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On the Oscillation of Solutions of Nonlinear Functional Differential Equations

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1. Introduction

This paper is concerned with nonlinear functional differential equations with deviating arguments of the form

$$\text{(A)}\qquad x^{(n)}(t)+f(t,\ x<\boldsymbol{g}_0(t)>,\ [x']^2<\boldsymbol{g}_1(t)>,\dots,\ [x^{(n-1)}]^2<\boldsymbol{g}_{n-1}(t)>)=0,$$

where $n\geqq 2$, $\boldsymbol{g}_i(t)=(g_{i1}(t),\dots,g_{im_i}(t))$, $i=0,1,\dots,n-1$,

$$x<\boldsymbol{g}_0(t)>\ =(x(g_{01}(t)),\dots,x(g_{0m_0}(t))),$$

and

$$[x^{(i)}]^2<\boldsymbol{g}_i(t)>\ =([x^{(i)}(g_{i1}(t))]^2,\dots,[x^{(i)}(g_{im_i}(t))]^2),\qquad i=1,\dots,n-1.$$

The conditions we always assume for f, g_{ij} are as follows:

(a) $f(t,\boldsymbol{y}_0,\boldsymbol{y}_1,\dots,\boldsymbol{y}_{n-1})$ is continuous on the set $[t_0,\infty)\times E$, where

$$E=R^{m_0}\times R_+^{m_1}\times\cdots\times R_+^{m_{n-1}}\quad (R=(-\infty,\infty),\ R_+=[0,\infty)),$$

$$f(t,\boldsymbol{y}_0,\boldsymbol{y}_1,\dots,\boldsymbol{y}_{n-1})>0 \text{ if } \boldsymbol{y}_0>0, \text{ and}$$

$$f(t,-\boldsymbol{y}_0,\boldsymbol{y}_1,\dots,\boldsymbol{y}_{n-1})=-f(t,\boldsymbol{y}_0,\boldsymbol{y}_1,\dots,\boldsymbol{y}_{n-1}).$$

(b) $g_{ij}(t)$, $j=1,\dots,m_i$, $i=0,1,\dots,n-1$, are continuous on $[t_0,\infty)$ and $\lim_{t\to\infty} g_{ij}(t)=\infty$.

In what follows we restrict our discussion to those solutions $x(t)$ of equation (A) which exist on some half-line $[T_x,\infty)$ and satisfy

$$\sup\{|x(t)|: t\geqq T\}>0$$

for every $t\geqq T_x$. Such a solution is called oscillatory if the set of its zeros is not bounded above. Otherwise the solution is called nonoscillatory. A nonoscillatory solution is said to be strongly nomotone if it tends monotonically to zero as $t\to\infty$ together with its first $n-1$ derivatives.

The objective of this paper is to study the oscillatory behavior of solutions of equation (A) with specific nonlinearity defined below. We provide conditions under which every (bounded) solution of (A) is oscillatory if n is even, and is either oscillatory or strongly monotone if n is odd. The results obtained prove to apply not only to the case where (A) is a *retarded* equation but also to the case where (A) is an *advanced* equation, and include recent results of Onose [4] and Grammatikopoulos [1] for retarded differential equations of the form less general than (A).

DEFINITION. (i) Equation (A) is called *superlinear* if there are nonnegative numbers $\rho_1,\dots,\rho_{m_0}$ with $\rho_1+\cdots+\rho_{m_0}=1$ such that for each $t\geqq t_0$ the function

$$(\prod_{j=1}^{m_0} y_{0j}^{-\rho_j})f(t, \boldsymbol{y}_0, \boldsymbol{y}_1,\dots, \boldsymbol{y}_{n-1})$$

is nondecreasing in $(\boldsymbol{y}_0, \boldsymbol{y}_1,\dots, \boldsymbol{y}_{n-1})$ on $E_+=(0, \infty)^{m_0}\times R_+^{m_1}\times\cdots\times R_+^{m_{n-1}}$.

(ii) Equation (A) is called *strongly superlinear* if there are nonnegative numbers $\sigma_1,\dots,\sigma_{m_0}$ with $\sigma_1+\cdots+\sigma_{m_0}>1$ such that for each $t\geqq t_0$ the function

$$(\prod_{j=1}^{m_0} y_{0j}^{-\sigma_j})f(t, \boldsymbol{y}_0, \boldsymbol{y}_1,\dots, \boldsymbol{y}_{n-1})$$

is nondecreasing in $(\boldsymbol{y}_0, \boldsymbol{y}_1,\dots, \boldsymbol{y}_{n-1})$ on E_+.

