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# *On Explicit One-step Methods Utilizing the Second Derivative*

Hisayoshi SHINTANI

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## 1. Introduction

Consider the initial value problem

$$y'=f(x, y), \qquad y(x_0)=y_0, \tag{1.1}$$

where the function

$$g(x, y)=f_x(x, y)+f(x, y)f_y(x, y) \tag{1.2}$$

is assumed to be sufficiently smooth. Let

$$x_1=x_0+h, \qquad y_1=y(x_1), \tag{1.3}$$

where $h$ is a small increment in $x$ and $y(x)$ is the solution to the given initial value problem. We are concerned with the case where the approximate value $z_1$ of $y_1$ is computed by means of the explicit one-step methods of the type

$$z_1=y_0+hk_0+h^2\sum_{i=1}^{r}p_il_i \qquad (p_r\neq 0), \tag{1.4}$$

and put

$$T=z_1-y_1=O(h^{p+1}), \tag{1.5}$$

where

$$k_0=f(x_0, y_0), \tag{1.6}$$

$$l_i=g(x_0+a_ih,\ y_0+a_ihk_0+h^2\sum_{j=1}^{i-1}b_{ij}l_j) \qquad (i=1, 2,\cdots, r). \tag{1.7}$$

In our previous paper [1][1)], we have shown that the formulas (1.4) of orders $p=r+2$ exist for $r=1$, 2, 3, 4 and 5. In this paper, together with (1.4), we consider the formulas

$$w_1=y_0+hk_0+h^2\sum_{j=1}^{r-1}q_jl_j, \tag{1.8}$$

1) Numbers in square brackets refer to the references listed at the end of this paper.

and put

$$S=w_1-y_1=O(h^{q+1}), \tag{1.9}$$

$$s=w_1-z_1=h^2\sum_{i=1}^{r} r_i l_i. \tag{1.10}$$

In the case where $p>q$, for sufficiently small $h$, the truncation error $S$ of $w_1$ will be approximated by $s$. Thus we are interested in the relations among $r$, $q$ and $p$. It will be shown that, for $r=2$, 3 and 4, the formulas of orders $q=r$ and $p=r+2$ exist, but those of orders $q=r+1$ and $p=r+2$ do not exist; for $r=5$, those of orders $q=4$ and $p=7$ and those of orders $q=5$ and $p=6$ exist, but those of orders $q=5$ and $p=7$ do not exist. Finally numerical examples are presented.

## 2. Preliminaries

Let $D$ be a differential operator defined by

$$D=\frac{\partial}{\partial x}+k_0\frac{\partial}{\partial y} \tag{2.1}$$

and put

$$D^i g(x_0, y_0)=Z_i,\ D^i g_y(x_0, y_0)=Y_i,\ D^i g_{yy}(x_0, y_0)=X_i, \tag{2.2}$$
$$D^i g_{yyy}(x_0, y_0)=W_i \qquad (i=0, 1, 2,\cdots).$$

Then $y_0^{(i)}=y^{(i)}(x_0)$ $(i=1, 2,\cdots)$ can be written as follows:

$$y_0^{(1)}=k_0,\ y_0^{(2)}=Z_0,\ y_0^{(3)}=Z_1,\ y_0^{(4)}=Z_2+Z_0Y_0, \tag{2.3}$$

$$y_0^{(5)}=Z_3+3Z_0Y_1+Z_1Y_0, \tag{2.4}$$

$$y_0^{(6)}=Z_4+6Z_0Y_2+4Z_1Y_1+Z_2Y_0+Z_0Y_0^2+3Z_0^2X_0, \tag{2.5}$$

$$y_0^{(7)}=Z_5+10Z_0Y_3+10Z_1Y_2+5Z_2Y_1+Z_3Y_0+8Z_0Y_0Y_1 \tag{2.6}$$
$$+Z_1Y_0^2+10Z_0Z_1X_0+15Z_0^2X_1,$$

$$y_0^{(8)}=Z_6+15Z_0Y_4+20Z_1Y_3+15Z_2Y_2+6Z_3Y_1+Z_4Y_0+21Z_0Y_0Y_2 \tag{2.7}$$
$$+10Z_1Y_0Y_1+18Z_0Y_1^2+Z_2Y_0^2+Z_0Y_0^3+18Z_0^2Y_0X_0+15Z_0Z_2X_0$$
$$+60Z_0Z_1X_1+10Z_1^2X_0+45Z_0^2X_2+15Z_0^3W_0.$$

Put for simplicity

$$d_{ij}=i(i+1)\sum_{k=1}^{j-1}a_k^{i-1}b_{jk} \qquad (i=1, 2,\cdots, r; j=2, 3,\cdots, r) \tag{2.8}$$

$$e_{ij}=(i+2)(i+3)\sum_{k=2}^{j-1}a_k^{i-1}d_{1k}b_{jk} \qquad (j=3,4,\cdots,r), \tag{2.9}$$

$$l_{ij}=(i+3)(i+4)\sum_{k=2}^{j-1}a_k^{i-1}d_{2k}b_{jk}, \tag{2.10}$$

$$m_{ij}=(i+4)(i+5)\sum_{k=2}^{j-1}a_k^{i-1}d_{3k}b_{jk}, \tag{2.11}$$

$$q_{ij}=(i+4)(i+5)\sum_{k=2}^{j-1}a_k^{i-1}d_{1k}^2b_{jk}, \tag{2.12}$$

$$r_{ij}=(i+4)(i+5)\sum_{k=3}^{j-1}a_k^{i-1}e_{1k}b_{jk} \qquad (j=4,5,\cdots,r). \tag{2.13}$$

