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# *Note on the Enumeration of Embeddings of Real Projective Spaces*

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## § 1. Introduction

Recently, Y. Nomura [12] has studied the enumeration problem of liftings of a given map to a fibration and its application to the enumeration problem of immersions of certain manifolds. In this note, using his results we enumerate the non-zero cross sections of certain vector bundles, and then study the embedding problem of the real projective spaces in the euclidean spaces.

Let $\xi$ be an orientable $n$-plane bundle over a $CW$-complex $X$ of dimension less than $n+2$, and let $w_2(\xi)$ be the second Stiefel-Whitney class of $\xi$. Consider the homomorphisms

$$(1.1)\qquad \begin{aligned} &\Theta^i_\xi\colon\ H^{i-1}(X; Z)\longrightarrow H^{i+1}(X; Z_2),\\ &\Gamma^i_\xi\colon\ H^i(X; Z_2)\longrightarrow H^{i+2}(X; Z_2),\end{aligned}$$

of the cohomology groups, defined by

$$\Theta^i_\xi(a)=Sq^2\rho_2 a+\rho_2 a\cdot w_2(\xi),$$

$$\Gamma^i_\xi(b)=Sq^2 b+b\cdot w_2(\xi),$$

where $\rho_2$ is the mod 2 reduction. Then we prove the following theorem in §§ 2–4, using Nomura's theorem [12, § 2] and the Postnikov factorization of the universal orientable $(n-1)$-sphere bundle $BSO(n-1)\rightarrow BSO(n)$.

THEOREM A. *Let $n\geq 6$ and let $\xi$ be an orientable n-plane bundle over a CW-complex X of dimension less than $n+2$ with a non-zero cross section. Then, the set cross $(\xi)$ of (free) homotopy classes of non-zero cross sections of $\xi$ is given by*

$$cross(\xi)=\begin{cases}\mathrm{Ker}\,\Theta^n_\xi\times\mathrm{Coker}\,\Theta^{n-1}_\xi, & \text{if } \Gamma^{n-1}_\xi \text{ is epimorphic},\\ \mathrm{Ker}\,\Theta^n_\xi\times\mathrm{Coker}\,\Theta^{n-1}_\xi\times\mathrm{Coker}\,\Gamma^{n-1}_\xi, & \text{if } \Theta^{n-1}_\xi \text{ is monomorphic},\end{cases}$$

*where $\Theta^i_\xi$, $\Gamma^i_\xi$ are the homomorphisms of* (1.1).

This is a generalization of a part of the theorem of I. M. James [8, Th. 5.1] for the case $\dim X \leq n$.

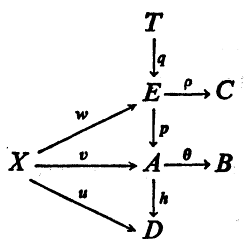
Applying the above theorem, we prove the following theorem in §§ 5–7, using the results of A. Haefliger [6].

THEOREM B. *Let $n$ be an even integer and let $n \geq 10$, $n \neq 2^r$. Then, there exists only one isotopy class of embeddings of the real $n$-dimensional projective space $RP^n$ in the real $(2n-2)$-space $R^{2n-2}$.*

## § 2. Nomura's theorem

Let $h$: $A \to D$ be a principal fibration with fiber $F$, and let $p$: $E \to A$ and $q$: $T \to E$ be the principal fibrations with the classifying maps $\theta$: $A \to B$ and $\rho$: $E \to C$, respectively. For a given $CW$-complex $X$ and a map $u$: $X \to D$, we assume that there are liftings $v$ and $w$ in the following commutative diagram:

(2.1)




and also we assume that $w$ has a lifting to $T$.

In this section, we consider the set $[X, T; u]$ of homotopy classes of liftings $X \to T$ of $u$, under the following stability condition (i)–(iii) for the sequence $\{h, p, q\}$ of fibrations:

(i) the spaces $B$ and $C$ are homotopy associative $H$-spaces,

(ii) there exists a map $d$: $F \times D \to B$ such that

$$\theta m \simeq d(id_F \times h) + \theta \pi_2 \quad \text{and} \quad di_2 \simeq 0,$$

(iii) there exists a map $c$: $\Omega B \times D \to C$ such that

$$\rho\mu \simeq c(id_{\Omega B} \times hp) + \rho\pi_2 \quad \text{and} \quad ci_2 \simeq 0,$$

where $m$: $F \times A \to A$ and $\mu$: $\Omega B \times E \to E$ are the actions of fibers in the principal fibrations $h$: $A \to D$ and $p$: $E \to A$, respectively, $\pi_2$ and $i_2$ denote the projection and the injection to the second factors, and $+$ denotes the multiplication of an $H$-space.

