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Classification of Free Involutions on Surfaces

Tohl ASOH

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§1. Introduction

A (continuous) map $\alpha: M \to M$ of a space M into itself is called an involution if $\alpha^2 = id$. We say that an involution α on M is equivalent to an involution α' on M' if there exists a homeomorphism $h: M \approx M'$ such that $\alpha' h = h\alpha$. The purpose of this note is to classify (fixed point) free invoultions on compact connected surfaces by this equivalence relation.

For an involution α on M, we obtain its orbit space M/α from M by identifying x with $\alpha(x)$ for $x \in M$. Then, we have the following

THEOREM 1.1. *Assume that X is a compact connected surface of genus g and the boundary ∂X consists of l components. Then the number n of equivalence classes of free involutions on connected surfaces, whose orbit spaces are homeomorphic to X, is given by*

$$n = \begin{cases} [l/2] + \min\{g, 1\} & \text{if } X \text{ is orientable,} \\ [l/2] + \min\{g, 3\} & \text{if } X \text{ is non-orientable.} \end{cases}$$

Now, we use the following notation:

(1.2) Let $\alpha: M \to M$ be an involution on a surface M of genus g such that the boundary ∂M has l components and the number of α invariant components is l_0 ($\leqq l$). Then, *the type* of such α is $(g, l, l_0, 1)$ if M is orientable and α preserves the orientation, $(g, l, l_0, -1)$ if M is orientable and α reverses the orientation, and $(g, l, l_0, 0)$ if M is non-orientable.

Then we have the following classification theorem of free involutions on compact connected surfaces.

THEOREM 1.3. (i) *There exists a free involution of type (g, l, l_0, ε) if and only if we have the following* (I), (II) *or* (III):

(I) $\varepsilon = 1$, $l_0 \geqq 0$ *is even*, $l \geqq l_0$ *is even and* $g + 2\min\{l_0, 1\} - l_0/2 \geqq 1$ *is odd*;

(II) $\varepsilon = -1$, $l_0 = 0$, $l \geqq 0$ *is even and* $g \geqq 0$;

(III) $\varepsilon = 0$, $l_0 \geqq 0$ *is even*, $l \geqq l_0$ *is even and* $g + 2\min\{l_0, 1\} - l_0 \geqq 2$ *is even.*

(ii) *There exist two free involutions of type (g, l, l_0, ε) up to equivalence if $\varepsilon = l_0 = 0$, $l \geqq 0$ is even and $g \geqq 4$ is even, and otherwise a free involution of type (g, l, l_0, ε) is unique up to equivalence.*

In §2, we construct some typical free involutions on compact connected surfaces, and prove Theorem 1.3 in Proposition 2.14, by assuming Theorem 1.1.

By using the well-known classification theorem of double coverings, we prove Theorem 1.1 for an orientable surface X in §3 and for a non-orientable surface X in §4.

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§2. Constructions of typical involutions

In this section, we construct some typical free involutions.

On the torus $S^1 \times S^1$, we have the free involution

(2.1) $$\alpha_0 : S_0 \longrightarrow S_0, \quad S_0 = S^1 \times S^1, \quad \alpha_0(z, z') = (-z, z').$$

We construct the free involution

(2.2) $$\alpha_i : S_i \longrightarrow S_i \qquad (i \geqq 1)$$

as follows: For $i=1$,

$$S_1 = S^1 \times I \ (I = [0, 1]), \quad \alpha_1(z, t) = (-z, t).$$

We consider the involution

$$\alpha_2 : S^1 \times S^1 \longrightarrow S^1 \times S^1, \quad \alpha_2(z, z') = (\bar{z}, \bar{z}') \ (^- \text{ denotes the conjugation}),$$

which has four fixed points $(\pm 1, \pm 1)$. Then, by removing the interiors of four disjoint α_2 invariant disks around these points, we obtain the desired surface S_2 and the restricted involution α_2. Inductively, we construct S_i and α_i from the disjoint union $S_{i-1} \cup S_2$ and the involutions α_{i-1} and α_2 by identifying one component of ∂S_{i-1} with one component of ∂S_2.