Then $z_1$ in (1.4) can be expanded as follows:

$$\begin{aligned} z_1 = {} & y_0+hk_0+h^2A_0Z_0+h^3A_1Z_1+\frac{1}{2!}h^4(A_2Z_2+A_3Z_0Y_0) \\ & +\frac{1}{3!}h^5(A_4Z_3+3A_5Z_0Y_1+A_6Z_1Y_0)+\frac{1}{4!}h^6(B_1Z_4+6B_2Z_0Y_2 \\ & +4B_3Z_1Y_1+B_4Z_2Y_0+B_5Z_0Y_0^2+3B_6Z_0^2X_0)+\frac{1}{5!}h^7(C_1Z_5 \\ & +10C_2Z_0Y_3+10C_3Z_1Y_2+5C_4Z_2Y_1+C_5Z_3Y_0+8C_6Z_0Y_0Y_1 \\ & +C_7Z_1Y_0^2+10C_8Z_0Z_1X_0+15C_9Z_0^2X_1)+\frac{1}{6!}h^8(D_1Z_6+15D_2Z_0Y_4 \\ & +20D_3Z_1Y_3+15D_4Z_2Y_2+6D_5Z_3Y_1+D_6Z_4Y_0+21D_7Z_0Y_0Y_2 \\ & +10D_8Z_1Y_0Y_1+18D_9Z_0Y_1^2+D_{10}Z_2Y_0^2+D_{11}Z_0Y_0^3+18D_{12}Z_0^2Y_0X_0 \\ & +15D_{13}Z_0Z_2X_0+60D_{14}Z_0Z_1X_1+10D_{15}Z_1^2X_0+45D_{16}Z_0^2X_2 \\ & +15D_{17}Z_0^3W_0)+\cdots, \end{aligned} \tag{2.14}$$

where

$$A_0=\sum_{i=1}^{r}p_i,\ A_1=\Sigma a_ip_i,\ A_2=\Sigma a_i^2p_i,\ A_3=\sum_{j=2}^{r}d_{1j}p_j, \tag{2.15}$$

$$A_4=\Sigma a_i^3p_i,\ A_5=\Sigma a_jd_{1j}p_j,\ A_6=\Sigma d_{2j}p_j, \tag{2.16}$$

$$\begin{aligned} & B_1=\Sigma a_i^4p_i,\ B_2=\Sigma a_j^2d_{1j}p_j,\ B_3=\Sigma a_jd_{2j}p_j,\ B_4=\Sigma d_{3j}p_j, \\ & B_5=\sum_{k=3}^{r}e_{1k}p_k,\ B_6=\Sigma d_{1j}^2p_j, \end{aligned} \tag{2.17}$$

$$C_1=\Sigma a_i^5p_i,\ C_2=\Sigma a_j^3d_{1j}p_j,\ C_3=\Sigma a_j^2d_{2j}p_j,\ C_4=\Sigma a_jd_{3j}p_j, \tag{2.18}$$

$$C_5=\Sigma d_{4j}p_j,\ 8C_6=5\Sigma a_k e_{1k}p_k+3\Sigma e_{2k}p_k,\ C_7=\Sigma l_{1k}p_k,$$

$$C_8=\Sigma d_{1j}d_{2j}p_j,\ C_9=\Sigma a_j d_{1j}^2 p_j,$$

$$D_1=\Sigma a_i^6 p_i,\ D_2=\Sigma a_j^4 d_{1j}p_j,\ D_3=\Sigma a_j^3 d_{2j}p_j,\ D_4=\Sigma a_j^2 d_{3j}p_j, \tag{2.19}$$

$$D_5=\Sigma a_j d_{4j}p_j,\ D_6=\Sigma d_{5j}p_j,\ 7D_7=5\Sigma a_k^2 e_{1k}p_k+2\Sigma e_{3k}p_k,$$

$$5D_8=3\Sigma a_k l_{1k}p_k+2\Sigma l_{2k}p_k,\ D_9=\Sigma a_k e_{2k}p_k,\ D_{10}=\Sigma m_{1k}p_k,$$

$$D_{11}=\sum_{l=4}^{r} r_{1l}p_l,\ 6D_{12}=5\Sigma d_{1k}e_{1k}p_k+\Sigma q_{1k}p_k,$$

$$D_{13}=\Sigma d_{1j}d_{3j}p_j,\ D_{14}=\Sigma a_j d_{1j}d_{2j}p_j,\ D_{15}=\Sigma d_{2j}^2 p_j,$$

$$D_{16}=\Sigma a_j^2 d_{1j}^2 p_j,\ D_{17}=\Sigma d_{1j}^3 p_j.$$

If we impose the condition that

$$a_1=0,\ d_{1j}=a_j^2 \qquad (j=2, 3,\cdots, r), \tag{2.20}$$

then it follows that

$$d_{j2}=0 \quad (j=2, 3,\cdots, r), \qquad l_{i3}=m_{i3}=0 \quad (i=1, 2,\cdots, r), \tag{2.21}$$

$$e_{ik}=d_{i+2,k},\ q_{ik}=d_{i+4,k},\ r_{il}=m_{il}, \tag{2.22}$$

$$A_3=A_2,\ A_5=A_4,\ B_2=B_6=B_1,\ B_5=B_4,\ C_2=C_9=C_1, \tag{2.23}$$

$$8C_6=5C_4+3C_5,\ C_8=C_3,\ D_2=D_{16}=D_{17}=D_1,$$

$$7D_7=5D_4+2D_6,\ D_9=D_5,\ D_{11}=D_{10},\ 6D_{12}=5D_4+D_6,$$

$$D_{13}=D_4,\ D_{14}=D_3.$$

We make use of the following notations:

$$V^{(n)}=\frac{1}{(n+1)(n+2)},\ W_i^{(n)}=V^{(n+1)}-a_i V^{(n)}, \tag{2.24}$$

$$X_{ij}^{(n)}=W_i^{(n+1)}-a_j W_i^{(n)},\ Y_{ijk}^{(n)}=X_{ij}^{(n+1)}-a_k X_{ij}^{(n)},$$

$$Z_{ijkl}^{(n)}=Y_{ijk}^{(n+1)}-a_l Y_{ijk}^{(n)},\ U^{(n)}=V^{(n+3)}-3a_1 V^{(n+2)} \qquad (n=0, 1,\cdots).$$

We denote by ( )$'$ the expression ( ) in which $p_r=0$ and $p_j$ $(j=1, 2,\cdots, r-1)$ are replaced by $q_j$ respectively.