(iii) Equation (A) is called *sublinear* if there are nonnegative numbers $\rho_1,\dots,\rho_{m_0}$ with $\rho_1+\cdots+\rho_{m_0}=1$ such that for each $t\geqq t_0$ the function defined in (i) is nonincreasing in $(\boldsymbol{y}_0, \boldsymbol{y}_1,\dots, \boldsymbol{y}_{n-1})$ on E_+.

(iv) Equation (A) is called *strongly sublinear* if there are nonnegative numbers $\tau_1,\dots,\tau_{m_0}$ with $\tau_1+\cdots+\tau_{m_0}<1$ such that for each $t\geqq t_0$ the function

$$(\prod_{j=1}^{m_0} y_{0j}^{-\tau_j})f(t, \boldsymbol{y}_0, \boldsymbol{y}_1,\dots, \boldsymbol{y}_{n-1})$$

is nonincreasing in $(\boldsymbol{y}_0, \boldsymbol{y}_1,\dots, \boldsymbol{y}_{n-1})$ on E_+.

REMARK. Inequality between vectors $\boldsymbol{y}=(y_1,\dots, y_m)$, $\boldsymbol{z}=(z_1,\dots, z_m)$ in R^m is defined as

$$\boldsymbol{y} < \boldsymbol{z} \qquad \text{equivalent to} \qquad y_j < z_j \qquad \text{for} \quad j = 1,\dots, m,$$

(and similarly for the symbols $\leqq$, $>$, $\geqq$). A function $h(\boldsymbol{y})$ defined on a set of R^m is said to be nondecreasing in $\boldsymbol{y}$ [or nonincreasing in $\boldsymbol{y}$] if $h(\boldsymbol{y})\leqq h(\boldsymbol{z})$ [or $h(\boldsymbol{y})\geqq h(\boldsymbol{z})$] for $\boldsymbol{y}\leqq\boldsymbol{z}$. We shall use the symbols $\mathbf{0}$ and $\mathbf{1}$ to denote the vectors $(0,\dots, 0)$ and $(1,\dots, 1)$ in R^m, respectively. We define: $\operatorname{sgn}\boldsymbol{y}=+1$ for $\boldsymbol{y}>\mathbf{0}$, and $\operatorname{sgn}\boldsymbol{y}=-1$ for $\boldsymbol{y}<\mathbf{0}$.

REMARK. The super- and sublinearity defined above extend the corresponding notions introduced by Onose [5] and Grammatikopoulos [1].

For closely related results we refer to Kusano [3], Sficas and Staikos [6] and Staikos and Sficas [7].

2. Oscillation of bounded solutions

In this section we study the oscillatory behavior of bounded solutions of equation (A) which is either superlinear or sublinear.

THEOREM 1. *Let equation* (A) *be superlinear. Suppose that*

$$\int^{\infty} t^{n-1}|f(t, \mu\mathbf{1}, \mathbf{0},\dots, \mathbf{0})|dt = \infty \tag{1}$$

for any $\mu \neq 0$. *Then, for n even, every bounded solution of* (A) *is oscillatory, while, for n odd, every bounded solution of* (A) *is either oscillatory or strongly monotone.*

THEOREM 2. *Let equation* (A) *be sublinear. Suppose that*

$$\int^{\infty} t^{n-1}\, |f(t, \mu_1\mathbf{1}, \mu_2^2\mathbf{1},\dots, \mu_2^2\mathbf{1})|dt = \infty \tag{2}$$

for any μ_1, μ_2 *with* $|\mu_1|>|\mu_2|$. *Then, for n even, every bounded solution of* (A) *is oscillatory, while, for n odd, every bounded solution of* (A) *is either oscillatory or strongly monotone.*

PROOF OF THEOREMS 1 AND 2. Let $x(t)$ be a bounded nonoscillatory solution of (A). Without loss of generality we may suppose that $x(t)$ is eventually positive. From (A) there is $t_1 \geqq t_0$ such that $x^{(n)}(t)<0$ for $t \geqq t_1$. Since $x(t)$ is bounded and positive, it follows that

$$(-1)^{k+1}x^{(n-k)}(t) > 0,\ t \geqq t_1, \quad \text{and} \quad \lim_{t\to\infty} x^{(n-k)}(t) = 0,\ k = 1,\dots, n-1. \tag{3}$$

In view of (3) $x'(t)$ is of fixed sign for $t \geqq t_1$, so that the limit $\lim_{t\to\infty} x(t)=x(\infty)\geqq 0$ exists and is finite. Observe that $x(\infty)>0$ if n is even, and that $x(\infty)=0$ may occur only if n is odd. We assume $x(\infty)>0$ and derive a contradiction. If $x(\infty)>0$, then there are positive numbers c, d $(c<d)$ and $t_2 \geqq t_1$ such that for $t \geqq t_2$

$$c < x(g_{0j}(t)) < d, \qquad j = 1,\dots, m_0, \tag{4}$$

$$|x^{(i)}(g_{ij}(t))| < c, \qquad j = 1,\dots, m_i, \quad i = 1,\dots, n-1. \tag{5}$$