The maps $d$ and $c$ define the maps $d'$: $\Omega F \times D \to \Omega B$ and $c'$: $\Omega^2 B \times D \to \Omega C$ by $d'(\lambda, x)(t) = d(\lambda(t), x)$ and $c'(v, y)(t) = c(v(t), y)$. These maps induce the maps between homotopy sets;

(2.2)
$$\Theta_u\colon [X, F] \longrightarrow [X, B], \qquad \Theta'_u\colon [X, \Omega F] \longrightarrow [X, \Omega B],$$
$$\Gamma_u\colon [X, \Omega B] \longrightarrow [X, C], \qquad \Gamma'_u\colon [X, \Omega^2 B] \longrightarrow [X, \Omega C],$$

by setting

$$\Theta_u(a) = d_*(a, u), \qquad \Theta'_u(a') = d'_*(a', u),$$
$$\Gamma_u(b) = c_*(b, u), \qquad \Gamma'_u(b') = c'_*(b', u),$$

where $u \in [X, D]$ is a given map, and $d_*\colon [X, F] \times [X, D] \to [X, B]$ is the induced map of $d$ and so on. Then it is easy to see that the maps of (2.2) are homomorphisms of groups, by the existence of a lifting of $u$ and the above stability condition (i)–(iii). Further, we define

(2.3) $$\varphi\colon \operatorname{Ker} \Theta_u \longrightarrow \operatorname{Coker} \Gamma_u$$

as follows: For a fixed lifting $v\colon X \to A$ of $u$, the correspondence $[X, F] \ni \sigma \to m_*(\sigma, v) \in [X, A; u]$ is, as is well-known, a bijection. We see easily that $\sigma \in \operatorname{Ker} \Theta_u$ if and only if $m_*(\sigma, v)$ has a lifting to $E$. Let $w_\sigma\colon X \to E$ be a lifting of $m_*(\sigma, v)$ and define

$$\varphi(\sigma) = \rho_*(w_\sigma) \mod \operatorname{Im} \Gamma_u.$$

It is easily shown that $\varphi$ is well-defined.

The following theorem is proved by Y. Nomura [12, Cor. 2.5–6].

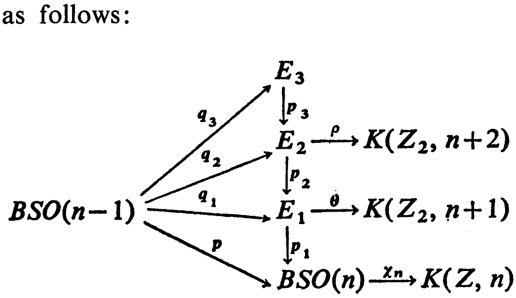
THEOREM. *Under the above assumptions and notations, we obtain, as a set,*

$$[X, T; u] = \begin{cases} \operatorname{Ker} \varphi \times (\operatorname{Ker} \Gamma_u / \operatorname{Im} \Theta'_u) & \text{if } \Gamma'_u \text{ is an epimorphism,} \\ \operatorname{Ker} \varphi \times (\operatorname{Ker} \Gamma_u / \operatorname{Im} \Theta'_u) \times \operatorname{Coker} \Gamma'_u & \text{if } \Theta'_u \text{ is a monomorphism.} \end{cases}$$

## § 3. The Postnikov factorization of the universal orientable $S^{n-1}$-bundle

Let $n \geq 6$. The Postnikov factorization for the fourth stage of the universal orientable $S^{n-1}$-bundle $BSO(n-1) \xrightarrow{p} BSO(n)$, induced by the inclusion $SO(n-1) \subset SO(n)$, is given as follows:

(3.1)



where $\chi_n \in H^n(BSO(n); Z)$ represents the Euler class, $p_1: E_1 \to BSO(n)$ is the principal fibration with the classifying map $\chi_n$, and $\theta$ and $\rho$ are the second and the third $k$-invariants, and $p_2: E_2 \to E_1$ and $p_3: E_3 \to E_2$ are the principal fibrations with the classifying maps $\theta$ and $\rho$, respectively. Furthermore $q_3: BSO(n-1) \to E_3$ is an $(n+2)$-equivalence, i.e., $q_{3*}: \pi_i(BSO(n-1)) \to \pi_i(E_3)$ is isomorphic for $i < n+2$ and epimorphic for $i = n+2$.

Let $m_1: K(Z, n-1) \times E_1 \to E_1$ be the action of fiber in $p_1: E_1 \to BSO(n)$ and consider the map $v_1 = m_1(id \times q_1): K(Z, n-1) \times BSO(n-1) \to E_1$. Then, by the results of E. Thomas [14, p. 21], the second $k$-invariant $\theta \in H^{n+1}(E_1; Z_2)$ is characterized by the equality

$$v_1^*\theta = Sq^2 \rho_2 \iota_1 \times 1 + \rho_2 \iota_1 \times p^* w_2, \tag{3.2}$$

where $v_1^*: H^{n+1}(E_1; Z_2) \to H^{n+1}(K(Z, n-1) \times BSO(n-1); Z_2)$ and $\iota_1 \in H^{n-1}(K(Z, n-1); Z)$ is the fundamental class and $w_2$ is the second universal Stiefel-Whitney class.

Now, consider the homomorphism

$$m_1^* - \pi_2^*: H^r(E_1; Z_2) \longrightarrow H^r(K(Z, n-1) \times E_1; Z_2),$$

where $\pi_2$ is the projection to the second factor. Since $(id \times q_1)^* \pi_2^*(\theta) = 1 \times q_1^*(\theta) = 0$, we have $(id \times q_1)^*(m_1^* - \pi_2^*)(\theta) = (id \times q_1)^* m_1^*(\theta) = v_1^*(\theta)$. On the other hand, $(id \times q_1)^*: \sum_{i=0}^{2} H^{n+1-i}(K(Z, n-1); Z_2) \otimes H^i(E_1; Z_2) \to \sum_{i=0}^{2} H^{n+1-i}(K(Z, n-1); Z_2) \otimes H^i(BSO(n-1); Z_2)$ is monomorphic, because $q_1^*: H^r(E_1; Z_2) \to H^r(BSO(n-1); Z_2)$ is so for $r \leq 2$. Therefore, (3.2) shows that

$$(m_1^* - \pi_2^*)(\theta) = Sq^2 \rho_2 \iota_1 \times 1 + \rho_2 \iota_1 \times p_1^* w_2. \tag{3.3}$$

Similarly, let $m_2: K(Z_2, n) \times E_2 \to E_2$ be the action of fiber in $p_2: E_2 \to E_1$, and consider the map $v_2 = m_2(id \times q_2): K(Z_2, n) \times BSO(n-1) \to E_2$. Then the third $k$-invariant $\rho \in H^{n+2}(E_2; Z_2)$ is characterized by

$$v_2^* \rho = Sq^2 \iota_2 \times 1 + \iota_2 \times p^* w_2,$$

where $\iota_2 \in H^n(K(Z_2, n); Z_2)$ is the fundamental class (cf. [15, Th. 3.5]). Therefore we have

$$(m_2^* - \pi_2^*)(\rho) = Sq^2 \iota_2 \times 1 + \iota_2 \times p_2^* p_1^* w_2, \tag{3.4}$$

by the same argument as above.