## 3. Case where r=2

Tha formulas of orders $q=2$ and $p=4$ exist. For instance, the choice

$a_1=\frac{1}{8}$ obtains the following results:

$$a_1=\frac{1}{8},\ a_2=\frac{3}{5},\ b_{21}=\frac{19}{100},\ p_1=\frac{16}{57},\ p_2=\frac{25}{114}, \tag{3.1}$$

$$q_1=\frac{1}{2},\ r_1=\frac{25}{114},\ r_2=-\frac{25}{114},$$

$$T=-\frac{1}{5!}h^5\left(\frac{1}{24}Z_3+\frac{3}{8}Z_1Y_0\right)+O(h^6), \tag{3.2}$$

$$s=-\frac{1}{3!}h^3\frac{8}{5}Z_1-\frac{1}{4!}h^4\left(\frac{29}{32}Z_2+Z_0Y_0\right)+O(h^5). \tag{3.3}$$

The formulas of orders $q=3$ and $p=4$ do not exist. For otherwise the equations

$$a_1=a_2=\frac{1}{3},\ a_2(a_2-a_1)p_2=\frac{1}{12}-\frac{1}{6}a_1 \tag{3.4}$$

must be satisfied.

## 4. Case where r=3

The formulas of orders $q=3$ and $p=5$ exist. For instance, the choice $a_1=\frac{1}{8}$ and $a_3=1$ obtains the following results:

$$a_1=\frac{1}{8},\ a_2=\frac{11}{20},\ b_{21}=\frac{17}{100},\ a_3=1,\ b_{31}=-\frac{7}{34}, \tag{4.1}$$

$$b_{32}=\frac{189}{340},\ p_1=\frac{32}{119},\ p_2=\frac{100}{459},\ p_3=\frac{5}{378},\ q_1=\frac{13}{51},$$

$$q_2=\frac{25}{102},\ r_1=-\frac{5}{357},\ r_2=\frac{25}{918},\ r_3=-\frac{5}{378},$$

$$T=-\frac{1}{6!}h^6\left(\frac{1}{320}Z_4+\frac{3}{10}Z_0Y_2-\frac{1}{2}Z_1Y_1+\frac{1}{160}Z_2Y_0+\frac{1}{10}Z_0Y_0^2+\right. \tag{4.2}$$

$$\left.\frac{3}{20}Z_0^2X_0\right)+O(h^7),$$

$$s=-\frac{1}{4!}h^4\frac{1}{16}Z_2-\frac{1}{5!}h^5\left(\frac{67}{384}Z_3+\frac{1}{4}Z_0Y_1+\frac{3}{8}Z_1Y_0\right)+O(h^6). \tag{4.3}$$

We shall show that the formulas of orders $q=4$ and $p=5$ do not exist.

Assume the contrary. Then the following equations must be satisfied:

$$(4.4) \qquad (a_3-a_1)(a_3-a_2)=0,$$

$$(4.5) \qquad X_{12}^{(n)}=0 \qquad (n=0,\ 1),$$

$$(4.6) \qquad (a_3-a_1)d_{12}=(a_2-a_1)d_{13},$$

$$(4.7) \qquad (a_3-a_2)d_{13}p_3=W_2^{(2)}.$$

The system (5) has the solution $a_1,\ a_2=(4\pm\sqrt{6})/10$. Hence $a_2\neq a_1$ and $W_2^{(2)}\neq 0$. Then, from the equation (7), it follows that $(a_3-a_2)d_{13}\neq 0$, and so $a_3\neq a_1$ by (6). This contradicts the condition (4), and our assertion is proved.

## 5. Case where r=4

We shall show first the following

LEMMA 1. *In order that the formulas of orders $q=4$ and $p=6$ may exist for $r=4$, the conditions*

$$(5.1) \qquad (a_2-a_1)(a_3-a_1)(a_3-a_2)\neq 0,$$

$$(5.2) \qquad a_1=0,\ d_{1j}=a_j^2 \qquad (j=2,\ 3,\ 4)$$

*must be valid.*

PROOF. Assume that such formulas exist. Then there must hold the following equations:

$$(5.3)_n \qquad \sum_{j=2}^{4} a_j^n(a_j-a_1)p_j=W_1^{(n)} \qquad (n=0,\ 1,\ 2),$$

$$(5.4)_n \qquad \sum_{k=j,4} a_k^n(a_k-a_1)(a_k-a_i)p_k=X_{1i}^{(n)} \qquad (j\neq i;\ i,\ j=2,\ 3),$$

$$(5.5)_n \qquad \sum_{j=2}^{4} a_j^n d_{1j}p_j=V^{(n+2)},$$

and $(4)_0'$, $(5)_0'$ and $(3)_n'$ $(n=0,\ 1)$.

Suppose that $(a_3-a_1)(a_3-a_2)=0$. Then, from $(4)_0'$ and $(4)_0$ $(j=3)$, it follows that

$$(a_4-a_1)(a_4-a_2)=0,\ X_{12}^{(0)}=0.$$

Hence, from $(4)_n$ $(n=1,\ 2)$, we obtain the equations $X_{12}^{(n)}=0$ $(n=1,\ 2)$, so that $a_1$ and $a_2$ must satisfy the system of equations $X_{12}^{(n)}=0$ $(n=0,\ 1,\ 2)$. As is easily checked, this system has no solution. Hence $a_3\neq a_1$ and $a_3\neq a_2$. Similarly it can be shown that $a_2\neq a_1$.