Also, on the 2-sphere S^2 and the torus $S^1 \times S^1$, we have the free involutions

(2.3) $$\beta : S^2 \longrightarrow S^2, \quad \beta(t_0, t_1, t_2) = (-t_0, -t_1, -t_2);$$

(2.4) $$\beta' : S^1 \times S^1 \longrightarrow S^1 \times S^1, \quad \beta'(z, z') = (-z, \bar{z}').$$

By (1.2) and the definitions, we have easily the following

LEMMA 2.5. *The types of these involutions* $\alpha_0, \alpha_i \ (i \geqq 1), \beta$ *and* β' *are* $(1, 0, 0, 1), (i-1, 2i, 2i, 1), (0, 0, 0, -1)$ *and* $(1, 0, 0, -1)$, *respectively.*

Now, let $\alpha : M \to M$ be a free involution on a compact connected surface M, and we consider the following constructions:

(2.6) Let $k \geqq 0$. By removing the interiors of $2k$ disjoint disks $D_1, \ldots, D_{2k}$

in $M-\partial M$ such that $\alpha D_{2i-1}=D_{2i}$ $(1\leqq i\leqq k)$, we obtain the surface $M_k=M-\bigcup_{i=1}^{2k}\operatorname{Int}D_i$ and the restricted free involution α: $M_k\to M_k$.

(2.7) Let $k\geqq j\geqq 1$. By attaching $2j$ Möbius bands on ∂D_i $(1\leqq i\leqq 2j)$ to the surface M_k of (2.6), we obtain the non-orientable surface $M_{k,j}^{\mathrm{o}}$ and the free involution α on $M_{k,j}^{\mathrm{o}}$ is induced from α on M_k.

(2.8) Consider the case that M is orientable, and let $k\geqq 2j\geqq 0$. Then, by attaching $2j$ handles on $\partial D_i\cup\partial D_{i+2}$ for $i\equiv 1, 2$ (4) and $1\leqq i\leqq 4j-2$ to the surface M_k of (2.6), we obtain the surface $M_{k,j}$ and the free involution α on $M_{k,j}$ is induced from α on M_k.

From these definitions, we have immediately the following

LEMMA 2.9. *Assume that* α *is a free involution on* M *of type* $(g, l, l_0, \pm 1)$.
(i) *The type of* α *on* $M_{k,j}^{\mathrm{o}}$ *of* (2.7) *is* $(2g+2j, l+2k-2j, l_0, 0)$ *for* $k\geqq j\geqq 1$.
(ii) *The type of* α *on* $M_{k,j}$ *of* (2.8) *is* $(g+2j, l+2k-4j, l_0, \pm 1)$ *for* $k\geqq 2j\geqq 0$.

By Lemmas 2.5 and 2.9, we have the following

LEMMA 2.10. (i) *By applying the construction* (2.8) *to the involution* α_i: $S_i\to S_i$ $(i\geqq 0)$ *of* (2.1), (2.2), *we obtain the involution*

$$\alpha_i \text{ on } (S_i)_{k,j} \qquad \text{for} \quad i\geqq 0,\ k\geqq 2j\geqq 0,$$

whose type is $(i+2j+1-2\min\{i, 1\}, 2i+2k-4j, 2i, 1)$.

(ii) *By applying* (2.8) *to the involutions* β: $S^2\to S^2$ *of* (2.3) *and* β': $S^1\times S^1\to S^1\times S^1$ *of* (2.4), *we obtain the involutions*

$$\beta \text{ on } (S^2)_{k,j} \text{ and } \beta' \text{ on } (S^1\times S^1)_{k,j} \qquad \text{for} \quad k\geqq 2j\geqq 0,$$

whose types are $(2j, 2k-4j, 0, -1)$ *and* $(2j+1, 2k-4j, 0, -1)$ *respectively.*

(iii) *By applying the constructions* (2.7) *to the involutions* α_i $(i\geqq 0)$ *of* (2.1), (2.2) *and* β *of* (2.3), *we obtain the involutions*

$$\alpha_i \text{ on } (S_i)_{k,j}^{\mathrm{o}} \text{ and } \beta \text{ on } (S^2)_{k,j}^{\mathrm{o}} \qquad \text{for} \quad i\geqq 0,\ k\geqq j\geqq 1,$$

whose types are $(2i+2j+2-4\min\{i, 1\}, 2i+2k-2j, 2i, 0)$ *and* $(2j, 2k-2j, 0, 0)$ *respectively.*

