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On Generators of Lie Algebras

Terukiyo SATÔ

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Introduction

The *nullity* of a Lie algebra is the minimum number of elements which generate the Lie algebra, and the *genus* is the difference between the dimension and the nullity. The concepts of genus and nullity seem to have originated with Knebelman [8]. He gives relations between the structure constants of Lie algebras and the genus, and classifies Lie algebras of genus zero and one. He states also that every perfect Lie algebra can be generated by two elements. However, Patterson [13] points out that the argument in [8] contains errors and he states some results concerning the genus of general algebras over an arbitrary field. Bond classifies Lie algebras of genus one and genus two ([1], [3]). On the other hand, it is proved by Kuranishi [9] that a semi-simple Lie algebra over an algebraically closed field of characteristic zero is generated by two elements. This result is generalized by Marshall [11] for a perfect Lie algebra of a certain type, and he constructs examples of perfect Lie algebras with arbitrarily given nullity more than 1, which prove the falsity of the statement in [8]. He states also an inequality between the nullity and the dimension of a perfect Lie algebra. However, this inequality needs to be slightly modified.

In this paper, we shall investigate the nullity of Lie algebras. Throughout the paper, every Lie algebra is finite-dimensional and over a field of characteristic zero. In § 1, it is shown that the nullity of a Lie algebra is invariant under the extension of the base field (Theorem 1), and we state Theorem 2 which gives a generalization of the first theorem in [11]. In § 2, two examples of perfect Lie algebras are given, one of which is a counter example to the inequality in [11]. In § 4, we give a sufficient condition for a perfect Lie algebra over an algebraically closed field to have the nullity two (Theorem 3), and an estimating formula for the nullity of perfect Lie algebras (Theorem 4). A corrected formula to the inequality in [11] is also given. We shall study in § 5 the nullity of a Lie algebra with the nilpotent radical (Theorem 5). In § 6, we treat solvable or non-solvable Lie algebras whose adjoint representations are splittable (Theorems 6, 7).

§ 1

First a base field of Lie algebras is assumed to be an arbitrary field of characteristic zero. The nullity of a Lie algebra $\mathfrak{g}$ is denoted by $\mathrm{Nul}(\mathfrak{g})$.

LEMMA 1. *Let $\mathfrak{g}$ be a Lie algebra generated by m elements over a field K, and let $\{f_1, \ldots, f_n\}$ be a basis for $\mathfrak{g}$. Let X_{ij} $(i=1, \ldots, m; j=1, \ldots, n)$ be the indeterminates and $P(X_{ij})$ a given non-zero polynomial. Then generators of $\mathfrak{g}$*

$$a_i = \sum_{j=1}^{n} \xi_{ij} f_j \qquad (i=1, \ldots, m)$$

may be chosen such as $P(\xi_{ij}) \neq 0$. Moreover, if K_0 is a subfield of K, then the coefficients ξ_{ij} of a_i may be taken in K_0.

PROOF. For $\mathfrak{g}$ to be generated by m elements $b_1, \ldots, b_m$, it is necessary and sufficient that the set of all the monomials in $b_1, \ldots, b_m$ contains n linearly independent elements. Now we assume that $b_1, \ldots, b_m$ generate $\mathfrak{g}$, and their monomials $c_1, \ldots, c_n$ are linearly independent. We extend the base field K to Ω by adjoining the indeterminates X_{ij} $(i=1, \ldots, m; j=1, \ldots, n)$. When b_i is expressed as $\Sigma \beta_{ij} f_j$, we consider the corresponding element $B_i = \Sigma X_{ij} f_j$ in $\mathfrak{g}_\Omega$, and corresponding to $c_k = [b_{i_1}, [\ldots, [b_{i_{p-1}}, b_{i_p}] \ldots]]$ we consider $C_k = [B_{i_1}, [\ldots, [B_{i_{p-1}}, B_{i_p}] \ldots]]$ in $\mathfrak{g}_\Omega$. When we express $C_1, \ldots, C_n$ as linear combinations of $f_1, \ldots, f_n$, the determinant of the matrix of coefficients is a non-zero polynomial in X_{ij}. We denote it by $Q(X_{ij})$. Then there exist $\xi_{ij} \in K$ such that $P(\xi_{ij}) Q(\xi_{ij}) \neq 0$. We can easily see that the elements

$$a_i = \sum_{j=1}^{n} \xi_{ij} f_j \qquad (i=1, \ldots, m)$$

generate the whole Lie algebra $\mathfrak{g}$.

The latter half of the lemma follows from the fact that the non-zero polynomial $P(X_{ij}) Q(X_{ij})$ with coefficients in K is not identically zero in K_0 since K_0 is an infinite field.

THEOREM 1. *Let $\mathfrak{g}$ be a Lie algebra over a field K, and let K' be an extension field of K. Then*

$$\mathrm{Nul}(\mathfrak{g}) = \mathrm{Nul}(\mathfrak{g}_{K'}).$$

PROOF. It is obvious that $\mathrm{Nul}(\mathfrak{g}) \geqq \mathrm{Nul}(\mathfrak{g}_{K'})$. We suppose that m elements generate $\mathfrak{g}_{K'}$ and $\{f_1, \ldots, f_n\}$ is a basis for $\mathfrak{g}$. By Lemma 1, $\mathfrak{g}_{K'}$ is generated by m elements $a_i = \Sigma \xi_{ij} f_j$ $(i=1, \ldots, m)$ where $\xi_{ij} \in K$. Hence $a_i \in \mathfrak{g}$, and $\mathrm{Nul}(\mathfrak{g}) \leqq m$. Thus we have the assertion.

The nullity of a semi-simple Lie algebra over an algebraically closed field is two ([9]). Hence by Theorem 1 we have the following:

The nullity of a semi-simple Lie algebra over an arbitrary field of characteristic zero is two.

Hereafter, we shall assume that the base field of a Lie algebra is algebraically closed. This gives no restrictions by Theorem 1.

LEMMA 2. *Let* $\mathfrak{h}$ *be an ideal of a Lie algebra* $\mathfrak{g}$. *Then,*

$$\mathrm{Nul}(\mathfrak{g}/\mathfrak{h}) \leqq \mathrm{Nul}(\mathfrak{g}) \leqq \mathrm{Nul}(\mathfrak{h}) + \mathrm{Nul}(\mathfrak{g}/\mathfrak{h}).$$

PROOF. Let $\{h_1, \ldots, h_m\}$ and $\{\bar{g}_1, \ldots, \bar{g}_n\}$ be systems of generators of $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ respectively. If we take $g_i \in \bar{g}_i$ $(i=1, \ldots, n)$, then $h_1, \ldots, h_m, g_1, \ldots, g_n$ generate $\mathfrak{g}$. Conversely, the generators of $\mathfrak{g}$ are considered as those of $\mathfrak{g}/\mathfrak{h}$ by taking the residue classes modulo $\mathfrak{h}$.

The following is an easy consequence of the above lemma.

LEMMA 3 (Knebelman [8]). *If* $\mathfrak{g}$ *is represented as a direct sum of two ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$, *then*

$$\begin{aligned} \mathrm{Max}\{\mathrm{Nul}(\mathfrak{a}), \mathrm{Nul}(\mathfrak{b})\} &\leqq \mathrm{Nul}(\mathfrak{g}) \\ &\leqq \mathrm{Nul}(\mathfrak{a}) + \mathrm{Nul}(\mathfrak{b}). \end{aligned}$$

LEMMA 4. *Let* $\mathfrak{n}$ *be a nilpotent ideal of a Lie algebra* $\mathfrak{g}$. *Then*

$$\mathrm{Nul}(\mathfrak{g}) = \mathrm{Nul}(\mathfrak{g}/[\mathfrak{n}, \mathfrak{n}]).$$

PROOF. It is obvious by Lemma 2 that $\mathrm{Nul}(\mathfrak{g}) \geqq \mathrm{Nul}(\mathfrak{g}/[\mathfrak{n}, \mathfrak{n}])$. For $g \in \mathfrak{g}$, we denote by $\bar{g}$ the class in $\mathfrak{g}/[\mathfrak{n}, \mathfrak{n}]$ which contains g. For n elements $g_1, \ldots, g_n$ in $\mathfrak{g}$, we suppose that $\bar{g}_1, \ldots, \bar{g}_n$ generate the Lie algebra $\mathfrak{g}/[\mathfrak{n}, \mathfrak{n}]$. Let $\mathfrak{g}_1$ be a sublgebra generated by $g_1, \ldots, g_n$. We suppose that $\mathfrak{n}^m = 0$. Then for k such that $1 \leqq k \leqq m$, we can show

$$\mathfrak{n}^k \subset \mathfrak{g}_1.$$

In fact, it is obvious for $k=m$. Therefore we assume $\mathfrak{n}^{k+1} \subset \mathfrak{g}_1$. For $n_1, \ldots, n_k \in \mathfrak{n}$, we set

$$n_i = a_i + n_i' \qquad (a_i \in \mathfrak{g}_1 \cap \mathfrak{n},\ n_i' \in [\mathfrak{n}, \mathfrak{n}]).$$

$$\begin{aligned} &[n_1, [n_2, [\ \ldots, [n_{k-1}, n_k] \ldots] \\ &\quad = [a_1 + n_1', [a_2 + n_2', [\ \ldots, [a_{k-1} + n_{k-1}', a_k + n_k'] \ldots] \\ &\quad \equiv [a_1, [a_2, [\ \ldots, [a_{k-1}, a_k] \ldots] \qquad \mathrm{mod}\ \mathfrak{n}^{k+1}, \end{aligned}$$

which implies that $\mathfrak{n}^k \subset \mathfrak{g}_1$.

Especially, $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{g}_1$, which implies that $\mathfrak{g} \subset \mathfrak{g}_1 + [\mathfrak{n}, \mathfrak{n}] = \mathfrak{g}_1$. Hence we get our assertion.

COROLLARY. *For a nilpotent Lie algebra* $\mathfrak{n}$,

$$\mathrm{Nul}(\mathfrak{n})=\dim \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}].$$

A Lie algebra $\mathfrak{g}$ is said to be *perfect* when $\mathfrak{g}=[\mathfrak{g}, \mathfrak{g}]$. The radical $\mathfrak{r}$ of a perfect Lie algebra $\mathfrak{g}$ is nilpotent, since $\mathfrak{r}=[\mathfrak{g}, \mathfrak{r}]$. Hence, by Lemma 4, to investigate the nullity of a perfect Lie algebra, we may assume that the radical $\mathfrak{r}$ is abelian ([11]).