Put

$$d_{1j}=(a_j-a_1)s_j \quad (j=2,\ 3), \qquad s_3-s_2=(a_3-a_2)r_3. \tag{5.6}$$

Then, from $(3)_0'$, $(4)_0'$, $(5)_0'$, $(3)_n$, $(4)_n$ and $(5)_n$ $(n=0,\ 1,\ 2)$, it follows that

$$d_{14}=(a_4-a_1)[s_2+(a_4-a_2)r_3], \tag{5.7}$$

$$V^{(n+2)}=W_1^{(n)}s_2+X_{12}^{(n)}r_3 \qquad (n=0,\ 1,\ 2). \tag{5.8$_n$}$$

Solving the system of equations $(8)_n$ $(n=0,\ 1,\ 2)$, we have the solution

$$a_1=0,\ s_2=a_2,\ r_3=1,$$

and the condition (2) follows from (6) and (7). This completes the proof.

The formulas of orders $q=4$ and $p=6$ exist. For instance, the choice $a_2=\dfrac{1}{5}$ and $a_4=1$ yields the following results:

$$\begin{aligned}
&a_1=0,\ a_2=\frac{1}{5},\ b_{21}=\frac{1}{50},\ a_3=\frac{3}{5},\ b_{31}=-\frac{1}{50},\\
&b_{32}=\frac{1}{5},\ a_4=1,\ b_{41}=\frac{13}{18},\ b_{42}=-\frac{2}{3},\ b_{43}=\frac{4}{9},\\
&p_1=\frac{1}{18},\ p_2=\frac{25}{96},\ p_3=\frac{25}{144},\ p_4=\frac{1}{96},\ q_1=\frac{1}{12},\\
&q_2=\frac{5}{24},\ q_3=\frac{5}{24},\ r_1=\frac{1}{36},\ r_2=-\frac{5}{96},\ r_3=\frac{5}{144},\\
&r_4=-\frac{1}{96},
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
T=-\frac{1}{7!}h^7\Big[&\frac{89}{125}(Z_5+10Z_0Y_3+15Z_0^2X_1)+\frac{1}{5}(Z_1Y_2+Z_0Z_1X_0)\\
&+\frac{4}{5}Z_2Y_1-\frac{2}{75}Z_3Y_0+\frac{18}{25}Z_0Y_0Y_1+\frac{1}{15}Z_1Y_0^2\Big]+O(h^8),
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
s=-\frac{1}{5!}h^5\frac{1}{15}(Z_3+3Z_0Y_1)-\frac{1}{6!}h^6\Big[&\frac{4}{25}(Z_4+6Z_0Y_2+4Z_1Y_1\\
&+3Z_0^2X_0)+\frac{2}{5}(Z_2Y_0+Z_0Y_0^2)\Big]+O(h^7).
\end{aligned} \tag{5.11}$$

Now we shall show that the formulas of orders $q=5$ and $p=6$ do not exist. Assume the contrary. Then the following equations must be satisfied:

$$a_4(a_4-a_2)(a_4-a_3)=0, \tag{5.12}$$

(5.13) $$X_{23}^{(1)}=0, \qquad X_{23}^{(2)}=0,$$

(5.14) $$a_3(a_3-a_2)p_3+a_4(a_4-a_2)p_4=W_2^{(1)},$$

(5.15) $$d_{23}p_3+d_{24}p_4=V^{(3)},$$

(5.16) $$(a_4-a_3)d_{24}p_4=W_3^{(3)},$$

and (14)$'$ and (15)$'$. Solving the system (13), we have the solution

(5.17) $$a_2,\ a_3=\frac{5\pm\sqrt{5}}{10}$$

Put $d_{23}=a_3(a_3-a_2)t_3$. Then, from (14)$'$, (15)$'$, (14) and (15) it follows that

(5.18) $$d_{24}=a_4(a_4-a_2)t_3.$$

By (18), (16) and (12) we have the equation $W_3^{(3)}=0$, from which follows that $a_3=\frac{2}{3}$. This contradicts the result (17). Hence such formulas do not exist.

Summarizing the results, we have the following

THEOREM 1. *For $r=2$, 3 and 4 the formulas of orders $q=r$ and $p=r+2$ exist, but those of orders $q=r+1$ and $p=r+2$ do not exist.*

## 6. Case where r = 5

We shall show the following

THEOREM 2. *For $r=5$, the formulas of orders $q=4$ and $p=7$ and those of orders $q=5$ and $p=6$ exist, but those of orders $q=5$ and $p=7$ do not exist.*

Assume that the formulas of orders $q=5$ and $p=7$ exist. Then there must hold the following equations:

$(6.1)_n$ $$\sum_{k=2}^{5} a_k^n(a_k-a_1)p_k=W_1^{(n)} \qquad (n=0, 1, 2, 3, 4),$$

$(6.2)_n$ $$\sum_{k=2}^{5} a_k^n d_{1k}p_k=V^{(n+2)} \qquad (n=0, 1, 2, 3),$$

$(6.3)_n$ $$\sum_{k=2, k\neq i}^{5} a_k^n(a_k-a_1)(a_k-a_i)p_k=X_{1i}^{(n)} \qquad (i=1, 2, 3, 4;\ n=0, 1, 2, 3),$$