## § 4. Proof of Theorem A

Continuing the previous section, we choose the maps

$$d\colon (K(Z, n-1)\times BSO(n), BSO(n))\longrightarrow (K(Z_2, n+1),*),$$

$$c\colon (K(Z_2, n)\times BSO(n), BSO(n))\longrightarrow (K(Z_2, n+2),*)$$

such that they represent the elements $d=Sq^2\rho_2\iota_1\times 1+\rho_2\iota_1\times w_2$ and $c=Sq^2\iota_2\times 1+\iota_2\times w_2$, respectively. Then from the equalities (3.3) and (3.4), it is easy to see that the sequence $\{p_1, p_2, p_3\}$ of principal fibrations in the diagram (3.1) satisfies the stability condition (i)–(iii) in § 2. Therefore, for a given map $\xi\colon X\to BSO(n)$ which has a lifting $X\to E_3$, we can define the homomorphisms

$$\Theta_\xi^i\colon H^{i-1}(X; Z)\longrightarrow H^{i+1}(X; Z_2) \qquad \textit{for } i=n, n-1,$$

$$\Gamma_\xi^i\colon H^i(X; Z_2)\longrightarrow H^{i+2}(X; Z_2) \qquad \textit{for } i=n, n-1,$$

corresponding to $\Theta_u$, $\Theta'_u$, $\Gamma_u$ and $\Gamma'_u$ of (2.2) and these are the homomorphisms of (1.1) by definition.

We now prove Theorem A in § 1.

Let $\xi$ be an orientable $n$-plane bundle over a $CW$-complex $X$ of dimension less than $n+2$ and suppose that $\xi$ has a non-zero cross section. Then the set cross ($\xi$) of homotopy classes of non-zero cross sections of $\xi$ is

$$cross(\xi)=[X, BSO(n-1); \xi]$$

by [9, Lemma 2.2], where $\xi\colon X\to BSO(n)$ denotes the classifying map of $\xi$. Since $\dim X<n+2$ and $q_3\colon BSO(n-1)\to E_3$ is an $(n+2)$-equivalence, we obtain

$$[X, BSO(n-1); \xi]=[X, E_3; \xi]$$

by [9, Th. 3.2]. Now we can apply the theorem in § 2. Since $\dim X<n+2$, we have $H^{n+2}(X; Z_2)=0$ and so $\operatorname{Ker}\Gamma_\xi^n=H^n(X; Z_2)$ and $\operatorname{Ker}(\varphi\colon \operatorname{Ker}\Theta_\xi^n\to \operatorname{Coker}\Gamma_\xi^n)=\operatorname{Ker}\Theta_\xi^n$. This completes the proof.

EXAMPLE. Let $\xi$ be a $(2n-1)$-plane bundle over the real $2n$-dimensional complex projective space $CP^n$ with a non-zero cross section. Then the set cross ($\xi$) is equal to $Z$, the set of integers. In fact, $\Theta_\xi^{2n-2}\colon H^{2n-3}(CP^n; Z)\to H^{2n-1}(CP^n; Z_2)$ is obviously monomorphic and $\operatorname{Coker}\Theta_\xi^{2n-2}=0$. Also $\operatorname{Ker}(\Theta_\xi^{2n-1}\colon H^{2n-2}(CP^n; Z)\to H^{2n}(CP^n; Z_2))$ is equal to $Z$ and $\operatorname{Coker}(\Gamma_\xi^{2n-2}\colon H^{2n-2}(CP^n; Z_2)\to H^{2n}(CP^n; Z_2))$ is $Z_2$ or 0.

## § 5. Enumeration of embeddings

Let $M$ be an $n$-dimensional differentiable closed manifold, $M^*$ be its reduced symmetric product obtained from $M\times M-\Delta$ ($\Delta$ is the diagonal of $M$) by identifying $(x, y)$ with $(y, x)$ and let $\eta$ be the real line bundle over $M^*$ associated with the double covering $M \times M - \Delta \to M^*$. Then the set $[M \subset R^{2n-2}]$ of isotopy classes of embeddings of $M$ into $R^{2n-2}$ for $n \geq 8$ is equal to the set of homotopy classes of cross sections of the associated $S^{2n-3}$-bundle $(M \times M - \Delta) \times_{Z_2} S^{2n-3} \to M^*$ and so equal to *cross* $((2n-2)\eta)$, by the theorem of A. Haefliger [6, § 1].

Since $M^*$ is an open $2n$-manifold, there is a proper Morse function on $M^*$ with no critical points of index $2n$ by [13, Lemma 1.1] and so $M^*$ has the homotopy type of a $CW$-complex of dimension less than $2n$ by [11, Th. 3.5]. Therefore we obtain the following proposition from Theorem A.