Let $\mathfrak{g}$ be a perfect Lie algebra, $\mathfrak{s}$ a maximal semisimple subalgabra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{s}$, and $\mathfrak{r}$ the abelian radical of $\mathfrak{g}$. We denote by Σ the root system of $\mathfrak{s}$ and by $\{\alpha_1, \ldots, \alpha_n\}$ a system of simple roots. We set $\Sigma_0=\{\alpha_1, \ldots, \alpha_n, -\alpha_1, \ldots, -\alpha_n\}$. These notations will be employed as far as §4.

It is proved in [11] that if there exists an element h in $\mathfrak{h}$ such that $\mathrm{ad}_{\mathfrak{g}}h$ has distinct non-zero latent roots and $\mathrm{ad}_{\mathfrak{r}}h$ has at most one zero latent root, then the nullity of $\mathfrak{g}$ is two. The following gives its generalization.

THEOREM 2. *Let* $\mathfrak{g}$ *be a perfect Lie algebra with the abelian radical* $\mathfrak{r}$, *and* $\mathfrak{r}$ *a direct sum of irreducible* $\mathfrak{s}$*-submodules*:

$$\mathfrak{r}=\mathfrak{r}_1+\cdots+\mathfrak{r}_m.$$

If each $\mathfrak{r}_i$ *has a weight* λ_i *such that* $\lambda_i\in\Sigma_0$ *and* $\lambda_1, \ldots, \lambda_m$ *are different from each other, then the nullity of* $\mathfrak{g}$ *is two.*

PROOF. *Let* f_{λ_i} be an element in $\mathfrak{r}_i$ which belongs to the weight λ_i, and e_α an element in $\mathfrak{s}$ which belongs to the root α.

$$P=\prod_{i\neq j}(\alpha_i^2-\alpha_j^2)\prod_{i,k}(\alpha_i^2-\lambda_k^2)\prod_{k\neq l}(\lambda_k-\lambda_l)$$

is a non-zero polynomial function defined in $\mathfrak{h}$, whence there exists an element $h\in\mathfrak{h}$ such that $P(h)\neq 0$. We set

$$x=e_{\alpha_1}+\cdots+e_{\alpha_n}+e_{-\alpha_1}+\cdots+e_{-\alpha_n}+f_{\lambda_1}+\cdots+f_{\lambda_m}.$$

We denote by $\mathfrak{g}_1$ the Lie subalgebra generated by h and x. As is easily seen,

$$(\mathrm{ad}h)^p x=\sum_i(\alpha_i(h))^p e_{\alpha_i}+\sum_i(-\alpha_i(h))^p e_{-\alpha_i}$$
$$+\sum_j(\lambda_j(h))^p f_{\lambda_j} \qquad (p=0, 1, \ldots, 2n+m-1).$$

By means of Vandermonde's formula, the matrix of coefficients of $e_{\alpha_i}, e_{-\alpha_i}, f_{\lambda_j}$ has the inverse. So e_{α_i}, $e_{-\alpha_i}$ and f_{λ_j} are contained in $\mathfrak{g}_1$. e_{α_i} and $e_{-\alpha_i}$ ($i=1, \ldots, n$) generate $\mathfrak{s}$ ([7], p. 123), and $\mathfrak{s}$ and f_{λ_j} generate the irreducible $\mathfrak{s}$-module $\mathfrak{r}_j$. Hence $\mathfrak{g}_1=\mathfrak{g}$, which proves the theorem.

§ 2

We show two examples of perfect Lie algebras, which give us a clue for considerations in § 4 on the nullity of perfect Lie algbras.

Example 1. Let $\mathfrak{s}$ be a three dimensional simple Lie algebra, whose basis is denoted by $\{h, e_\alpha, e_{-\alpha}\}$ as usual. For any integer $n\geqq 1$, let $\mathfrak{r}_1, \dots, \mathfrak{r}_{2n+1}$ be two dimensional irreducible $\mathfrak{s}$-modules, and we consider the Lie algebra

$$\mathfrak{g}=\mathfrak{s}+\mathfrak{r}_1+\cdots+\mathfrak{r}_{2n+1},$$

where $[\mathfrak{r}_i, \mathfrak{r}_j]=0$. Each $\mathfrak{r}_i$ has the basis $\{f_{i,+}, f_{i,-}\}$ such that

$$[h, f_{i,+}]=f_{i,+} \qquad [h, f_{i,-}]=-f_{i,-}$$

$$[e_\alpha, f_{i,-}]=f_{i,+} \qquad [e_{-\alpha}, f_{i,+}]=f_{i,-},$$

and other Lie multiplications are zero. Then we assert that $\mathrm{Nul}(\mathfrak{g})=n+2$.

First we consider the case $n=1$, and show that $\mathrm{Nul}(\mathfrak{g})\geqq 3$. Suppose that this is false, i.e., that $\mathfrak{g}$ is generated by two elements $\gamma_0 h+\gamma_\alpha e_\alpha+\gamma_{-\alpha}e_{-\alpha}+r$ and $\gamma'_0 h+\gamma'_\alpha e_\alpha+\gamma'_{-\alpha}e_{-\alpha}+r'$, where $r, r'\in\mathfrak{r}_1+\mathfrak{r}_2+\mathfrak{r}_3$. By means of Lemma 1, we may assume that $\gamma_0(\gamma_0\, \gamma'_\alpha-\gamma_\alpha\gamma'_0)\neq 0$. Hence, eliminating the terms $\gamma_\alpha e_\alpha$ and $\gamma'_0 h$, we can take the generators

$$x=h+\delta e_{-\alpha}+a_{1,+}+b_{1,-}$$

$$y=e_\alpha+\delta' e_{-\alpha}+a_{2,+}+b_{2,-},$$

where $a_{i,+}$ and $b_{i,-}$ are elements in $\mathfrak{r}_1+\mathfrak{r}_2+\mathfrak{r}_3$ which belong to the weights $\frac{\alpha}{2}$ and $-\frac{\alpha}{2}$ respectively ($i=1, 2$). We set $a_{i,-}=[e_{-\alpha}, a_{i,+}]$ and $b_{i,+}=[e_\alpha, b_{i,-}]$. Then it is easily verified that

$$\begin{aligned}\mathfrak{g}_1=\{\{\, & h+a_{1,+}-a_{2,-},\quad e_\alpha+a_{2,+},\quad e_{-\alpha}+a_{1,-},\\ & a_{2,+}+b_{1,+}-\delta a_{1,+},\quad a_{2,-}+b_{1,-}-\delta a_{1,-},\\ & b_{2,-}-\delta' a_{1,-},\quad b_{2,+}-\delta' a_{1,+}\,\}\}\end{aligned}$$

is a subalgebra of dimension 7 which contains x and y (where $\{\{\,*\,\}\}$ means a subspace spanned by the elements $*$). This contradicts $\dim\mathfrak{g}=9$. Hence in this case $\mathfrak{g}$ cannot be generated by two elements, i.e. $\mathrm{Nul}(\mathfrak{g})\geqq 3$.

In the case $n\geqq 2$, we may conclude that $\mathrm{Nul}(\mathfrak{g})\geqq n+2$ by the similar argument. In fact, we suppose that there exist $n+1$ generators as follows:

$$\begin{aligned}& h+a_{1,+}+b_{1,-},\quad e_\alpha+a_{2,+}+b_{2,-},\quad e_{-\alpha}+a_{3,+}+b_{3,-},\\ & a_{4,+}+b_{4,-},\quad \dots,\quad a_{n+1,+}+b_{n+1,-}.\end{aligned}$$

We set again $a_{i,-}=[e_{-\alpha}, a_{i,+}]$, $b_{i,+}=[e_\alpha, b_{i,-}]$ and consider the following submodule of dimension $4n+3$:

$$\begin{aligned}\mathfrak{g}_1=\{\{&h+a_{1,+}+b_{1,-},\quad e_\alpha+a_{2,+},\quad e_{-\alpha}+b_{3,-},\\ &a_{1,+}-b_{3,+},\quad a_{1,-}-b_{3,-},\quad b_{1,+}+a_{2,+},\quad b_{1,-}+a_{2,-},\\ &b_{2,+},\quad b_{2,-},\quad a_{3,+},\quad a_{3,-},\quad a_{4,+},\quad a_{4,-},\quad b_{4,+},\\ &b_{4,-},\quad \dots,\quad a_{n+1,+},\quad a_{n+1,-},\quad b_{n+1,+},\quad b_{n+1,-}\}\}.\end{aligned}$$

Then it is easy to see that $\mathfrak{g}_1$ is a subalgebra and contains the above generators. $\mathfrak{g}_1$ cannot be the whole Lie algebra $\mathfrak{g}$ of dimension $4n+5$. This is a contradiction. Hence $\mathrm{Nul}(\mathfrak{g})\geqq n+2$.

Using Theorem 4 to be shown in § 4, we may ascertain easily that the nullity of $\mathfrak{g}$ is exactly $n+2$.

Since these Lie algebras are of dimension $4n+5$ and its nullity is $n+2$, we get counter examples to the following theorem in [11]:

"If $\mathfrak{g}$ is a perfect Lie algebra of dimension d (>8), then

$$\mathrm{Nul}(\mathfrak{g})\leqq\frac{1}{4}(d-1)."$$

A corrected form to this inequality is given in § 4.

Example 2. Let $\mathfrak{s}$ be a three dimensional simple Lie algebra, and $\mathfrak{r}_1$, $\mathfrak{r}_2$, $\mathfrak{r}_3$ three dimensional $\mathfrak{s}$-modules isomorphic to $\mathfrak{s}$. We consider the Lie algebra $\mathfrak{g}=\mathfrak{s}+\mathfrak{r}_1+\mathfrak{r}_2+\mathfrak{r}_3$, where $[\mathfrak{r}_i, \mathfrak{r}_j]=0$. The basis $\{f_{i,+}, f_{i,0}, f_{i,-}\}$ for $\mathfrak{r}_i$ has the following multiplication rules:

$$\begin{aligned}&[h, f_{i,+}]=2f_{i,+} &&[h, f_{i,-}]=-2f_{i,-}\\ &[e_\alpha, f_{i,-}]=f_{i,0} &&[e_\alpha, f_{i,0}]=-2f_{i,+}\\ &[e_{-\alpha}, f_{i,0}]=2f_{i,-} &&[e_{-\alpha}, f_{i,+}]=-f_{i,0}\end{aligned}$$

and all other products are zero. We take two elements in $\mathfrak{g}$ as follows:

$$x=h+f_{1,0}$$

$$y=e_\alpha+e_{-\alpha}+f_{2,0}+f_{3,-}.$$

Then by easy computations we know that x and y generate $\mathfrak{g}$, that is, $\mathrm{Nul}(\mathfrak{g})=2$.

§ 3

We prove several lemmas for the later use.