$(6.4)_n$ $$\sum_{k=3}^{5} a_k^n c_k p_k=U^{(n)} \qquad (n=0, 1, 2),$$

$(6.5)_n$ $$\sum_{k=2, k\neq 4}^{5} a_k^n(a_k-a_4)d_{1k}p_k=W_4^{(n+2)} \qquad (n=0, 1, 2),$$

$$(6.6)_n \qquad \sum_{k=3,5} a_k^n(a_k-a_4)c_k p_k = W_4^{(n+3)} - 3a_1 W_4^{(n+2)} \qquad (n=0,\ 1),$$

$$(6.7)_n \qquad \sum_{l=k,5} a_l^n(a_l-a_1)(a_l-a_i)(a_l-a_j)p_l = Y_{1ij}^{(n)}$$
$$(i\neq j,\ k;\ j\neq k;\ i,\ j,\ k=2,\ 3,\ 4;\ n=0,\ 1,\ 2),$$

$$(6.8)_n \qquad a_5^n(a_5-a_1)(a_5-a_2)(a_5-a_3)(a_5-a_4)p_5 = Z_{1234}^{(n)} \qquad (n=0,\ 1),$$

$$(6.9) \qquad \sum_{k=4,5}(a_k-a_2)(a_k-a_3)d_{1k}p_k = X_{23}^{(2)},$$

$$(6.10) \qquad (a_k-a_3)(a_k-a_j)c_k p_k = X_{3j}^{(3)} - 3a_1 X_{3j}^{(2)} \qquad (j\neq k;\ j,\ k=4,\ 5),$$

$$(6.11) \qquad (a_4-a_2)(a_4-a_3)(a_4-a_5)d_{14}p_4 = Y_{235}^{(2)},$$

$$(6.12) \qquad \sum_{k=3}^{5} e_{1k}p_k = V^{(4)},$$

$$(6.13) \qquad \sum_{k=4}^{5} f_k p_k = P,$$

$$(6.14) \qquad 20(a_4-a_1)(a_4-a_2)(a_4-a_3)b_{54}p_5 = Q,$$

$$(6.15) \qquad (a_4-a_5)f_4 p_4 = R,$$

and $(1)_n'$ $(n=0,\ 1,\ 2)$, $(2)_m'$, $(3)_m'$ $(m=0,\ 1)$, $(4)_0'$, $(5)_0'$ and $(7)_0'$, where

$$(6.16) \qquad c_k = 6\sum_{j=2}^{k-1}(a_j-a_1)b_{kj} \qquad (k=3,\ 4,\ 5),$$

$$(6.17) \qquad f_k = 12\sum_{j=3}^{k-1}(a_j-a_1)(a_j-a_2)b_{kj} \qquad (k=4,\ 5),$$

$$(6.18) \qquad P = \frac{1}{30} - \frac{1}{10}(a_1+a_2) + \frac{1}{2}a_1a_2,$$

$$(6.19) \qquad Q = \frac{1}{42} - \frac{1}{18}(a_1+a_2) + \frac{1}{6}a_1a_2 - \frac{5}{3}a_3P,$$

$$(6.20) \qquad R = \frac{1}{42} - \frac{1}{15}(a_1+a_2) + \frac{3}{10}a_1a_2 - a_5P.$$

Consider the following system of equations:

$$(6.21) \qquad Y_{ijk}^{(n)} = 0 \qquad (n=0,\ 1,\ 2;\ i\neq j,\ k;\ j\neq k).$$

Then it follows that

$$(6.22) \qquad a_i+a_j+a_k = \frac{9}{7},\ a_ia_j+a_ia_k+a_ja_k = \frac{3}{7},\ a_ia_ja_k = \frac{1}{35},$$

so that $a_i$, $a_j$ and $a_k$ are the roots of the equation

$$P(x)=35x^3-45x^2+15x-1=0. \tag{6.23}$$

This equation has three real distinct roots and they can be expressed as follows:

$$\frac{3}{7}+\frac{4\sqrt{2}}{343}\cos\frac{1}{3}(\varphi+2k\pi) \qquad (k=0, 1, 2),$$

where $\tan\varphi=7$. Hence these roots do not satisfy any quadratic equation with rational coefficients, and they lie in the interval (0, 1).

LEMMA 2. *Let $a_i$, $a_j$ and $a_k$ be the solution of the system* (6.21). *Then*

$$X_{ij}^{(0)}\neq 0, \qquad X_{ij}^{(2)}\neq 0, \tag{6.24}$$

*and $\frac{1}{3}a_k$ is not a root of the equation* (6.23).

PROOF. Suppose that $X_{ij}^{(2)}=0$. Then, from the equation $Y_{ijk}^{(2)}=0$, it follows that $X_{ij}^{(3)}=0$. Hence $a_i$ and $a_j$ must satisfy the equation $7x^2-8x+2=0$. But this is impossible, and so $X_{ij}^{(2)}\neq 0$.

Suppose that $X_{ij}^{(0)}=0$. Then $X_{ij}^{(1)}=0$ by the equation $Y_{ijk}^{(0)}=0$, and $X_{ij}^{(2)}=0$ from $Y_{ijk}^{(1)}=0$. This contradiction shows that $X_{ij}^{(0)}\neq 0$.

Assume that $P(a_k/3)=0$. Since $a_k\neq 0$, evidently $a_k\neq a_k/3$. Hence suppose that $a_i=a_k/3$. Then by (22) we have

$$4a_i+a_j=\frac{9}{7}, \qquad 3a_i^2+4a_ia_j=\frac{3}{7},$$

so that $a_i$ must satisfy the equation $91x^2-36x+3=0$. But this is impossible and so $a_i\neq a_k/3$. Similarly it can be shown that $a_j\neq a_k/3$. Hence $P(a_k/3)\neq 0$ and the lemma is proved.

LEMMA 3. *Under the assumption that the formulas of orders $q=5$ and $p=7$ exist for $r=5$, let $i$, $j$, $k$ and $l$ be a permutation of* 1, 2, 3 *and* 4. *If $a_l=a_k$, then $a_i$, $a_j$ and $a_k$ satisfy the system* (6.21),

$$(a_5-a_1)(a_5-a_2)(a_5-a_3)(a_5-a_4)=0, \tag{6.25}$$

*and*

$$X_{ij}^{(3)}-3a_1X_{ij}^{(2)}\neq 0. \tag{6.26}$$

PROOF. Suppose that $a_l=a_k$. From $(7)_0'$ and $(7)_0$ follow (25) and $Y_{ijk}^{(0)}=0$. Then by $(7)_n$ $(n=1, 2)$ we have $Y_{ijk}^{(n)}=0$ $(n=1, 2)$.