LEMMA 2.11. *In the above lemma, the involutions* α_0 *on* $(S_0)_{k,j}^{\mathrm{o}}$ *and* β *on* $(S^2)_{k+1,j+1}^{\mathrm{o}}$ *of* (iii) *have the same type* $(2j+2, 2k-2j, 0, 0)$ *for* $k\geqq j\geqq 1$, *but these are not equivalent.*

PROOF. Consider the involution α_0 on $M=(S_0)_{j,j}^{\mathrm{o}}$. From the definitions of (2.1) and (2.7), we see easily that the homology group $H_1(M; Z)$ is generated by cycles a, b (of $S_0=S^1\times S^1$) and $c_1,\dots, c_{2j}$ (of the attaching Möbius bands) with the relation $2(c_1+\cdots+c_{2j})=0$, and that the induced automorphism α_{0*}

of $H_1(M; Z)$ is given by

$$\alpha_{0*}(a) = a, \quad \alpha_{0*}(b) = b+2\sum_{i=1}^{j} c_{2i-1}, \quad \alpha_{0*}(c_{2i-1}) = c_{2i},$$
$$\alpha_{0*}(c_{2i}) = c_{2i-1}.$$

Therefore, for odd prime p, the induced automorphism α_{0*} of the vector space $H_1(M; Z_p)=Z_p\{a, b, c_1,\dots, c_{2j-1}\}$ is given by the above equalities with $c_{2j}=-\sum_{i=1}^{2j-1} c_i$, and so its determinant $\det \alpha_{0*}$ is equal to $(-1)^j$.

For the involution β on $N=(S^2)^0_{j+1,j+1}$, we see by the same way from the definitions of (2.3) and (2.7) that the induced automorphism β_* of the vector space $H_1(N; Z_p)=Z_p\{c_1,\dots, c_{2j+1}\}$ is given by

$$\beta_*(c_{2i-1})=c_{2i},\ \beta_*(c_{2i})=c_{2i-1}\ (i\leqq j),\ \beta_*(c_{2j+1})=-\sum_{i=1}^{2j+1} c_i,$$

and so its determinant $\det\beta_*$ is equal to $(-1)^{j+1}$.

By the above facts, there is not a homeomorphism $h\colon M\approx N$ satisfying $\beta_*h_*=h_*\alpha_{0*}$, and we have the desired result for $k=j$.

Assume that there is a homeomorphism $h\colon (S_0)^0_{k,j}\approx(S^2)^0_{k+1,j+1}$ such that $\beta h=h\alpha_0$ for $k>j$. Then, by attaching $2k-2j$ disks, we can extend h to a homeomorphism $h\colon (S_0)^0_{j,j}\approx(S^2)^0_{j+1,j+1}$ such that $\beta h=h\alpha_0$, which contradicts the above result. *q.e.d.*

For the orbit space of a free involution, we have

Lemma 2.12. *Let α be a free involution on M of type (g, l, l_0, ε). Then the boundary of the orbit space M/α consists of $(l+l_0)/2$ components, and M/α is orientable for $\varepsilon=1$ and non-orientable for $\varepsilon=-1$, 0. Furthermore, the genus of M/α is equal to $(2+2g-l_0)/4$, $(2+2g-l_0)/2$ and $(2+g-l_0)/2$, according to $\varepsilon=1$, -1 and 0.*

Proof. The first half is clear. Since $M\to M/\alpha$ is a double covering, we have the equality $2\chi(M/\alpha)=\chi(M)$ of the Euler characteristics. By this equality, we see easily the second half of the lemma. *q.e.d.*

By this lemma, we have immediately the following

Lemma 2.13. (i) *The orbit space $(S_i)_{k,j}/\alpha_i$ of the involution α_i of Lemma* 2.10 (i) *is orientable, its genus is $j+1-\min\{i, 1\}$, and its boundary consists of $2i+k-2j$ components.*

(ii) *The orbit spaces $(S^2)_{k,j}/\beta$ and $(S^1\times S^1)_{k,j}/\beta'$ of Lemma* 2.10 (ii) *are non-orientable, their genuses are $2j+1$ and $2j+2$, respectively, and their boundaries consist of $k-2j$ components.*