LEMMA 5. *Let m and n be natural numbers such that $m\geqq n$. Then,*

$$\begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ x_1 & \cdots & x_m & 2x_1 & \cdots & 2x_n \\ x_1^2 & \cdots & x_m^2 & 3x_1^2 & \cdots & 3x_n^2 \\ \vdots & & \vdots & \vdots & & \vdots \\ x_1^N & \cdots & x_m^N & (N+1)x_1^N & \cdots & (N+1)x_n^N \end{vmatrix}$$

$$=(-1)^{\frac{m(m-1)}{2}} \prod_{i=1}^{n} x_i \prod_{i<j\leqq n} (x_i-x_j)^4$$

$$\prod_{\substack{i\leqq n \\ n<j\leqq m}} (x_i-x_j)^2 \prod_{n<i<j} (x_i-x_j),$$

where $N=m+n-1$.

Proof. We consider $x_1, \ldots, x_m$ as indeterminates. We denote by Δ the determinant on the left hand side and D the polynomial on the right hand side except the sign factor. Then,

$$\Delta=\begin{vmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ x_1 & \cdots & x_m & 1 & \cdots & 1 \\ x_1^2 & \cdots & x_m^2 & 2x_1 & \cdots & 2x_n \\ \vdots & & \vdots & \vdots & & \vdots \\ x_1^N & \cdots & x_m^N & Nx_1^{N-1} & \cdots & Nx_n^{N-1} \end{vmatrix} \prod_{i=1}^{n} x_i.$$

Let Δ_1 be the part of determinant on the right hand side. Δ_1 has factors x_i-x_j $(1\leqq i<j\leqq m)$, since $\Delta_1=0$ by substituting $x_i=x_j$ into Δ_1. We differentiate partially Δ_1 by x_i for i such that $i\leqq n$, and substitute $x_i=x_j$ for $i<j\leqq m$. Then it becomes zero. Hence Δ_1 has a factor $(x_i-x_j)^2$ for i and j such that $i\leqq n$ and $i<j\leqq m$. On the other hand, for i and j such that $i<j\leqq n$,

$$\frac{\partial^3 \Delta_1}{\partial x_i^3}=\begin{vmatrix} \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & 0 & \cdots & 1 & \cdots & 1 & \cdots \\ \cdots & 0 & \cdots & 2x_i & \cdots & 2x_j & \cdots \\ \cdots & 6 & \cdots & 3x_i^2 & \cdots & 3x_j^2 & \cdots \\ & \vdots & & \vdots & & \vdots & \end{vmatrix} +3 \begin{vmatrix} \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & 2 & \cdots & 2 & \cdots \\ \cdots & 6x_i & \cdots & 6x_i & \cdots \\ & \vdots & & \vdots & \end{vmatrix}$$

$$+3\begin{vmatrix} \vdots & 0 & \vdots & 0 & \vdots & 0 & \vdots \\ \vdots & 1 & \vdots & 0 & \vdots & 1 & \vdots \\ \vdots & 2x_i & \vdots & 0 & \vdots & 2x_j & \vdots \\ \vdots & 3x_i^2 & \vdots & 6 & \vdots & 3x_j^2 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} + \begin{vmatrix} \vdots & 1 & \vdots & 1 & \vdots & 0 & \vdots \\ \vdots & x_i & \vdots & x_j & \vdots & 0 & \vdots \\ \vdots & x_i^2 & \vdots & x_j^2 & \vdots & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \vdots \\ \vdots & & \vdots & & \vdots & 24 & \vdots \\ \vdots & & \vdots & & \vdots & \vdots & \vdots \end{vmatrix}$$

The second term of the right hand side is zero, and other terms become zero if wes ubstitute $x_i=x_j$. Similarly $\frac{\partial^2 \Delta_1}{\partial x_i^2}=0$ by substituting $x_i=x_j$. Hence Δ_1 has factors $(x_i-x_j)^4$ for i and j such that $i<j\leqq n$. Hence Δ is divisible by D. Since Δ and D have the same order $(m+n-1)(m+n)/2$, its quotient factor K must be a constant. For $n=0$, $K=(-1)^{\frac{m(m-1)}{2}}$ is a direct consequence from Vandermonde's formula. So we use the mathematical induction, and assume the result for $n-1$. Now we introduce a lexicographic ordering in the set of monomials in $x_1, \dots, x_m$ as $x_1 \succ x_2 \succ \cdots \succ x_m$. Then the coefficient of the highest term of D is 1, whence that of Δ is K. On the other hand, as easily verified, the coefficient of x_1^{2N-1} in Δ is

$$(-1)^{m+1}\begin{vmatrix} 1 & \vdots & 1 & \vdots \\ x_2 & \vdots & 2x_2 & \vdots \\ x_2^2 & \vdots & 3x_2^2 & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

By induction hypothesis,

$$K=(-1)^{m+1}(-1)^{\frac{(m-1)(m-2)}{2}}=(-1)^{\frac{m(m-1)}{2}}.$$

Hence the lemma is proved.

LEMMA 6. *Let $\lambda=\Sigma m_i\alpha_i$ be the highest weight of an irreducible representation of a semi-simple Lie algebra. Then it does not occur that only one m_i is negative.*

PROOF. Suppose that $m_i<0$ and $m_j\geqq 0$ for $j\neq i$. Then

$$(\lambda, \alpha_i)=m_i(\alpha_i, \alpha_i)+\sum_{j\neq i} m_j(\alpha_j, \alpha_i)<0,$$

which contradicts the fact that λ is dominant.

LEMMA 7. *Let $\mathfrak{s}$ be a semi-simple Lie algebra, ρ an irreducible representation of $\mathfrak{s}$ and $\mathfrak{m}$ the representation space of ρ. Let λ be the highest weight of ρ and μ an arbitrary weight. The weight spaces of λ and μ are denoted by $\mathfrak{m}_\lambda$ and $\mathfrak{m}_\mu$ respectively. Let v_μ be an arbitrary non-zero element of $\mathfrak{m}_\mu$. Then there exists a sequence of simple roots $\{\alpha_{i_1}, \dots, \alpha_{i_p}\}$ such that $\rho(e_{\alpha_{i_1}}) \dots \rho(e_{\alpha_{i_p}})v_\mu$ is a non-zero element of $\mathfrak{m}_\lambda$, and μ is represented as*

$$\mu = \lambda - \alpha_{i_1} - \cdots - \alpha_{i_p},$$

where $\lambda - \alpha_{i_1} - \cdots - \alpha_{i_q}$ is also a weight for every $q \leqq p$.

PROOF. Since $\mathfrak{m}$ is an irreducible $\mathfrak{s}$-space, v_μ generates the $\mathfrak{s}$-space $\mathfrak{m}$. Hence there exists a non-zero element in $\mathfrak{m}_\lambda$ of the form $\rho(e_{\beta_1}) \dots \rho(e_{\beta_p})v_\mu$, where $\beta_1, \dots, \beta_p \in \Sigma_0$. Obviously $\mu + \Sigma\beta_i = \lambda$. Among such elements, let

$$v_\lambda = \rho(e_{\beta_1}) \dots \rho(e_{\beta_p})v_\mu$$

have the minimum length p. Then we assert that every β_i is a simple root ($i=1, \dots, p$). For this we show by mathematical induction on r that $\beta_1, \dots, \beta_r$ are simple roots. First, if $\beta_1 = -\alpha_{i_1}$, then $\rho(e_{\beta_2}) \dots \rho(e_{\beta_p})v_\mu \in \mathfrak{m}_{\lambda+\alpha_{i_1}} = \{0\}$, which contradicts $v_\lambda \neq 0$. Hence $\beta_1 = \alpha_{i_1}$. Now we suppose that $\beta_1 = \alpha_{i_1}, \dots, \beta_r = \alpha_{i_r}$ have been shown and prove that $\beta_{r+1} = \alpha_{i_{r+1}}$. Suppose that it is false, i.e., that $\beta_{r+1} = -\alpha_{i_{r+1}}$. Then

$$\begin{aligned} v_\lambda = {} & \rho(e_{\alpha_{i_1}}) \dots \rho([e_{\alpha_{i_r}}, e_{-\alpha_{i_{r+1}}}]) \dots \rho(e_{\beta_p})v_\mu \\ & + \rho(e_{\alpha_{i_1}}) \dots \rho(e_{-\alpha_{i_{r+1}}})\rho(e_{\alpha_{i_r}}) \dots \rho(e_{\beta_p})v_\mu. \end{aligned}$$

If $i_r \neq i_{r+1}$, then $[e_{\alpha_{i_r}}, e_{-\alpha_{i_{r+1}}}] = 0$. Hence the first term of the right hand side is zero. In the case $i_r = i_{r+1}$, the first term is a scalar multiple of $\rho(e_{\alpha_{i_1}}) \dots \rho(e_{\alpha_{i_{r-1}}})\rho(e_{\beta_{r+2}}) \dots \rho(e_{\beta_p})v_\mu$, which vanishes, for its length is less than p. However,

$$v_\lambda = \rho(e_{\alpha_{i_1}}) \dots \rho(e_{-\alpha_{i_{r+1}}})\rho(e_{\alpha_{i_r}}) \dots \rho(e_{\beta_p})v_\mu$$

contradicts our induction hypothesis. Hence we have proved the first half of the lemma. Then the latter half will be evident.

LEMMA 8. *Let $\mathfrak{s}$ be a semi-simple Lie algebra, and $\mathfrak{m}$ a non-trivial. irreducible $\mathfrak{s}$-space. Donote by $\{\alpha_1, \dots, \alpha_n\}$ the system of simple roots and by Σ_0 the set $\{\alpha_1, \dots, \alpha_n, -\alpha_1, \dots, -\alpha_n\}$. Then there exist at least three weights of $\mathfrak{m}$ which do not belong to Σ_0, except for the case where $\mathfrak{s}$ has a 3-dimensional simple ideal $\mathfrak{s}_1$, and $\mathfrak{m}$ is $\mathfrak{s}$-isomorphic to $\mathfrak{s}_1$ or $\mathfrak{s}$-isomorphic to an irreducible 2-dimensional $\mathfrak{s}_1$-space considered as an $\mathfrak{s}$-space.*

PROOF. Let Σ be the system of roots. We denote by Λ the set of weights of $\mathfrak{m}$ and by λ the highest weight in Λ. We mean by S_α the reflection in the hyperplane orthogonal to the root α. Let μ be an arbitrary weight in Λ and let M be the collection of weights of the form $\mu+i\alpha$, i an integer. Then M is an arithmetic progression with first term $\mu-p\alpha$, difference α, and last term $\mu+q\alpha$ and we have

$$2\,\frac{(\mu,\,\alpha)}{(\alpha,\,\alpha)}=p-q$$

([7]). This sequence is called the α-series of the weight μ. Since $\lambda\neq 0$, there exists α_i such that $(\lambda,\,\alpha_i)>0$. Then $\lambda-\alpha_i\in\Lambda$.

