HIROSHIMA MATH. J.
4 (1974), 211-228

Some Remarks on the Cauchy Problem for p-Parabolic Equations

Mitsuyuki ITANO
(Received September 17, 1973)

In his paper [11] S. Mizohata gave a semi-group theoretic treatment of the Cauchy problem for a regularly p-parabolic equation. This was successfully done with the aid of an operator matrix $H_q(t)=H_q(x, t, D_x)$ introduced therein. Recently D. Ellis [2] developed a Hilbert space approach to the Cauchy problem for a uniformly p-parabolic equation, following in rough outline the method explored by S. Kaplan [9] in his treatment of the Cauchy problem for a parabolic operator $\frac{\partial}{\partial t}-L(t)$, where $L(t)$ is uniformly strongly elliptic. Generally, in such an approach, special attention has been paid to find out energy estimates appropriate to the problem. As for the Cauchy problem for a specified parabolic system (§ 6 in [7]), the present author, in collaboration with K. Yoshida, has tried a generalization of Kaplan's treatment indicated above by introducing a certain type of energy estimates.

The main purpose of this paper is to investigate the uniqueness and existence theorems of a solution to the Cauchy problem for a regularly p-parabolic equation from a Hilbert space approach as done by D. Ellis [2], relying upon another type of energy estimates which will be established with the aid of a prescribed operator matrix $H_q(t)$, and following the same arguments as in our treatment (§ 6 in [7]) of a parabolic system.

By the Cauchy problem we shall always mean a fine Cauchy problem as described in paper [7]. With this in mind, in Section 1, some notations and functional spaces are introduced with a precise formulation of such a Cauchy problem for a regularly p-parabolic equation, where the notions of the $\mathscr{D}'_{L^2}$-boundary value and the $\mathscr{D}'_{L^2}$-canonical extension of a distribution are discussed in some detail. In Section 2 the energy inequalities (cf. Theorems 1 and 2 below) for a regularly p-parabolic operator and for its dual operator are derived by making use of the operator matrix $H_q(t)$, which was introduced by S. Mizohata [11]. The former estimate will be of a type very similar to the one obtained in [7, Theorem 8]. These estimates enable us to apply a Hilbert space approach to our problem. Finally in Section 3 the uniqueness and existence theorems for our problem are discussed along this line of thought. We improve some of the results obtained by D. Ellis [2]. Combining Corollary 4 with Proposition 5 below, we have a refinement of Theorem 9 in his paper [2]. This, in a sense, is a generalization of a result of S. Mizohata [11, Proposition 5]. We add here that the improvement itself has been announced in his paper [2] without proof.

1. Preliminaries

We denote by $R_{n+1}=R_n\times R$ an $(n+1)$-dimensional Euclidean space with a generic point $(x, t)=(x_1,\dots, x_n, t)$ and write $D_x=(D_1,\dots, D_n)$, $D_j=\frac{1}{i}\frac{\partial}{\partial x_j}$, $D_t=\frac{1}{i}\frac{\partial}{\partial t}$ and $D_x^\alpha=D_1^{\alpha_1}\dots D_n^{\alpha_n}$ with $\alpha=(\alpha_1,\dots, \alpha_n)$. For a point $\xi=(\xi_1,\dots, \xi_n)$ of the dual Euclidean space Ξ_n we write $|\xi|=(\xi_1^2+\cdots+\xi_n^2)^{1/2}$ and $\xi^\alpha=\xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}$.

Let p be a positive integer and let $a_{\alpha,j}\in\mathscr{B}(H)$ for α, j with $|\alpha|\leqq jp$, $j=1, 2,\dots, m$, where by $\mathscr{B}(H)$ we mean the space of C^∞ functions a on $H=R_n\times[0, T]$ such that a is bounded with its derivatives of every order. In the present paper we shall consider the differential operator

$$P=D_t^m+\sum_{j=1}^m\sum_{|\alpha|\leqq jp}a_{\alpha,j}(x, t)D_x^\alpha D_t^{m-j}, \qquad (m\geqq 1)$$

satisfying the following condition: for every root $\tau=\tau(x, t, \xi)$, $\xi\in\Xi_n$, of the polynomial

$$P_0(x, t, \xi, \tau)=\tau^m+\sum_{j=1}^m\sum_{|\alpha|=jp}a_{\alpha,j}(x, t)\xi^\alpha\tau^{m-j}$$

in τ there exists a positive constant δ, independent of x, t and ξ but depending on T, such that $\operatorname{Im}\tau\geqq\delta$ for $(x, t)\in H$ and $\xi\in\Xi_n$ with $|\xi|=1$. Then P is called a regularly p-parabolic in $0\leqq t\leqq T$ [11, p. 269]. Let $P=D_t^m+\sum_{j=1}^m\sum_{|\alpha|\leqq jp}a_{\alpha,j}(x, t)\,D_x^\alpha D_t^{m-j}$, where $a_{\alpha,j}\in C^\infty(\bar{R}_{n+1}^+)$, $R_{n+1}^+=R_n\times(0, \infty)$, and their restrictions $a_{\alpha,j}|H_T=R_n\times[0, T]$, belong to the space $\mathscr{B}(H_T)$ for any $T>0$. If for any $T>0$ there exists a positive constant δ_T such that $\operatorname{Im}\tau\geqq\delta_T$ for $(x, t)\in H_T$ and $\xi\in\Xi_n$ with $|\xi|=1$, then P is called a regularly p-parabolic operator in $0\leqq T<\infty$. It is known that p must be a positive even integer. In what follows, we write $p=2p'$.

By $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ we mean the ε-product $\mathscr{D}'_t\varepsilon(\mathscr{D}'_{L^2})_x$ and by $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$ the space of distributions $\in\mathscr{D}'(\mathring{H})$ which can be extended to distributions $\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$. The quotient topology is introduced in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$. Similarly the space $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(R_n\times(-\infty, T])$ will be defined.

Let $u\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$ and suppose $u(x, \varepsilon t)$ converges in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$ to a distribution v as $\varepsilon\downarrow 0$. Then we see that v is independent of t and it can be written in the form $\alpha_0\otimes Y_t$, where $\alpha_0\in(\mathscr{D}'_{L^2})_x$ and Y_t is a Heaviside function [6, p. 375]. α_0 is called the $\mathscr{D}'_{L^2}$-boundary value of u and denoted by $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} u$ [6, p. 375].

Let $\phi \in \mathscr{D}(R_t^+)$ be such that $\phi \geqq 0$ and $\int_0^\infty \phi dt = 1$, and put $\rho = Y * \phi$. Consider a $u \in \mathscr{D}_t'((\mathscr{D}'_{L^2})_x)(H)$. Then $\rho(t/\varepsilon)u$ may be regarded as an element of $\mathscr{D}_t'((\mathscr{D}'_{L^2})_x)(R_n \times (-\infty, T])$ for any $\varepsilon > 0$. If $\rho(t/\varepsilon)u$ converges in $\mathscr{D}_t'((\mathscr{D}'_{L^2})_x)(R_n \times (-\infty, T])$ to v_ϕ as $\varepsilon \downarrow 0$, then v_ϕ does not depend on the choice of ϕ. The limit element is called the $\mathscr{D}'_{L^2}$-canonical extension of u over $t=0$. The $\mathscr{D}'_{L^2}$-canonical extension $u_\sim$ exists whenever $\mathscr{D}'_{L^2}\text{-}\lim\limits_{t \downarrow 0} u$ exists.

In the present paper we shall consider the fine Cauchy problem

$$(1) \qquad \begin{cases} Pu = f & \text{in } \overset{\circ}{H} \\ u_0 \equiv \mathscr{D}'_{L^2}\text{-}\lim\limits_{t \downarrow 0} (u, D_t u, \dots, D_t^{m-1} u) = \alpha \end{cases}$$

for preassigned $f \in \mathscr{D}_t'((\mathscr{D}'_{L^2})_x)(H)$ and $\alpha = (\alpha_0, \dots, \alpha_{m-1})$, $\alpha_j \in (\mathscr{D}'_{L^2})_x$. Suppose there exists a solution $u \in \mathscr{D}_t'((\mathscr{D}'_{L^2})_x)(H)$ of (1). Then f and u must have the $\mathscr{D}'_{L^2}$-canonical extensions $f_\sim$ and $u_\sim$ over $t=0$ [5, p. 82; 7, p. 404].

If we put $F = (0, \dots, 0, f)'$ and $U = (u_1, \dots, u_m)'$ with $u_j = D_t^{j-1} u$, where V' means the transposed vector of V, we can rewrite (1) in vector form

$$(2) \qquad \begin{cases} LU \equiv D_t U - A(t) U = F & \text{in } \overset{\circ}{H}, \\ \mathscr{D}'_{L^2}\text{-}\lim\limits_{t \downarrow 0} U = \alpha \end{cases}$$

with

$$A(x, t, D_x) = \begin{pmatrix} 0 & 1 & & & \\ & \cdot & \cdot & & \\ \cdot & & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & 0 & 1 \\ -a_m & \cdot & \cdot & \cdot & -a_1 \end{pmatrix}, \quad a_j = \sum_{|\alpha| \leq jp} a_{\alpha, j}(x, t) D_x^\alpha .$$

We shall write by $M(x, t, \xi)$ the matrix $A(x, t, \xi)$ with $a_j(x, t, \xi)$ replaced by $a_j^0(x, t, \xi) = \sum_{|\alpha| = jp} a_{\alpha, j}(x, t) \xi^\alpha$.

