\right\|^2\right)^{1/2} ds \leqq Ml,$$

where M is a constant and $l\leqq \pi|\xi'-\xi''|$.

Let us consider the set

$$\mathfrak{S}=\{(\hat{A}_1^0(t, x, \cdot),\cdots, \hat{A}_m^0(t, x, \cdot)): t\in[0, T], x\in R_n\},$$

where $\hat{A}_j^0(t, x, \cdot)$ are continuous functions of $\xi\in \Xi_n$ on $|\xi|=1$ with parameter (t, x), and equip $\mathfrak{S}$ with the uniform convergence topology. Since the set $\mathfrak{S}$ is equicontinuous and uniformly bounded, its closure $\overline{\mathfrak{S}}$ is compact. For any $(\hat{B}_1, \cdots, \hat{B}_m)\in\overline{\mathfrak{S}}$ the polynomial $Q(\tau, \xi)=\tau^m+\sum_{j=1}^m \hat{B}_j(\xi)\tau^{m-j}$ in τ have simple zeros only. Let Δ_Q be its discriminant. Since it is a continuous function of $(\hat{B}_1,\cdots, \hat{B}_m)\in\overline{\mathfrak{S}}$ and $\xi\in\Xi_n$ with $|\xi|=1$, it follows that $\Delta_Q(\xi)\geqq d_T>0$ for a constant d_T depending on T.

Conversely, suppose P is regularly hyperbolic. By means of the Calderón transformation $v_j=S^{m-j}D_t^{j-1}u$, $j=1, 2,\cdots, m$, we are reduced to consider the system of linear operators

$$L\vec{v}\equiv D_t\vec{v}+\vec{A}^0(t)\Lambda\vec{v}+\vec{B}(t)\vec{v}, \qquad \vec{v}=(v_1,\cdots, v_m),$$

where the eigenvalues of the matrix $\hat{\vec{A}}^0(t, x, \xi)$ are $\lambda_j(t, x, \xi)$, $j=1, 2,\cdots, m$. Owing to Theorem 25 in [3, p. 110], we see that there exists an $N\times N$ matrix $\vec{N}(t)$ whose elements are continuous linear operators of the space $\mathscr{H}_{(0)}(R_n)$ into itself for every t and which satisfies the following properties:

(i) $\vec{N}(t)$ is a positive definite Hermitian matrix and $t\to\vec{N}(t)$ is continuous.

(ii) $\|(\vec{N}(t)\vec{A}^0(t)S-S\vec{A}^{0*}(t)\vec{N}(t))\vec{\chi}\|_{(0)}\leqq M\|\vec{\chi}\|_{(0)}$, $0\leqq t\leqq T$, $\vec{\chi}\in C_0^\infty(R_n)$ with a constant M.

By Proposition 22 below (which will be proved in the next section) we see that $(E_{(0)}^1\uparrow\downarrow)$ holds for L and therefore $[E_{(0)}^1\uparrow\downarrow]$ holds for $P(D)$. Thus the proof is complete.

COROLLARY 5. *Assume that* $\hat{A}_j^0(t, x, \xi)$, $j=1, 2,\cdots, m$, *are real. If* $[\tilde{E}_{(0)}^1\uparrow]$ *holds for* $P(D)$, *then* P *is regularly hyperbolic.*

If $A_j^0(t)$ is $\mathfrak{B}$-valued C^{m-j} function of t and $B_j(t)\in\mathfrak{C}_{(j-1)}^{m-j}$ for each j, then we can consider the formal adjoint operator $P^*(D)$. Suppose $[E_{(0)}^2\uparrow]$ holds for $P(D)$ and $P^*(D)$. Then in the same way as in the proof of Proposition 8 in [9, p. 100] we can show that $[\tilde{E}_{(0)}^1\uparrow]$ holds for $P(D)$ and $[\tilde{E}_{(0)}^1\downarrow]$ for $P_{(t_0,x_0)}^*(D)$. Thus we have the following

COROLLARY 6. *If* $[E^2_{(0)}\uparrow]$ *holds for* $P(D)$ *and* $P^*(D)$, *then* P *is regularly hyperbolic.*

6. A generalization of Kaplan's treatment on parabolic operators

Let $\vec{A}(t)\in\mathfrak{C}_{(r)}$. We shall first give a sufficient condition in order that $(\tilde{E}^1_{(0)}\uparrow\downarrow)$ may hold for $L=D_t+\vec{A}(t)$.

PROPOSITION 22. *Let* $\vec{H}(t)$, $0\leqq t\leqq\infty$, *be an* $N\times N$ *positive definite matrix whose elements are continuous linear operators of the space* $\mathscr{H}_{(0)}(R_n)$ *into itself for each* t *and suppose that for any* $T>0$

(i) *There exists a constant* γ_T *such that*

$$\frac{1}{\gamma_T}\|\vec{\chi}\|^2_{(0)}\leqq(\vec{H}(t)\vec{\chi},\vec{\chi})\leqq\gamma_T\|\vec{\chi}\|^2_{(0)},\quad 0\leqq t\leqq T,\quad \vec{\chi}\in C_0^\infty(R_n),$$

(ii) $\vec{H}(t)$ *is locally Lipsitzian*:

$$\|(\vec{H}(t)-\vec{H}(t'))\vec{\chi}\|_{(0)}\leqq C_T\|\vec{\chi}\|_{(0)}|t-t'|,\quad 0\leqq t,t'\leqq T,\quad \vec{\chi}\in C_0^\infty(R_n),$$

(iii) *There exists a constant* C_T *such that*

$$\|(\vec{H}(t)\vec{A}(t)-\vec{A}^*(t)\vec{H}(t))\vec{\chi}\|_{(0)}\leqq C_T\|\vec{\chi}\|_{(0)},\quad 0\leqq t\leqq T,\quad \vec{\chi}\in C_0^\infty(R_n).$$

Then $(\tilde{E}^1_{(0)}\uparrow\downarrow)$ *holds for* L.

PROOF. For any $\vec{u}\in C_0^\infty(R_{n+1})$ we put $\vec{f}=L\vec{u}$, $h^2(t)=(\vec{H}(t)\vec{u}(t,\cdot),\vec{u}(t,\cdot))$ and consider Dini's derivates $D_\pm(h^2(t))$. Then we have

$$\begin{aligned}D_\pm(h^2(t))&\leqq C_T\|\vec{u}(t,\cdot)\|^2_{(0)}+|(\vec{H}(t)D_t\vec{u}(t,\cdot),\vec{u}(t,\cdot))-(\vec{H}(t)\vec{u}(t,\cdot),D_t\vec{u}(t,\cdot))|\\&\leqq C_T\|\vec{u}(t,\cdot)\|^2_{(0)}+|((\vec{H}(t)\vec{A}(t)-\vec{A}^*(t)\vec{H}(t))\vec{u}(t,\cdot),\vec{u}(t,\cdot))|+\\&\qquad\qquad+2\|\vec{f}(t,\cdot)\|_{(0)}\|\vec{H}(t)\vec{u}(t,\cdot)\|_{(0)}\\&\leqq(C_T+C'_T)\|\vec{u}(t,\cdot)\|^2_{(0)}+2\|\vec{f}(t,\cdot)\|_{(0)}\|\vec{H}(t)\vec{u}(t,\cdot)\|_{(0)}\\&\leqq\gamma_T(C_T+C'_T)h^2(t)+2(\gamma_T)^{1/2}\|\vec{f}(t,\cdot)\|_{(0)}h(t)\end{aligned}$$

with a constant C'_T. Put $2C=\gamma_T(C_T+C'_T)$. From the fact that $D_-h^2=2hD_-h$ for $h(t)>0$ and $D_-h^2\leqq 0$ for $h(t)=0$ we obtain

$$D_-(e^{-Ct}h)\leqq e^{-Ct}(\gamma_T)^{1/2}\|\vec{f}(t,\cdot)\|_{(0)},$$

and therefore

$$\|\vec{u}(t_1,\cdot)\|_{(0)}\leqq\gamma_Te^{C(t_1-t_0)}\|\vec{u}(t_0,\cdot)\|_{(0)}+\gamma_T\int_{t_0}^{t_1}e^{-C(t-t_1)}\|\vec{f}(t,\cdot)\|_{(0)}dt$$

$$0\leqq t_0\leqq t_1\leqq T.$$

On the other hand, from the inequality

$$-D_+h(t) \leqq C(h(t)+(\gamma_T)^{1/2}\|\vec{f}(t, \cdot)\|_{(0)})$$

we obtain

$$\|\vec{u}(t_0, \cdot)\|_{(0)} \leqq \gamma_T e^{C(t_1-t_0)}\|\vec{u}(t, \cdot)\|_{(0)} + \gamma_T\int_{t_0}^{t_1} e^{C(t-t_0)}\|\vec{f}(t, \cdot)\|_{(0)}dt,$$

which completes the proof.