Suppose that (26) is not true. Then, since $X_{ij}^{(3)}-a_kX_{ij}^{(2)}=0$ by (21), we have $(a_k-3a_1)X_{ij}^{(2)}=0$. By (24) it follows that $a_k=3a_1$. Since $a_1$ and $a_k$ are roots of the equation (23), this contradicts the lemma 2. Thus the proof is complete.

LEMMA 4. *In order that the formulas of orders $q=5$ and $p=7$ may exist for $r=5$, it is necessary that*

$$a_l \neq a_k \qquad (l \neq k;\ k,\ l=1,\ 2,\ 3,\ 4).$$

PROOF. Suppose first that $a_2=a_1$. Then $c_3=0$ and $a_1$ must satisfy the equation (23) by the lemma 3. If we put $c_4=(a_4-a_1)(a_4-a_3)t_4$, from $(4)_0'$, $(3)_0'$, $(4)_0$ and $(3)_0$ $(i=3)$, it follows that

$$c_5=(a_5-a_1)(a_5-a_3)t_4. \tag{6.27}$$

By (27), (25) and $(6)_n$ $(n=0, 1)$ we have

$$W_4^{(n+3)}-3a_1 W_4^{(n+2)}=0 \quad (n=0,\ 1).$$

Solving this system, we have $a_1=(4\pm\sqrt{2})/21$. But this value does not satisfy the equation (23). Hence $a_2 \neq a_1$.

Suppose next that $(a_3-a_1)(a_3-a_2)=0$. Then $f_4=0$ and $R=0$ by (15). Since by (22)

$$7(a_1+a_2)=9-7a_4, \qquad 7a_1a_2=7a_4^2-9a_4+3,$$

from the equation $R=0$ we have

$$a_5=\frac{63a_4^2-67a_4+14}{105a_4^2-114a_4+25}.$$

By (25) $a_5$ must be equal to one of $a_1$, $a_2$ and $a_4$, so that it must satisfy the equation (23). But, as is easily checked, it is impossible. Hence $a_3 \neq a_1$ and $a_3 \neq a_2$.

Suppose that $a_4=a_1$ and put $d_{13}-d_{12}=(a_3-a_2)w$. Then, from $(5)_0'$, $(1)_0'$, $(3)_0'$, $(5)_0$, $(1)_0$ and $(3)_0$ $(i=2)$, it follows that

$$(a_5-a_1)[d_{15}-d_{12}-(a_5-a_2)w]=0, \tag{6.28}$$

$$W_1^{(2)}=W_1^{(0)}d_{12}+X_{12}^{(0)}w. \tag{6.29}$$

By $(5)_1$, $(1)_1$, $(3)_1$ and (28) we have

$$W_1^{(3)}=W_1^{(1)}d_{12}+X_{12}^{(1)}w. \tag{6.30}$$

Since by (21) and (24)

$$X_{12}^{(1)}=a_3X_{12}^{(0)},\ X_{13}^{(2)}=a_2^2X_{13}^{(0)},\ X_{13}^{(0)}\neq 0,$$

we have $d_{12}=a_2^2$ from (29) and (30). Similarly $d_{13}=a_3^2$ can be obtained.

By (26) and (10) $(k=5)$ we have

$$(a_5-a_1)(a_5-a_3)c_5\neq 0. \tag{6.31}$$

Hence it must hold that $a_5=a_2$ by (25) and then $d_{15}=d_{12}$ by (28). Put $c_3=(a_3-a_1)(a_3-a_2)t_3$. Then, from $(4)_0'$, $(3)_0'$, $(4)_0$ and $(3)_0$ $(i=2)$, it follows that

(6.32) $$c_4q_4=U^{(0)}-X_{12}^{(0)}t_3=c_4p_4+c_5p_5.$$

From $(2)_0'$, $(3)_0'$, $(2)_0$ and $(3)_0$ $(i=2, 3)$ we have

$$d_{14}q_4=d_{14}p_4=a_1^2X_{23}^{(0)}/(a_2-a_1)(a_3-a_1)\neq 0.$$

Hence $q_4=p_4\neq 0$, and $c_5p_5=0$ by (32). Since $p_5\neq 0$, we must have $c_5=0$, which contradicts (31). Hence $a_4\neq a_1$.

Suppose that $(a_4-a_2)(a_4-a_3)=0$. Then we have $X_{23}^{(3)}=a_5X_{23}^{(2)}$ by (11) and $X_{23}^{(3)}=a_1X_{23}^{(2)}$ by $Y_{123}^{(2)}=0$. Hence $a_5=a_1$ by (24). Assume first that $a_4=a_3$. Then $X_{13}^{(3)}=3a_1X_{13}^{(2)}$ by (10) $(k=4)$. This contradicts (26), so that $a_4\neq a_3$. Next suppose that $a_4=a_2$. Then $(a_5-a_2)d_{15}=0$ from $(5)_0'$, $(3)_0'$, $(5)_0$ and $(3)_0$ $(i=2)$. Since $a_5-a_2=a_1-a_2\neq 0$, it follows that $d_{15}=0$ and $X_{23}^{(2)}=0$ by (9). This contradicts (24). Hence $a_4\neq a_2$. Thus the lemma has been proved.