PROPOSITION. *Let $n \geq 8$ and let $M$ be an $n$-dimensional differentiable closed manifold which is embedded in $R^{2n-2}$. Then the set $[M \subset R^{2n-2}]$ of isotopy classes of embeddings of $M$ into $R^{2n-2}$ is given by*

$$[M \subset R^{2n-2}] = \begin{cases} \operatorname{Ker} \Theta^{2n-2} \times \operatorname{Coker} \Theta^{2n-3}, & \text{if } \Gamma \text{ is epimorphic,} \\ \operatorname{Ker} \Theta^{2n-2} \times \operatorname{Coker} \Theta^{2n-3} \times \operatorname{Coker} \Gamma, & \text{if } \Theta^{2n-3} \text{ is monomorphic,} \end{cases}$$

*where the homomorphisms*

$$\Theta^i \colon H^{i-1}(M^*; Z) \longrightarrow H^{i+1}(M^*; Z_2) \qquad \text{for } i = 2n-2,\ 2n-3,$$

$$\Gamma \colon H^{2n-3}(M^*; Z_2) \longrightarrow H^{2n-1}(M^*; Z_2),$$

*are defined by*

$$\Theta^i(a) = Sq^2 \rho_2 a + (n-1)\rho_2 a \cdot v^2,$$

$$\Gamma(b) = Sq^2 b + (n-1) b \cdot v^2,$$

*and $v \in H^1(M^*; Z_2)$ is the first Stiefel-Whitney class of the double covering $M \times M - \Delta \to M^*$.*

COROLLARY. *In addition to the conditions of the above proposition, we assume that $H_1(M; Z_2) = 0$. Then we have*

$$[M \subset R^{2n-2}] = H^{2n-3}(M^*; Z) \times \operatorname{Coker} \Theta^{2n-3}.$$

PROOF. Since $H_1(M; Z_2) = 0$, we have $H_1(M \times M, \Delta; Z_2) = 0$ by the exact sequence of the pair $(M \times M, \Delta)$ and so $H^{2n-1}(M \times M - \Delta; Z_2) = H_1(M \times M, \Delta; Z_2) = 0$ by the Poincaré duality. Therefore, the Thom-Gysin exact sequence of the double covering $M \times M - \Delta \to M^*$:

$$\cdots \to H^{2n-1}(M \times M - \Delta; Z_2) \to H^{2n-1}(M^*; Z_2) \to H^{2n}(M^*; Z_2) \quad (=0)$$

shows that $H^{2n-1}(M^*; Z_2) = 0$ and we have the desired result by the above proposition.

## §6. Remarks on the cohomology of $(RP^n)^*$

Let $G_{n+1,2}$ be the Grassmann manifold of 2-planes in $R^{n+1}$. By [2, Th. 11], the mod 2 cohomology of $G_{n+1,2}$ is given by

$$H^*(G_{n+1,2}; Z_2)=Z_2[x, y]/(a_n, a_{n+1}),$$

where $\deg x=1$, $\deg y=2$ and $a_r=\sum_i \binom{r-i}{i} x^{r-2i}y^i$ $(r=n, n+1)$.

S. Feder [4], [5] and D. Handel [7] investigated the mod 2 cohomology of the reduced symmetric product $(RP^n)^*$ of the $n$-dimensional real projective space $RP^n$ and they showed that

(6.1) *$H^*((RP^n)^*; Z_2)$ has $\{1, v\}$ as basis of $H^*(G_{n+1,2}; Z_2)$-module, where $v\in H^1((RP^n)^*; Z_2)$ is the first Stiefel-Whitney class of the double covering $RP^n \times RP^n - \Delta \to (RP^n)^*$ and there are the relations*

$$v^2=vx,\ Sq^1y=xy, \text{ and } x^{2^{r+1}-1}=0 \quad \text{for } n=2^r+s,\ 0\le s<2^r.$$