(iii) *The orbit spaces $(S_i)^0_{k,j}/\alpha_i$ and $(S^2)^0_{k,j}/\beta$ of Lemma* 2.10 (iii) *are non-orientable, their genuses are $j+2-2\min\{i, 1\}$ and $j+1$, and their boundaries* *consist of* $2i+k-j$ *and* $k-j$ *components, respectively.*

Now, we are ready to prove the following

PROPOSITION 2.14. *Theorem* 1.3 *holds if Theorem* 1.1 *is valid.*

PROOF. By Lemmas 2.10 and 2.11, the free involutions in Lemma 2.10 show the existence in Theorem 1.3. Also by Lemma 2.13, we see easily that there are n equivalence classes of these involutions whose orbit spaces are homeomorphic to X, where n is the number given in the end of Theorem 1.1. Therefore we have Theorem 1.3, if Theorem 1.1 is valid. *q.e.d.*

§3. Orientable surfaces

Let α be a free involution on a connected surface M, whose orbit space M/α is homeomorphic to a given compact connected surface X. Then, we have a double covering $M \to M/\alpha \approx X$ and its Stiefel-Whitney class $w_1(\alpha) \neq 0$ in the cohomology group $H^1(X; Z_2)$. Furthermore, if α' is equivalent to α, then we see easily that $w_1(\alpha') \sim w_1(\alpha)$ by the definition of the equivalence of involutions, where $\sim$ is the equivalence relation defined by

(3.1) $x \sim y$ in $H^1(X; Z_2)$ if $x = t^*(y)$ by the induced automorphism $t^*: H^1(X; Z_2) \to H^1(X; Z_2)$ of some homeomorphism $t: X \overset{\approx}{\to} X$.

Therefore, by the classification theorem of double coverings, we have the following

LEMMA 3.2. *The equivalence classes of free involutions on connected surfaces, whose orbit spaces are homeomorphic to* X*, are in one-to-one correspondence with the equivalence classes of* $H^1(X; Z_2) - \{0\}$ *under* $\sim$ *of* (3.1).

Now assume that X is a compact connected orientable surface of genus g and the boundary ∂X has l components. Then X has a cellular decomposition induced from the convex region with the boundary polygon

$$a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} w_1 d_1 w_1^{-1} \dots w_l d_l w_l^{-1} .$$

Therefore, the (co)homology groups of X are given by

$$(3.3) \quad \begin{aligned} H_1(X; Z_2) &= Z_2\{a_1, b_1, \dots, a_g, b_g, d_1, \dots, d_{l-1}\}, \quad d_1 + \cdots + d_l = 0; \\ H^1(X; Z_2) &= Z_2\{a_1^\#, b_1^\#, \dots, a_g^\#, b_g^\#, d_1^\# - d_l^\#, \dots, d_{l-1}^\# - d_l^\#\} . \end{aligned}$$

To determine the equivalence relation $\sim$ of (3.1) on $H^1(X; Z_2)$, we use the following results of P. A. Smith [2, (8.1)].

(3.4) There exist homeomorphisms t_{1i}, t_{2i} $(1 \leqq i \leqq g)$, t_{3ij}, t_{4ij} $(1 \leqq i, j \leqq g, i \neq j)$, t_k $(1 \leqq k \leqq l-1)$ of X onto itself, and the induced automorphisms of

$H_1(X; Z_2)$ satisfy the following conditions:

$$t_{1i*}(a_i) = a_i + b_i;$$

$$t_{2i*}(b_i) = a_i + b_i;$$

$$t_{3ij*}(a_i) = a_i + a_j, \quad t_{3ij*}(b_j) = b_i + b_j;$$

$$t_{4ij*}(a_i) = a_i + b_j, \quad t_{4ij*}(a_j) = a_j + b_i;$$

$$t_{k*}(a_1) = a_1 + d_k;$$

where these conditions mean that the remaining basis elements in (3.3) are unchanged.

The following lemma is clear, since t^* is given by the transposed matrix of t_*.