PROOF of the theorem. Assume that the formulas of orders $q=5$ and $p=7$ exist and put

(6.33) $$d_{1k}=(a_k-a_1)s_k \qquad (k=2, 3, 4)$$

(6.34) $$s_j-s_2=(a_j-a_2)r_j \quad (j=3, 4), \qquad r_4-r_3=(a_4-a_3)u.$$

Then, from $(2)_0'$, $(1)_0'$, $(3)_0'$, $(7)_0'$, $(2)_n$, $(1)_n$, $(3)_n$ $(i=2)$, $(7)_n$ $(k=4)$ $(n=0, 1, 2)$, it follows that

(6.35) $$d_{15}=(a_5-a_1)[s_2+(a_5-a_2)(r_3+(a_5-a_3)u)],$$

(6.36) $$V^{(n+2)}=W_1^{(n)}s_2+X_{12}^{(n)}r_3+Y_{123}^{(n)}u \qquad (n=0, 1, 2).$$

Also from $(2)_0'$, $(1)_0'$, $(3)_0'$, $(2)_n$, $(1)_n$, $(3)_n$ $(i=2)$ $(n=0, 1, 2, 3)$, we have

(6.37) $$(a_5-a_1)(a_5-a_2)(a_5-a_3)(a_5-a_4)u=0,$$

(6.38) $$W_4^{(n+2)}=X_{14}^{(n)}s_2+Y_{124}^{(n)}r_3 \qquad (n=0, 1, 2).$$

From (12), $(4)_0$, (13) and (14) it follows that

(6.39) $$V^{(4)}=2U^{(0)}s_2+Pr_3+Qu.$$

The system (36) can be solved as follows:

(6.40) $$u=35a_1^2/d, \qquad r_3=[1-15a_1+35a_1^2(a_2+a_3)]/d,$$

$$s_2=(a_1+a_2-15a_1a_2+35a_1^2a_2^2)/d, \qquad d=1-15a_1+45a_1^2-35a_1^3.$$

Put

(6.41) $$c_j=(a_j-a_1)(a_j-a_2)t_j \quad (j=3, 4), \qquad t_4-t_3=(a_4-a_3)v.$$

Then from $(4)_0'$, $(3)_0'$, $(7)_0'$, $(4)_n$, $(3)_n$ $(i=2)$, and $(7)_n$ $(k=4)$ $(n=0, 1, 2)$ we have

(6.42) $$c_5=(a_5-a_1)(a_5-a_2)[t_3+(a_5-a_3)v],$$

(6.43) $$U^{(n)}=X_{12}^{(n)}t_3+Y_{123}^{(n)}v \qquad (n=0,\ 1,\ 2).$$

Suppose that $a_1\neq 0$. Then (25) must be valid by (40) and (37), so that we have by $(8)_n$

(6.44) $$Z_{1234}^{(n)}=0 \qquad (n=0,\ 1).$$

Since $Y_{124}^{(n+1)}=a_3Y_{124}^{(n)}$ $(n=0,\ 1)$ by (44), from (38) it follows that

$$X_{34}^{(n+2)}=Y_{134}^{(n)}s_2 \qquad (n=0,\ 1),$$

and from this we have

(6.45) $$X_{34}^{(3)}=a_2X_{34}^{(2)},$$

because $Y_{134}^{(1)}=a_2Y_{134}^{(0)}$ by (44). Similarly from (43) it follows that

$$W_4^{(n+3)}-3a_1W_4^{(n+2)}=Y_{124}^{(n)}t_3 \qquad (n=0,\ 1),$$

and from this we have

(6.46) $$X_{34}^{(3)}=3a_1X_{34}^{(2)}.$$

From (45) we have $Y_{234}^{(2)}=0$ and so by (44).

(6.47) $$Y_{234}^{(n)}=0 \qquad (n=0,\ 1,\ 2),$$

because $a_1\neq 0$ by the assumption. From (45) and (46) it follows that $a_1=a_2/3$. Then by the lemma 2 $a_1$ is not a root of $P(x)=0$, so that $a_4\neq a_1$. Substituting (40) into (39), we have

$$6-14(a_1+a_2+a_3)+42(a_1a_2+a_1a_3+a_2a_3)-210a_1a_2a_3=0.$$

On the other hand, by (47) and (22) there holds

$$6-14(a_2+a_3+a_4)+42(a_2a_3+a_2a_4+a_3a_4)-210a_2a_3a_4=0.$$

From these we have

(6.48) $$(a_4-a_1)[1-3(a_2+a_3)+15a_2a_3]=0.$$

Since $a_4\neq a_1$ and

$$7[1-3(a_2+a_3)+15a_2a_3]=105a_4^2-114a_4+25\neq 0,$$

(48) can not be satisfied. Hence $a_1=0$.

When $a_1=0$, the system of equations (43) has the solution

$$a_2=0,\ v=1,\ t_3=a_3,$$

which contradicts $a_2 \neq a_1$. Thus the last part of the theorem has been proved.

The formulas of orders $q=5$ and $p=6$ exist. For instance, we have the following formulas:

$$a_1=0,\ a_2=\frac{1}{5},\ b_{21}=\frac{1}{50},\ a_3=\frac{1}{2},\ b_{31}=0,\ b_{32}=\frac{1}{8}, \tag{6.49}$$

$$a_4=\frac{3}{5},\ b_{41}=\frac{1}{70},\ b_{42}=\frac{1}{7},\ b_{43}=\frac{4}{175},\ a_5=1,$$

$$b_{51}=\frac{337}{1050},\ b_{52}=-\frac{44}{315},\ b_{53}=\frac{472}{1575},\ b_{54}=\frac{2}{105},$$

$$p_1=\frac{1}{18},\ p_2=\frac{25}{96},\ p_3=0,\ p_4=\frac{25}{144},\ p_5=\frac{1}{96},$$

$$q_1=\frac{1}{36},\ q_2=\frac{25}{72},\ q_3=-\frac{2}{9},\ q_4=\frac{25}{72},$$

$$r_1=-\frac{1}{36},\ r_2=\frac{25}{288},\ r_3=-\frac{2}{9},\ r_4=\frac{25}{144},\ r_5=-\frac{1}{96},$$