PROPOSITION 23. *If we assume in Proposition 22 that*

$$\vec{H}^{-1}(t)((\mathscr{D}_{L^2})) \subset \mathscr{H}_{(r)}(R_n) \qquad \textit{for each } t,\ 0 \leqq t < \infty,$$

then $(\tilde{E}^1_{(0)}\uparrow\downarrow)$ *holds for* L^*.

PROOF. From the condition (i), it follows that $\vec{H}^{-1}(t)$ exists and has the property (i). For any t, t' with $0 \leqq t, t' \leqq T$, we obtain

$$\begin{aligned}\|(\vec{H}^{-1}(t)-\vec{H}^{-1}(t'))\vec{\chi}\|_{(0)} &= \|(\vec{H}^{-1}(t)(\vec{H}(t')-\vec{H}(t))\vec{H}^{-1}(t'))\vec{\chi}\|_{(0)}\\ &\leqq C_T\gamma_T^2|t'-t|\,\|\vec{\chi}\|_{(0)}.\end{aligned}$$

If we put $\vec{\phi}(t, \cdot)=\vec{H}^{-1}(t)\vec{\chi} \in \mathscr{H}_{(r)}(R_n)$ for any $\vec{\chi} \in C_0^\infty(R_n)$, then we have

$$\begin{aligned}\|(\vec{H}^{-1}(t)\vec{A}^*(t)-\vec{A}(t)\vec{H}^{-1}(t))\vec{\chi}\|_{(0)} &= \|\vec{H}^{-1}(t)(\vec{A}^*(t)\vec{H}(t)-\vec{H}(t)\vec{A}(t))\vec{\phi}(t, \cdot)\|_{(0)}\\ &\leqq \gamma_T C_T\|\vec{\phi}(t, \cdot)\|_{(0)}\\ &\leqq \gamma_T^2 C_T\|\vec{\chi}\|_{(0)}.\end{aligned}$$

Applying the preceding proposition, we see that $(\tilde{E}^1_{(0)}\uparrow\downarrow)$ holds for L^*.

Let $\vec{A}(t)$ be an element of $\mathfrak{C}_{(r)}$, $r>0$, satisfying the pseudo-commutativity $(*)$ and assume there exists an $N\times N$ positive definite matrix $\vec{H}(t)$, $0 \leqq t \leqq \infty$, whose elements are continuous linear operators of $\mathscr{H}_{(-r/2)}(R_n)$ into itself for each t, and assume $\vec{H}(t)$ has the following properties: (i) There exists a constant γ_T such that $\frac{1}{\gamma_T}\|\vec{\chi}\|^2_{(0)} \leqq (\vec{H}(t)\vec{\chi}, \vec{\chi}) \leqq \gamma_T\|\vec{\chi}\|^2_{(0)}$, $0 \leqq t \leqq T$, for $\vec{\chi} \in C_0^\infty(R_n)$. (ii) There exists a constant C_T such that $\|(\vec{H}(t)-\vec{H}(t'))\vec{\chi}\|_{(0)} \leqq C_T\|\vec{\chi}\|_{(0)}|t-t'|$, $0 \leqq t, t' \leqq T$, for $\vec{\chi} \in C_0^\infty(R_n)$ and (iii)′ $(\vec{H}(t)\vec{A}(t)\vec{\chi}, \vec{\chi})$ is coercive in the sense:

$$\operatorname{Im}(\vec{H}(t)\vec{A}(t)\vec{\chi}, \vec{\chi}) \leqq \mu_0\|\vec{\chi}\|^2_{(0)} - \mu_1\|\vec{\chi}\|^2_{(r/2)},\ 0 \leqq t \leqq T,\ \vec{\chi} \in C_0^\infty(R_n)$$

with constants $\mu_1=\mu_1(T)$ and $\mu_0=\mu_0(T)>0$.

S. Kaplan [11] has investigated the Cauchy problem for the parabolic operator with the form:

$$\frac{\partial}{\partial t}-M(t)=\frac{\partial}{\partial t}-\sum_{|\alpha|\leqq 2m} a_\alpha(t, x)D_x^\alpha, \qquad a_\alpha \in \mathscr{B}(R_{n+1})$$

and $M(t)$ is assumed to be uniformly strongly elliptic, where m is a positive integer. The operator $M(t)$ satisfies the condition (iii)$'$ with $N=1$, $H(t)=1$ and $r=2m$.

We shall first note that the following energy inequality holds true for $L=D_t+\vec{A}(t)$.

THEOREM 8. *For any* $\vec{u}\in\mathscr{H}_{(0,s+r)}(H)$, $H=[0, T]\times R_n$, *there exists a constant* C_T *such that*

$$(23)\qquad \|\vec{u}(t_1,\cdot)\|^2_{(s+r/2)}+\int_{t_0}^{t_1}\|\vec{u}(t,\cdot)\|^2_{(s+r)}dt\leqq C_T(\|\vec{u}(t_0,\cdot)\|^2_{(s+r/2)}+ +\int_{t_0}^{t_1}\|L\vec{u}(t,\cdot)\|^2_{(s)}dt),\qquad 0\leqq t_0\leqq t_1\leqq T.$$

PROOF. For any $\vec{u}\in C_0^\infty(R_{n+1})$, if we put $\vec{f}=L\vec{u}$, then we have

$$\begin{aligned}
&D_-(\vec{H}(t)\vec{u}(t,\cdot),\ \vec{u}(t,\cdot))\\
&\quad\leqq K\|\vec{u}(t,\cdot)\|^2_{(0)}+i\{(\vec{H}(t)D_t\vec{u}(t,\cdot),\ \vec{u}(t,\cdot))-(\vec{H}(t)\vec{u}(t,\cdot),\ D_t\vec{u}(t,\cdot))\}\\
&\quad=K\|\vec{u}(t,\cdot)\|^2_{(0)}+2\mathrm{Im}(\vec{u}(t,\cdot),\ \vec{H}(t)\vec{f}(t,\cdot))+2\mathrm{Im}(\vec{H}(t)\vec{A}(t)\vec{u}(t,\cdot),\ \vec{u}(t,\cdot))\\
&\quad\leqq 2|(\vec{H}(t)\vec{f}(t,\cdot),\ \vec{u}(t,\cdot))|-2\mu_1\|\vec{u}(t,\cdot)\|^2_{(r/2)}+(2\mu_0+K)\|\vec{u}(t,\cdot)\|^2_{(0)}
\end{aligned}$$

with a constant K. Putting $h^2(t)=(\vec{H}(t)\vec{u}(t,\cdot),\ \vec{u}(t,\cdot))$, we obtain

$$h^2(t_1)-h^2(t_0)\leqq 2\int_{t_0}^{t_1}|(\vec{H}(t)\vec{f}(t,\cdot),\ \vec{u}(t,\cdot))|\,dt- -2\mu_1\int_{t_0}^{t_1}\|\vec{u}(t,\cdot)\|^2_{(r/2)}dt+(2\mu_0+K)\int_{t_0}^{t_1}\|\vec{u}(t,\cdot)\|^2_{(0)}dt,$$

and therefore if we put $\vec{v}=S^{-s-r/2}\vec{u}$, then we can write

$$\frac{1}{\gamma}\|\vec{v}(t_1,\cdot)\|^2_{(s+r/2)}-\gamma_T\|\vec{v}(t_0,\cdot)\|^2_{(s+r/2)}\leqq 2\int_{t_0}^{t_1}|(\vec{H}(t)LS^{s+r/2}\vec{v},\ S^{s+r/2}\vec{v})|\,dt- -2\mu_1\int_{t_0}^{t_1}\|\vec{v}(t,\cdot)\|^2_{(s+r)}dt+(2\mu_0+K)\int_{t_0}^{t_1}\|\vec{v}(t,\cdot)\|^2_{(s+r/2)}dt,$$

where

$$\begin{aligned}
&|(\vec{H}(t)LS^{s+r/2}\vec{v},\ S^{s+r/2}\vec{v})|\\
&\leqq|(\vec{H}(t)S^{s+r/2}Lv,\ S^{s+r/2}\vec{v})|+|(\vec{H}(t)(\vec{A}(t)S^{s+r/2}-S^{s+r/2}\vec{A}(t))\vec{v},\ S^{s+r/2}\vec{v})|\\
&\leqq C_1\|(L\vec{v})(t,\cdot)\|_{(s)}\|\vec{v}(t,\cdot)\|_{(s+r)}+C_2\|\vec{v}(t,\cdot)\|_{(s+r-1)}\|\vec{v}(t,\cdot)\|_{(s+r)}.
\end{aligned}$$

