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Nonoscillation in Linear Second Order Ordinary Differential Equations

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Let q be a continuous function from $[0, \infty)$ to $(0, \infty)$, and consider

$$u''+qu = 0 \tag{1}$$

on $[0, \infty)$. It has long been known that if (1) is nonoscillatory then

$$\int_0^\infty q(t)dt < \infty \tag{2}$$

and

$$\limsup_{t\to\infty} t\int_t^\infty q(s)ds \leq 1 \tag{3}$$

(see [1] and [2], also [3, Chapter 2]). From (3) it is clear that if (1) is nonoscillatory then

$$\int_0^\infty\left(\int_t^\infty q(s)ds\right)^2 dt < \infty . \tag{4}$$

Under the assumption that (1) is nonoscillatory we shall obtain a result which shows that (2), (3), and (4) can be extended to

$$\limsup_{t\to\infty} t\left(\int_t^\infty q(s)ds+\int_t^\infty\left(\int_s^\infty q(\xi)d\xi\right)^2 ds\right) \leq 1 , \tag{5}$$

$$\int_0^\infty q(t)\exp\left(\int_0^t sq(s)ds\right)dt < \infty , \tag{6}$$

and

$$\int_0^\infty\left(\int_t^\infty q(s)ds\right)^2 \exp\left(\int_0^t sq(s)ds\right)dt < \infty . \tag{7}$$

It is clear that (5) is an extension of (3), and since nonoscillation does not imply

$$\int_0^\infty tq(t)dt < \infty ,$$

(6) and (7) are extensions of (2) and (4).

THEOREM. *Suppose* (1) *is nonoscillatory, and let f be given on* $[0, \infty)$ *by*

$$f(t) = \int_t^\infty q(s)ds.$$

Then there is a number $a \geq 0$ and a sequence $\{w_k\}_{k=0}^\infty$, each value of which is a continuous nondecreasing function from $[a, \infty)$ to $(0, \infty)$, such that each of (i), (ii), (iii), (iv), (v), *and* (vi) *is true.*

(i) $w_1 = f$

(ii) *If k is a positive integer and $t \geq a$ then $w_k(t) \leq w_{k+1}(t)$.*

(iii) *If k is a nonnegative integer then*

$$\int_a^\infty f(s)w_k(s)ds < \infty$$

and if $k \geq 1$ then

$$w_{k+1}(t) = f(t) + \int_t^\infty f(s)w_k(s)ds$$

whenever $t \geq a$.

(iv) *If $t \geq a$ then $w_0(t) = \lim_{k\to\infty} w_k(t)$, and the convergence is uniform in compact subsets of $[a, \infty)$.*

(v) *If k is a nonnegative integer then*

$$\limsup_{t\to\infty} t w_k(t) \leq 1.$$

(vi) *If $t \geq a$ then*

$$w_0(t) = f(t) + \int_t^\infty f(s)w_0(s)ds.$$

Note that (5) is an immediate consequence of the theorem since w_2 is given by

$$w_2(t) = f(t) + \int_t^\infty f(s)^2 ds.$$

Also, from (vi) we have, if $t > a$,

$$w_0'(t) = f'(t) - f(t)w_0(t) = -q(t) - f(t)w_0(t), \tag{8}$$

$$w_0(a) - \int_a^t q(s)\exp\left(\int_a^s f(\xi)d\xi\right)ds = w_0(t)\exp\left(\int_a^t f(s)ds\right) > 0,$$

$$w_0(a) > \int_a^t q(s)\exp\left(\int_a^s f(\xi)d\xi\right)ds.$$

Thus

$$\int_a^\infty q(s)\exp\left(\int_a^s f(\xi)d\xi\right)ds < \infty.$$

Since (2) is already known, this last inequality is equivalent to

$$\int_0^\infty q(t)\exp\left(\int_0^t f(s)ds\right)dt < \infty. \tag{9}$$

But if $t \geq 0$ then

$$\int_0^t f(s)ds = t\int_t^\infty q(s)ds + \int_0^t sq(s)ds \geq \int_0^t sq(s)ds,$$

so (9) implies (6). Let z be given on $[a, \infty)$ by

$$z(t) = \int_t^\infty f(s)w_0(s)ds.$$

Now, if $t \geq a$,

$$z'(t) = -f(t)w_0(t) = -f(t)^2 - f(t)z(t). \tag{10}$$

But (10) can be used to obtain (7) in much the same way that (8) was used to obtain (6), and we shall not include the details.