Let (A) be superlinear. Then, using (4), (5) and the superlinearity, we have

$$f(t,\, x<\mathbf{g}_0(t)>,\, [x']^2<\mathbf{g}_1(t)>,\dots,\, [x^{(n-1)}]^2<\mathbf{g}_{n-1}(t)>)$$

$$= \prod_{j=1}^{m_0} [x(g_{0j}(t))]^{\rho_j} \cdot \prod_{j=1}^{m_0} [x(g_{0j}(t))]^{-\rho_j} f(t, x<\boldsymbol{g}_0(t)>,$$

$$[x']^2 <\boldsymbol{g}_1(t)>, \dots, [x^{(n-1)}]^2 <\boldsymbol{g}_{n-1}(t)>)$$

$$\geqq c^{\rho_1+\cdots+\rho_{m_0}} \cdot c^{-\rho_1-\cdots-\rho_{m_0}} f(t, c\mathbf{1}, \mathbf{0}, \dots, \mathbf{0})$$

$$= f(t, c\mathbf{1}, \mathbf{0}, \dots, \mathbf{0}), \qquad t \geqq t_2.$$

From this and (A) we obtain

$$x^{(n)}(t) + f(t, c\mathbf{1}, \mathbf{0}, \dots, \mathbf{0}) \leqq 0, \qquad t \geqq t_2. \tag{6}$$

We multiply (6) by t^{n-1} and integrate it from t_2 to t:

$$\int_{t_2}^{t} s^{n-1} x^{(n)}(s) ds + \int_{t_2}^{t} s^{n-1} f(s, c\mathbf{1}, \mathbf{0}, \dots, \mathbf{0}) ds \leqq 0. \tag{7}$$

It is easy to verify that

$$\int_{t_2}^{t} s^{n-1} x^{(n)}(s) ds = P(t) - P(t_2) + (-1)^{n+1} n! [x(t) - x(t_2)], \tag{8}$$

where $P(t) = \sum_{k=1}^{n} (-1)^{k+1} (n-1)(n-2)\cdots(n-k+1) t^{n-k} x^{(n-k)}(t)$. Noting that $P(t) > 0$ by (3), we see from (7) and (8) that

$$\int_{t_2}^{\infty} s^{n-1} f(s, c\mathbf{1}, \mathbf{0}, \dots, \mathbf{0}) ds < \infty,$$

which contradicts (1).

Let (A) be sublinear. Then, by (4), (5) and the sublinearity, we have

$$f(t, x<\boldsymbol{g}_0(t)>, [x']^2 <\boldsymbol{g}_1(t)>, \dots, [x^{(n-1)}]^2 <\boldsymbol{g}_{n-1}(t)>)$$

$$= \prod_{j=1}^{m_0} [x(g_{0j}(t))]^{\rho_j} \cdot \prod_{j=1}^{m_0} [x(g_{0j}(t))]^{-\rho_j} f(t, x<\boldsymbol{g}_0(t)>,$$

$$[x']^2 <\boldsymbol{g}_1(t)>, \dots, [x^{(n-1)}]^2 <\boldsymbol{g}_{n-1}(t)>)$$

$$= c^{\rho_1+\cdots+\rho_{m_0}} \cdot d^{-\rho_1-\cdots-\rho_{m_0}} f(t, d\mathbf{1}, c^2\mathbf{1}, \dots, c^2\mathbf{1})$$

$$= (c/d) f(t, d\mathbf{1}, c^2\mathbf{1}, \dots, c^2\mathbf{1}), \qquad t \geqq t_2.$$

Therefore we obtain

$$x^{(n)}(t) + (c/d) f(t, d\mathbf{1}, c^2\mathbf{1}, \dots, c^2\mathbf{1}) \leqq 0, \qquad t \geqq t_2,$$

from which, proceeding as in the superlinear case, we arrive at

$$\int_{t_2}^{\infty} s^{n-1} f(s, d\mathbf{1}, c^2\mathbf{1},\dots, c^2\mathbf{1})ds < \infty,$$

a contradiction to (2). This completes the proof.

We now present conditions which guarantee the existence of a bounded nonoscillatory solution of equation (A).

THEOREM 3. *Let equation* (A) *be superlinear. Suppose there exist* $\mu_1 \neq 0$, $\mu_2 \neq 0$ *such that*

$$(9) \qquad \int^{\infty} t^{n-1} |f(t, \mu_1\mathbf{1}, \mu_2^2\mathbf{1},\dots, \mu_2^2\mathbf{1})| dt < \infty.$$

Then, for n even, (A) *has a bounded nonoscillatory solution, while, for n odd,* (A) *has a bounded nonoscillatory solution* $x(t)$ *with* $\lim_{t\to\infty} x(t) \neq 0$.