We shall next introduce some spaces. Let σ, s be any real numbers. By $\mathscr{H}_s = \mathscr{H}_s(R_n)$ [8, p. 45] we mean the set of all distributions $u \in \mathscr{S}'(R_n)$ such that $\hat{u}$ is a function and

$$\|u\|_s^2 = \frac{1}{(2\pi)^n} \int |\hat{u}(\xi)|^2 (1 + |\xi|^2)^s d\xi < \infty ,$$

and by $\mathscr{K}_{\sigma, s} = \mathscr{K}_{\sigma, s}(R_{n+1})$ [9, p. 172] the space of all distributions $u \in \mathscr{S}'(R_{n+1})$ such that $\hat{u}$ is a function and

$$\|u\|_{\sigma, s}^2 = \frac{1}{(2\pi)^{n+1}} \iint |\hat{u}(\xi, \tau)|^2 (\tau^2 + (1 + |\xi|^2)^p)^{\sigma/p} (1 + |\xi|^2)^s d\xi d\tau < \infty .$$

In what follows, we shall use the notations

$$D_s(R_n)=\mathscr{H}_s(R_n)\times\mathscr{H}_{s-p}(R_n)\times\cdots\times\mathscr{H}_{s-(m-1)p}(R_n),$$

$$D_{\sigma,s}(R_{n+1})=\mathscr{K}_{\sigma,s}(R_{n+1})\times\mathscr{K}_{\sigma,s-p}(R_{n+1})\times\cdots\times\mathscr{K}_{\sigma,s-(m-1)p}(R_{n+1}),$$

where the norms $\|\cdot\|_{D_s}$ and $\|\cdot\|_{D_{\sigma,s}}$ are defined by $\{\|\cdot\|_s^2+\cdots+\|\cdot\|_{s-(m-1)p}^2\}^{1/2}$ and $\{\|\cdot\|_{\sigma,s}^2+\cdots+\|\cdot\|_{\sigma,s-(m-1)p}^2\}^{1/2}$ respectively. We shall denote by $D^{\#}_{-s}(R_n)$ and $D^{\#}_{-\sigma,-s}(R_{n+1})$ the dual spaces of $D_s(R_n)$ and $D_{\sigma,s}(R_{n+1})$ respectively. By $\mathscr{K}_{\sigma,s}(H)$ we mean the set of all $u\in\mathscr{D}'(\mathring{H})$ such that there exists a distribution $v\in\mathscr{K}_{\sigma,s}(R_{n+1})$ with $u=v$ in $\mathring{H}$. The norm of u is defined by $\|u\|_{\sigma,s}=\inf\|v\|_{\sigma,s}$, the infimum being taken over all such v. Especially, the space $\mathscr{K}_{kp,s}(H)$, k being a non-negative integer, has the equivalent norm

$$\left(\sum_{j=0}^{k}\int_0^T\|D_t^j u(\cdot,t)\|_{s+(k-j)p}^2dt\right)^{1/2},$$

which will also be denoted by the symbol $\|u\|_{kp,s}$. We shall consider the space $\mathring{\mathscr{K}}_{\sigma,s}(H)$, the space of all $u\in\mathring{\mathscr{K}}_{\sigma,s}(R_{n+1})$ with supp $u\subset H$. Then $\mathscr{K}_{\sigma,s}(H)$ and $\mathring{\mathscr{K}}_{-\sigma,-s}(H)$ are anti-dual Hilbert spaces with respect to an extension of the sesquilinear form $\int_{R_n}\int_0^T u\bar{v}dxdt$, $u\in C_0^\infty(H)$, $v\in C_0^\infty(\mathring{H})$ [7, p. 51]. The spaces $D_{\sigma,s}(H)$, $\mathring{D}_{\sigma,s}(H)$ and the like are similarly defined.

Consider the space $\mathscr{K}_{\sigma,s}(H)$. The $\mathscr{D}'_{L^2}$-boundary value $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} u$ exists for every $u\in\mathscr{K}_{\sigma,s}(H)$ if and only if $\sigma>p'$. If this is the case, $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} u$ must belong to the space $\mathscr{H}_{\sigma+s-p'}(R_n)$. The $\mathscr{D}'_{L^2}$-canonical extension $u_\sim$ exists for every $u\in\mathscr{K}_{\sigma,s}(H)$ if and only if $\sigma>-p'$. It is also noticed that $\mathscr{K}_{\sigma,s}(H)$ and $\mathring{\mathscr{K}}_{\sigma,s}(H)$ may be identified for $|\sigma|<p'$ [4, p. 416]. Let k be a positive integer such that $|\sigma-k|<p'$. Then $u_\sim\in\mathscr{K}_{\sigma,s}(R_n\times(-\infty,T])$ for every $u\in\mathscr{K}_{\sigma,s}(H)$ if and only if $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} u=\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} D_tu=\cdots=\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} D_t^{k-1}u=0$ [4, p. 419], where the $\mathscr{D}'_{L^2}$-boundary value coincides with the distributional boundary value [3, p. 12].

2. Energy inequalities

Let P be a regularly p-parabolic operator in $0\leqq t\leqq T$. We shall derive energy inequalities for P and for its dual operator P^* by making use of the operator matrix $H_q(t)$, which was constructed by S. Mizohata [11]. He starts for the construction of $H_q(t)$ with the following consideration.

Let $P_0(\tau)=\tau^m+\sum_{j=1}^m a_j^0(x,t,\xi)\tau^{m-j}$ and consider the symmetric polynomial in τ and τ':

$$K(P_0;\tau,\tau')=\frac{P_0(\tau)P_0^*(\tau')-P_0(\tau')P_0^*(\tau)}{\tau-\tau'}=\sum_{h,k=1}^{m}A_{hk}\tau^{h-1}\tau'^{k-1},$$

where $P_0^*(\tau)$ stands for $\overline{P_0(\bar{\tau})}$. Then $-iA_{hk}$ is real and coincides with $-iA_{kh}$. Since all roots of the polynomial $P_0(\tau)$ lie in the half-plane $\mathrm{Im}\,\tau\geqq\delta_T>0$ for $(x,t)\in H=R_n\times[0,T]$ and $\xi\in\Xi_n$ with $|\xi|=1$, the Hermitian form

$$H(P_0;u_1,\dots,u_m)=-i\sum_{h,k=1}^{m}A_{hk}u_h\bar{u}_k$$

is positive definite [1, p. 64]. Let B be the real symmetric matrix (b_{hk}) with $b_{hk}=-iA_{hk}$. Then it follows that $-i(BM-(BM)^*)\geqq 0$ for $M=M(x,t,\xi)$ stated in Section 1, where $(x,t)\in H$ and $\xi\in\Xi_n$ with $|\xi|=1$. On the basis of these facts and applying the method of J. Leray [10, pp. 121–127] in connection with hyperbolic operators to the parabolic case, S. Mizohata has obtained the proposition below.

Let us denote by $E_s=E_s(D_x)$ the operator matrix (e_{hk}), $e_{hh}=S^{2s-2(h-1)p}$, $h=1,\dots,m$ and $e_{hk}=0$ otherwise, where S is a pseudo-differential operator with symbol $\lambda(\xi)=(1+|\xi|^2)^{1/2}$. For two Hermitian matrices $C_1(x,t,D_x)$ and $C_2(x,t,D_x)$ whose components are differential operators with coefficients $\in\mathscr{B}(H)$, the inequality $C_1(x,t,D_x)\leqq C_2(x,t,D_x)$ means that $(C_1(x,t,D_x)\phi,\phi)\leqq(C_2(x,t,D_x)\phi,\phi)$ for any $\phi\in(\mathscr{D}'_{L^2})_x$ and $t\in[0,T]$, where $(\ ,\ )$ means the inner product in L_x^2. For the system of operators $L=D_t-A(t)$ stated in Section 1 we have

PROPOSITION 1 (S. Mizohata). *Let q be any integer. Then there exists an Hermitian matrix $H_q(t)=H_q(x,t,D_x)$ such that*

$$\alpha E_q\leqq H_q(t)\leqq\alpha_q E_q,$$

$$-i(H_q(t)A(t)-(H_q(t)A(t))^*)\geqq\frac{\varepsilon}{2}E_{q+p'}-\gamma_q E_q$$

with positive constants ε, α, α_q and γ_q, which are independent of $(x,t)\in H$, and $H_q(t)$ is an $\mathscr{L}(D_s,D_{s-2q})$-valued C^∞ function of $t\in[0,T]$ for any real s.