For any given $\varepsilon>0$ there exists a constant $C(\varepsilon)$ such that

$$\|\vec{v}(t,\cdot)\|^2_{(s+r-1)}\leqq\varepsilon^2\|\vec{v}(t,\cdot)\|^2_{(s+r)}+C(\varepsilon)\|\vec{v}(t,\cdot)\|^2_{(s)}.$$

Applying the Schwarz inequality, we have the estimate with a constant $C'(\varepsilon)$

$$2\int_{t_0}^{t_1}|(\vec{H}(t)LS^{s+r/2}\vec{v},\ S^{s+r/2}\vec{v})|\,dt \leqq 3\varepsilon\int_{t_0}^{t_1}\|\vec{v}(t,\ \cdot)\|_{(s+r)}^2 dt +$$

$$+C'(\varepsilon)\int_{t_0}^{t_1}\|L\vec{v}(t,\ \cdot)\|_{(s)}^2 dt + C'(\varepsilon)\int_{t_0}^{t_1}\|\vec{v}(t,\ \cdot)\|_{(s)}^2 dt.$$

Taking $\varepsilon=\frac{1}{3}\mu_1$ and applying Lemma 3 in Section 2, we obtain the inequality with a constant C_T such that

$$\|\vec{v}(t_1,\ \cdot)\|_{(s+r/2)}^2 + \int_{t_0}^{t_1}\|\vec{v}(t,\ \cdot)\|_{(s+r)}^2 dt \leqq C_T(\|\vec{v}(t_0,\ \cdot)\|_{(s+r/2)}^2 +$$

$$+\int_{t_0}^{t_1}\|L\vec{v}(t,\ \cdot)\|_{(s)}^2 dt,$$

which will yield the estimate (23) since $C_0^\infty(H)$ is dense in $\mathscr{H}_{(0,s+r)}(H)$. Thus the proof is complete.

By modifying the method developed in Sections 2 and 3 we shall show the uniqueness and existence theorems for the Cauchy problem:

$$\text{(24)}\qquad \begin{cases} L\vec{u}=\vec{f} & \text{in } \mathring{H}, \\ \mathscr{D}'_{L^2}\text{-}\lim\limits_{t\downarrow 0}\ \vec{u}=\vec{\alpha} \end{cases}$$

for any preassigned $\vec{\alpha}\in(\mathscr{D}'_{L^2})_x$ and $\vec{f}\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$, where $\vec{f}$ is assumed to have the $\mathscr{D}'_{L^2}$-canonical extension $\vec{f}_\sim$.

From now on, we assume that $\vec{A}(t)\in\mathfrak{C}_{(r)}^\infty$.

THEOREM 9. *If $\vec{u}\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$ and $\vec{u}$ is a solution of the Cauchy problem $L\vec{u}=0$ in $\mathring{H}$ with initial condition $\mathscr{D}'_{L^2}\text{-}\lim\limits_{t\downarrow 0}\vec{u}=0$, then $\vec{u}=0$.*

PROOF. There exists real numbers σ, s such that $\vec{u}\in\mathscr{K}^{(\sigma,s)}(H)$. From the equation $D_t\vec{u}=-\vec{A}(t)\vec{u}\in\mathscr{K}^{(\sigma,s-r)}(H)$ we see that $\vec{u}\in\mathscr{K}^{(\sigma+r,s-r)}(H)$. Thus we may assume that $\sigma\geqq 0$. From the energy inequality in the preceding theorem we can conclude that $\vec{u}=0$.

We shall say that $(\mathrm{CP})'_{(s)}$ holds for L if the Cauchy problem (24) has a solution $\vec{u}\in\mathscr{H}_{(0,s+r)}(H)$ for any given $\vec{f}\in\mathscr{H}_{(0,s)}(H)$ and $\vec{\alpha}\in\mathscr{H}_{(s+r/2)}(R_n)$. Then, in the same way as in the proof of Propositions 4 and 5 we have

PROPOSITION 4′. *If $(\mathrm{CP})'_{(s)}$ holds for L, then it also holds for $L^1=L+\vec{B}(t)$, $\vec{B}(t)\in\mathfrak{C}_{(r-1)}$.*

PROPOSITION 5′. *If $(\mathrm{CP})'_{(s)}$ holds for some s, then it does for any s.*

Next we shall show an analogue of Proposition 7.

PROPOSITION 7'. $(\mathrm{CP})'_s$ *holds for L if and only if the conditions that* $\vec{w} \in \mathscr{K}_{(0,-s)}(H)$, $L^*\vec{w}=0$ *in* $\mathring{H}$ *and* $\mathscr{D}'_{L^2}\text{-}\lim_{t\uparrow T} \vec{w}=0$ *imply* $\vec{w}=0$ *in* $\mathring{H}$.

PROOF. Let $(\mathrm{CP})'_{(s)}$ hold for L and $\vec{w} \in \mathscr{H}_{(0,-s)}(H)$ and assume that $L^*\vec{w}=0$ in $\mathring{H}$ with $\mathscr{D}'_{L^2}\text{-}\lim_{t\uparrow T} \vec{w}=0$. For any $\vec{f} \in C_0^\infty(\mathring{H})$, let $\vec{u} \in \mathscr{H}_{(0,s+r)}(H)$ be a solution of $L\vec{u}=\vec{f}$. From the fact that the energy inequality (23) holds true, there exists a sequence $\{\vec{\phi}_j\}$, $\vec{\phi}_j \in C_0^\infty(H)$ vanishing near $t=0$ and we have $\int_0^T (L\vec{\phi}_j(t, \cdot), \vec{w}(t, \cdot))dt=0$. Thus $\vec{w}=0$ in $\mathring{H}$.

To prove the converse, we first show that $A=\{(\vec{\phi}(0, \cdot), L\vec{\phi}): \vec{\phi} \in C_0^\infty(H)\}$ is dense in $\mathscr{H}_{(s+r/2)}(R_n)\times\mathscr{H}_{(0,s)}(H)$. Let $(i\vec{\beta}, \vec{w}) \in \mathscr{H}_{(-s-r/2)}(R_n)\times\mathscr{H}_{(0,-s)}(H)$ such that

$$\int_0^T (L\vec{\phi}(t, \cdot), \vec{w}(t, \cdot))dt - i(\vec{\phi}_0, \vec{\beta})=0, \qquad \vec{\phi} \in C_0^\infty(H),$$

which implies $L^*\vec{w}=0$ in $\mathring{H}$, and $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} \vec{w}=0$. Thus we see that $\vec{w}=0$ in $\mathring{H}$ and $\vec{\beta}=0$. For any $\vec{f} \in \mathscr{H}_{(0,s)}(H)$ and $\vec{\alpha} \in \mathscr{H}_{(s+r/2)}(R_n)$ there exists a sequence $\{\vec{\phi}_j\}$, $\vec{\phi}_j \in C_0^\infty(H)$, such that $\vec{\phi}_j(0, \cdot)$, $L\vec{\phi}_j$ converge in $\mathscr{H}_{(s+r/2)}(R_n)$, $\mathscr{H}_{(0,s)}(H)$ to $\vec{\alpha}$, $\vec{f}$ respectively as $j\to\infty$. From the energy inequality (23) we see that $\{\vec{\phi}_j\}$ is a Cauchy sequence in $\mathscr{H}_{(0,s+r)}(H)$. Let $\vec{u}$ be the limit of the sequence $\{\vec{\phi}_j\}$. Then $\vec{u}$ satisfies the equation $L\vec{u}=\vec{f}$ in $\mathring{H}$ with $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} \vec{u}=\vec{\alpha}$. From the fact that $\vec{u} \in \mathscr{H}_{(0,s+r)}(H)$ and $D_t\vec{u} \in \mathscr{H}_{(0,s)}(H)$ we see that $\vec{u} \in \mathscr{K}^{(r,s)}(H)$.