For the convenience of the proof, we divide the proof into four cases.

i) *The case* $\lambda=\alpha_i$. Since λ is the highest weight, $(\alpha_i,\,\alpha_j)\geqq 0$ for any α_j. On the other hand, by the property of simple roots $(\alpha_i,\,\alpha_j)\leqq 0$ for any $j\neq i$. Hence we get $(\alpha_i,\,\alpha_j)=0$ for $j\neq i$, i.e. Σ is decomposed into the union of mutually orthogonal two subsystems, that is, $\{\alpha_i,\,-\alpha_i\}$ and the collection of all other roots. Hence in this case $\mathfrak{s}$ has a 3-dimensional simple ideal $\mathfrak{s}_1$ and $\mathfrak{m}$ is $\mathfrak{s}$-isomorphic to $\mathfrak{s}_1$.

ii) *The case* $\lambda\in\Sigma-\Sigma_0$. In this case, $-\lambda=S_\lambda\lambda\in\Lambda-\Sigma_0$. Then the λ-series of the weight λ contains the weight 0. Hence we may take the weights λ, 0 and $-\lambda$.

iii) *The case where* $\lambda^{-}\Sigma$ and $\lambda-\alpha_i\in\Lambda\cap\Sigma$. We denote the α_i-series of the weight λ by $\{\lambda-p\alpha_i,\,\dots,\,\lambda-\alpha_i,\,\lambda\}$ and the α_i-series of the root $\lambda-\alpha_i$ by $\{\lambda-p'\alpha_i,\,\dots,\,\lambda-\alpha_i\}$. Then,

$$2\,\frac{(\lambda-\alpha_i,\,\alpha_i)}{(\alpha_i,\,\alpha_i)}=p-2=p'-1$$

implies $p\geqq 2$ and $\lambda-p\alpha_i\in\Lambda-\Sigma$. $\lambda-\alpha_i\in\Lambda\cap\Sigma$ means $0\in\Lambda$, considering the $(\lambda-\alpha_i)$-series of the weight $\lambda-\alpha_i$. $\lambda-p\alpha_i\notin\Sigma$ implies that $\lambda-p\alpha_i\neq 0$. Hence, in this case we may take the weights λ, $\lambda-p\alpha_i$ and 0.

iv) *The case where* $\lambda\notin\Sigma$ *and* $\lambda-\alpha_i\notin\Sigma$. If there exists α_j such as $(\alpha_i,\,\alpha_j)<0$, then $(\lambda-\alpha_i,\,\alpha_j)=(\lambda,\,\alpha_j)-(\alpha_i,\,\alpha_j)>0$ implies $\lambda-\alpha_i-\alpha_j\in\Lambda$. When $\lambda-\alpha_i-\alpha_j\notin\Sigma_0$, we may take λ, $\lambda-\alpha_i$ and $\lambda-\alpha_i-\alpha_j$. $\lambda-\alpha_i-\alpha_j$ coincides with neither $-\alpha_i$ nor $-\alpha_j$, and $\lambda-\alpha_i-\alpha_j$ cannot be $-\alpha_k$ by Lemma 6 $(k\neq i,\,j)$. If $\lambda-\alpha_i-\alpha_j=\alpha_k$, then $-\alpha_k$ is also a weight. Therefore 0 is a weight and we may take λ, $\lambda-\alpha_i$ and 0. On the other hand, if $\lambda-2\alpha_i\in\Lambda-\Sigma$, we may take λ, $\lambda-\alpha_i$ and $\lambda-2\alpha_i$. If $\lambda-2\alpha_i\in\Lambda\cap\Sigma$, there exists a positive integer p (>3) such that $\lambda-p\alpha_i\in\Lambda-\Sigma$ as in iii). Hence there remains only the case where $\lambda-\alpha_i\in\Lambda-\Sigma$, $\lambda-2\alpha_i\notin\Lambda$ and $(\alpha_i,\,\alpha_j)=0$ for every $j\neq i$. Then $\mathfrak{s}$ contains a 3-dimensional simple ideal. If there exists $j(\neq i)$ such that $\lambda-\alpha_j\in\Lambda\cap\Sigma$, it may be reduced to the case iii) by replacing α_i by α_j. Hence we may suppose that $\lambda-\alpha_j\notin\Lambda$ or $\lambda-\alpha_j\in\Lambda-\Sigma$ for every j $(\neq i)$. If there exists j such that $\lambda-\alpha_j\in\Lambda-\Sigma$, we may take λ, $\lambda-\alpha_i$ and $\lambda-\alpha_j$. When $\lambda-\alpha_j\notin\Lambda$ for any $j(\neq i)$, then $(\lambda,\,\alpha_j)=0$. Let

λ be represented as $\Sigma m_k\alpha_k$. Since the α_i-series of the weight λ is $\{\lambda-\alpha_i, \lambda\}$,

$$2\frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}=2m_i=1.$$

On the other hand, for $j\neq i$,

$$0=(\lambda, \alpha_j)=\left(\frac{1}{2}\alpha_i+\sum_{k\neq i}m_k\alpha_k, \alpha_j\right)$$
$$=(\sum_{k\neq i}m_k\alpha_k, \alpha_j).$$

Hence $(\sum_{k\neq i}m_k\alpha_k, \sum_{k\neq i}m_k\alpha_k)=0$ implies that $m_k=0$ for $k\neq i$ and $\lambda=\frac{1}{2}\alpha_i$. Then $\mathfrak{m}$ is a two-dimensional space with two weights $\frac{\alpha_i}{2}$ and $-\frac{\alpha_i}{2}$. This completes the proof.

§ 4

Let $\mathfrak{g}$ be a perfect Lie algebra whose radical $\mathfrak{r}$ is abelian. We denote by $\mathfrak{s}$ a maximal semi-simple subalgebra of $\mathfrak{g}$. Let $\mathfrak{s}$ be decomposed into a direct sum of simple ideals of $\mathfrak{s}$ as follows:

$$\mathfrak{s}=\sum_{i=1}^{s}\mathfrak{s}^{(i)}+\sum_j\mathfrak{t}_j,$$

where $\mathfrak{s}^{(i)}$ is a 3–dimensional simple ideal and $\mathfrak{t}_j$ is a simple ideal of other type. We consider the radical $\mathfrak{r}$ as an $\mathfrak{s}$-space and decompose $\mathfrak{r}$ into a direct sum of $\mathfrak{s}$-irreducible subspaces. Among them, we denote by $\mathfrak{u}_1^{(i)}, \dots, \mathfrak{u}_{p_i}^{(i)}$ the subspaces isomorphic to the $\mathfrak{s}$-space $\mathfrak{s}^{(i)}$. We take a basis $\{u_{j,+}^{(i)}, u_{j,0}^{(i)}, u_{j,-}^{(i)}\}$ for $\mathfrak{u}_j^{(i)}$ as follows:

$$[h^{(i)}, u_{j,+}^{(i)}]=2u_{j,+}^{(i)} \qquad [h^{(i)}, u_{j,-}^{(i)}]=-2u_{j,-}^{(i)}$$
$$[e_\alpha^{(i)}, u_{j,-}^{(i)}]=u_{j,0}^{(i)} \qquad [e_\alpha^{(i)}, u_{j,0}^{(i)}]=-2u_{j,+}^{(i)}$$
$$[e_{-\alpha}^{(i)}, u_{j,+}^{(i)}]=-u_{j,0}^{(i)} \qquad [e_{-\alpha}^{(i)}, u_{j,0}^{(i)}]=2u_{j,-}^{(i)}$$

(cf. Example 2). We consider a two dimensional irreducible $\mathfrak{s}^{(i)}$-space $\mathfrak{v}^{(i)}$, which is also considered as an $\mathfrak{s}$-space. We denote by $\mathfrak{v}_1^{(i)},\dots, \mathfrak{v}_{q_i}^{(i)}$ the irreducible components of $\mathfrak{r}$ isomorphic to $\mathfrak{v}^{(i)}$ (cf. Example 1). Besides, we denote by $\mathfrak{w}_{kl}$'s irreducible components of other types, where $\mathfrak{w}_{k1},\dots, \mathfrak{w}_{kr_k}$ are $\mathfrak{s}$-isomorphic to each other and $\mathfrak{w}_{kl}$ and $\mathfrak{w}_{k'l'}$ are not $\mathfrak{s}$-isomorphic for $k\neq k'(k, k'=1, \dots, t)$. Namely, $\mathfrak{r}$ is decomposed as follows:

$$\mathfrak{r}=\sum_{i=1}^{s}\Big(\sum_{j=1}^{p_i}\mathfrak{u}_j^{(i)}+\sum_{j=1}^{q_i}\mathfrak{v}_j^{(i)}\Big)+\sum_{k=1}^{t}\sum_{l=1}^{r_k}\mathfrak{w}_{kl} \tag{1}$$

The following is a generalization of Theorem 2.

THEOREM 3. *Let the abelian radical* $\mathfrak{r}$ *of a perfect Lie algebra* $\mathfrak{g}$ *be decomposed into* $\mathfrak{s}$*-irreducible subspaces as* (1). *Let* Λ_k *be a set of weights of* $\mathfrak{w}_{kl}$ *and let*

$$u_k=\mathrm{Card}\,(\Lambda_k-\Sigma_0).$$

Then $u_k\geqq 3$. *If*

$$p_i\leqq 3 \qquad q_i\leqq 2 \qquad r_k\leqq u_k \qquad (i=1,\dots,s;\ k=1,\dots,t),$$

then the nullity of $\mathfrak{g}$ *is two.*

PROOF. We can assume that $p_1=\cdots=p_s=3$, $q_1=\cdots=q_s=2$, $r_1=u_1,\dots,r_t=u_t$. For otherwise we adjoin a suitable $\mathfrak{s}$-space $\mathfrak{r}^*$ to $\mathfrak{g}$ such that the Lie algebra $\mathfrak{g}^*=\mathfrak{g}+\mathfrak{r}^*$ satisfies our assumption. Then by Lemma 2 $\mathrm{Nul}(\mathfrak{g}^*)\geqq\mathrm{Nul}(\mathfrak{g})$. Hence we have only to prove that $\mathrm{Nul}(\mathfrak{g}^*)=2$.