We shall give an energy inequality for L. We need the following lemma (cf. Lemma 3 in [7, p. 405]).

LEMMA 1. *Let $r(t)$ and $\rho(t)$ be two real-valued functions defined in the interval $0\leqq t\leqq T$ and assume that r is continuous and ρ is non-decreasing. Then the inequality*

$$r(t)\leqq C(\rho(t)+\int_0^t r(t')dt')\qquad (C>0\ \textit{is a constant})$$

implies $r(t) \leqq Ce^{Ct}\rho(t)$.

THEOREM 1. *Let s be any real number. Then there exists a constant* C_T, *independent of U and* t_0, t_1 *but depending on s, such that*

$$(E_s)\quad \|U(t_1)\|^2_{D_{s+p'}} + \int_{t_0}^{t_1} \|U(t)\|^2_{D_{s+p}} dt \leqq C_T(\|U(t_0)\|^2_{D_{s+p'}} + \int_{t_0}^{t_1} \|LU(t)\|^2_{D_s} dt)$$

for any t_0, t_1 *with* $0 \leqq t_0 < t_1 \leqq T$ *and any* $U = (u_1, \ldots, u_m)'$, $u_j \in C_0^\infty(R_{n+1})$.

PROOF. Let $U = (u_1, \ldots, u_m)'$ with $u_j \in C_0^\infty(R_{n+1})$ and put $F = LU$ and $h^2(t) = (H_0(t)U(t), U(t))$. Then we have

$$\begin{aligned}
\frac{d}{dt} h^2(t) &= i(H_0(t)D_t U(t), U(t)) - i(H_0(t)U(t), D_t U(t)) + \left(\frac{d}{dt} H_0(t) \cdot U(t), U(t)\right) \\
&= i(H_0(t)A(t)U(t), U(t)) - i(H_0(t)U(t), A(t)U(t)) + \\
&\quad + i(H_0(t)F(t), U(t)) - i(H_0(t)U(t), F(t)) + \left(\frac{d}{dt} H_0(t) \cdot U(t), U(t)\right) \\
&\leqq -\frac{\varepsilon}{2}(E_{p'}U(t), U(t)) + \gamma_0(E_0 U(t), U(t)) + \\
&\quad + 2|\mathrm{Im}(H_0(t)F(t), U(t))| + \left|\left(\frac{d}{dt} H_0(t) \cdot U(t), U(t)\right)\right|
\end{aligned}$$

and therefore

$$\begin{aligned}
h^2(t_1) - h^2(t_0) \leqq -\frac{\varepsilon}{2}\int_{t_0}^{t_1} \|U(t)\|^2_{D_{p'}} dt + (\gamma_0 + \gamma_0')\int_{t_0}^{t_1} \|U(t)\|^2_{D_0} dt + \\
+ 2\int_{t_0}^{t_1} |(H_0(t)F(t), U(t))| dt
\end{aligned}$$

with a constant γ_0' such that $\frac{d}{dt} H_0(t) \leqq \gamma_0' E_0$, $0 \leqq t \leqq T$. Put $V = S^{-s-p'}U$. Then each component v_j of V is a $(\mathscr{D}_{L^2})_x$-valued C^∞ function of $t \in [0, T]$ and

$$\begin{aligned}
\alpha\|V(t_1)\|^2_{D_{s+p'}} - \alpha_0\|V(t_0)\|^2_{D_{s+p'}} \leqq -\frac{\varepsilon}{2}\int_{t_0}^{t_1} \|V(t)\|^2_{D_{s+p}} dt + \\
+ (\gamma_0 + \gamma_0')\int_{t_0}^{t_1} \|V(t)\|^2_{D_{s+p'}} dt + 2\int_{t_0}^{t_1} |(H_0 L S^{s+p'} V(t), S^{s+p'} V(t))| dt,
\end{aligned}$$

$$\begin{aligned}
&|(H_0 L S^{s+p'} V(t), S^{s+p'} V(t))| \\
&\leqq |(H_0 S^{s+p'} L V, S^{s+p'} V)| + |H_0(A S^{s+p'} - S^{s+p'} A) V, S^{s+p'} V)|.
\end{aligned}$$

Since $H_0(t)$ is a continuous operator from D_s into $D_s^\#$ for each $t \in [0, T]$ and

$D^{\#}_{-p'}$ is the dual space of $D_{p'}$, we have the following estimates:

$$\begin{aligned}|(H_0S^{s+p'}LV,\ S^{s+p'}V)| &= |(H_0S^{s+p'}LV,\ S^{s+p'}V)_{D^{\#}_{-p'},D_{p'}}|\\ &\leqq \|H_0S^{s+p'}LV\|_{D^{\#}_{-p'}}\|S^{s+p'}V\|_{D_{p'}}\\ &\leqq C_1\|S^{s+p'}LV\|_{D_{-p'}}\|V\|_{D_{s+p}}\\ &= C_1\|LV\|_{D_s}\|V\|_{D_{s+p}},\end{aligned}$$

$$\begin{aligned}&|(H_0(AS^{s+p'}-S^{s+p'}A)V,\ S^{s+p'}V)|\\ &= |(H_0(AS^{s+p'}-S^{s+p'}A)V,\ S^{s+p'}V)_{D^{\#}_{-p'},D_{p'}}|\\ &\leqq \|H_0(AS^{s+p'}-S^{s+p'}A)V\|_{D^{\#}_{-p'}}\|V\|_{D_{s+p}}\\ &\leqq C_2\|(AS^{s+p'}-S^{s+p'}A)V\|_{D_{-p'}}\|V\|_{D_{s+p}}\end{aligned}$$

with constants C_1 and C_2. Here the operator matrix $AS^{s+p'}-S^{s+p'}A$ has the form $(\alpha_{hk}(t))$ with $\alpha_{hk}(t)=0$ for $h\neq m$. In virtue of Proposition 15 in [6, p. 387] we see that $\alpha_{mk}(t)$ is the operator of order $\leqq(m-k+1)p+s+p'-1$. Thus there exists a constant C_3 such that

$$\|(AS^{s+p'}-S^{s+p'}A)V\|_{D_{-p'}}\leqq C_3\|V\|_{D_{s+p-1}}$$

and therefore we have

$$\begin{aligned}&|(H_0LS^{s+p'}V(t),\ S^{s+p'}V(t))|\\ &\leqq C_1\|LV(t)\|_{D_s}\|V(t)\|_{D_{s+p}}+C_2C_3\|V(t)\|_{D_{s+p-1}}\|V(t)\|_{D_{s+p}}.\end{aligned}$$

Let ε' be any positive number. Then there exists a constant $C_4(\varepsilon')$ such that

$$\|V\|_{D_{s+p-1}}\leqq\varepsilon'\|V\|_{D_{s+p}}+C_4(\varepsilon')\|V\|_{D_s}$$

and we have the inequalities

$$\begin{aligned}\|V\|_{D_{s+p-1}}\|V\|_{D_{s+p}} &\leqq(\varepsilon'\|V\|_{D_{s+p}}+C_4(\varepsilon')\|V\|_{D_s})\|V\|_{D_{s+p}}\\ &\leqq 2\varepsilon'\|V\|^2_{D_{s+p}}+C_5(\varepsilon')\|V\|^2_{D_s}\end{aligned}$$

with a constant $C_5(\varepsilon')$ and consequently

$$\begin{aligned}\int_{t_0}^{t_1}|(H_0LS^{s+p'}V(t),\ S^{s+p'}V(t))|dt &\leqq\varepsilon'(1+2C_2C_3)\int_{t_0}^{t_1}\|V(t)\|^2_{D_{s+p}}dt+\\ &+C_6(\varepsilon')\int_{t_0}^{t_1}\|LV(t)\|^2_{D_s}dt+C_7(\varepsilon')\int_{t_0}^{t_1}\|V(t)\|^2_{D_s}dt\end{aligned}$$

Lemma 1 we obtain (E_s) for V. Thus the proof is complete.

For the regularly p-parabolic operator P we have the following energy inequality.

COROLLARY 1. *Let s be any real number. Then there exists a constant C_T, independent of u and t_0, t_1 but depending on s, such that*

$$\sum_{j=0}^{m-1}\|D_t^j u(\cdot, t_1)\|_{s+p'-jp}^2+\sum_{j=0}^{m-1}\int_{t_0}^{t_1}\|D_t^j u(\cdot, t)\|_{s-(j-1)p}^2 dt$$

$$\leqq C_T\Big(\sum_{j=0}^{m-1}\|D_t^j u(\cdot, t_0)\|_{s+p'-jp}^2+\int_{t_0}^{t_1}\|Pu(\cdot, t)\|_{s-(m-1)p}^2 dt\Big)$$

for any t_0, t_1 with $0\leqq t_0<t_1\leqq T$ and any $u\in C_0^\infty(R_{n+1})$.