PROPOSITION 24. *Suppose* $(\mathrm{CP})'_{(0)}$ *holds for L. For any* $\vec{f} \in \mathscr{K}^{(kr,s)}(H)$, *k being any non-negative integer, and* $\vec{\alpha} \in \mathscr{H}_{((k+1/2)r+s)}(R_n)$ *there exists a unique solution* $\vec{u} \in \mathscr{K}^{((k+1)r,s)}(H)$ *of the Cauchy problem* (24). *Furthermore* $\vec{u}$ *satisfies the inequality*

$$(25)\qquad \sum_{j=0}^{k} \|D_t^j\vec{u}(t, \cdot)\|^2_{(s+(k-j+1/2)r)} + \sum_{j=0}^{k}\int_0^t \|D_t^j\vec{u}(t', \cdot)\|^2_{(s+(k-j+1)r)}dt'$$
$$\leqq C_T(\|\vec{\alpha}\|^2_{(s+(k+1/2)r)} + \sum_{j=0}^{k-1}\|D_t^j\vec{f}(0, \cdot)\|^2_{(s+(k-1/2)r)} +$$
$$+\sum_{j=0}^{k}\int_0^t \|D_t^j\vec{f}(t', \cdot)\|^2_{(s+(k-j)r)}dt')$$

with a constant C_T.

PROOF. If $k=0$ this result is already shown in Theorem 8 and Proposition 7'. Let $\vec{f} \in \mathscr{K}^{(r,s)}(H)$ and $\vec{\alpha} \in \mathscr{H}_{(s+(3/2)r)}(R_n)$. Then the Cauchy problem (24) has a unique solution $\vec{u} \in \mathscr{K}^{(2r,s)}(H)$. Furthermore $\vec{u}$ satisfies the inequality:

$$(26)\qquad \|\vec{u}(t, \cdot)\|^2_{(s+(3/2)r)} + \int_0^t \|\vec{u}(t', \cdot)\|^2_{(s+2r)}dt \leqq C_1(\|\vec{u}(0, \cdot)\|^2_{(s+(3/2)r)} +$$
$$+\int_0^t \|Lu(t', \cdot)\|^2_{(s+r)}dt').$$

Put $\vec{v}=D_t\vec{u}$. Then $\vec{v}\in\mathscr{K}^{(r,s)}(H)$ and we have $L\vec{v}=D_t\vec{f}+i\vec{A}'(t)\vec{u}\in\mathscr{H}_{(0,s)}(H)$, and therefore

$$\|\vec{v}(t,\cdot)\|^2_{(s+r/2)}+\int_0^t\|\vec{v}(t',\cdot)\|^2_{(s+r)}dt\leqq C_2(\|\vec{v}(0,\cdot)\|^2_{(s+r/2)}+$$
$$+\int_0^t\|D_t\vec{f}(t',\cdot)\|^2_{(s)}dt'+\int_0^t\|\vec{A}'(t')\vec{u}(t',\cdot)\|^2_{(s)}dt'),$$

where C_2 is a constant. Applying Lemma 3 in Section 2, we obtain with a constant C_3

$$\|\vec{v}(t,\cdot)\|^2_{(s+r/2)}+\int_0^t\|\vec{v}(t',\cdot)\|^2_{(s+r)}dt'\leqq C_3(\|\vec{v}(0,\cdot)\|^2_{(s+r/2)}+ \tag{27}$$
$$+\int_0^t\|D_t\vec{f}(t',\cdot)\|^2_{(s)}dt').$$

From (26) and (27) we have with a constant C_T

$$\|\vec{u}(t,\cdot)\|^2_{(s+(3/2)r)}+\|D_t\vec{u}(t,\cdot)\|^2_{(s+r/2)}+\int_0^t\|\vec{u}(t',\cdot)\|^2_{(s+2r)}dt+$$
$$+\int_0^t\|D_t\vec{u}(t',\cdot)\|^2_{(s+r)}dt\leqq C_T(\|\vec{u}(0,\cdot)\|^2_{(s+(3/2)r)}+\|\vec{f}(0,\cdot)\|^2_{(s+r/2)}+$$
$$+\int_0^t\|\vec{f}(t',\cdot)\|^2_{(s+r)}dt'+\int_0^t\|D_t\vec{f}(t',\cdot)\|^2_{(s)}dt').$$

Repeating this procedure, we obtain the inequality (25).

We note here that $\mathscr{K}^{(kr,s)}(H)$, k being non-negetive integer, has the equivalent norm

$$\Big(\sum_{j=0}^{k}\int_0^T D_t^j\|u(t',\cdot)\|^2_{(s+(k-j)r)}dt'\Big)^{1/2}.$$

With the aid of the interpolation theorem for the Hilbert scales, we can show

COROLLARY 7, *Suppose* $(\mathrm{CP})'_{(0)}$ *holds for* L. *For any* $\vec{f}\in\mathscr{K}^{(\sigma,s)}(H)$, $\sigma\geqq 0$ *and* $\vec{\alpha}\in\mathscr{H}_{(\sigma+s+r/2)}(R_n)$ *there exists a unique solution of the Cauchy problem* (24) *and* $(\vec{\alpha},\vec{f})\to\vec{u}$ *is a continuous map of* $\mathscr{H}_{(\sigma+s+r/2)}(R_n)\times\mathscr{K}^{(\sigma,s)}(H)$ *into* $\mathscr{K}^{(\sigma+r,s)}(H)$.

We shall denote by $\mathring{\mathscr{K}}^{(\sigma,s)}(H_-)$ the space which is a restriction of the space $\mathring{\mathscr{K}}^{(\sigma,s)}(\bar{R}^+_{n+1})$ to $(-\infty, T]\times R_n$ and similarly for $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H_-)$. By Proposition 7 in [8, p. 416] we see that for every $\vec{u}\in\mathscr{K}^{(\sigma,s)}(H)$ with $|\sigma|<\frac{r}{2}$ its canonical extension $\vec{u}_\sim$ over $t=0$ belongs to the space $\mathring{\mathscr{K}}^{(\sigma,s)}(H_-)$.

PROPOSITION 25. *Suppose* $(\mathrm{CP})'_{(0)}$ *holds for* L. *For any* $\vec{f}\in\mathscr{K}^{(\sigma,s)}(H)$, $-\frac{r}{2}<\sigma<0$ *and* $\vec{\alpha}\in\mathscr{H}_{(\sigma+s+r/2)}(R_n)$ *there exists a unique solution* $\vec{u}\in\mathscr{K}^{(\sigma+r,s)}(H)$ *of the Cauchy problem* (24).

PROOF. For any given $\vec{f}\in\mathscr{K}^{(\sigma,s)}(H)$ we shall consider $\vec{g}$ satisfying the equation

$$D_t\vec{g}-i\lambda^r(D_x)\vec{g}=\vec{f}_{\sim}.$$

Then $\vec{g}\in\mathring{\mathscr{K}}^{(\sigma+r,s)}(H_-)$ and therefore $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}\vec{g}=0$. The Cauchy problem (24) can be written in the form

$$(28)\qquad \begin{cases} D_t(\vec{u}-\vec{g})+\vec{A}(t)(\vec{u}-\vec{g})=-i\lambda^r(D_x)\vec{g}-\vec{A}(t)\vec{g} & \text{in } \mathring{H},\\ \mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(\vec{u}-\vec{g})=\vec{\alpha}, \end{cases}$$

where $-i\lambda^r(D_x)\vec{g}-\vec{A}(t)\vec{g}\in\mathring{\mathscr{K}}^{(\sigma+r,s-r)}(H_-)$, $\frac{r}{2}<\sigma+r<r$. Thus there exists a unique solution $\vec{v}=\vec{u}-\vec{g}\in\mathscr{K}^{(\sigma+2r,s-r)}(H)$ and therefore $\vec{u}=\vec{v}+\vec{g}\in\mathscr{K}^{(\sigma+r,s)}(H)$.