Let $\lambda_{k1},\dots,\lambda_{ku_k}$ be weights of $\mathfrak{w}_{kl}$ which are not contained in Σ_0. Let $\{\alpha_1,\dots,\alpha_n\}$ be the system of the simple roots, and let $\alpha^{(i)}$ be the simple root corresponding to a simple ideal $\mathfrak{s}^{(i)}$. Then

$$P=\prod_{i\neq j}(\alpha_i^2-\alpha_j^2)\prod_{i\neq j}(4\alpha_i^2-\alpha_j^2)\prod_{k=1}^{t}\prod_{i<j\leqq u_k}(\lambda_{ki}-\lambda_{kj})\prod_{i,j,k}(\alpha_i^2-\lambda_{jk}^2)\prod_{i=1}^{n}\alpha_i$$

is a non-zero polynomial. We take an element $h\in\mathfrak{h}$ such that $P(h)\neq 0$. From the set of weights $\left\{\frac{\alpha^{(1)}}{2},\dots,\frac{\alpha^{(s)}}{2},-\frac{\alpha^{(1)}}{2},\dots,-\frac{\alpha^{(s)}}{2},\lambda_{11},\dots,\lambda_{1u_1},\dots,\lambda_{t1},\dots,\lambda_{tu_t}\right\}$ we pick up only the mutually different ones and denote them by $\mu_0=0$, $\mu_1,\dots,\mu_d$ (where $\mu_0=0$ should be omitted if unnecessary). Let w_{ki} be an element of $\mathfrak{w}_{ki}$ which belongs to the weight $\lambda_{ki}(k=1,\dots,t;\ i=1,\dots,u_k)$, and let $v_{1,+}^{(i)}\in\mathfrak{v}_1^{(i)}$ and $v_{2,-}^{(i)}\in\mathfrak{v}_2^{(i)}$ be elements which belong to the weights $\frac{\alpha^{(i)}}{2}$ and $-\frac{\alpha^{(i)}}{2}$ respectively. Among such weight vectors, we take the ones which belong to the same weight μ_j. Let f_j be the sum of them $(j=0,1,\dots,d)$. Now we take the following two elements:

$$x=h+\sum_{i=1}^{s}u_{1,0}^{(i)}$$

$$y=\sum_{\beta\in\Sigma_0}e_\beta+\sum_{i=1}^{s}(u_{2,0}^{(i)}+u_{3,-}^{(i)})+f_0+\sum_{j=1}^{d}f_j$$

We shall show that the Lie algebra $\mathfrak{g}_1$ generated by x and y coincides with $\mathfrak{g}$. First we can easily show the following $(l\geqq 1)$.

$$
\begin{aligned}
&(\mathrm{ad}\,x)^l y \\
&= \sum_{\substack{\beta\in\Sigma_0\\ \beta\neq-\alpha^{(i)}}} (\beta(h))^l e_\beta + \sum_i (-\alpha^{(i)}(h))^l (e_{-\alpha^{(i)}} + u_{3,-}^{(i)}) \\
&\quad + \sum_j (\mu_j(h))^l f_j + 2l\sum_i (\alpha^{(i)}(h))^{l-1} u_{1,+}^{(i)} \\
&\quad - 2l\sum_i (-\alpha^{(i)}(h))^{l-1} u_{1,-}^{(i)}
\end{aligned}
$$

Then by Lemma 5, the following determinant is not zero:

$$
\begin{vmatrix}
\vdots & \beta(h) & \vdots & -\alpha^{(i)}(h) & \vdots & \mu_j(h) & \vdots & 2 & \vdots & -2 & \vdots \\
\vdots & (\beta(h))^2 & \vdots & (-\alpha^{(i)}(h))^2 & \vdots & (\mu_j(h))^2 & \vdots & 4\alpha^{(i)}(h) & \vdots & -4(-\alpha^{(i)}(h)) & \vdots \\
\vdots & (\beta(h))^3 & \vdots & (-\alpha^{(i)}(h))^3 & \vdots & (\mu_j(h))^3 & \vdots & 6(\alpha^{(i)}(h))^2 & \vdots & -6(-\alpha^{(i)}(h))^2 & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{vmatrix}
$$

Hence all the following elements are contained in $\mathfrak{g}_1$:

$$
e_\beta,\quad e_{-\alpha^{(i)}} + u_{3,-}^{(i)},\quad f_j,\quad u_{1,+}^{(i)},\quad u_{1,-}^{(i)},\quad f_0 + \sum_{k=1}^{s} u_{2,0}^{(k)}
$$

$$
(\beta\in\Sigma_0,\ \beta\neq-\alpha^{(i)},\ i=1,\dots,s;\ j=1,\dots,d)
$$

Moreover, since the elements

$$
[e_{\alpha^{(i)}},\ e_{-\alpha^{(i)}} + u_{3,-}^{(i)}] = h^{(i)} + u_{3,0}^{(i)}
$$

and

$$
[h^{(i)} + u_{3,0}^{(i)},\ e_{-\alpha^{(i)}} + u_{3,-}^{(i)}] = -2e_{-\alpha^{(i)}} - 4u_{3,-}^{(i)}
$$

belong to $\mathfrak{g}_1$, $e_{-\alpha^{(i)}}$ also belongs to $\mathfrak{g}_1$. Hence $\mathfrak{g}_1$ contains $\mathfrak{s}$, $\mathfrak{u}_1^{(i)}$ and $\mathfrak{u}_3^{(i)}$, because $\mathfrak{u}_1^{(i)}$ and $\mathfrak{u}_3^{(i)}$ are irreducible $\mathfrak{s}$-spaces. Now we decompose f_j again as follows $(j\neq 0)$:

$$
f_j = w_{i_1 l_1} + \cdots + w_{i_m l_m}, \tag{2}
$$

where $w_{i_\nu l_\nu}\in\mathfrak{w}_{i_\nu l_\nu}$ belong to the same weight μ_j and $i_1,\dots,i_m$ are defferent from each other (In certain cases some $w_{i_\nu l_\nu}$ should be replaced by $v_{k,+}^{(j)}$ or $v_{k,-}^{(j)}$). Among the $\mathfrak{s}$-spaces $\mathfrak{w}_{i_\nu l_\nu}$'s we assume that $\mathfrak{w}_{i_1 l_1}$ has the highest weight λ. Then Lemma 7 implies that there exists a sequence of simple roots $\{\alpha_{j_1},\dots,\alpha_{j_k}\}$ such that $w_\lambda=[e_{\alpha_{j_1}}, [\dots, [e_{\alpha_{j_k}}, w_{i_1 l_1}]\dots]]$ is not zero and belongs to the highest weight λ. We operate $(\mathrm{ad}\,e_{\alpha_{j_1}})\dots(\mathrm{ad}\,e_{\alpha_{j_k}})$ to the both sides of (2). Since λ cannot be a weight of $\mathfrak{w}_{i_\nu l_\nu}(\nu\neq 1)$, all the terms $w_{i_\nu l_\nu}$ vanish except $w_{i_1 l_1}$. Hence $w_\lambda\in\mathfrak{g}_1$, which implies that $\mathfrak{w}_{i_1 l_1}$ is contained in $\mathfrak{g}_1$. By induction we may show that other $\mathfrak{w}_{i_\nu l_\nu}(\nu\geqq 2)$ or $\mathfrak{v}_k^{(j)}$ are contained in $\mathfrak{g}_1$. Quite similarly $\mathfrak{g}_1$ contains $\mathfrak{w}_{i_\nu l_\nu}$ corresponding to μ_0 and $\mathfrak{u}_2^{(i)}$. Thus $\mathfrak{g}_1$ must coincide with $\mathfrak{g}$. This completes the proof.

Furthermore we can prove the following

THEOREM 4. *Let the abelian radical* $\mathfrak{r}$ *of a perfect Lie algebra* $\mathfrak{g}$ *be decomposed into* $\mathfrak{s}$*-irreducible spaces as* (1). *Let* v_k *be the number of distinct weights in* $\mathfrak{w}_{kl}$, *and let* u_k *be the number of distinct weights in* $\mathfrak{w}_{kl}$ *which are not contained in* $\Sigma_0(k=1, \dots, t;\ l=1, \dots, r_k)$. *Then*

$$\mathrm{Nul}(\mathfrak{g})\leqq \operatorname*{Max}_{\substack{i=1,\dots,s\\ k=1,\dots,t}}\left\{\frac{p_i-1}{3},\ \frac{q_i-1}{2},\ \frac{r_k+v_k-u_k-1}{v_k},\ 0\right\}+2 \tag{3}$$

PROOF. By the same reason as remarked in the proof of the previous theorem, we may increase the values of p_i, q_i and r_k unless the integral part of the maximum value in the above inequality (3) is altered. Hence we can assume that there exists an integer K such as $p_i=3K$, $q_i=2K$ and $r_k+v_k-u_k=Kv_k$, i.e. the integral part of the maximum value is equal to $K-1$. Now we divide the family of subspaces $\mathfrak{u}_\iota^{(i)}$, $\mathfrak{v}_\kappa^{(i)}$, $\mathfrak{w}_{kl}$ $(i=1, \dots, s;\ \iota=4, \dots, 3K;\ \kappa=3, \dots, 2K;\ k=1, \dots, t;\ l=u_k+1, \dots, (K-1)v_k+u_k)$ into $K-1$ groups as follows:

$$\{\mathfrak{u}_4^{(i)}, \mathfrak{u}_5^{(i)}, \mathfrak{u}_6^{(i)}, \mathfrak{v}_3^{(i)}, \mathfrak{v}_4^{(i)}, \mathfrak{w}_{ku_k+1}, \dots, \mathfrak{w}_{ku_k+v_k}\},$$

$$\dots\dots\dots\dots\dots\dots,$$

$$\{\mathfrak{u}_{3K-2}^{(i)}, \mathfrak{u}_{3K-1}^{(i)}, \mathfrak{u}_{3K}^{(i)}, \mathfrak{v}_{2K-1}^{(i)}, \mathfrak{v}_{2K}^{(i)}, \mathfrak{w}_{ku_k+(K-2)v_k+1}, \dots,$$
$$\mathfrak{w}_{ku_k+(K-1)v_k}\}.$$

Let $u_{j,+}^{(i)}$, $u_{j,0}^{(i)}$ and $u_{j,-}^{(i)}$ be elements in $\mathfrak{u}_j^{(i)}$ which belong to the weights $\alpha^{(i)}$, 0, $-\alpha^{(i)}$ respectively, and let $v_{j,+}^{(i)}$ and $v_{j,-}^{(i)}$ be elements in $\mathfrak{v}_j^{(i)}$ which belong to the weights $\frac{\alpha^{(i)}}{2}$ and $-\frac{\alpha^{(i)}}{2}$ respectively. Now we denote by $\{\lambda_{k1}, \dots, \lambda_{kv_k}\}$ the system of weights of $\mathfrak{w}_{kj}$, and we choose an element $w_{ku_k+(\nu-1)v_k+l} \in \mathfrak{w}_{ku_k+(\nu-1)v_k+l}$ which belongs to the weight λ_{kl} $(k=1, \dots, t;\ \nu=1, \dots, K-1;\ l=1, \dots, v_k)$. We take the following additional $K-1$ elements together with the generators x and y in the proof of Theorem 3:

$$z_\nu=\sum_{i=1}^{s}(u_{3\nu+1,+}^{(i)}+u_{3\nu+2,0}^{(i)}+u_{3\nu+3,-}^{(i)}$$
$$+v_{2\nu+1,+}^{(i)}+v_{2\nu+2,-}^{(i)})+\sum_{k=1}^{t}\sum_{l=1}^{v_k}w_{ku_k+(\nu-1)v_k+l}$$
$$(\nu=1, \dots, K-1).$$

Theorem 3 shows that x and y generate a subalgebra containing $\mathfrak{s}$. Hence it is easy to see that $x, y, z_1, \ldots, z_{K-1}$ generate $\mathfrak{g}$. Thus $\mathrm{Nul}(\mathfrak{g}) \leqq K+1$, which proves the theorem.