Let us consider the formal adjoint operator of P:

$$P^*=D_t^m+\sum_{j=1}^{m} D_t^{m-j}a_j^*(x, t, D_x)=D_t^m+\sum_{j=1}^{m} c_j(x, t, D_x)D_t^{m-j},$$

where

$$a_j^*(x, t, D_x)=\sum_{|\alpha|\leq jp} D_x^\alpha \bar{a}_{\alpha, j}, \quad c_j(x, t, D_x)=\sum_{|\alpha|=jp}\bar{a}_{\alpha, j}D_x^\alpha+\sum_{|\alpha|<jp}c_{\alpha, j}D_x^\alpha.$$

Let $v\in C_0^\infty(R_{n+1})$ and put $g=P^*v$. If we write $V=(v_1,\dots, v_m)'$ with $v_j=D_t^{j-1}v$, $j=1, 2,\dots, m$ and $G=(0,\dots, 0, g)'$, then $P^*v=g$ can be rewritten in the vector form

$$\tilde{L}V\equiv D_tV-C(t)V=G,$$

where $C(t)=C(x, t, D_x)$ is the operator matrix $A(x, t, D_x)$ with $a_j(x, t, D_x)$ replaced by $c_j(x, t, D_x)$. Following the method of construction of $H_q(t)=H_q(x, t, D_x)$ obtained by S. Mizohata with the matrix $M(x, t, \xi)$ replaced by $\overline{M}(x, t, \xi)$, we can find an operator matrix $\tilde{H}_q(t)$, q being any integer, such that

$$\beta E_q\leqq\tilde{H}_q(t)\leqq\beta_qE_q,$$

$$-i(\tilde{H}_q(t)C(t)-(\tilde{H}_q(t)C(t))^*)\leqq-\varepsilon E_{q+p'}+\gamma_qE_q$$

with positive constants ε, β, β_q and γ_q, which are independent of $(x, t)\in H$. $\tilde{H}_q(t)$ is an $\mathscr{L}(D_s, D_{s-2q}^\sharp)$-valued C^∞ function of $t\in[0, T]$ for any real s.

We shall derive the following energy inequality for $\tilde{L}$ by making use of $\tilde{H}_q(t)$.

THEOREM 2. *Let q be any integer. Then there exists a constant C_T, independent of u and t_0, t_2 but depending on q, such that*

$$\|V(t_0)\|_{D_q}\leqq C_T(\|V(t_1)\|_{D_q}+\int_{t_0}^{t_1}\|\tilde{L}V(t)\|_{D_q}dt)$$

for any t_0, t_1 *with* $0 \leqq t_0 < t_1 \leqq T$ *and any* $V=(v_1,\dots, v_m)'$, $v_j \in C_0^\infty(R_{n+1})$.

PROOF. Let $V=(v_1,\dots, v_m)'$ with any $v_j \in C_0^\infty(R_{n+1})$ and put $G=\tilde{L}V$ and $h^2(t)=(\tilde{H}_q(t)V(t), V(t))$. There exists a positive constant β'_q, independent of $(x, t)\in H$, such that $\frac{d}{dt}\tilde{H}_q(t) \leqq \beta'_q E_q$, $0 \leqq t \leqq T$. In the same way as in the proof of Theorem 1 we have

$$\frac{d}{dt}h^2(t) \geqq \varepsilon(E_{q+p'}V, V) - (\gamma_q + \beta'_q)(E_q V, V) - 2\|G\|_{D_q}\|\tilde{H}_q V\|_{D^*_{-q}}$$

$$\geqq -2C_1 h^2(t) - 2C_2\|G\|_{D_q} h(t)$$

with $C_1 = (\gamma_q + \beta'_q)/(2\beta)$ and a positive constant C_2, which implies

$$\frac{d}{dt}(e^{C_1 t}h(t)) \geqq -C_2 e^{C_1 t}\|G(t)\|_{D_q}.$$

Thus we obtain

$$h(t_0) \leqq e^{C_1(t_1-t_0)}h(t_1) + C_2\int_{t_0}^{t_1} e^{C_1(t-t_0)}\|G(t)\|_{D_q} dt.$$

Since we have the inequalities $\beta\|V(t)\|^2_{D_q} \leqq h^2(t) \leqq \beta_q\|V(t)\|^2_{D_q}$, our proof is complete.

For the formal adjoint operator P^* we have the following

COROLLARY 2. *Let* q *be any integer. Then there exists a constant* C_T, *independent of* v *and* t_0, t_1 *but depending on* q, *such that*

$$\sum_{j=0}^{m-1}\|D_t^j v(\cdot, t_0)\|_{q-jp} \leqq C_T\left(\sum_{j=0}^{m-1}\|D_t^j v(\cdot, t_1)\|_{q-jp} + \int_{t_0}^{t_1}\|P^* v(\cdot, t)\|_{q-(m-1)p} dt\right)$$

for any t_0, t_1 *with* $0 \leqq t_0 < t_1 \leqq T$ *and any* $v \in C_0^\infty(R_{n+1})$.

3. Cauchy problem

Let us consider the fine Cauchy problem (1):

$$\begin{cases} Pu \equiv D_t^m u + \sum_{j=1}^{m} a_j(x, t, D_x) D_t^{m-j} u = f & \text{in } \mathring{H}, \\ u_0 \equiv \mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(u, D_t u, \dots, D_t^{m-1}u) = \alpha \end{cases}$$

for preassigned $f \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$ and $\alpha = (\alpha_0, \dots, \alpha_{m-1})$, $a_j \in (\mathscr{D}'_{L^2})_x$, where $a_j(x, t, D_x) = \sum_{|\alpha| \leq jp} a_{\alpha,j} D_x^\alpha$ with $a_{\alpha,j} \in \mathscr{B}(H)$. As noted in [5, p. 78], $a_{\alpha,j}$ can be extended to a function $\in \mathscr{B}(R_{n+1})$. We assume that $a_{\alpha,j} \in \mathscr{B}(R_{n+1})$.

Suppose there exists a solution $u \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$ of (1). Then f, u have the $\mathscr{D}'_{L^2}$-canonical extensions $u_\sim$, $f_\sim$ as noted in Section 1. In addition, $u_\sim$ and $f_\sim$ satisfy the equation

$$P(u_\sim)=f_\sim+\sum_{k=0}^{m-1} D_t^k\delta\otimes\gamma_k(0) \qquad \text{in } R_n\times(-\infty, T],$$

where

$$\gamma_k(t)=-i\sum_{j=k+1}^{m}\sum_{l=1}^{j-k}(-1)^{j-l-k}\binom{j-l}{k} D_t^{j-l-k}a_{m-j}(x, t, D_x)\alpha_{l-1}.$$

For example, $\gamma_{m-1}=-i\alpha_0$, $\gamma_{m-2}=-ia_1\alpha_0-ia_1$, $\gamma_{m-3}=-i(a_2-(m-2)D_ta_1)\alpha_0-$ $-ia_1\alpha_1-ia_2$, ...[5, p. 82]. In what follows, we shall use the notation $\Gamma_t(\alpha)=$ $(\gamma_0(t),\dots, \gamma_{m-1}(t))$. Then Γ_t is an isomorphism of D_s onto $D^\#_{s-(m-1)p}$ for any real s.

Conversely, if $v\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(R_n\times(-\infty, T])$ with support in $R_n\times[0, T]$ is a solution of the equation

$$Pv=f_\sim+\sum_{k=0}^{m-1} D_t^k\delta\otimes\gamma_k(0),$$

that is,

$$(3) \qquad ((v, P^*w))=((f_\sim, w))+(\Gamma_0(\alpha), w_0), \qquad w\in C_0^\infty(R_n\times(-\infty, T)),$$

where by $((\ ,\ ))$ we mean the scalar product between $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(R_n\times(-\infty, T])$ and $\mathscr{D}((-\infty, T))\hat{\otimes}_\iota(\mathscr{D}_{L^2})_x$, then the restriction $u=v|\overset{\circ}{H}$ is a solution of the Cauchy problem (1) and $v=u_\sim$. The equation (3) implies Green's formula:

$$(((Pu)_\sim, w))-((u_\sim, P^*w))=-(\Gamma_0(u_0), w_0).$$

Similarly we have the equation

$$(((Pu)_\sim^\sim, w))-((u_\sim^\sim, P^*w))=(\Gamma_T(u_T), w_T)-(\Gamma_0(u_0), w_0),$$

where $w_T=\mathscr{D}'_{L^2}\text{-}\lim_{t\uparrow T}(w, D_tw,\dots, D_t^{m-1}w)$, $u^\sim$ is the $\mathscr{D}'_{L^2}$-canonical extension of u over $t=T$ and $((\ ,\ ))$ means the scalar product between $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ and $\mathscr{D}_t\hat{\otimes}_\iota(\mathscr{D}_{L^2})_x$.