We study $H^*((RP^n)^*; Z)$ for even $n$. According to [7, (3.4)], there exists a fibration

$$V_{n+1,2}\longrightarrow SZ_{n+1,2}\longrightarrow BG,$$

such that $V_{n+1,2}$ is the Stiefel manifold of 2-frames in $R^{n+1}$, $SZ_{n+1,2}$ is a $(2n-1)$-dimensional closed manifold having the homotopy type of $(RP^n)^*$ and $BG$ is the classifying space of a group $G$ of order 8 (as a matter of fact, $G$ is the dihedral group $D_4$). Let $p$ be an odd prime. The $E_2$-term of the mod $p$ cohomology spectral sequence of the above fibration is given by

$$E_2^{s,t}=H^s(BG; \underline{H}^t(V_{n+1,2}; Z_p)),$$

which is the cohomology with local coefficients $\{H^t(V_{n+1,2}; Z_p)\}$. Since $H^*(V_{n+1,2}; Z_p)=H^*(S^{2n-1}; Z_p)$ for even $n$ by [1, (10.5)], we have

$$E_2^{s,t}=\begin{cases} H^s(BG; \underline{H}^0(V_{n+1,2}; Z_p)) & \text{for } t=0 \\ H^s(BG; \underline{H}^{2n-1}(V_{n+1,2}; Z_p)) & \text{for } t=2n-1 \\ 0 & \text{for } t\neq 0, 2n-1. \end{cases}$$

Since the action of $\pi_1(BG)$ on $H^0(V_{n+1,2}; Z_p)$ is trivial and $H^i(BG; Z_p)=0$ for $i>0$ by [3, Chap. 12, Cor. 2.7], we have

$$E_2^{s,0}=H^s(BG; Z_p)=\begin{cases} Z_p & s=0 \\ 0 & s\neq 0. \end{cases}$$

These imply that $H^s((RP^n)^*; Z_p)=0$ for $0<s<2n-1$ and so

(6.2) *the orders of elements of* $H^s((RP^n)^*; Z)$ *for* $0<s<2n-1$ *are powers of* 2.

Using the above facts, we determine the groups $H^{2n-3}((RP^n)^*; Z)$ and $\rho_2 H^{2n-4}((RP^n)^*; Z)$. Let $n=2^r+s$, $0<s<2^r$ and $s$ be even. By (6.1) and the Poincaré duality for the manifold $SZ_{n+1,2}$,

(6.3) *the* mod 2 *cohomology groups* $H^t((RP^n)^*; Z_2)$ *for* $2n-4\leq t\leq 2n-1$ *are given as follows*:

| $t$ | $H^t((RP^n)^*; Z_2)$ | basis |
|---|---|---|
| $2n-1$ | $Z_2$ | $vx^{2^{r+1}-2}y^s$ |
| $2n-2$ | $Z_2+Z_2$ | $vx^{2^{r+1}-3}y^s$, $x^{2^{r+1}-2}y^s$ |
| $2n-3$ | $Z_2+Z_2+Z_2$ | $vx^{2^{r+1}-4}y^s$, $x^{2^{r+1}-3}y^s$, $vx^{2^{r+1}-2}y^{s-1}$ |
| $2n-4$ | $Z_2+Z_2+Z_2+Z_2$ | $vx^{2^{r+1}-5}y^s$, $x^{2^{r+1}-4}y^s$, $vx^{2^{r+1}-3}y^{s-1}$, $x^{2^{r+1}-2}y^{s-1}$ |

Consider the exact seuqence associated with $0\to Z\xrightarrow{\times 2}Z\xrightarrow{\rho_2}Z_2\to 0$:

$$\cdots\to H^{2n-4}((RP^n)^*; Z)\xrightarrow{\rho_2}H^{2n-4}((RP^n)^*; Z_2)\xrightarrow{\beta_2}H^{2n-3}((RP^n)^*; Z)\xrightarrow{\times 2}$$
$$H^{2n-3}((RP^n)^*; Z)\xrightarrow{\rho_2}H^{2n-3}((RP^n)^*; Z_2)\xrightarrow{\beta_2}H^{2n-2}((RP^n)^*; Z)\to\cdots,$$

where $\beta_2$ is the Bockstein homomorphism. By simple calculations, we have the following relations for the elements of $H^{2n-3}((RP^n)^*; Z_2)$ by (6.1):

$$Sq^1(vx^{2^{r+1}-4}y^s)=vx^{2^{r+1}-3}y^s,\quad Sq^1(x^{2^{r+1}-3}y^s)=x^{2^{r+1}-2}y^s,$$
$$vx^{2^{r+1}-2}y^{s-1}=Sq^1(vx^{2^{r+1}-3}y^{s-1})=\rho_2\beta_2(vx^{2^{r+1}-3}y^{s-1}).$$

These imply that $\rho_2 H^{2n-3}((RP^n)^*; Z)$ is $Z_2$ generated by $vx^{2^{r+1}-2}y^{s-1}$. Hence we have

(6.4) $H^{2n-3}((RP^n)^*; Z)=Z_2$ *generated by* $\beta_2(vx^{2^{r+1}-3}y^{s-1})$

by (6.2) and the above exact sequence.