The following is a rectification of Marshall's inequality.

COROLLARY. *Let d be the dimension of a perfect Lie algebra $\mathfrak{g}$. Then,*

$$\mathrm{Nul}(\mathfrak{g}) \leqq \mathrm{Max}\left\{\frac{d+3}{4}, 2\right\}.$$

PROOF. Set $n = \mathrm{Nul}(\mathfrak{g})$. In the case $n=2$, there is nothing to be proved. Hence we suppose $n \geqq 3$. Example 1 shows the existence of a perfect Lie algebra of dimension $4n-3$ whose nullity is n. Suppose that $\mathfrak{g}$ has the least dimension among perfect Lie algebras with nullity n. We have only to show that $\dim \mathfrak{g} = 4n-3$. Obviously the radical $\mathfrak{r}$ of $\mathfrak{g}$ is abelian. Let $\mathfrak{r}$ be decomposed as in (1), and we denote by $K-1$ the integral part of the maximum value in (3). Then Theorem 4 shows $n \leqq K+1$. If there exists i_0 such that $K-1 = \left[\frac{p_{i_0}-1}{3}\right]$, then $p_{i_0} \geqq 3n-5$, and

$$d \geqq \dim \mathfrak{s} + 3(3n-5) \geqq 9n-12 > 4n-3,$$

which is impossible. Similarly $K-1 = \left[\frac{r_{k_0}+v_{k_0}-u_{k_0}-1}{v_{k_0}}\right]$ is also a contradiction. In fact, in this case,

$$d \geqq 3 + r_{k_0}v_{k_0} \geqq 3 + (nv_{k_0} + u_{k_0} - 3v_{k_0} + 1)v_{k_0}.$$

The right hand side is greater than $4n-3$ since $n \geqq 3$ and $v_{k_0} \geqq u_{k_0} \geqq 3$, which is also impossible. Hence we have $K-1 = \left[\frac{q_{i_0}-1}{2}\right] \geqq n-2$, i.e. if q_{i_0} is even, $q_{i_0} \geqq 2n-2$, and if q_{i_0} is odd, $q_{i_0} \geqq 2n-3$. Then $d \geqq 3+2(2n-2)$ and $d \geqq 3+2(2n-3)$ respectively. Hence only the latter case is possible. Thus the Lie algebra of the least dimension is such one satisfying that $\dim \mathfrak{s} = 3$, $s=1$, $p_1 = t = 0$ and $q_1 = 2n-3$. This is nothing but the one stated in Example 1. Hence the proof is complete.

§5

In this section we discuss the nulity of a Lie algebra whose radical is nilpotent. We begin with the following lemmas.

LEMMA 9. *If $\mathfrak{g}$ is represented as a direct sum of two ideals $\mathfrak{a}$ and $\mathfrak{b}$, then*

$$\mathrm{Nul}(g) \leqq \mathrm{Max}\{\mathrm{Nul}(\mathfrak{a}/[\mathfrak{a}, \mathfrak{a}]), \mathrm{Nul}([\mathfrak{b}, \mathfrak{b}])\}$$

$$+ \mathrm{Nul}([\mathfrak{a}, \mathfrak{a}]) + \mathrm{Nul}(\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]).$$

PROOF. Let $\{\bar{a}_1, \dots, \bar{a}_n\}$ and $\{\bar{b}_1, \dots, \bar{b}_q\}$ be sets of generators of $\mathfrak{a}/[\mathfrak{a}, \mathfrak{a}]$ and $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ respectively. We take representative elements $a_1, \dots, a_n$, $b_1, \dots, b_q$ contained in these residue classes. Let $[\mathfrak{a}, \mathfrak{a}]$, $[\mathfrak{b}, \mathfrak{b}]$ be generated by $\{a'_1, \dots, a'_p\}$, $\{b'_1, \dots, b'_n\}$ respectively. We set $c_i = a_i + b'_i$ $(i = 1, \dots, n)$ and denote by $\mathfrak{g}_1$ the subalgebra generated by $a'_1, \dots, a'_p$, $b_1, \dots, b_q$, $c_1, \dots, c_n$. $\mathfrak{b}$ is generated by $b_1, \dots, b_q$, $b'_1, \dots, b'_n$. Hence an arbitrary element of $[\mathfrak{b}, \mathfrak{b}]$ is represented as a linear combination of monomials in these elements of order not less than 2. Then, since $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{g}_1$,

$$[b'_i, b'_j] = [c_i, c_j] - [a_i, a_j] \in \mathfrak{g}_1 \cap \mathfrak{b}$$

$$[b_i, \mathfrak{g}_1 \cap \mathfrak{b}] \subset \mathfrak{g}_1 \cap \mathfrak{b}$$

$$[b'_i, \mathfrak{g}_1 \cap \mathfrak{b}] = [c_i, \mathfrak{g}_1 \cap \mathfrak{b}] \subset \mathfrak{g}_1 \cap \mathfrak{b}.$$

Hence $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{g}_1$, which implies that b'_i and a_i are contained in $\mathfrak{g}_1$. Thus we obtain $\mathfrak{g} = \mathfrak{g}_1$, which proves our statement.

LEMMA 10. *If a Lie algebra* $\mathfrak{g}$ *is a direct sum of an abelian ideal* $\mathfrak{a}$ *and a perfect ideal* $\mathfrak{b}$, *then*

$$\mathrm{Nul}(\mathfrak{g}) = \mathrm{Max}\{\dim \mathfrak{a}, \mathrm{Nul}(\mathfrak{b})\}.$$

PROOF. It is obvious by Lemma 3 that $\mathrm{Nul}(\mathfrak{a}) = \dim \mathfrak{a}$ and $\mathrm{Nul}(\mathfrak{g}) \geqq \mathrm{Max}\{\mathrm{Nul}(\mathfrak{a}), \mathrm{Nul}(\mathfrak{b})\}$. In the previous lemma, we put $\mathrm{Nul}([\mathfrak{a}, \mathfrak{a}]) = \mathrm{Nul}(\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]) = 0$. Then we have the assertion.

Now let $\mathfrak{g}$ be a Lie algebra whose radical $\mathfrak{r}$ is nilpotent, and $\mathfrak{s}$ a maximal semi-simple subalgebra of $\mathfrak{g}$. The radical $\mathfrak{r}$, considered as an $\mathfrak{s}$-space, is represented as a direct sum of $\mathfrak{s}$-spaces as follows:

$$\mathfrak{r} = [\mathfrak{r}, \mathfrak{r}] + \mathfrak{a} + \mathfrak{b},$$

where $[\mathfrak{s}, \mathfrak{a}] = 0$ and $[\mathfrak{s}, \mathfrak{b}] = \mathfrak{b}$. Then

$$\bar{\mathfrak{b}} = (\mathfrak{s} + \mathfrak{b} + [\mathfrak{r}, \mathfrak{r}])/[\mathfrak{r}, \mathfrak{r}]$$

is a perfect ideal of $\mathfrak{g}/[\mathfrak{r}, \mathfrak{r}]$ and

$$\bar{\mathfrak{a}} = (\mathfrak{a} + [\mathfrak{r}, \mathfrak{r}])/[\mathfrak{r}, \mathfrak{r}]$$

is an abelian ideal. $\mathfrak{g}/[\mathfrak{r}, \mathfrak{r}]$ is a direct sum of these two ideals. From Lemma 4, $\mathrm{Nul}(\mathfrak{g}) = \mathrm{Nul}(\mathfrak{g}/[\mathfrak{r}, \mathfrak{r}])$. Then the following theorem immediately follows from the previous lemma.

THEOREM 5. *Let* $\mathfrak{g}$ *be a Lie algebra whose radical* $\mathfrak{r}$ *is nilpotent, and* $\mathfrak{s}$ *its maximal semi-simple subalgebra. Let* $\mathfrak{r}$ *be decomposed as follows:*

$$\mathfrak{r}=[\mathfrak{r}, \mathfrak{r}]+\mathfrak{a}+\mathfrak{b},$$

where $[\mathfrak{s}, \mathfrak{a}]=0$ *and* $[\mathfrak{s}, \mathfrak{b}]=\mathfrak{b}$. *Then,* $(\mathfrak{s}+\mathfrak{b}+[\mathfrak{r}, \mathfrak{r}])/[\mathfrak{r}, \mathfrak{r}]$ *is a perfect Lie algebra and*

$$\mathrm{Nul}(\mathfrak{g})=\mathrm{Max}\{\mathrm{Nul}((\mathfrak{s}+\mathfrak{b}+[\mathfrak{r}, \mathfrak{r}])/[\mathfrak{r}, \mathfrak{r}]), \dim \mathfrak{a}\}.$$

§6

In this last section, we investigate the nullity of a Lie algebra whose adjoint representation is splittable. A Lie algebra $\mathfrak{g}$ is said to be splittable if it is a linear Lie algebra and a nilpotent component of an arbitrary element of $\mathfrak{g}$ also belongs to $\mathfrak{g}$ (Malcev [10]). This notion is an extension of Chevalley's notion of an algebraic Lie algebra ([4]). A Lie algebra whose adjoint representation is algebraic has a decomposition of a special type (Gotô[6], Matsushima [12], Chevalley [5]). A Lie algebra whose adjoint representation is splittable has also a similar decomposition (Tôgô [14]). That is, let ad $\mathfrak{g}$ is splittable for a Lie algebra $\mathfrak{g}$. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$ and $\mathfrak{n}$ the largest nilpotent ideal. Then there exist a maximal semi-simple subalgebra $\mathfrak{s}$ and an abelian subalgebra $\mathfrak{a}$ such that

$$\mathfrak{g}=\mathfrak{s}+\mathfrak{r}, \qquad \mathfrak{r}=\mathfrak{a}+\mathfrak{n}, \tag{4}$$

where $[\mathfrak{s}+\mathfrak{a}, \mathfrak{a}]=0$ and $\mathrm{ad}_{\mathfrak{g}}\mathfrak{a}$ consists of semi-simple matrices. Conversely every Lie algebra admitting such a decomposition has a splittable adjoint representation.