Let σ, s be any real numbers and write $\sigma=kr+\sigma'$ with integer k and $-\frac{r}{2}<\sigma'\leqq\frac{r}{2}$. We are now prepared to show the following theorem, a generalization of a result of S. Kaplan [11, p. 180].

THEOREM 10. *Suppose* $(\mathrm{CP})'_{(0)}$ *holds for* L. *For any* $\vec{\alpha}\in\mathscr{H}_{(\sigma+s+r/2)}(R_n)$ *and* $\vec{f}\in\mathscr{K}^{(\sigma,s)}(H)$ *with* $\vec{f}_{\sim}\in\mathring{\mathscr{K}}^{(\sigma,s)}(H_-)$, *there exists a unique solution* $\vec{u}\in\mathscr{K}^{(\sigma+r,s)}(H)$ *of the Cauchy problem* (24). *In particular, if* $\vec{\alpha}=0$ *then* $\vec{u}_{\sim}\in\mathring{\mathscr{K}}^{(\sigma+r,s)}(H_-)$.

PROOF. Consider the case $k\geqq 0$. In Corollary 7 we have shown that there exists a solution $\vec{u}\in\mathscr{K}^{(\sigma+r,s)}(H)$. Since a solution of the Cauchy problem is unique in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$, we have only to show that if $\vec{\alpha}=0$ then $\vec{u}_{\sim}\in\mathring{\mathscr{K}}^{(\sigma+r,s)}(H_-)$. Suppose $\vec{\alpha}=0$. Then $\lim_{t\downarrow 0}(\vec{u},\cdots,D_t^k\vec{u})=0$. In fact, if $k=0$ then $\lim_{t\downarrow 0}\vec{u}=\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}\vec{u}=0$. Let $k>0$. Then the condition $\vec{f}_{\sim}\in\mathring{\mathscr{K}}^{(kr+\sigma',s)}(H_-)$ implies $\lim_{t\downarrow 0}(\vec{f},\cdots,D_t^{k-1}\vec{f})=\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(\vec{f},\cdots,D_t^{k-1}\vec{f})=0$ (cf. Theorem 3 in [8, p. 419]). In the same way as in the proof of Proposition 13 we can prove that $\lim_{t\downarrow 0}(\vec{u},\cdots,D_t^k\vec{u})=0$. In the case $\sigma'<\frac{r}{2}$, Theorem 3 in [8, p. 419] implies immediately $\vec{u}\in\mathring{\mathscr{K}}^{(\sigma+r,s)}(H_-)$. Let $\sigma'=\frac{r}{2}$. Then $\vec{u}_{\sim}\in\mathring{\mathscr{K}}^{(\sigma+r-\varepsilon,s+\varepsilon)}(H_-)\subset\mathring{\mathscr{K}}^{(\sigma,s+r)}(H_-)$, $0<\varepsilon\leqq r$. Combining with the relation $D_t(\vec{u}_{\sim})=\vec{f}_{\sim}-\vec{A}(t)\vec{u}_{\sim}\in\mathring{\mathscr{K}}^{(\sigma,s)}(H_-)$ shows that $\vec{u}_{\sim}\in\mathring{\mathscr{K}}^{(\sigma+r,s)}(H_-)$.

Consider the case where $k<0$. We shall reason by descending induction over k. Assume that the results hold true of any $k+1$. Let $\vec{f}_{\sim}\in\mathring{\mathscr{K}}^{(\sigma,s)}(H_-)$,

$\sigma=kr+\sigma'$, and $\vec{\alpha}\in\mathscr{H}_{(\sigma+s+r/2)}(R_n)$. Let $\vec{g}\in\mathring{\mathscr{K}}^{(\sigma+r,s)}(H_-)$ be a solution of the equation

$$D_t\vec{g}-i\lambda^r(D_x)\vec{g}=\vec{f}_\sim.$$

Then $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}\vec{g}=0$. The Cauchy problem (24) can be written in the form (28) and $-i\lambda^r(D_x)\vec{g}-\vec{A}(t)\vec{g}\in\mathring{\mathscr{K}}^{(\sigma+r,s-r)}(H_-)$, and therefore there exists a solution $\vec{v}=\vec{u}-\vec{g}\in\mathscr{K}^{(\sigma+2r,s-r)}(H)$ and $\vec{u}=\vec{v}+\vec{g}\in\mathscr{K}^{(\sigma+r,s)}(H)$. Especially, if $\vec{\alpha}=0$ then $\vec{u}_\sim\in\mathring{\mathscr{K}}^{(\sigma+r,s)}(H_-)$.

Along the same line as in the proof of the preceding theorem we can prove the following

PROPOSITION 26. *Suppose* $(\mathrm{CP})'_{(0)}$ *holds for* L. *For any* $\vec{h}\in\mathring{\mathscr{K}}^{(\sigma,s)}(H_-)$ *there exists a unique solution* $\vec{v}\in\mathring{\mathscr{K}}^{(\sigma+r,s)}(H_-)$ *of* $L\vec{v}=\vec{h}$.

The following theorems are the analogues of Theorems 4 and 5 and can be proved in a similar way, so the proofs are omitted.

THEOREM 11. *Suppose* $(\mathrm{CP})'_{(0)}$ *holds for* L. *Then for any* $\vec{h}\in(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ *there exists a unique solution* $\vec{v}\in(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ *of* $L\vec{v}=\vec{h}$ *and* $\vec{h}\to\vec{v}$ *is a continuous map of* $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ *onto itself.*

THEOREM 12. *Suppose* $(\mathrm{CP})'_{(0)}$ *holds for* L. *Then for any* $\vec{\alpha}\in(\mathscr{D}'_{L^2})_x$ *and* $\vec{f}\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$, *which has the* $\mathscr{D}'_{L^2}$*-canonical extension* $\vec{f}_\sim\in(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$, *the Cauchy problem* (24) *has a unique solution* $\vec{u}\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$ *and* $(\vec{\alpha},\vec{f}_\sim)\to\vec{u}_\sim$ *is a continuous map under the topology of* $(\mathscr{D}'_{L^2})_x\times(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ *and the topology of* $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$.

7. Notes on a system of ordinary differential operators

Let L be a system of ordinary differential operators of the form $L=D_t+\vec{A}(t)$, where $\vec{A}(t)$ is an $N\times N$ matrix of C^∞ functions on R_t and consider the Cauchy problem:

$$(29)\qquad\begin{cases} L\vec{u}=\vec{f} \quad \text{in } R^+,\\ \lim_{t\downarrow 0}\vec{u}=\vec{\alpha}\end{cases}$$

for any preassigned $\vec{f}\in\mathscr{D}'(R^+)$ and $\vec{\alpha}\in\boldsymbol{C}^N$. If $\vec{u}\in\mathscr{D}'(R^+)$ exists, then $\vec{f}$ has the canonical extension $\vec{f}_\sim\in\mathscr{D}'_+$ and

$$(30)\qquad L(\vec{u}_\sim)=\vec{f}_\sim-i\vec{\alpha}\delta.$$

Conversely, if $\vec{v}\in\mathscr{D}'_+$ satisfies the equation (30), then the restriction $\vec{u}=\vec{v}\,|\,R^+$ is a solution of the Cauchy problem (29) and $\vec{v}=\vec{u}_\sim$, where by $\mathscr{D}'_+$ we mean the closed subspace of $\mathscr{D}'(R)$ with support $\subset \bar{R}^+$.

Now we shall show that our considerations in the present paper can also be applied to the Cauchy problem for ordinary differential system as a special case.

For any $\vec{\phi} \in C_0^\infty(R)$

$$\vec{\phi}(t_1) = \vec{\phi}(t_0) + \int_{t_0}^{t_1} \vec{\phi}'(t)dt,$$

whence for any $T>0$

$$|\vec{\phi}(t_1)| \leqq |\vec{\phi}(t_0)| + \int_{t_0}^{t_1} |L\vec{\phi}(t)|\, dt + \max_{0\leqq t\leqq T} |\vec{A}(t)| \int_{t_0}^{t_1} |\vec{\phi}(t)|\, dt, \quad 0 \leqq t_0 \leqq t_1 \leqq T.$$

By Lemma 3 in Section 2, we can find a constant C_T such that

$$(31) \qquad |\vec{\phi}(t_1)| \leqq C_T(|\vec{\phi}(t_0)| + \int_{t_0}^{t_1} |L\vec{\phi}(t)|\, dt), \quad 0 \leqq t_0 \leqq t_1 \leqq T, \quad \vec{\phi} \in C_0^\infty(R).$$

Similarly, for the formal adjoint L^* of L we have

$$(32) \qquad |\vec{\phi}(t_0)| \leqq C_T(|\vec{\phi}(t_1)| + \int_{t_0}^{t_1} |L^*\vec{\phi}(t)|\, dt), \quad 0 \leqq t_0 \leqq t_1 \leqq T, \quad \vec{\phi} \in C_0^\infty(R).$$

We shall first show the following

THEOREM 12. *If* $\vec{u} \in \mathscr{D}'(R^+)$ *satisfies* $L\vec{u}=0$ *in* $t>0$ *and* $\lim_{t\downarrow 0} \vec{u}=0$, *then* $\vec{u}=0$.