Let s be any real number and let L, $\tilde{L}$ be the differential operator systems that correspond to the operators P, P^* respectively, which are defined in Section 1. Then we have

PROPOSITION 2. *If* $U\in D_{0,s+p}(H)$, $LU=F\in D_{0,s}(H)$ *and* $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} U=\alpha\in D_{s+p'}(R_n)$, *then* $U\in D_{p,s}(H)$ *and* U *satisfies the inequality*

$$\|U(t)\|^2_{D_{s+p'}}+\int_0^t\|U(t)\|^2_{D_{s+p}}dt\leqq C_T(\|\alpha\|^2_{D_{s+p'}}+\int_0^t\|F(t)\|^2_{D_s}dt), \qquad 0\leqq t\leqq T.$$

In particular, if $F=0$ *and* $\alpha=0$, *then* $U=0$.

PROOF. From the relation $D_tU=F+A(t)U\in D_{0,s}(H)$ we see that $U\in D_{p,s}(H)$. There exists a sequence $\{\Phi_k\}$, $\Phi_k\in C_0^\infty(R_{n+1})\times\cdots\times C_0^\infty(R_{n+1})$, such that $\{\Phi_k\}$ converges in $D_{p,s}(H)$ to U. The sequences $\{\Phi_k(\cdot,0)\}$ and $\{L\Phi_k\}$ converge in $D_{s+p'}$ and $D_{0,s}(H)$ to α and F respectively. Owing to the energy inequality (E_s), we see that $\{\Phi_k\}$ is a Cauchy sequence in $D_{0,s+p}(H)$. Let V be the limit of $\{\Phi_k\}$. Clearly V coincides with U as a distribution and U satisfies the above inequality.

THEOREM 3. *If* $U\in\mathscr{D}_t'((\mathscr{D}_{L^2}')_x)(H)\times\cdots\times\mathscr{D}_t'((\mathscr{D}_{L^2}')_x)(H)$, $LU=0$ *in* $\mathring{H}$ *and* $\mathscr{D}_{L^2}'\text{-}\lim_{t\downarrow 0} U=0$, *then* $U=0$ *in* $\mathring{H}$.

PROOF. There exist two integers k, l such that $U\in D_{k,l}(H)$. Suppose $k<p$. From the relation $D_tU=A(t)U\in D_{k,l-p}(H)$ it follows that $U\in D_{k+p,l-p}(H)$. Repeating the procedure, we see that $U\in D_{p,k+l-p}(H)$. Thus Proposition 2 implies $U=0$.

Let us denote by $\mathscr{E}_t^0(\mathscr{H}_s)$(resp. $\mathscr{E}_t^0(D_s)$), $0\leqq t<T$, the space of $\mathscr{H}_s(R_n)$-valued (resp. $D_s(R_n)$-valued) continuous functions of $t\in[0,T)$. Along the same line as in the proof of the preceding theorem, we have

PROPOSITION 3. *If* $V\in\mathscr{D}_t'((\mathscr{D}_{L^2}')_x)(H)\times\cdots\times\mathscr{D}_t'((\mathscr{D}_{L^2}')_x)(H)$, $\tilde{L}V=0$ *in* $\mathring{H}$ *and* $\mathscr{D}_{L^2}'\text{-}\lim_{t\uparrow T} V=0$, *then* $V=0$ *in* $\mathring{H}$.

PROOF. We can find a real s such that $V\in D_{p,s}(H)$. There exists a sequence $\{\Phi_k\}$, $\Phi_k\in C_0^\infty(R_{n+1})\times\cdots\times C_0^\infty(R_{n+1})$, such that $\{\Phi_k\}$ converges in $D_{p,s}(H)$ to V. The sequence $\{\Phi_k(\cdot,T)\}$ converges in $D_{s+p'}$ to 0 and therefore it converges in D_s to 0. On the other hand the sequence $\{\tilde{L}\Phi_k\}$ converges in $D_{0,s}(H)$ to 0. In virtue of Theorem 2 we have

$$\|\Phi_k(\cdot,t)\|_{D_s}\leqq C_T(\|\Phi_k(\cdot,T)\|_{D_s}+\int_t^T\|\tilde{L}\Phi_k(t)\|_{D_s}dt)$$

and therefore $\{\Phi_k\}$ converges in $\mathscr{E}_t^0(D_s)$, $0\leqq t<T$, to 0. Thus V vanishes in $\mathring{H}$.

COROLLARY 3. *If* $v\in\mathscr{D}_t'((\mathscr{D}_{L^2}')_x)(H)$, $P^*v=0$ *in* $\mathring{H}$ *and* $\mathscr{D}_{L^2}'\text{-}\lim_{t\uparrow T}(v, D_tv,\dots,D_t^{m-1}v)=0$, *then* $v=0$ *in* $\mathring{H}$.

THEOREM 4. *For any* $f\in\mathscr{K}_{0,s}(H)$ *and* $\alpha\in D_{s+(m-1)p+p'}$ *there exists a unique solution* $u\in\mathscr{K}_{mp,s}(H)$ *of the Cauchy problem* (1) *and* u *satisfies the inequality*

$$(4)\qquad \sum_{j=0}^{m-1}\|D_t^ju(\cdot,t)\|_{s+(m-j-1)p+p'}^2+\sum_{j=0}^{m-1}\int_0^t\|D_t^ju(\cdot,t)\|_{s+(m-j)p}^2dt$$

$$\leqq C_T(\|\alpha\|^2_{D_{s+(m-1)p+p'}}+\int_0^t\|f(\cdot,\ t)\|_s^2dt)$$

with a constant C_T.

Proof. We shall first show that $A=\{(P\phi,\ \Gamma_0(\phi_0)):\ \phi\in C_0^\infty(R_{n+1})\}$ is dense in $\mathscr{K}_{0,s}(H)\times D^{\#}_{s+p'}(R_n)$. Let $w\in\mathscr{K}_{0,-s}(H)$ and $\beta\in D_{-s-p'}(R_n)$ such that

$$\int_0^T(P\phi(\cdot,\ t),\ w(\cdot,\ t))dt+(\Gamma_0(\phi_0),\ \beta)=0$$

for any $\phi\in C_0^\infty(R_{n+1})$. If we take $\phi\in C_0^\infty(\overset{\circ}{H})$, then the relation is reduced to

$$\int_0^T(P\phi(\cdot,\ t),\ w(\cdot,\ t))dt=0,$$

which means $P^*w=0$ in $\overset{\circ}{H}$. If we take $\phi\in C_0^\infty(H)$ such that $\phi=0$ near $t=0$, then

$$0=\int_0^T(P\phi(\cdot,\ t),\ w(\cdot,\ t))dt=(\Gamma_T(\phi_T),\ w_T),$$

where $\phi_T=(\phi(\cdot,\ T),\ D_t\phi(\cdot,\ T),\ \dots,\ D_t^{m-1}\phi(\cdot,\ T))$. Since $\Gamma_T(\phi_T)$ may be arbitrarily taken, it follows that $w_T=0$. By Corollary 3 w must vanish in $\overset{\circ}{H}$ and therefore $(\Gamma_0(\phi_0),\ \beta)=0$ for any $\phi\in C_0^\infty(H)$, which implies $\beta=0$.

For any given $f\in\mathscr{K}_{0,s}(H)$ and $\alpha\in D_{s+(m-1)p+p'}$ there exists a sequence $\{\phi_k\}$, $\phi_k\in C_0^\infty(H)$, such that $(\phi_k(\cdot,0),\dots,D_t^{m-1}\phi_k(\cdot,0))$ converges in $D_{s+(m-1)p+p'}$ to α and $\{P\phi_k\}$ converges in $\mathscr{K}_{0,s}(H)$ to f. In virtue of the energy inequality

$$\sum_{j=0}^{m-1}\|D_t^j\phi_k(\cdot,\ t)\|^2_{s+(m-j-1)p+p'}+\sum_{j=0}^{m-1}\int_0^t\|D_t^j\phi_k(\cdot,\ t)\|^2_{s+(m-j)p}dt$$
$$\leqq C_T\Big(\sum_{j=0}^{m-1}\|D_t^j\phi_k(\cdot,\ 0)\|^2_{s+(m-j-1)p+p'}+\int_0^t\|P\phi_k(\cdot,\ t)\|_s^2dt\Big),$$

we see that $(\phi_k,\dots,D_t^{m-1}\phi_k)$ is a Cauchy sequence in $D_{0,s+mp}(H)$. Let $(v_1,\dots,v_m)$ be the limit. From the fact that $D_t^j\phi_k$ converges in $\mathscr{K}_{-jp,s+mp}(H)$ to $D_t^jv_1$ and the space $\mathscr{K}_{0,s+(m-j)p}$ belongs to the space $\mathscr{K}_{-jp,s+mp}(H)$ it follows that $v_{j+1}=D_t^jv_1,\ j=1,\dots,m-1$, and $Pv_1=f$ in $\overset{\circ}{H}$ with $(v_1)_0=\alpha$. Since $(v_1,\ D_tv_1,\dots,\ D_t^{m-1}v_1)\in D_{0,s+mp}(H)$ and $D_t^mv_1=f-\sum_{j=1}^m a_j(x,\ t,\ D_x)D_t^{m-j}v_1\in\mathscr{K}_{0,s}(H)$ we see that $(v_1,\ D_tv_1,\dots,\ D_t^{m-1}v_1)\in D_{p,s+(m-1)p}(H)$ and therefore $v_1\in\mathscr{K}_{mp,s}(H)$, which is a unique solution of the Cauchy problem (1) (Theorem 3) and satisfies the above inequality (4).