PROPOSITION 1. *Let* $\mathfrak{g}$ *be a Lie algebra whose nullity is* m. *Let* $\{f_1, \dots, f_n\}$ *be an arbitrary basis for* $\mathfrak{g}$. *Then there exist generators of* $\mathfrak{g}$ *such that*

$$a_i=f_i+\sum_{j=m+1}^{n}\alpha_{ji}f_j \qquad (i=1, \dots, m)$$

Moreover, let r *be the dimension of* $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. *For given* r *elements* $g_1, \dots, g_r$ *linearly independent modulo* $[\mathfrak{g}, \mathfrak{g}]$, *there exist generators of* $\mathfrak{g}$ *of the following forms*:

$$g_1+h_1, \;\dots,\; g_r+h_r, \;\; h_{r+1}, \;\dots,\; h_m \qquad (h_1, \dots, h_m\in[\mathfrak{g}, \mathfrak{g}]).$$

PROOF. We take generators

$$b_i=\sum_{j=1}^{n}\beta_{ji}f_j \qquad (i=1, \dots, m).$$

We may assume by Lemma 1 that

$$P=\begin{pmatrix} \beta_{11} & \vdots & \beta_{1m} \\ \vdots & \vdots & \vdots \\ \beta_{m1} & \vdots & \beta_{mm} \end{pmatrix}$$

is a regular matrix. Then

$$(a_1 \dots a_m)=(b_1 \dots b_m)P^{-1}=(f_1 \dots f_n)\begin{pmatrix} E \\ * \end{pmatrix}$$

also generate $\mathfrak{g}$. The latter half is evident, since $m \geqq r$ by Lemma 2.

Now we suppose that $\mathfrak{g}$ is a solvable Lie algebra and ad$\mathfrak{g}$ is splittable. Then $\mathfrak{g}$ is decomposed into a direct sum of subalgebras such as $\mathfrak{g}=\mathfrak{a}+\mathfrak{n}$, where $\mathfrak{a}$ is an abelian subalgabra, $\mathfrak{n}$ is the largest nilpotent ideal and $\mathrm{ad}_{\mathfrak{g}}\mathfrak{a}$ consists of semi-simple elements. Considering $\mathfrak{n}$ as an ad $\mathfrak{a}$-space, we decompose it into a sum of ad $\mathfrak{a}$-spaces as follows:

$$\mathfrak{n}=\mathfrak{n}_0+\mathfrak{n}_1+\cdots+\mathfrak{n}_r+[\mathfrak{n}, \mathfrak{n}],$$

where $[\mathfrak{a}, \mathfrak{n}_0]=0$, $[\mathfrak{a}, \mathfrak{n}_i]=\mathfrak{n}_i$ and $[a, n_i]=\beta_i(a)n_i$ for $a \in \mathfrak{a}$, $n_i \in \mathfrak{n}_i$ $(i \geqq 1)$. Then we have

THEOREM 6. *Let $\mathfrak{g}$ be a solvable Lie algebra whose adjoint representation is splittable. Under the above notations, let*

$$M=\operatorname*{Max}_{1 \leqq i \leqq r} \dim \mathfrak{n}_i.$$

Then

$$\mathrm{Nul}(\mathfrak{g})=\mathrm{Max}\{\dim(\mathfrak{a}+\mathfrak{n}_0), M+1\}.$$

PROOF. For $i \geqq 1$, we suppose that

$$\mathfrak{n}_i=\{\{n_{i1}, \dots, n_{iM}\}\}.$$

We chose an element a_1 in $\mathfrak{a}$ such that

$$\prod_{1 \leqq i<j \leqq r}(\beta_i-\beta_j)\prod_{1 \leqq i \leqq r}\beta_i$$

does not vanish at a_1. Let $\{a_1, a_2, \dots, a_p\}$ be a basis for $\mathfrak{a}$ and $\{n_{01}, \dots, n_{0q}\}$ a basis for $\mathfrak{n}_0$. Let $L=\mathrm{Max}\{p+q-1, M\}$. Then it is easily proved that the following $L+1$ elements generate $\mathfrak{g}$:

$$\left.\begin{array}{llll} a_1, & a_2+\sum_{1 \leqq i \leqq r} n_{i1}, & \dots, & a_p+\sum_{1 \leqq i \leqq r} n_{ip-1}, \\ n_{01}+\sum_{1 \leqq i \leqq r} n_{ip}, & \dots, & n_{0q}+\sum_{1 \leqq i \leqq r} n_{ip+q-1}, & \end{array}\right\} \qquad (5)$$

$$\sum_{1 \leqq i \leqq r} n_{ip+q}, \quad \ldots, \quad \sum_{1 \leqq i \leqq r} n_{iL}, \qquad \bigg\rfloor$$

where we consider that $n_{ij}=0$ if $j>M$. Hence we get

$$\mathrm{Nul}(\mathfrak{g}) \leqq \mathrm{Max}\{\dim(\mathfrak{a}+\mathfrak{n}_0),\ M+1\}.$$

To prove the reversed inequality we set $N=\mathrm{Nul}(\mathfrak{g})$. Let $a_1, \ldots, a_p, n_{01}, \ldots, n_{0q}$ be the same as above. Since $[\mathfrak{g}, \mathfrak{g}]=\sum_{i=1}^{r}\mathfrak{n}_i+[\mathfrak{n}, \mathfrak{n}]$, $N \geqq p+q$ follows from Lemma 2. We can assume that $[\mathfrak{n}, \mathfrak{n}]=0$. Proposition 1 implies that there exist N generators of $\mathfrak{g}$ as follows:

$$\begin{aligned}
b_1 &= a_1+\sum_{i \geqq 1} m_{i1} \\
&\ \ \vdots \\
b_p &= a_p+\sum_{i \geqq 1} m_{ip} \\
b_{p+1} &= n_{01}+\sum_{i \geqq 1} m_{ip+1} \\
&\ \ \vdots \\
b_{p+q} &= n_{0q}+\sum_{i \geqq 1} m_{ip+q} \\
b_{p+q+1} &= \sum_{i \geqq 1} m_{ip+q+1} \\
&\ \ \vdots \\
b_N &= \sum_{i \geqq 1} m_{iN},
\end{aligned}$$

where m_{ij} are some elements in $\mathfrak{n}_i$. As is easily verified, for $1 \leqq i<j \leqq p$ and $l \geqq 0$,

$$\{\mathrm{ad}(b_1)\}^l[b_i, b_j]=\sum_{k=1}^{r}\{\beta_k(a_1)\}^l\{\beta_k(a_i)m_{kj}-\beta_k(a_j)m_{ki}\}.$$

Let $\mathfrak{b}$ be the subalgebra generated by $b_1, \ldots, b_p$. Then

$$\mathfrak{b}=\{\{\, b_1, \quad \ldots, \quad b_p, \quad \beta_k(a_i)m_{kj}-\beta_k(a_j)m_{ki} \quad (k=1, \ldots, r;\ 1 \leqq i<j \leqq p)\}\}.$$

Since $\mathfrak{g}$ is generated by $b_1, \ldots, b_N$,

$$\mathfrak{g}=\mathfrak{b}+\mathfrak{n}_0+\{\{\, m_{kl}\ (k=1, \ldots, r;\ p+1 \leqq l \leqq N)\,\}\}.$$

Let $\dim \mathfrak{n}_{k_0}=M$. Then the system of linear equations

$$\beta_{k_0}(a_j)x_i - \beta_{k_0}(a_i)x_j = 0 \qquad (1 \leqq i < j \leqq p)$$

has a solution $x_i = \beta_{k_0}(a_i)$. Hence the $p \times \frac{p(p-1)}{2}$ matrix

$$\begin{array}{r} \\ \\ i) \\ \\ \\ \\ j) \\ \\ \\ \end{array} \begin{pmatrix} \vdots & 0 & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \beta_{k_0}(a_j) & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & 0 & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & -\beta_{k_0}(a_i) & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & 0 & \vdots \end{pmatrix}$$

has a rank at most $p-1$. Let

$$\mathfrak{b}_{k_0} = \{\{\beta_{k_0}(a_j)m_{k_0 i} - \beta_{k_0}(a_i)m_{k_0 j} \qquad (1 \leqq i < j \leqq p)\}\}.$$

Then $\dim \mathfrak{b}_{k_0}$ is at most $p-1$. Since

$$\mathfrak{n}_{k_0} = \mathfrak{b}_{k_0} + \{\{m_{k_0 l} \quad (p+1 \leqq l \leqq N)\}\},$$

$$\dim \mathfrak{n}_{k_0} \leqq (p-1) + (N-p) = N-1,$$

which implies that

$$N = \mathrm{Nul}(\mathfrak{g}) \geqq M+1.$$

Thus we have proved the theorem.

Remark. Knebelman [8] and Bond [2] state the following:

If the genus of $\mathfrak{g}$ is zero, then $\mathfrak{g}$ is either abelian or $\mathfrak{g} = \{\{a\}\} + \mathfrak{n}$, where $[\mathfrak{n}, \mathfrak{n}] = 0$ and $[a, n] = n$ for any $n \in \mathfrak{n}$.

We give here another proof in our words. First it is obvious that $[a, b] \in \{\{a, b\}\}$ for $a, b \in \mathfrak{g}$. Let $\mathfrak{r}$ be the radical and $\mathfrak{s}$ a maximal semi-simple subalgebra. If $\mathfrak{s} \neq 0$, then

$$\dim \mathfrak{g} = \mathrm{Nul}(\mathfrak{g}) \leqq \mathrm{Nul}(\mathfrak{s}) + \mathrm{Nul}(\mathfrak{r}) = 2 + \mathrm{Nul}(\mathfrak{r})$$

$$< \dim \mathfrak{s} + \dim \mathfrak{r} = \dim \mathfrak{g},$$

which is a contradiction. Hence $\mathfrak{s} = 0$, that is, $\mathfrak{g}$ is solvable. Let a, b be two elements which are linearly independent modulo $[\mathfrak{g}, \mathfrak{g}]$. Then $[a, b] \in \{\{a, b\}\} \cap [\mathfrak{g}, \mathfrak{g}]$ implies $[a, b]=0$. Hence $\mathfrak{g}$ splits into $\mathfrak{a}+[\mathfrak{g}, \mathfrak{g}]$, where $\mathfrak{a}$ is an abelian subalgebra. Since $[a, n] \in \{\{n\}\}$ for $a \in \mathfrak{a}$ and $n \in [\mathfrak{g}, \mathfrak{g}]$, $\operatorname{ad} \mathfrak{g}$ is splittable. Then under the same notations as in Theorem 6,

$$\dim \mathfrak{g}=\operatorname{Nul}(\mathfrak{g})=\operatorname{Max}\{\dim(\mathfrak{a}+\mathfrak{n}_0), M+1\}.$$

If $\dim(\mathfrak{a}+\mathfrak{n}_0)=\dim \mathfrak{g}$, $\mathfrak{g}$ is abelian. If $\dim \mathfrak{g}=M+1$, then $\dim \mathfrak{a}=1$ and $\mathfrak{g}=\{\{a\}\}+\mathfrak{n}_1$, where $[a, n_1]=\beta(a) n_1$ for $n_1 \in \mathfrak{n}_1$. We can assume that $\beta(a)=1$, and the proof is complete.