LEMMA 3.5. *These homeomorphisms induce the automorphisms of $H^1(X; Z_2)$ given by*

$$t_{1i}^*(b_i^\#) = a_i^\# + b_i^\# ;$$

$$t_{2i}^*(a_i^\#) = a_i^\# + b_i^\# ;$$

$$t_{3ij}^*(a_j^\#) = a_i^\# + a_j^\# , \; t_{3ij}^*(b_i^\#) = b_i^\# + b_j^\# ;$$

$$t_{4ij}^*(b_j^\#) = a_i^\# + b_j^\# , \; t_{4ij}^*(b_i^\#) = a_j^\# + b_i^\# ;$$

$$t_k^*(d_k^\# - d_l^\#) = a_1^\# + (d_k^\# - d_l^\#) ;$$

where these conditions mean that the remaining basis elements in (3.3) *are unchanged.*

Also, we have the following by [2, (8.1)].

LEMMA 3.6. *For any permutation τ of $\{1,\dots, l\}$, there exists a homeomorphism t_τ of X onto itself, satisfying*

$$t_{\tau*}(a_i) = a_i, \quad t_{\tau*}(b_i) = b_i, \quad t_{\tau*}(d_k) = d_{\tau(k)} \quad \text{in} \quad H_1(X; Z_2).$$

Thus we have $t_\tau^*(a_i^\#) = a_i^\#$, $t_\tau^*(b_i^\#) = b_i^\#$ and

$$t_\tau^*(d_k^\# - d_l^\#) = (d_{\tau^{-1}(k)}^\# - d_l^\#) - (d_{\tau^{-1}(l)}^\# - d_l^\#) \text{ in } H^1(X; Z_2).$$

By using these results, we prove the following lemmas for the equivalence relation $\sim$ of (3.1) on $H^1(X; Z_2)$.

LEMMA 3.7. (i) $a \sim a_1^\#$ *for any* $a \in A - \{0\}$.
(ii) $a + d \sim d$ *for any* $a \in A$ *and* $d \in D - \{0\}$.

(iii) *Any* $d \in D-\{0\}$ *is not equivalent to* $a_1^\sharp$.
Here $A=Z_2\{a_1^\sharp, b_1^\sharp,\dots, a_g^\sharp, b_g^\sharp\}$, $D=Z_2\{d_1^\sharp-d_l^\sharp,\dots, d_{l-1}^\sharp-d_l^\sharp\}\subset H^1(X; Z_2)$.

PROOF. (i) Let $a=a_{i_1}^\sharp+\cdots+a_{i_p}^\sharp+b_{j_1}^\sharp+\cdots+b_{j_q}^\sharp$, $i_1<\cdots<i_p$, $j_1<\cdots<j_q$.
If $q=0$ and $i_1=1$, then $a\sim a_1^\sharp$ by the composition of t_{3i1}^* $(i=i_2,\dots, i_p)$ of Lemma 3.5. If $q=0$ and $i_1>1$, then $a\sim a_1^\sharp+a$ by $t_{31i_1}^*$.

Assume $q\geqq 1$. By applying t_{1j}^* of Lemma 3.5, we may assume that $\{i_1,\dots, i_p\}$ and $\{j_1,\dots, j_q\}$ are disjoint. Then, by the composition of t_{4ij}^* $(i=i_1,\dots, i_p;\ j=j_1)$ we see that $a\sim b_{j_1}^\sharp+\cdots+b_{j_q}^\sharp$, which is equivalent to $b_1^\sharp$ by the similar proof to the above. Also, $b_1^\sharp\sim a_1^\sharp$ by using t_{11}^* and t_{21}^*.

(ii) If $a\neq 0$, we see that $a+d\sim a_1^\sharp+d$ by the same proof as in (i). Also, $a_1^\sharp+d\sim d$ by t_k^*.

(iii) Any homeomorphism $t: X\approx X$ satisfies $t(\partial X)=\partial X$, and so $t^*(A)\subset A$ since $A=\operatorname{Im}\{H^1(X, \partial X; Z_2)\to H^1(X; Z_2)\}$. This shows (iii). *q.e.d.*

LEMMA 3.8. (i) $(d_{k_1}^\sharp-d_l^\sharp)+\cdots+(d_{k_n}^\sharp-d_l^\sharp)\sim(d_1^\sharp-d_l^\sharp)+\cdots+(d_n^\sharp-d_l^\sharp)$, *for any* $1\leqq k_1<\cdots<k_n<l$.