$$T=\frac{1}{7!}h^7\Big[\frac{1}{125}(Z_5+10Z_0Y_3+15Z_0^2X_1)-\frac{1}{5}(Z_1Y_2+Z_0Z_1X_0) \tag{6.50}$$

$$-\frac{1}{15}Z_0Y_0^2\Big]+O(h^8),$$

$$s=-\frac{1}{6!}h^6\Big[\frac{1}{20}(Z_4+6Z_0Y_2+3Z_0^2X_0)+\frac{1}{35}(Z_2Y_0+Z_0Y_0^2)\Big] \tag{6.51}$$

$$-\frac{1}{7!}h^7\Big[\frac{161}{1000}(Z_5+10Z_0Y_3+15Z_0^2X_1)+\frac{7}{10}(Z_1Y_2+Z_0Z_1X_0)$$

$$+\frac{2}{5}Z_2Y_1+\frac{1}{50}Z_3Y_0+\frac{23}{50}Z_0Y_0Y_1-\frac{1}{15}Z_0Y_0^2\Big]+O(h^8).$$

The formulas of orders $q=4$ and $p=7$ exist. For instance, we have the formulas as follows:

$$a_1=0,\ a_2=\frac{1}{7},\ b_{21}=\frac{1}{98},\ a_3=\frac{2}{5},\ b_{31}=-\frac{1}{250}, \tag{6.52}$$

$$b_{32}=\frac{21}{250},\ a_4=\frac{5}{7},\ b_{41}=\frac{235}{2058},\ b_{42}=-\frac{10}{1323},$$

$$b_{43}=\frac{1375}{9261},\ a_5=1,\ b_{51}=-\frac{47}{55},\ b_{52}=\frac{56}{33},\ b_{53}=-\frac{425}{726},$$

$$b_{54}=\frac{147}{605},\ p_1=\frac{13}{300},\ p_2=\frac{2401}{12960},\ p_3=\frac{625}{3564},\ p_4=\frac{2401}{26400},$$

$$p_5=\frac{11}{2160},\ q_1=\frac{1}{40},\ q_2=\frac{49}{216},\ q_3=\frac{325}{2376},\ q_4=\frac{49}{440},$$

$$r_1=-\frac{11}{600},\ r_2=\frac{539}{12960},\ r_3=-\frac{25}{648},\ r_4=\frac{49}{2400},\ r_5=-\frac{11}{2160},$$

$$(6.53)\quad T=\frac{1}{8!}h^8\Big[\frac{1}{525}(Z_6+15Z_0Y_4+45Z_0^2X_2+15Z_0^3W_0)+\frac{4}{35}(Z_1Y_3+3Z_0Z_1X_1 - Z_0^2Y_0X_0)-\frac{1}{7}(Z_2Y_2+Z_0Z_2X_0)+\frac{6}{35}(Z_3Y_1+3Z_0Y_1^2)+\frac{1}{105}Z_4Y_0 -\frac{3}{35}Z_0Y_0Y_2+\frac{26}{105}Z_1Y_0Y_1-\frac{1}{21}(Z_2Y_0^2+Z_0Y_0^3)+\frac{158}{1155}Z_1^2X_0\Big] +O(h^9),$$

$$(6.54)\quad s=-\frac{1}{5!}h^5\frac{1}{42}Z_1Y_0-\frac{1}{6!}h^6\Big[\frac{11}{490}(Z_4+6Z_0Y_2+3Z_0^2X_0)+\frac{46}{245}Z_1Y_1 -\frac{3}{98}(Z_2Y_0+Z_0Y_0^2)\Big]-\frac{1}{7!}h^7\Big[\frac{869}{12250}(Z_5+10Z_0Y_3+15Z_0^2X_1) +\frac{1012}{1225}(Z_1Y_2+Z_0Z_1X_0)+\frac{8}{245}Z_2Y_1+\frac{25}{294}Z_3Y_0+\frac{141}{490}Z_0Y_0Y_1\Big] +O(h^8).$$

Thus the theorem has been proved.

## 7. Numerical examples

The initial value problem

$$(7.1)\qquad y'=y,\qquad y(0)=1$$

is solved numerically by means of the formulas for $r=2$, 3 and 4 with the step-size $h=0.25$. At each step of integration $z_1$ is accepted as the approximate value of $y_1$. The values of $s$ and $S$ are listed in the table 1 for comparison.

Table 1.

| | r=2 | | r=3 | | r=4 | |
|---|---|---|---|---|---|---|
| X | s | S | s | S | s | S |
| 0.25 | −1.80E−3 | −1.80E−3 | −3.37E−6 | −3.40E−6 | −1.47E−7 | −1.48E−7 |
| 0.50 | −2.31E−3 | −2.31E−3 | −4.32E−6 | −4.37E−6 | −1.89E−7 | −1.90E−7 |
| 0.75 | −2.96E−3 | −2.97E−3 | −5.55E−6 | −5.61E−6 | −2.43E−7 | −2.44E−7 |
| 1.00 | −3.80E−3 | −3.81E−3 | −7.13E−6 | −7.20E−6 | −3.12E−7 | −3.14E−7 |
| 1.25 | −4.88E−3 | −4.89E−3 | −9.15E−6 | −9.25E−6 | −4.00E−7 | −4.03E−7 |
| 1.50 | −6.27E−3 | −6.28E−3 | −1.18E−5 | −1.19E−5 | −5.14E−7 | −5.17E−7 |
| 1.75 | −8.05E−3 | −8.06E−3 | −1.51E−5 | −1.53E−5 | −6.60E−7 | −6.64E−7 |
| 2.00 | −1.03E−2 | −1.04E−2 | −1.94E−5 | −1.96E−5 | −8.47E−7 | −8.53E−7 |

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*Department of Mathematics,*
*Faculty of Science,*
*Hiroshima University*