Now, let $\mathfrak{g}$ be a non-solvable Lie algebra whose adjoint representation is splittable. $\mathfrak{g}$ has a decomposition as in (4). $\mathfrak{n}$ is a completely reducible $(\mathfrak{s}+\mathfrak{a})$-space ([5]), and $\{n \in \mathfrak{n};\ [n, \mathfrak{s}]=0\}$ is $(\mathfrak{s}+\mathfrak{a})$-stable since $[\mathfrak{s}, \mathfrak{a}]=0$, whence $\mathfrak{n}$ is represented as follows:

$$\mathfrak{n}=[\mathfrak{n}, \mathfrak{n}]+\mathfrak{n}_0+\mathfrak{n}_1+\cdots+\mathfrak{n}_r+\tilde{\mathfrak{m}}_1+\cdots+\tilde{\mathfrak{m}}_s,$$

where

$$[\mathfrak{s}+\mathfrak{a}, \mathfrak{n}_0]=0, \qquad [\mathfrak{s}, \mathfrak{n}_i]=0 \qquad (i=1, \ldots, r),$$

$\tilde{\mathfrak{m}}_1, \ldots, \tilde{\mathfrak{m}}_s$ are non-trivial $(\mathfrak{s}+\mathfrak{a})$-irreducible spaces and $\mathfrak{n}_1, \ldots, \mathfrak{n}_r$ are weight spaces which belong to non-zero weights of $\mathfrak{a}$. We may assume $[\mathfrak{n}, \mathfrak{n}]=0$ by Lemma 4. Let $\mathfrak{m}_i$ be an irreducible $\mathfrak{s}$-space contained in $\tilde{\mathfrak{m}}_i$. For $a \in \mathfrak{a}$, $[a, \mathfrak{m}_i]$ is either $\mathfrak{s}$-isomorphic to $\mathfrak{m}_i$ or zero. Hence there exist $a_{i1}, \ldots, a_{ik_i}$ in $\mathfrak{a}$ such that

$$\tilde{\mathfrak{m}}_i=\mathfrak{m}_i+a_{i1} \mathfrak{m}_i+\cdots+a_{ik_i} \mathfrak{m}_i.$$

We divide the family of irreducible $\mathfrak{s}$-spaces $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ into classes as follows:

$$\{\mathfrak{m}_{11}, \ldots, \mathfrak{m}_{1t_1}\},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\{\mathfrak{m}_{u1}, \ldots, \mathfrak{m}_{ut_u}\},$$

where $\Sigma t_k=s$, $\mathfrak{m}_{ij} \cong \mathfrak{m}_{ij'}$ and $\mathfrak{m}_{ij} \ncong \mathfrak{m}_{i'j'}$ for $i \neq i'$. Let the $\mathfrak{s}$-space $\mathfrak{m}_{ij}$ have d_i different weights. We take an element h_0 in a Cartan subalgebra of $\mathfrak{s}$ which separates all the distinct roots of $\mathfrak{s}$ and all the distinct weights in $\mathfrak{m}_{11}+\cdots+\mathfrak{m}_{u1}$, and let h_0 and s_0 generate $\mathfrak{s}$. Let

$$K=\operatorname*{Max}_{1 \leqq i \leqq u}\left[\frac{t_i-1}{d_i}\right]+1.$$

Then, by the almost same argument as in the proof of Theorem 4, the perfect Lie algebra $\mathfrak{s}+\mathfrak{m}_1+\cdots+\mathfrak{m}_s$ is generated by the following $K+2$ elements:

$$h_0, s_0, m_1, \dots, m_K,$$

where m_i are suitable elements in $\mathfrak{m}_1+\cdots+\mathfrak{m}_s$. On the other hand let

$$p=\dim \mathfrak{a}, \qquad q=\dim \mathfrak{n}_0, \qquad M=\operatorname*{Max}_{i\geqq 1} \dim \mathfrak{n}_i,$$

$$N=\operatorname{Max}\{M+1,\ p+q,\ K+2\}.$$

Then, under the same notations as in the proof of Theorem 6, the solvable Lie algebra $\mathfrak{a}+\mathfrak{n}_0+\mathfrak{n}_1+\cdots+\mathfrak{n}_r$ is generated by the elements in (5). Moreover, it is also verified that the following N elements generate $\mathfrak{g}$:

$$\begin{array}{ll}
a_1 & +h_0 \\
a_2+\Sigma n_{i1} & +s_0 \\
a_3+\Sigma n_{i2} & +m_1 \\
\cdots\cdots\cdots\cdots & \\
a_p+\Sigma n_{ip-1} & +m_{p-2} \\
n_{01}+\Sigma n_{ip} & +m_{p-1} \\
\cdots\cdots\cdots\cdots & \\
n_{0q}+\Sigma n_{ip+q-1} & +m_{p+q-2} \\
\Sigma n_{ip+q} & +m_{p+q-1} \\
\cdots\cdots\cdots\cdots & \\
\Sigma n_{iN-1} & +m_{N-2},
\end{array}$$

where Σ means a sum on i from 1 to r, and we consider that $n_{ij}=0$ if $j>M$ and $m_i=0$ if $i>K$. The nullity of the solvable Lie algebra $\mathfrak{a}+\mathfrak{n}_0+\mathfrak{n}_1+\cdots+\mathfrak{n}_r$ is Max $\{M+1, p+q\}$ and it is not greater than Nul($\mathfrak{g}$) since $\mathfrak{s}+\tilde{\mathfrak{m}}_1+\cdots+\tilde{\mathfrak{m}}_s$ is an ideal of $\mathfrak{g}$. We can summarize our results in the following

THEOREM 7. *Let $\mathfrak{g}$ be a Lie algebra whose adjoint representation is splittable. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$, and $\mathfrak{n}$ the largest nilpotent ideal. Then $\mathfrak{g}$ is decomposed into a direct sum of subalgebras as follows:*

$$\mathfrak{g}=\mathfrak{s}+\mathfrak{r}, \qquad \mathfrak{r}=\mathfrak{a}+\mathfrak{n},$$

$$\mathfrak{n}=[\mathfrak{n},\mathfrak{n}]+\mathfrak{n}_0+\tilde{\mathfrak{n}}+\tilde{\mathfrak{m}}_{11}+\cdots+\tilde{\mathfrak{m}}_{1t_1}+\cdots+\tilde{\mathfrak{m}}_{u1}+\cdots+\tilde{\mathfrak{m}}_{ut_u},$$

where $\mathfrak{s}$ is a maximal semi-simple subalgebra, $\mathfrak{a}$ is an abelian subalgebra, $\mathrm{ad}_{\mathfrak{g}}\mathfrak{a}$ consists of semi-simple matrices, $\tilde{\mathfrak{m}}_{ij}$ is a non-trivial irreducible $(\mathfrak{s}+\mathfrak{a})$-subspace and

$$[\mathfrak{s}, \mathfrak{a}]=0, \qquad [\mathfrak{s}+\mathfrak{a}, \mathfrak{n}_0]=0,$$

$$[\mathfrak{s}, \tilde{\mathfrak{n}}]=0, \qquad [\mathfrak{a}, \tilde{\mathfrak{n}}]=\tilde{\mathfrak{n}}.$$

Moreover $\tilde{\mathfrak{m}}_{ij}$ *contains an* $\mathfrak{s}$*-irreducible space* $\mathfrak{m}_{ij}$, *and*

$$\mathfrak{m}_{ij}\cong\mathfrak{m}_{ij'}, \qquad \mathfrak{m}_{ij}\not\cong\mathfrak{m}_{i'j'} \qquad \textit{for } i\neq i'.$$

Let the $\mathfrak{s}$*-space* $\mathfrak{m}_{ij}$ *have* d_i *distinct weights. Then* $(\mathfrak{a}+\mathfrak{n}_0+\tilde{\mathfrak{n}}+[\mathfrak{n}, \mathfrak{n}])/[\mathfrak{n}, \mathfrak{n}]$ *is a solvable Lie algebra whose adjoint representation is splittable, and its nullity is given by Theorem* 6. *We denote it by P. Then*

$$P\leqq \mathrm{Nul}(\mathfrak{g})\leqq \mathrm{Max}\left\{P, \left[\frac{t_i-1}{d_i}\right]+3 \qquad (i=1,\dots, u)\right\}.$$

References

[1] James Bond, The structure of Lie algebras with large minimal generating sets, Math. Ph. D. thesis, Univ. Notre Dame, 1964.

[2] James Bond, Weak minimal generating set reduction theorems for associative and Lie algebras, Illinois J. Math., **10** (1966), 579–591.

[3] James Bond, Lie algebras of genus one and genus two, Pacific J. Math., **37** (1971), 591–616.

[4] C. Chevalley, A new kind of relationship between matrices, Amer. J. Math., **65** (1943), 521–531.

[5] C. Chevalley, Théorie des groupes de Lie, Tome III, Paris, 1955.

[6] M. Gotô, On algebraic Lie algebras, J. Math. Soc. Japan, **1** (1948), 29–45.

[7] N. Jacobson, Lie algebras, New York, 1962.

[8] M. S. Knebelman, Classification of Lie algebras, Ann. of Math., **36** (1935), 46–56.

[9] M. Kuranishi, On everywhere dense imbedding of free groups in Lie groups, Nagoya Math. J., **2** (1951), 63–71.

[10] A. Malcev, On solvable Lie algebras, Izvest. Akad. Nauk SSSR, Ser. Mat., **9** (1945), 329–356.

[11] E. I. Marshall, The genus of a perfect Lie algebra, J. London Math. Soc., **40** (1965), 276–282.

[12] Y. Matsushima, On algebraic Lie groups and algebras, J. Math. Soc. Japan, **1** (1948), 46–57.

[13] E. M. Patterson, Generators of linear algebras, Proc. London Math. Soc., (3) **7** (1957), 467–480.

[14] S. Tôgô, On splittable linear Lie algebras, J. Sci. Hiroshima Univ. Ser. A, **18** (1955), 289–306.

Faculty of Engineering,
Ibaraki University