Remark. Theorem 4 is in a sense a generalization of a result of S. Mizohata [11, Proposition 5].

Proposition 4. *Let k be any non-negative integer. For any $f\in\mathscr{K}_{kp,s}(H)$*

and $\alpha\in D_{s+(m+k)p-p'}$ *there exists a unique solution* $u\in\mathscr{K}_{(m+k)p,s}(H)$ *of the Cauchy problem* (1) *and* u *satisfies the inequality*

$$(5)\quad \sum_{j=0}^{m+k-1}\|D_t^j u(\cdot,\ t)\|^2_{s+(m+k-j)p-p'}+\sum_{j=0}^{m+k-1}\int_0^t\|D_t^j u(\cdot,\ t)\|^2_{s+(m+k-j)p}dt$$
$$\leqq C_T(\|\alpha\|^2_{D_{s+(m+k)p-p'}}+\sum_{j=0}^{k-1}\|D_t^j f(\cdot,\ 0)\|^2_{s+(k-j)p-p'}+$$
$$+\sum_{j=0}^{k}\int_0^t\|D_t^j f(\cdot,\ t)\|^2_{s+(k-j)p}dt),\qquad 0\leqq t\leqq T,$$

with a constant C_T.

PROOF. In the case where $k=0$, the statement coincides with Theorem 4. Let us consider the case $k\geqq 1$. Since $f\in\mathscr{K}_{kp,s}(H)\subset\mathscr{K}_{0,s+kp}(H)$ it follows from Theorem 4 that there exists a unique solution $u\in\mathscr{K}_{mp,s+kp}(H)$ of (1). $u\in\mathscr{K}_{mp,s+kp}(H)$ means $U=(u,\ D_t u,\dots,\ D_t^{m-1}u)'\in D_{p,s+(m+k-1)p}(H)$. Then $D_t U=A(t)U+F\in D_{p,s+(m+k-2)p}(H)$ with $F=(0,\dots,0,f)'$ and therefore $U\in D_{2p,s+(m+k-2)p}(H)$. Repeating this procedure, we see that $U\in D_{(k+1)p,s+(m-1)p}(H)$, that is, $u\in\mathscr{K}_{(m+k)p,s}(H)$.

Let $k=1$. For any $f\in\mathscr{K}_{p,s}(H)$ and $\alpha\in D_{s+mp+p'}$ the unique solution $u\in\mathscr{K}_{(m+1)p,s}(H)$ satisfies

$$(6)\quad \|U(t)\|^2_{D_{s+mp+p'}}+\int_0^t\|U(t)\|^2_{D_{s+(m+1)p}}dt\leqq C_T(\|\alpha\|^2_{D_{s+mp+p'}}+\int_0^t\|f(t)\|^2_{s+p}dt)$$

with a constant C_T. Put $V=D_t U$. Then $V\in D_{0,s+mp}(H)$, $D_t V-A(t)V=D_t F+D_t A(t)\cdot U\in D_{0,s+(m-1)p}(H)$, $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} V\in D_{s+(m-1)p+p'}$ and therefore V satisfies

$$(7)\quad \|V(t)\|^2_{D_{s+(m-1)p+p'}}+\int_0^t\|V(t)\|^2_{D_{s+mp}}dt$$
$$\leqq C'_T(\|V(0)\|^2_{D_{s+(m-1)p+p'}}+\int_0^t\|D_t f(t)\|^2_s dt+\int_0^t\|U(t)\|^2_{D_{s+mp}}dt)$$

with constant C'_T, where

$$\|V(0)\|^2_{D_{s+(m-1)p+p'}}\leqq C_1\|\alpha\|^2_{D_{s+mp+p'}}+C_2\|f(\cdot,\ 0)\|^2_{s+p'}$$

with constants C_1 and C_2. Summing (6) and (7) and applying Lemma 1 to the result, we have

$$\sum_{j=0}^{m}\|D_t^j u(\cdot,\ t)\|^2_{s+(m-j)p+p'}+\sum_{j=0}^{m}\int_0^t\|D_t^j u(\cdot,\ t)\|^2_{s+(m-j+1)p}dt$$
$$\leqq C''_T(\|\alpha\|^2_{D_{s+mp+p'}}+\|f(\cdot,\ 0)\|^2_{s+p'}+\sum_{j=0}^{1}\int_0^t\|D_t^j f(\cdot,\ t)\|^2_{s+(1-j)p}dt)$$

with a constant C''_T. Repeating this procedure we obtain (5).

Let k be a positive integer and put $\mathfrak{y}_0=\mathscr{K}_{0,s}(H)$, $\mathfrak{y}_1=\mathscr{K}_{kp,s}(H)$. Then $\mathfrak{y}_1$ is dense in $\mathfrak{y}_0$ and $\|u\|_{0,s}\leqq\|u\|_{kp,s}$ for any $u\in\mathfrak{y}_1$ and therefore there exists an unbounded self-adjoint operator J in $\mathfrak{y}_0$ with domain $\mathfrak{y}_1$, which generates a Hilbert scale $\{\mathfrak{y}_\lambda\}_{-\infty<\lambda<\infty}$. In the same way as in the proof of Corollary 4 in [5, p. 97] we see that $\mathfrak{y}_\lambda=\mathscr{K}_{\lambda kp,s}(H)$ within the equivalent norms. From the preceding proposition the map $(f, \alpha)\to u$ which assignes a unique solution u to the data (f, α) is continuous from $\mathscr{K}_{0,s}(H)\times D_{s+mp-p'}$ into $\mathscr{K}_{mp,s}(H)$ and from $\mathscr{K}_{kp,s}(H)\times D_{s+(m+k)p-p'}$ into $\mathscr{K}_{(m+k)p,s}(H)$. By the interpolation theorem we obtain

COROLLARY 4. *Let σ be any non-negative number. For any $f\in\mathscr{K}_{\sigma,s}(H)$ and $\alpha\in D_{\sigma+s+mp-p'}$ there exists a unique solution $u\in\mathscr{K}_{\sigma+mp,s}(H)$ of the Cauchy problem* (1) *and $(f, \alpha)\to u$ is a continuous map from $\mathscr{K}_{\sigma,s}(H)\times D_{\sigma+s+mp-p'}$ into $\mathscr{K}_{\sigma+mp,s}(H)$.*

We shall denote by $\mathring{\mathscr{K}}_{\sigma,s}(H_-)$ the space which is a restriction of the space $\mathring{\mathscr{K}}_{\sigma,s}(\bar{R}^+_{n+1})$ to $R_n\times(-\infty, T)$ and similarly $\mathring{D}_{\sigma,s}(H_-)$ is defined.

PROPOSITION 5. *Let σ be a real number with $-p'<\sigma<0$. For any $f\in\mathscr{K}_{\sigma,s}(H)$ and $\alpha\in D_{\sigma+s+mp-p'}$ there exists a unique solution $u\in\mathscr{K}_{\sigma+mp,s}(H)$ of the Cauchy problem* (1) *and $(f, \alpha)\to u$ is a continuous map from $\mathscr{K}_{\sigma,s}(H)\times D_{\sigma+s+mp-p'}$ into $\mathscr{K}_{\sigma+mp,s}(H)$.*

PROOF. Let $f\in\mathscr{K}_{\sigma,s}(H)$ and $\alpha\in D_{\sigma+s+mp-p'}$. Since $-p'<\sigma<0$ the $\mathscr{D}'_{L^2}$-canonical extension $f_\sim$ belongs to the space $\mathring{\mathscr{K}}_{\sigma,s}(H_-)$. Let $g\in\mathring{\mathscr{K}}_{\sigma+mp,s}(H_-)$ be such that $(D_t-i\lambda^p(D_x))^m g=f_\sim$, where $\lambda(D_x)$ is the operator with symbol $\lambda(\xi)=(1+|\xi|^2)^{1/2}$. Then it follows from Corollary 3 in [6, p. 393] that $\mathscr{D}'_{L^2}$-$\lim_{t\downarrow 0}(g, D_tg,\dots, D_t^{m-1}g)=0$. The Cauchy problem (1) is reduced to

$$
(8)\qquad
\begin{cases}
P(D)(u-g)=\sum_{j=1}^{m}((-i)^j\binom{m}{j}\lambda^{jp}(D_x)-a_j(x, t, D_x))D_t^{m-j}g \quad \text{in } \mathring{H} \\
\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}\ ((u-g), D_t(u-g),\dots, D_t^{m-1}(u-g))=\alpha,
\end{cases}
$$

where $\sum_{j=1}^{m}((-i)^j\lambda^{jp}(D_x)-a_j(x, t, D_x))D_t^{m-j}g\in\mathscr{K}_{\sigma+p,s-p}(H_-)$ with $\sigma+p>p'$. It follows from Corollary 4 that there exists a unique solution $v\in\mathscr{K}_{\sigma+(m+1)p,s-p}(H)$ of the Cauchy problem (8). Thus $u=v+g\in\mathscr{K}_{\sigma+mp,s}(H)$ is a unique solution of the Cauchy problem (1). In view of the closed graph theorem it follows that $(f, \alpha)\to u$ is a continuous map from $\mathscr{K}_{\sigma,s}(H)\times D_{\sigma+s+mp-p'}$ into $\mathscr{K}_{\sigma+mp,s}(H)$.