PROOF. In the same way as in the proof of Proposition 8 in [7, p. 22] we see that $\vec{u} \in \mathscr{E}_t^\infty(R^+)$. By the energy inequality (31) we have immediately $\vec{u}=0$ in R^+.

In what follows we shall show the existence theorems for the Cauchy problem (29).

PROPOSITION 27. *Let* $\sigma > -\frac{1}{2}$. *For any* $\vec{\alpha} \in \boldsymbol{C}^N$ *and* $\vec{f} \in \tilde{\mathscr{H}}_{(\sigma)}(\bar{R}^+)$ *there exists a unique solution* $\vec{u} \in \tilde{\mathscr{H}}_{(\sigma+1)}(\bar{R}^+)$ *of the Cauchy problem* (29).

PROOF. (1) Let $\sigma=k$ be a non-negative integer. First consider the case $k=0$. The set $A=\{(\vec{\phi}(0), L\vec{\phi}): \vec{\phi} \in C_0^\infty(\bar{R}^+)\}$ is dense in $\boldsymbol{C}^N \times \tilde{\mathscr{H}}_{(0)}(\bar{R}^+)$. In fact, let $(i\vec{\beta}, \vec{w}) \in \boldsymbol{C}^N \times \tilde{\mathscr{H}}^*_{(0)}(\bar{R}^+)$ such that

$$(L\vec{\phi}, \vec{w}) - i(\vec{\phi}(0), \vec{\beta}) = 0, \qquad \vec{\phi} \in C_0^\infty(\bar{R}^+).$$

Then we see that $L^*\vec{w}=0$ in R^+. By the energy inequality (32) we conclude that $\vec{w}=0$ in R^+ and therefore $\vec{\beta}=0$.

For any $\vec{\alpha} \in \boldsymbol{C}^N$ and $\vec{f} \in \tilde{\mathscr{H}}_{(k)}(\bar{R}^+)$ there exists a sequence $\{\vec{\phi}_j\}$, $\vec{\phi}_j \in C_0^\infty(\bar{R}^+)$, such that $\vec{\phi}_j(0)$, $L\vec{\phi}_j$ converge in $\boldsymbol{C}^N$ and $\tilde{\mathscr{H}}_{(k)}(\bar{R}^+)$ to $\vec{\alpha}$, $\vec{f}$ respectively as $j \to \infty$.

By the energy inequality (31) we see that $\vec{\phi}_j$ is a Cauchy sequence in $\mathscr{E}_t^0(\bar{R}^+)$. Let $\vec{u}$ be the limit of $\vec{\phi}_j$. Then $\vec{u} \in \mathscr{E}_t^0(\bar{R}^+)$ is a solution of the Cauchy problem (24). From the equation $D_t\vec{u}=\vec{f}-\vec{A}(t)\vec{u} \in \mathscr{H}_{(0)}(\bar{R}^+)$ we see that $\vec{u} \in \mathscr{H}_{(1)}(\bar{R}^+)$.

Let $k=1$ and $\vec{f} \in \mathscr{H}_{(1)}(\bar{R}^+)$. Then $\vec{v}=D_t\vec{u} \in \tilde{\mathscr{H}}_{(0)}(\bar{R}^+)$ and $L\vec{v}=D_t\vec{f}+iA'(t)\vec{u} \in \mathscr{H}_{(0)}(\bar{R}^+)$, and therefore $\vec{v} \in \tilde{\mathscr{H}}_{(1)}(\bar{R}^+)$, which implies $\vec{u} \in \tilde{\mathscr{H}}_{(2)}(\bar{R}^+)$.

In the case where $\sigma=k\geqq 2$, repeating this procedure, we see that $\vec{u} \in \tilde{\mathscr{H}}_{(k+1)}(\bar{R}^+)$.

(2) Let σ be a non-negative real number. For any $\vec{\alpha} \in C^N$ and $\vec{f} \in \tilde{\mathscr{H}}_{(\sigma)}(\bar{R}^+)$ there exists a solution $\vec{u} \in \mathscr{H}_{(k+1)}(\bar{R}^+)$ of the Cauchy problem (29), where $k=[\sigma]$. Since $D_t\vec{u}=\vec{f}-\vec{A}(t)\vec{u} \in \tilde{\mathscr{H}}_{(\sigma)}(\bar{R}^+)$ we see that $\vec{u} \in \tilde{\mathscr{H}}_{(\sigma+1)}(\bar{R}^+)$.

(3) Let σ be such that $-\frac{1}{2}<\sigma<0$. For any $\vec{f} \in \tilde{\mathscr{H}}_{(\sigma)}(\bar{R}^+)$ if we define $\vec{g}$ by the equation $(D_+-i)\vec{g}=\vec{f}_\sim$, then $\vec{g} \in \mathring{\tilde{\mathscr{H}}}_{(\sigma+1)}(\bar{R}^+)$ and $\lim_{t\downarrow 0}\vec{g}=0$. The Cauchy problem (29) can be written in the form

$$\begin{cases} D_t(\vec{u}-\vec{g})+\vec{A}(t)(\vec{u}-\vec{g})=-i\vec{g}-\vec{A}(t)\vec{g}, \\ \lim_{t\downarrow 0}(\vec{u}-\vec{g})=\vec{\alpha}, \end{cases}$$

where $-i\vec{g}-\vec{A}(t)\vec{g} \in \mathring{\tilde{\mathscr{H}}}_{(\sigma+1)}(\bar{R}^+)$. Thus there exists a solution $\vec{v}=\vec{u}-\vec{g} \in \tilde{\mathscr{H}}_{(\sigma+2)}(\bar{R}^+)$ and therefore $\vec{u}=\vec{v}+\vec{g} \in \tilde{\mathscr{H}}_{(\sigma+1)}(\bar{R}^+)$. Thus the proof is complete.

In the same way as in the proof of Proposition 13 we can prove the following

PROPOSITION 28. *Let σ be any real number. For any $\vec{\alpha} \in \boldsymbol{C}^N$ and $\vec{f} \in \tilde{\mathscr{H}}_{(\sigma)}(\bar{R}^+)$ with $\vec{f}_\sim \in \mathring{\tilde{\mathscr{H}}}_{(\sigma)}(\bar{R}^+)$ there exists a unique solution $\vec{u} \in \tilde{\mathscr{H}}_{(\sigma+1)}(\bar{R}^+)$. In particular, if $\vec{\alpha}=0$ then $u \in \mathring{\tilde{\mathscr{H}}}_{(\sigma+1)}(\bar{R}^+)$.*

As an extension of Theorem 37 in E. Berz [2, p. 32] we have

PROPOSITION 29. *Let σ be any real number. For any $\vec{h} \in \mathring{\tilde{\mathscr{H}}}_{(\sigma)}(\bar{R}^+)$ there exists a unique solution $\vec{v} \in \mathring{\tilde{\mathscr{H}}}_{(\sigma+1)}(\bar{R}^+)$.*

The following two theorems are the analogues of Theorems 4 and 5 and these can be proved in a similar way.