(ii) *For integers n and m such that* $l>n>m\geqq 1$,

$$(d_1^\sharp-d_l^\sharp)+\cdots+(d_n^\sharp-d_l^\sharp)\sim(d_1^\sharp-d_l^\sharp)+\cdots+(d_m^\sharp-d_l^\sharp)$$

if and only if $n=m+1$ *is even.*

PROOF. (i) It is evident by considering t_τ^* of Lemma 3.6.

(ii) (Sufficiency) By t_τ of Lemma 3.6 for the transposition $\tau=(n, l)$, we see that

$$(d_1^\sharp-d_l^\sharp)+\cdots+(d_n^\sharp-d_l^\sharp)\sim(d_1^\sharp-d_l^\sharp)+\cdots+(d_{n-1}^\sharp-d_l^\sharp)+n(d_n^\sharp-d_l^\sharp),$$

which is equal to $(d_1^\sharp-d_l^\sharp)+\cdots+(d_m^\sharp-d_l^\sharp)$ since $n=m+1$ is even.

(Necessity) Any homeomorphism $t: X\approx X$ induces a permutation of components of ∂X, and so $t^*(d_{\tau(k)})=d_k$ in $H_1(X; Z_2)$ for some permutation τ of $\{1,\dots, l\}$. Then we have

$$t^*(d_k^\sharp-d_l^\sharp)=x_k+(d_{\tau(k)}^\sharp-d_l^\sharp)-(d_{\tau(l)}^\sharp-d_l^\sharp) \text{ in } H^1(X; Z_2)$$

for some $x_k\in A$, and so

$$t^*(\textstyle\sum_{k=1}^m(d_k^\sharp-d_l^\sharp))=\sum_{k=1}^m x_k+\sum_{k=1}^m(d_{\tau(k)}^\sharp-d_l^\sharp)-m(d_{\tau(l)}^\sharp-d_l^\sharp)\,.$$

If this is equal to $\sum_{k=1}^n(d_k^\sharp-d_l^\sharp)$ $(l>n>m)$, we have $m+1=n$ and m is odd by (3.3). *q.e.d.*

Now, we obtain the first half of Theorem 1.1.

PROPOSITION 3.9. *Theorem* 1.1 *holds for an orientable surface X.*

Proof. By Lemmas 3.7 and 3.8, the equivalence classes of $H^1(X; Z_2) - \{0\}$ are represented by $a_1^\#$ (when $g>0$) and $\sum_{k=1}^{2n+1}(d_k^\# - d_l^\#)$ for $1 \leqq 2n+1 < l$. Therefore, we have the desired result by Lemma 3.2. *q.e.d.*

§4. Non-orientable surfaces

In this section, we assume that X is a compact connected non-orientable surface of genus g and the boundary ∂X has l components.

Then, X has a cellular decomposition induced from the convex region with the boundary polygon

$$c_1c_1 \dots c_gc_gw_1d_1w_1^{-1} \dots w_ld_lw_l^{-1},$$

and the (co) homology groups of X are given by

$$(4.1) \quad \begin{aligned} H_1(X; Z_2) &= Z_2\{c_1,\dots, c_g, d_1,\dots, d_{l-1}\}, \quad d_1 + \cdots + d_l = 0; \\ H^1(X; Z_2) &= Z_2\{c_1^\#,\dots, c_g^\#, d_1^\# - d_l^\#,\dots, d_{l-1}^\# - d_l^\#\}. \end{aligned}$$

To determine the equivalence relation $\sim$ of (3.1) on $H^1(X; Z_2)$, we construct some homeomorphisms of X onto itself as follows.

Lemma 4.2. *For each permutation σ of $\{1,\dots, g\}$, τ of $\{1,\dots, l\}$ and $1 \leqq k < l$, there exist homeomorphisms s_σ, t_τ and t_k of X onto itself such that*

$$s_{\sigma*}(c_i) = c_{\sigma(i)} \qquad (1 \leqq i \leqq g);$$

$$t_{\tau*}(d_k) = d_{\tau(k)} \qquad (1 \leqq k \leqq l);$$

$$t_{k*}(c_1) = c_1 + d_k;$$

and the remaining basis elements in $H_1(X; Z_2)$ of (4.1) are unchanged.