Let σ, s be any real numbers and write $\sigma=kp+\sigma'$ with integer k and $-p'<\sigma'\leqq p'$. Then we have the following

THEOREM 5. *For any* $\alpha \in D_{\sigma+s+mp-p'}$ *and* $f \in \mathscr{H}_{\sigma,s}(H)$ *with* $f_{\sim} \in \mathring{\mathscr{H}}_{\sigma,s}(H_-)$ *there exists a unique solution* $u \in \mathscr{H}_{\sigma+mp,s}(H)$ *of the Cauchy problem* (1). *In particular, if* $\alpha = 0$ *then* $u_{\sim} \in \mathring{\mathscr{H}}_{\sigma+mp,s}(H_-)$.

PROOF. Consider the case where $k \geqq 0$. By Proposition 5 and Corollary 4 it suffices to show that $u_{\sim} \in \mathring{\mathscr{H}}_{\sigma+mp,s}(H_-)$ for $\alpha=0$. Suppose $\alpha=0$, that is, $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(u,\dots, D_t^{m-1}u)=0$. If $k>0$ then $f_{\sim} \in \mathring{\mathscr{H}}_{kp+\sigma',s}(H_-)$ implies $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(f,\dots, D_t^{k-1}f)=0$. From the equation $P(D)u=f$ we obtain $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(u,\dots, D_t^{m+k-1}u)=0$ for $k \geqq 0$. If $\sigma' < p'$ then $u_{\sim} \in \mathring{\mathscr{H}}_{\sigma+mp,s}(H_-)$. If $\sigma'=p'$ then $u_{\sim} \in \mathring{\mathscr{H}}_{\sigma+(m-1)p,s+p}(H_-)$. Since $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(u,\dots, D_t^{m-1}u)=0$, if we put $V=(u_{\sim}, D_t(u_{\sim}),\dots, D_t^{m-1}(u_{\sim}))'$, $F=(0,\dots,0,f_{\sim})'$, then $V \in \mathring{D}_{\sigma,s+mp}(H_-)$ and $D_tV=A(t)V+F \in \mathring{D}_{\sigma,s+(m-1)p}(H_-)$ and therefore $V \in \mathring{D}_{\sigma+p,s+(m-1)p}(H_-)$, that is, $u_{\sim} \in \mathring{\mathscr{H}}_{\sigma+mp,s}(H_-)$.

Consider the case where $k<0$. Assume that the results hold true of any $k+1$. Let $f_{\sim} \in \mathring{\mathscr{H}}_{\sigma,s}(H_-)$, $\sigma = kp+\sigma'$ and $\alpha \in D_{\sigma+s+mp-p'}$. Let $g \in \mathring{\mathscr{H}}_{\sigma+mp,s}(H_-)$ be such that $(D_t - i\lambda^p(D_x))^m g = f_{\sim}$. Then $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(g,\dots, D_t^{m-1}g)=0$ and the Cauchy problem (1) is reduced to (8), where $\sum_{j=1}^{m}((-i)^j\binom{m}{j}\lambda^{jp}(D_x)-a_j(x,t,D_x))\cdot D_t^{m-j}g \in \mathring{\mathscr{H}}_{\sigma+p,s-p}(H_-)$ and $\sigma+p=(k+1)p+\sigma'$. Thus there exists a unique solution $v \in \mathscr{H}_{\sigma+(m+1)p,s-p}(H)$. Consequently, $u=v+g \in \mathscr{H}_{\sigma+mp,s}(H)$. Since $v_{\sim} \in \mathring{\mathscr{H}}_{\sigma+(m+1)p,s-p}(H_-)$ for $\alpha=0$ we can conclude that $u_{\sim}=v_{\sim}+g \in \mathring{\mathscr{H}}_{\sigma+mp,s}(H_-)$.

PROPOSITION 6. *For any* $h \in \mathring{\mathscr{H}}_{\sigma,s}(H_-)$ *there exists a unique solution* $v \in \mathring{\mathscr{H}}_{\sigma+mp,s}(H_-)$ *of* $Pv=h$.

PROOF. In the case where $\sigma > -p'$, the problem to find a solution v of $Pv=h$ is equivalent to the problem to find a solution u of the Cauchy problem $Pu=h|\mathring{H} \in \mathscr{H}_{\sigma,s}(H)$ with $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(u,\dots, D_t^{m-1}u)=0$. Thus there exists a unique solution $u \in \mathscr{H}_{\sigma+mp,s}(H)$ and $u_{\sim} \in \mathring{\mathscr{H}}_{\sigma+mp,s}(H_-)$.

In the case where $\sigma \leqq -p'$, our assertion will follow in the same way as in the proof of Theorem 5.

Let P be a regularly p-parabolic operator in $0 \leqq T < \infty$ and consider the Cauchy problem

$$(9) \qquad \begin{cases} Pu=f & \text{in } R^+_{n+1}, \\ \mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(u, D_tu,\dots, D_t^{m-1}u)=\alpha \end{cases}$$

for given $\alpha \in (\mathscr{D}'_{L^2})_x \times \cdots \times (\mathscr{D}'_{L^2})_x$ and $f \in \mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x) = \mathscr{D}'(R^+_t)\varepsilon(\mathscr{D}'_{L^2})_x$ which has the $\mathscr{D}'_{L^2}$-canonical extension $f_{\sim}$. From the fact that Theorem 3 holds true of any $H_T = R_n \times [0, T]$, the Cauchy problem (9) is unique in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(\bar{R}^+_{n+1})$.

The spaces $\mathscr{K}_{\sigma,s}(\bar{R}^+_{n+1})$ and $\mathring{\mathscr{K}}_{\sigma,s}(\bar{R}^+_{n+1})$ are defined in the same way as $\mathscr{K}_{\sigma,s}(H)$ and $\mathring{\mathscr{K}}_{\sigma,s}(H)$. By $\mathscr{K}^{\sim}_{\sigma,s}(\bar{R}^+_{n+1})$ we mean the space of $u \in \mathscr{D}'(R^+_{n+1})$ such that $\phi u \in \mathscr{K}_{\sigma,s}(\bar{R}^+_{n+1})$ for any $\phi \in \mathscr{D}(R_t)$ and the topology is defined by the semi-norms $u \to \|\phi u\|_{\sigma,s}$. Along the same way as in the proof of Theorem 5 and Proposition 6 we have the following

THEOREM 5′. *For any* $\alpha \in D_{\sigma+s+mp-p'}$ *and* $f \in \mathscr{K}^{\sim}_{\sigma,s}(\bar{R}^+_{n+1})$ *with* $f_\sim \in \mathring{\mathscr{K}}^{\sim}_{\sigma,s}(\bar{R}^+_{n+1})$ *there exists a unique solution* $u \in \mathscr{K}^{\sim}_{\sigma+mp,s}(\bar{R}^+_{n+1})$ *of the Cauchy problem* (9). *In particular, if* $\alpha=0$ *then* $u_\sim \in \mathring{\mathscr{K}}^{\sim}_{\sigma+mp,s}(\bar{R}^+_{n+1})$.

PROPOSITION 6′. *For any* $h \in \mathring{\mathscr{K}}^{\sim}_{\sigma,s}(\bar{R}^+_{n+1})$ *there exists a unique solution* $v \in \mathring{\mathscr{K}}^{\sim}_{\sigma+mp,s}(\bar{R}^+_{n+1})$ *of* $Pv=h$.

Let us denote by $\mathscr{D}'_+$ the subspace of $\mathscr{D}'_t$ which consists of all one-dimentional distributions with support contained in $[0, \infty)$ and by $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ the ε-product $\mathscr{D}'_t\varepsilon(\mathscr{D}'_{L^2})_x$, which is a reflexive, ultrabornological Souslin space [6, p. 372]. In the same way as in the proof of Theorem 5 [7, p. 415] we have

THEOREM 6. *For any* $h \in (\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ *there exists a unique solution* $v \in (\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ *of* $Pv=h$ *and* $h \to v$ *is a continuous map from* $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ *onto itself.*

PROOF. Take a sequence $\{t_j\}$ of real numbers such that $t_0<0<t_1<t_2<\cdots$, $\lim_{j\to\infty} t_j=\infty$ and put $U_j=(t_j, t_{j+2})$. Let $\{\phi_j\}$ be a partition of unity subordinate to the covering $\{U_j\}_{j=0,1,\dots}$ of (t_0, ∞) and consider the equations $Pv_j=\phi_j f$, $j=0, 1,\dots$, where $\phi_j f \in \mathring{\mathscr{K}}^{\sim}_{\sigma_j,s_j}(\bar{R}^+_{n+1})$. In virtue of Proposition 6′ there exists a unique solution $v_j \in \mathring{\mathscr{K}}^{\sim}_{\sigma_j+mp,s_j}(R^+_{n+1}) \subset (\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$. By our energy inequality (E_s) we see that $v_j=0$ for $t<t_j$. Thus $v=\sum v_j$ is well defined in $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ and v is unique in $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$.