THEOREM 13. *For any $\vec{h} \in \mathscr{D}'_+$, there exists a unique solution $\vec{v} \in \mathscr{D}'_+$ of the equation $L\vec{u}=\vec{h}$ and $\vec{h}\to\vec{v}$ is a continuous map of $\mathscr{D}'_+$ onto itself.*

THEOREM 14. *For any $\vec{\alpha} \in \boldsymbol{C}^N$ and $\vec{f} \in \mathscr{D}'(R^+)$ with the canonical extension $\vec{f}_\sim$ there exists a unique solution $\vec{u} \in \mathscr{D}'(R^+)$ of the Cauchy problem (29) and $(\vec{\alpha}, \vec{f}_\sim)\to\vec{u}_\sim$ is a continuous map under the topology $\boldsymbol{C}^N\times\mathscr{D}'_+$ and the topology $\mathscr{D}'_+$.*

Let us consider an ordinary differential operator

$$P(D)=D_t^m+\sum_{j=1}^m a_j(t)D_t^{m-j}, \qquad a_j \in C^\infty(R).$$

Substituting $u_j=D_t^{j-1}u$, $j=1, 2,\cdots, m$, then we obtain the equivalent system:

$$\begin{cases} D_t u_j - u_{j+1}=0 \quad \text{for} \quad j=1, 2,\cdots, m-1, \\ D_t u_m + \sum_{j=1}^m a_j(t) u_{m-j+1}=f. \end{cases}$$

Thus we have

COROLLARY 8. *For any $h \in \mathscr{D}'_+$, there exists a unique solution $v \in \mathscr{D}'_+$ of the equation $Pv=h$.*

COROLLARY 9. *For any $(\alpha_0,\cdots, \alpha_{m-1}) \in \boldsymbol{C}^m$ and $f \in \mathscr{D}'(R^+)$ with the canonical extension $f_\sim$, there exists a unique solution $u \in \mathscr{D}'(R^+)$ of the Cauchy problem:*

$$\begin{cases} Pu=f \quad \text{in } R^+, \\ \lim_{t\downarrow 0}(u, D_t u,\cdots, D_t^{m-1}u)=(\alpha_0, \alpha_1,\cdots, \alpha_{m-1}). \end{cases}$$

We can prove an analogue of Theorem 37 in E. Berz [2, p. 32].

PROPOSITION 30. *Let l be a non-negative integer such that $l \leqq m$ and let $h \in \mathscr{D}'_+$. Then the unique solution $v \in \mathscr{D}'_+$ of $Pv=h$ is a canonical distribution and $\lim_{t\downarrow 0}(v\,|\,R^+)=\cdots=\lim_{t\downarrow 0} D_t^{m-1-l}(v\,|\,R^+)=0$ when $l<m$, if and only if h can be written in the form $h=D_t^l g$, where $g \in \mathscr{D}'_+$ is a canonical distribution.*

PROOE. Let h be written in the form $h=D_t^l g$, $g \in \mathscr{D}'_+$ being canonical. Suppose $l=0$. Then, in virtue of Corollary 1 in [7, p. 19], the restriction $u=v\,|\,R^+$ is a solution of the Cauchy problem:

$$\begin{cases} Pu=h \qquad \text{in } R^+, \\ \lim_{t\downarrow 0}\,(u, D_t u,\cdots, D_t^{m-1}u)=0 \end{cases}$$

and $v=u_\sim$.

By the induction on l we shall prove the v is a canonical distribution and $\lim_{t\downarrow 0}(u,\cdots, D_t^{m-1-l}u)=0$. Let $l>0$ and suppose the assertion is true for $l-1$, $0<l<m$. Consider the equation $Pw=D_t^{l-1}g$. Then $w \in \mathscr{D}'_+$ is canonical and $\lim_{t\downarrow 0}(w\,|\,R^+)=\cdots=\lim_{t\downarrow 0} D_t^{m-l}(w\,|\,R^+)=0$, and therefore $w,\cdots, D_t^{m-1-l}w$ are canonical. If we put $v=D_t w+\chi$, then

$$P\chi=-i\sum_{j=1}^{m}{}' a_j'(t)D_t^{m-j}w.$$

Thus $\chi\in\mathscr{D}'_+$ is canonical and $\lim_{t\downarrow 0}(\chi|R^+)=\cdots=\lim_{t\downarrow 0}D_t^{m-l}(\chi|R^+)=0$ and therefore $v=D_tw+\chi$ is canonical and $\lim_{t\downarrow 0}(u,\cdots, D_t^{m-1-l}u)=0$ for $l<m$.

Conversely, let v be canonical and $\lim_{t\downarrow 0}(u,\cdots, D_t^{m-1-l}u)=0$ when $l<m$. Put $Y_k=\frac{1}{(k-1)!}t_+^{k-1}$. Owing to the relation (16) in [10, p. 392], we have

$$(-i)^lD_t^{m-l}v+\sum_{j=1}^{m}\sum_{k=1}^{l}(-i)^k\binom{l}{k}Y_k*(D_t^ka_j(Y_l*D_t^{m-j}v))=Y_l*h.$$

Since v is canonical and $\lim_{t\downarrow 0}(u,\cdots, D_t^{m-1-l}u)=0$, $l<m$, we see that v, $D_tv,\cdots$, $D_t^{m-1}v$ are canonical and therefore the left hand side of the equation is canonical. Thus Y_l*h is a canonical distribution, which implies that we can write $h=D_t^lg$, with a canonical g. The proof is thus complete.

PROPOSITION 31. *Let $\{\vec{u}_\iota\}_{\iota\in I}$ be a directed set in $\mathscr{E}_t^0(\bar{R}^+)$ and put $\vec{f}_\iota=L\vec{u}_\iota$ in $\mathscr{D}'(R^+)$. If $\vec{f}_\iota$ can be written in the form $\vec{f}_\iota=D_t\vec{g}_\iota$ in R^+, where $\vec{g}_\iota\in\mathscr{E}_t^0(\bar{R}^+)$ and $\vec{g}_\iota(0)=0$, and if $\vec{u}_\iota(0)$, $\vec{g}_\iota$ converge in $\boldsymbol{C}^N$, $\mathscr{E}_t^0(\bar{R}^+)$ to $\vec{\alpha}$, $\vec{g}$ respectively, then $\vec{u}_\iota$ converges in $\mathscr{E}_t^0(\bar{R}^+)$ to $\vec{u}$ and $\vec{u}$ satisfies the equation $L\vec{u}=\vec{f}$ in R^+ and $\vec{u}(0)=\vec{\alpha}$, where $\vec{f}=D_t\vec{g}$.*

PROOF. Consider the Cauchy problem:

$$\begin{cases} L\vec{v}_\iota=\vec{g}_\iota \quad \text{in } R^+,\\ \lim_{t\downarrow 0}\vec{v}_\iota=0.\end{cases}$$

There exists a unique solution $\vec{v}_\iota\in\mathscr{E}_t^1(\bar{R}^+)$ and $\vec{v}_\iota$ converges in $\mathscr{E}_t^1(\bar{R}^+)$ to $\vec{v}$ when ι run through I. Then $\vec{v}$ is a solution of the Cauchy problem $L\vec{v}=\vec{g}$ in R^+ with $\vec{v}(0)=0$. On the other hand, from the equation $D_t\vec{v}_\iota=\vec{g}_\iota-\vec{A}(t)\vec{v}_\iota$ we have $D_t\vec{v}_\iota(0)=0$. If we put $\vec{u}_\iota=D_t\vec{v}+\vec{w}_\iota$, then

$$\begin{cases} L\vec{w}_\iota=i\vec{A}'(t)\vec{v}_\iota \quad \text{in} \quad R^+,\\ \lim_{t\downarrow 0}\vec{w}_\iota=\vec{u}(0),\end{cases}$$

where $\vec{A}'(t)\vec{v}\in\mathscr{E}_t^1(\bar{R}^+)$. Thus there exists a unique solution $\vec{w}_\iota\in\mathscr{E}_t^2(\bar{R}^+)$. Since $\vec{u}_\iota(0)$, $\vec{A}(t)\vec{v}_\iota$ converge in $\boldsymbol{C}^N$, $\mathscr{E}_t^1(\bar{R}^+)$ to α, $\vec{A}(t)\vec{v}$ respectively, $\vec{w}_\iota$ converges in $\mathscr{E}_t^2(\bar{R}^+)$ to $\vec{w}$. Consequently $\vec{u}_\iota=D_t\vec{v}_\iota+\vec{w}_\iota$ converges in $\mathscr{E}_t^0(\bar{R}^+)$ to $\vec{u}=D_t\vec{v}+\vec{w}$ and $\vec{u}$ satisfies $L\vec{u}=\vec{f}$ in R^+ and $\vec{u}(0)=\vec{\alpha}$, completing the proof.