Proof. To prove the existence of s_σ, we may assume that σ is a transposition $(i, i+1)$. Write the surface symbol of X by $aabbP$ $(a=c_i,\ b=c_{i+1})$. By the equivalences

$$aabbP \sim a_1b^{-1}b^{-1}a_1P\ (a_1 = abb) \sim ba_2b^{-1}a_2P\ (a_2 = b^{-1}a_1) \sim bba_3a_3P$$
$$(a_3 = b^{-1}a_2)$$

of surface symbols, we have a homeomorphism $s: X \approx X$ such that $s_*(a)=b$, $s_*(b)=a_3$ and the other basis cycles are unchanged. In $H_1(X; Z_2)$, we have $a_1=a+b+b=a$, $a_2=b+a_1$, $a_3=b+a_2$ and so $a_3=a$ as desired.

By the same way, we can prove the existences of t_τ and t_k, by using the equivalences

$$wdw^{-1}vev^{-1}P \sim vev^{-1}w_1^{-1}dw_1P \ (w_1 = w^{-1}vev^{-1}),$$

$$ccwdw^{-1}P \sim c_1wd^{-1}w^{-1}c_1P \ (c_1 = cwdw^{-1}) \sim c_1c_1w_1^{-1}d^{-1}w_1P$$

$$(w_1 = w^{-1}c_1)$$

of surface symbols. *q.e.d.*

LEMMA 4.3. *Assume $g \geqq 4$. Then there exists a homeomorphism $s: X \approx X$ such that $s_*: H_1(X; Z_2) \to H_1(X; Z_2)$ satisfies*

$$s_*(c_1) = c_1+c_2+c_4, \quad s_*(c_2) = c_1+c_2+c_3,$$

$$s_*(c_3) = c_2+c_3+c_4, \quad s_*(c_4) = c_1+c_3+c_4,$$

$$s_*(c_i) = c_i \ (i > 4), \quad s_*(d_k) = d_k.$$

PROOF. Consider the following equivalences of surface symbols:

$$c_1c_1c_2c_2c_3c_3c_4c_4P \sim c'_1c_2^{-1}c'_1c_2c_3c'_4c_3^{-1}c'_4P \qquad (c'_1 = c_1c_2,\ c'_4 = c_3c_4)$$

$$\sim c_2c''_1c_3^{-1}c_2^{-1}c''_1c'_4c_3^{-1}c'_4P \qquad (c''_1 = c_2^{-1}c'_1c_2c_3)$$

$$\sim c_2c''_4c''^{-1}_1c_2c_3c''^{-1}_1c''_4c_3P \qquad (c''_4 = c''_1c_3^{-1}c_2^{-1}c''_1c'_4c_3^{-1})$$

$$\sim c'_2c'_2c''_1c''^{-1}_4c''^{-1}_4c''_1c'_3c'_3P \qquad (c'_2 = c_2c''_4c''^{-1}_1,\ c'_3 = c''^{-1}_1c''_4c_3)$$

$$\sim c'_2c'_2c'''_1c'''_1c''_4c''_4c'_3c'_3P \qquad (c'''_1 = c''_1c''^{-1}_4c''^{-1}_4).$$

Then, $c'_2 = c_1+c_2+c_4$, $c'''_1 = c_1+c_2+c_3$, $c''_4 = c_2+c_3+c_4$ and $c'_3 = c_1+c_3+c_4$ in $H_1(X; Z_2)$, and we have the desired result in the same way as the proof of the above lemma. *q.e.d.*

By using these lemmas, we have the following lemmas for the equivalence $\sim$ of (3.1) in $H^1(X; Z_2)$, where

$$C = Z_2\{c_1^\#,\dots, c_g^\#\},\ D = Z_2\{d_1^\# - d_l^\#,\dots, d_{l-1}^\# - d_l^\#\}\ .$$

LEMMA 4.4. *Any element $c \in C - \{0\}$ is equivalent to*

$$c_1^\#,\ c_1^\# + c_2^\# \ \text{if } g \geqq 2, \text{ or } c_1^\# + \cdots + c_g^\# \ \text{if } g \geqq 3,$$

and these are not equivalent to each other.