Proof of the Theorem. Suppose (1) is nonoscillatory, and let u be an eventually positive solution of (1). Find $a \geq 0$ such that $u(t) > 0$ if $t \geq a$. Let v be given on $[a, \infty)$ by $v(t) = u'(t)/u(t)$. It is known [2] (see also [3, Chapter 2, § 5]) that

$$\int_a^\infty v(s)^2 ds < \infty \tag{11}$$

and

$$v(t) = f(t) + \int_t^\infty v(s)^2 ds \tag{12}$$

whenever $t \geq a$. It is clear from (12) that $v \geq f$ on $[a, \infty)$ and thus (11) and (12) say that

$$\int_a^\infty f(s)v(s)ds < \infty \tag{13}$$

and

$$v(t) \geq f(t) + \int_t^\infty f(s)v(s)ds \tag{14}$$

whenever $t \geq a$. Also, since u' is positive and decreasing on $[a, \infty)$,

$$\frac{u(t)}{tu'(t)} = \frac{u(a) + \int_a^t u'(s)ds}{tu'(t)} \geq \frac{u(a)+(t-a)u'(t)}{tu'(t)}$$

$$= 1 + \frac{u(a) - au'(t)}{tu'(t)} \geq 1 - a/t$$

whenever $t > a$, so

$$tv(t) = tu'(t)/u(t) \leq t/(t-a)$$

and

(15) $$\limsup_{t\to\infty} tv(t) \leq 1 .$$

Let $w_1(t) = f(t)$ if $t \geq a$, and let

$$w_2(t) = f(t) + \int_t^\infty f(s)w_1(s)ds$$

if $t \geq a$. If $t \geq a$ then

$$f(t) + \int_t^\infty f(s)w_1(s)ds \leq f(t) + \int_t^\infty f(s)v(s)ds \leq$$

so, $w_2(t) \leq v(t)$. Also, $w_2(t) \geq w_1(t)$ if $t \geq a$, and (15) says tha
Suppose n is a positive integer and $w_1, \dots, w_n$ are defined su
$w_n \leq v$ on $[a, \infty)$,

$$\int_a^\infty f(s)w_k(s)ds < \infty$$

whenever $1 \leq k \leq n$, and

$$w_k(t) = f(t) + \int_t^\infty f(s)w_{k-1}(s)ds$$

whenever $2 \leq k \leq n$ and $t \geq a$. Let w_{n+1} be given by

$$w_{n+1}(t) = f(t) + \int_t^\infty f(s)w_n(s)ds .$$

Now

$$w_n(t) = f(t) + \int_t^\infty f(s)w_{n-1}(s)ds \leq f(t) + \int_t^\infty f(s)w_n(s)ds$$

$$\int_a^\infty f(s)w_{n+1}(s)ds \leq \int_a^\infty f(s)v(s)ds < \infty .$$

We now see that there is a sequence $\{w_k\}_{k=1}^\infty$ satisfying (i), (ii), (iii), and

$$w_k(t) \leq v(t) \tag{16}$$

whenever $k \geq 1$ and $t \geq a$. Now (16) and (15) give (v). From (iii) we see that the family $\{w_1, w_2, \ldots\}$ is equicontinuous, so (16) says there is a subsequence $\{w_{k_j}\}_{j=1}^\infty$ with a locally uniform limit on $[a, \infty)$. This and (ii) say that $\{w_k\}_{k=1}^\infty$ has a locally uniform limit, say w_0, on $[a, \infty)$. Clearly $w_0 \leq v$, so

$$\int_a^\infty f(s)w_0(s)ds < \infty .$$

Now Lebesgue's Dominated Convergence Theorem yields

$$\int_t^\infty f(s)w_0(s)ds = \lim_{k\to\infty}\int_t^\infty f(s)w_k(s)ds$$

whenever $t \geq a$. But this implies (vi), and (iv) is clear from the above discussion, so the proof is complete.

References

[1] W. B. Fite, Concerning the zeros of the solutions of certain differential equations, Trans. Amer. Math. Soc., **19** (1917), 341–352.

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[3] C. A. Swanson, Comparison and oscillation theory of linear differential equations, Academic Press, New York, 1968.

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