Consider the map

$$l : \mathscr{D}'_t((\mathscr{D}'_{L^2})_x) \ni v \to Pv \in (\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x),$$

which is linear, continuous and onto. Since the space $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ is ultrabornological and Souslin it follows from the open mapping theorem that l is an epimorphism. Thus the proof is complete.

As a consequence of Theorem 6 we can state the following

THEOREM 7. *For any* $\alpha \in (\mathscr{D}'_{L^2})_x \times \cdots \times (\mathscr{D}'_{L^2})_x$ *and* $f \in \mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$ *with* $f_\sim \in (\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$, *the fine Cauchy problem* (9) *has a unique solution* $u \in \mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$ *and* $(f_\sim, \alpha) \to u_\sim$ *is a continuous map under the topology of* $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x) \times ((\mathscr{D}'_{L^2})_x \times \cdots \times (\mathscr{D}'_{L^2})_x)$ *and the topology of* $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$.

We shall close this paper with some remarks on the Cauchy problem (2):

$$\begin{cases} LU \equiv D_t U - A(t)U = F & \text{in } \mathring{H}, \\ \mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} U = \alpha \end{cases}$$

for preassigned $F \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H) \times \cdots \times \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$ with $\mathscr{D}'_{L^2}$-canonical extension $F_\sim$ and $\alpha \in (\mathscr{D}'_{L^2})_x \times \cdots \times (\mathscr{D}'_{L^2})_x$. As shown in Theorem 1 the energy inequality (E_s) holds true for any $U = (u_1, \dots, u_m)'$, $u_j \in C_0^\infty(R_{n+1})$.

Let s be any real number. If for any $F \in D_{0,s}(H)$ and $\alpha \in D_{s+p'}(R_n)$ there exists a solution $U \in D_{0,s+p}(H)$ of the Cauchy problem (2) we shall say that $(\mathrm{CP})_s$ holds for L. As shown in Theorem 3, U is uniquely defined if it exists. In the same way as in the proof of Proposition 7′ in [7, p. 434] we have

PROPOSITION 7. *$(\mathrm{CP})_s$ holds for L if and only if the conditions that $W \in D^*_{0,-s}(H)$, $L^*W = 0$ in $\mathring{H}$ and $\mathscr{D}'_{L^2}\text{-}\lim_{t\uparrow T} W = 0$ imply $W = 0$ in $\mathring{H}$.*

LEMMA 2. *Suppose $(\mathrm{CP})_s$ holds for some s. Then, for any $F \in C_0^\infty(H) \times \cdots \times C_0^\infty(H)$ and $\alpha \in C_0^\infty(R_n) \times \cdots \times C_0^\infty(R_n)$ a unique solution U of the Cauchy problem (2) belongs to the space $D_{0,s'}(H)$ for any s'.*

PROOF. From our assumption it follows that $U \in D_{0,s+p}(H)$. If we put $V_1 = \lambda(D_x)U$, then

$$\begin{cases} D_t V_1 + A(t)V_1 = \lambda(D_x)F + (A(t)\lambda(D_x) - \lambda(D_x)A(t))U & \text{in } \mathring{H}, \\ \mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} V_1 = \lambda(D_x)\alpha, \end{cases}$$

where $\lambda(D_x)F \in \mathscr{S}(H)$, $\lambda(D_x)\alpha \in \mathscr{S}(R_n)$ and $(A(t)\lambda(D_x) - \lambda(D_x)A(t))U \in D_{0,s}(H)$ [6, p. 387]. From our assumption it follows that $V_1 = \lambda(D_x)U \in D_{0,s+p}(H)$ and therefore $U \in D_{0,s+p+1}(H)$.

If we put $V_2 = \lambda^2(D_x)U$, then

$$\begin{cases} D_t V_2 + A(t)V_2 = \lambda^2(D_x)F + (A(t)\lambda^2(D_x) - \lambda^2(D_x)A(t))U \in D_{0,s}(H), \\ \mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} V_2 = \lambda^2(D_x)\alpha \in \mathscr{S}(R_n). \end{cases}$$

Thus $V_2 = \lambda^2(D_x)U \in D_{0,s+p}(H)$ and therefore $U \in D_{0,s+p+2}(H)$. Repeating this procedure, we see that $U \in \bigcap_s D_{0,s}(H)$.

PROPOSITION 8. *If $(\mathrm{CP})_s$ holds for some s, then it does also for any s'.*

PROOF. For any given $F \in D_{0,s'}(H)$ and $\alpha \in D_{s'+p'}(R_n)$ there exist two sequences $\{F_j\}$, $F_j \in C_0^\infty(H) \times \cdots \times C_0^\infty(H)$ and $\{\alpha_j\}$, $\alpha_j \in C_0^\infty(R_n) \times \cdots \times C_0^\infty(R_n)$ such that $\{F_j\}$ and $\{\alpha_j\}$ converge in $D_{0,s}(H)$ and $D_{s+p'}(R_n)$ respectively. Let U_j be a unique solution of the Cauchy problem (2) for L associated with F_j and α_j.

Then U_j belongs to the space $\bigcap_s D_{0,s}(H)$ and it satisfies the energy inequality

$$\|U_j(t)\|^2_{D_{s'+p'}}+\int_0^t\|U_j(t)\|^2_{D_{s'+p}}dt\leqq C_T(\|\alpha_j\|^2_{D_{s'+p'}}+\int_0^t\|F_j(t)\|^2_{D_{s'}}dt),\ 0\leqq t\leqq T,$$

with a constant C_T, which implies that $\{U_j\}$ is a Cauchy sequence in $D_{0,s'+p}(H)$. By the relation $D_tU_j=F_j-A(t)U_j\in D_{0,s'}(H)$ we see that $\{U_j\}$ is also a Cauchy sequence in $D_{p,s'}(H)$. Let U be the limit of U_j in $D_{p,s'}(H)$. Then $U\in D_{p,s'}(H)$ satisfies $LU=F$ in $\mathring{H}$ and $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} U=\alpha$, which means that $(\mathrm{CP})_{s'}$ holds true.

From the energy inequality (E_s) and Proposition 8 we can prove the following proposition in the same arguments as used in [7, Proposition 6].

PROPOSITION 9. *If for any $F\in D_{0,s}(H)$ and $\alpha\in D_{s+p'}(R_n)$ the Cauchy problem* (2) *has a solution $U\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)\times\cdots\times\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(H)$, then $U\in D_{p,s}(H)$.*

If we suppose $(\mathrm{CP})_0$ for L, then our discussions on the Cauchy problem for a specified parabolic system given in Section 6 of [7] can be applied also to the Cauchy problem for L.

References

[1] J. Dieudonné, *Calcul infinitésimal,* Hermann, 1968.

[2] D. Ellis, *An energy inequality for higher order linear parabolic operators and its applications,* Trans. Amer. Math. Soc. **165** (1972), 167–206.

[3] M. Itano, *On the fine Cauchy problem for the system of linear partial differential equations,* J. Sci. Hiroshima Univ. Ser. A–I **33** (1969), 11–27.

[4] M. Itano, *Note on the canonical extensions and the boundary values for distributions in the space* H^μ, Hiroshima Math. J. **1** (1971), 405–425.

[5] M. Itano and K. Yoshida, *Energy inequalities and Cauchy problem for a system of linear partial differential equations,* Hiroshima Math. J. **1** (1971), 75–108.

[6] M. Itano and K. Yoshida, *A study of $\mathscr{D}'_{L^2}$-valued distributions on a semi-axis in connection with the Cauchy problem for a pseudo-differential system,* Hiroshima Math. J. **2** (1972), 369–396.

[7] M. Itano and K. Yoshida, *Energy inequalities and the Cauchy problem for a pseudo-differential system,* Hiroshima Math. J. **2** (1972), 397–444.

[8] L. Hörmander, *Linear partial differential operators,* Springer, 1969.

[9] S. Kaplan, *An analogue of Gårding's inequality for parabolic operators and its applications,* J. Math. Mech. **19** (1969), 171–187.

[10] J. Leray, *Hyperbolic differential equations,* The Institute for Advanced Study, Princeton, N. J., 1953. Reprinted November, 1955.

[11] S. Mizohata, *Le problème de Cauchy pour les équations paraboliques,* J. Math. Soc. Japan **8** (1956), 269–299.

Department of Mathematics,
Faculty of General Education,
Hiroshima University