This shows that $\rho_2$: $H^{2n-3}((RP^n)^*; Z)\to H^{2n-3}((RP^n)^*; Z_2)$ is a monomorphism. Furthermore $\rho_2 H^{2n-4}((RP^n)^*; Z)=\mathrm{Ker}\,\beta_2=\mathrm{Ker}\,(Sq^1\colon H^{2n-4}((RP^n)^*; Z_2)\to H^{2n-3}((RP^n)^*; Z_2))$, because $Sq^1=\rho_2\beta_2$. On the other hand, we have the relations:

$$Sq^1(vx^{2^{r+1}-5}y^s)=0,\qquad Sq^1(x^{2^{r+1}-4}y^s)=0,$$
$$Sq^1(vx^{2^{r+1}-3}y^{s-1})=vx^{2^{r+1}-2}y^{s-1},\quad Sq^1(x^{2^{r+1}-2}y^{s-1})=0.$$

Therefore, by (6.3), we have
(6.5) $\rho_2 H^{2n-4}((RP^n)^*; Z) = Z_2 + Z_2 + Z_2$ *generated by* $\{vx^{2^{r+1}-5}y^s, x^{2^{r+1}-4}y^s, x^{2^{r+1}-2}y^{s-1}\}$.

## § 7. Proof of Theorem B

We now prove Theorem B in § 1.

The existence of embeddings of $RP^n$ in $R^{2n-2}$ is shown in [7, Th. 4.1] and [10, Th. 7.2.2]. To prove that any two embeddings of $RP^n$ in $R^{2n-2}$ are isotopic, we apply the proposition in § 5 for $M = RP^n$, where the homomorphisms

$$\Theta^i: H^{i-1}((RP^n)^*; Z) \longrightarrow H^{i+1}((RP^n)^*; Z_2) \quad \textit{for } i = 2n-2, 2n-3,$$

$$\Gamma: H^{2n-3}((RP^n)^*; Z_2) \longrightarrow H^{2n-1}((RP^n)^*; Z_2)$$

are defined by $\Theta^i(a) = Sq^2 \rho_2 a + \rho_2 a v^2$ and $\Gamma(b) = Sq^2 b + bv^2$. We see that $\Theta^{2n-2}$ is a monomorphism by (6.4) and the following relations:

$$\Theta^{2n-2}(\beta_2(vx^{2^{r+1}-3}y^{s-1})) = Sq^2(vx^{2^{r+1}-2}y^{s-1}) + vx^{2^{r+1}-2}y^{s-1}v^2$$

$$= vx^{2^{r+1}-2}y^s \neq 0 \ (by\ (6.3)).$$

Also, the equation $\Gamma(vx^{2^{r+1}-2}y^{s-1}) = vx^{2^{r+1}-2}y^s$ and (6.3) imply that $\Gamma$ is an epimorphism. Consider the homomorphism $\Theta'$: $\rho_2 H^{2n-4}((RP^n)^*; Z) \to H^{2n-2}((RP^n)^*; Z_2)$ defined by $\Theta'(a) = Sq^2 a + av^2$. Then we have the relations

$$\Theta'(x^{2^{r+1}-2}y^{s-1}) = x^{2^{r+1}-2}y^s,$$

$$\Theta'(x^{2^{r+1}-4}y^s) = vx^{2^{r+1}-3}y^s + \binom{s}{2}x^{2^{r+1}-2}y^s.$$

These and (6.3), (6.5) show that $\Theta'$ is an epimorphism, and so is $\Theta^{2n-3} = \Theta' \rho_2$. This completes the proof of Theorem B.

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