Consider the Cauchy problem

$$(33)\qquad \begin{cases} L\vec{v}=\vec{h} \quad \text{in} \quad R,\\ \vec{v}(0)=\vec{\alpha}\end{cases}$$

for any preassigned $\vec{\alpha} \in C^N$ and $\vec{h} \in \mathscr{D}'(R)$, where $\vec{v}(0)$ is the value of $\vec{v}$ in the sense of S. Łojasiewicz. By Theorem 5 in [10, p. 392] and Theorem 14, if $\vec{h} \in \mathscr{D}'(R)$ has no mass on $t=0$ and the restrictions $\vec{h}_1=\vec{h}\,|\,R^+$ and $\vec{h}_2=\vec{h}\,|\,R^-$ have the canonical extensions $\vec{h}_{1\sim}$ and $\vec{h}_2^{\sim}$ over $t=0$, then, owing to Theorem 5 in [10, p. 392] and Theorem 14, there exists a unique solution $\vec{v} \in \mathscr{D}'(R)$ of the Cauchy problem (33), and $\vec{v}_1=\vec{v}\,|\,R^+$, $\vec{v}_2=\vec{v}\,|\,R^-$ satisfy the equations

$$L(\vec{v}_{1\sim})=\vec{h}_{1\sim}-i\vec{\alpha}\delta,$$

$$L(\vec{v}_2^{\sim})=\vec{h}_2^{\sim}+i\vec{\alpha}\delta,$$

Thus we have the following

THEOREM 15. *Let $\{\vec{v}_\iota\}_{\iota\in I}$ be a directed set in $\mathscr{E}_t^0(R)$ and put $\vec{h}_\iota=L\vec{v}_\iota$ in $\mathscr{D}'(R)$. If $\vec{h}_\iota$ can be written in the form $\vec{h}_\iota=D_t\vec{g}_\iota$, where $\vec{g}_\iota \in \mathscr{E}_t^0(R)$ and $\vec{g}_\iota(0)=0$, and if $\vec{v}_\iota(0)$, $\vec{g}_\iota$ converge in C^N, $\mathscr{E}_t^0(R)$ to $\vec{\alpha}$, $\vec{g}$ respectively, then $\vec{v}_\iota$ converges in $\mathscr{E}_t^0(R)$ to $\vec{v}$ and $\vec{v}$ satisfies the equation $L\vec{v}=\vec{h}$ and $\vec{v}(0)=\vec{\alpha}$, where $\vec{h}=D_t\vec{g}$.*

Let us again consider the differential operator $P(D)$. The discussions made for a system will allow to show the following

THEOREM 16. *Let $\{v_\iota\}_{\iota\in I}$ be a directed set in $\mathscr{E}_t^0(R)$ and put $h_\iota=Pv_\iota$ in $\mathscr{D}'(R)$. If the values $(v_\iota(0), D_tv_\iota(0),\cdots, D_t^{m-1}v_\iota(0))=\vec{\alpha}_\iota$ exist and if h_ι can be written in the form $h_\iota=D_t^lg_\iota$, $0\leqq l\leqq m$, where $g_\iota \in \mathscr{E}_t^0(R)$ and $(g_\iota(0), D_tg_\iota(0),\cdots, D_t^{l-1}g_\iota(0))=0$, then v_ι belongs to the space $\mathscr{E}_t^{m-l}(R)$. If $\vec{\alpha}_\iota$, g_ι converge in C^m, $\mathscr{E}_t^0(R)$ to $\vec{\alpha}$, g respectively, then v_ι converges in $\mathscr{E}_t^{m-l}(R)$ to v and v satisfies the equation $Pv=h$ and $(v(0), D_tv(0),\cdots, D_t^{m-1}v(0))=\vec{\alpha}$.*

PROOF. Since the values $(g_\iota(0),\cdots, D_t^{l-1}g_\iota(0))$ exist, $D_t^jg_\iota$ has no mass on $t=0$ and the restrictions $(D_t^jg_\iota)\,|\,R^+$, $(D_t^jg_\iota)\,|\,R^-$ have the canonical extensions over $t=0$ for each j, $0\leqq j\leqq l$. Put $v_\iota^+=v_\iota\,|\,R^+$, $v_\iota^-=v_\iota\,|\,R^-$, $h_\iota^+=h_\iota\,|\,R^+$ and $h_\iota^-=h_\iota\,|\,R^-$ and consider the Cauchy problems

$$\begin{cases} Pv_\iota^+=h_\iota^+ \quad \text{in } R^+, \\ \lim\limits_{t\downarrow 0}(v_\iota^+,\cdots, D_t^{m-1}v_\iota^+)=\vec{\alpha}_\iota, \end{cases} \qquad \begin{cases} Pv_\iota^-=h_\iota^- \quad \text{in } R^-, \\ \lim\limits_{t\uparrow 0}(v_\iota^-,\cdots, D_t^{m-1}v_\iota^-)\vec{\alpha}_\iota. \end{cases}$$

The reasonings made in the proofs of Propositions 30, 31 will show that $v_\iota=v_\iota^++v_\iota^- \in \mathscr{E}_t^{m-l}(R)$ and that v_ι converges in $\mathscr{E}_t^{m-l}(R)$ to v, which satisfies $Pv=h$ and $(v(0), D_tv(0),\cdots, D_t^{m-l}v(0))=0$. This completes the proof.

In closing this paper let us add a comment on the coerciveness considered in Section 6. Let us consider the operator $L=D_t+\vec{A}$, where $\vec{A}$ is an $N\times N$ matrix of convolution operators and $\vec{A} \in \mathrm{OP}_r$, $r>0$, and assume that L is parabolic in the sense of I.G. Petrowski. If we let $\varkappa_j(\xi)$ be the characteristic roots of $\hat{\vec{T}}_A(\xi)$, where $\vec{A}\vec{u}=\vec{T}_A*\vec{u}$ with $\vec{u} \in (\mathscr{D}'_{L^2})_x$, then there exist constants $C>0$ and C_0 such that

(34) $$\operatorname{Im} \varkappa_j(\xi) \leqq -C|\xi|^r + C_0, \qquad \xi \in \Xi_n.$$

We shall show that L satisfies the energy inequality (23) in Section 6.

The inequality (34) is equivalent to the inequality

(35) $$\operatorname{Im} \varkappa_j(\xi) \leqq -C'(1+|\xi|^2)^{r/2} + C_1$$

with constants $C'>0$ and C_1. Let us consider an $N\times N$ matrix $\vec{B}=\left(\frac{i}{2}C'S^r\right)\vec{E}$ with an $N\times N$ unit matrix $\vec{E}$. Then the operator $\tilde{L}=L+\vec{B}=D_t+\vec{\tilde{A}}$ is also parabolic. As noted in Section 4, $\tilde{L}$ is well posed in the L^2 norm. Owing to Corollary 3 in Section 4 it follows that there exists a positive definite Hermitian matrix $\vec{H}(\xi)$ such that

$$-i(\vec{H}(\xi)\hat{\vec{T}}_{\tilde{A}}(\xi) - \hat{\vec{T}}^*_{\tilde{A}}(\xi)\vec{H}(\xi)) \leqq C_2 \qquad \text{a.e. on } \Xi_n$$

with a constant C_2. Since $\hat{\vec{T}}_{\tilde{A}}(\xi)=\hat{\vec{T}}_A(\xi)+i\frac{C'}{2}(1+|\xi|^2)^{r/2}$, we have

$$\begin{aligned}
\operatorname{Im}(\vec{H}L\vec{u}, \vec{u}) &= -\frac{i}{2}\{(\vec{H}L\vec{u}, \vec{u})-(\vec{u}, \vec{H}L\vec{u})\} \\
&= -\frac{i}{2}((\vec{H}\vec{A}-\vec{A}^*\vec{H})\vec{u}, \vec{u}) \\
&\leqq C_2\|\vec{u}\|^2 - C'(\vec{H}\vec{u}, S^r\vec{u}) \\
&= C_2\|\vec{u}\|^2 - C'(\vec{H}S^{r/2}\vec{u}, S^{r/2}\vec{u}) \\
&\leqq C_2\|\vec{u}\|^2 - C''\|\vec{u}\|^2_{(r/2)}, \qquad \vec{u} \in C_0^\infty(R_n)
\end{aligned}$$

with a constant C''. In virtue of Proposition 22 we see that L satisfies the energy inequality (23).

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