PROOF. We have

$$s_\sigma^*(c_i^\#) = c_{\sigma^{-1}(i)}^\#(1 \leqq i \leqq g),\ s_\sigma^*(d_k^\# - d_l^\#) = d_k^\# - d_l^\#(1 \leqq k < l)\ ,$$

$$s^*(c_1^\#) = c_1^\# + c_2^\# + c_4^\#, \quad s^*(c_2^\#) = c_1^\# + c_2^\# + c_3^\#\ ,$$

$$s^*(c_3^\#) = c_2^\# + c_3^\# + c_4^\#,\ s^*(c_4^\#) = c_1^\# + c_3^\# + c_4^\#,$$

$$s^*(c_i^\#) = c_i^\#(i > 4),\ s^*(d_k^\# - d_l^\#) = d_k^\# - d_l^\#$$

for s_σ of Lemma 4.2 and s of Lemma 4.3. By using these homeomorphisms, we see easily that

$$c_{i_1}^\# + \cdots + c_{i_p}^\# \sim c_{j_1}^\# + \cdots + c_{j_q}^\# \qquad (1 \leqq i_1 < \cdots < i_p \leqq g,\ 1 \leqq j_1 < \cdots < j_q \leqq g)$$

if $p<g$, $q<g$ and $p \equiv q$ (2). This shows the first half.

Now, we can consider $c_i^\# \in H^1(X, \partial X; Z_2)$ and the cup product $c_i^\# c_j^\#$ is equal to $\delta_{ij}[X]$, where $[X] \in H^2(X, \partial X; Z_2)$ is the fundamental class (cf. [1, §75.2, p. 270]).

Assume $c_1^\# + c_2^\# \sim c_1^\#$. Then $c_1^\# + c_2^\# = t^*(c_1^\#)$ in $H^1(X, \partial X; Z_2)$ for some homeomorphism $t: (X, \partial X) \to (X, \partial X)$, and so

$$0 = (c_1^\# + c_2^\#)(c_1^\# + c_2^\#) = t^*(c_1^\# c_1^\#) = t^*[X] \neq 0 \qquad \text{in } H^2(X, \partial X; Z_2),$$

which is a contradiction.

We notice that $c_1 + \cdots + c_g \in H_1(X, \partial X; Z)$ is the unique element of order 2. Therefore, for any homeomorphism $t: X \to X$, $t_*(c_1 + \cdots + c_g) = c_1 + \cdots + c_g$ in $H_1(X, \partial X; Z)$, and so in $H_1(X, \partial X; Z_2)$. This shows that $t^*(c_1^\# + \cdots + c_g^\#) = c_1^\# + \cdots + c_g^\#$ by the Poincaré duality. Thus, $c_1^\# + \cdots + c_p^\# = t^*(c_1^\# + \cdots + c_g^\#)$ implies $p = g$. *q.e.d.*

Lemma 4.5. (i) *$c + d \sim d$ for any $c \in C$ and $d \in D - \{0\}$.*
(ii) *Any $c \in C - \{0\}$ and $d \in D - \{0\}$ are not equivalent.*

Proof. (i) is proved by using t_k and s_σ of Lemma 4.2. (ii) is proved by the same way as Lemma 3.7 (iii). *q.e.d.*

By the exactly same proof, we have

Lemma 4.6. *Lemma 3.8 holds also for a non-orientable surface X.*

By Lemmas 4.4–4.6 and 3.2, we have immediately

Proposition 4.7. *Theorem 1.1 holds for a non-orientable surface X.*

Combining Propositions 3.9 and 4.7, we have completely shown Theorem 1.1, and so Theorem 1.3 by Proposition 2.14.

References

[1] H. Seifert und W. Threlfall: *Lehrbuch der Topologie*, B. G. Teubner, 1934, (Chelsea Publ. Co. (1963)).
[2] P. A. Smith: *Abelian actions on 2-manifolds*, Michigan Math. J., **14** (1967), 257–275.

Department of Mathematics,
Faculty of Science,
Hiroshima University