THEOREM 4. *Let equation* (A) *be sublinear. Suppose there exists* $\mu \neq 0$ *such that*

$$(10) \qquad \int^{\infty} t^{n-1} |f(t, \mu\mathbf{1}, \mathbf{0},\dots, \mathbf{0})| dt < \infty.$$

Then, for n even, (A) *has a bounded nonoscillatory solution, while, for n odd,* (A) *has a bounded nonoscillatory solution* $x(t)$ *with* $\lim_{t\to\infty} x(t) \neq 0$.

PROOF OF THEOREM 3 AND 4. Without loss of generality we may suppose that μ_1, μ_2 in (9) and μ in (10) are positive. When (A) is superlinear we take c_0 such that $\max\{\mu_1/2, \mu_1 - \mu_2\} \leqq c_0 < \mu_1$ and put $\delta = \mu_1 - c_0$. When (A) is sublinear we take c_0 such that $0 < \mu < c_0$ and put $\delta = c_0 - \mu$.

For simplicity of exposition we restrict our consideration to the case where all the $g_{ij}(t)$ are advanced arguments, that is, $g_{ij}(t) \geqq t$ for $t \geqq t_0$. Choose T so large that

$$(11) \qquad \int_T^{\infty} (s-T)^{n-1-i} f(s, \mu\mathbf{1}, \mu_2^2\mathbf{1},\dots, \mu_2^2\mathbf{1}) ds \leqq \delta, \qquad i = 1,\dots, n-1,$$

if (A) is superlinear, and

$$(12) \qquad \int_T^{\infty} (s-T)^{n-1-i} f(s, \mu\mathbf{1}, \mathbf{0},\dots, \mathbf{0}) ds \leqq \delta, \qquad i = 1,\dots, n-1,$$

if (A) is sublinear, and consider the integral equation

$$(13) \qquad x(t) = (\Phi x)(t), \qquad t \geqq T,$$

where

$$(14) \qquad (\Phi x)(t)=c_0+\frac{(-1)^{n-1}}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}f(s,\ x<\boldsymbol{g}_0(s)>,\ [x']^2<\boldsymbol{g}_1(s)>,\ \ldots,\ [x^{(n-1)}]^2<\boldsymbol{g}_{n-1}(s)>)ds.$$

In view of (14) it is clear that a solution of (13) is a bounded nonoscillatory solution of (A) and tends monotonically to c_0 as $t\to\infty$. To solve (13) with the aid of Tychonoff's fixed point theorem we introduce the Fréchet space Y of all $n-1$ times continuously differentiable functions $x(t)$ on $[T, \infty)$ endowed with the topology induced by the sequence of seminorms $\{p_\nu\}$:

$$p_\nu(x) = \max_{t\in[T,T+\nu]}|x^{(n-1)}(t)|+\sum_{i=0}^{n-2}|x^{(i)}(T)|, \qquad \nu = 1, 2,\ldots.$$

Let X denote the set of all $x\in Y$ such that

$$(15) \qquad |x(t)-c_0| \leqq \delta, \quad |x^{(i)}(t)| \leqq \delta, \qquad t \geqq T, \quad i = 1,\ldots, n-1.$$

Obviously, X is a nonempty, closed and convex subset of Y. We shall show that Φ is continuous and maps X into a compact subset of X. We shall do this only for the superlinear equation (A), since a similar argument holds if (A) is sublinear.

i) *Φ maps X into X.* If $x\in X$, then, by (15),

$$0 < x(g_{0j}(t)) \leqq c_0+\delta = \mu_1,$$
$$|x^{(i)}(g_{ij}(t))| \leqq \delta = \mu_1-c_0 \leqq \mu_2.$$

Using the above inequalities and the superlinearity, we have

$$\begin{aligned} & f(t,\ x<\boldsymbol{g}_0(t)>,\ [x']^2<\boldsymbol{g}_1(t)>,\ldots,\ [x^{(n-1)}]^2<\boldsymbol{g}_{n-1}(t)>) \\ &= \prod_{j=1}^{m_0}[x(g_{0j}(t))]^{\rho_j}\cdot\prod_{j=1}^{m_0}[x(g_{0j}(t))]^{-\rho_j}f(t,\ x<\boldsymbol{g}_0(t)>, \\ (16) \qquad & \qquad [x']^2<\boldsymbol{g}_1(t)>,\ldots,\ [x^{(n-1)}]^2<\boldsymbol{g}_{n-1}(t)>) \\ &\leqq \mu_1^{\rho_1+\cdots+\rho_{m_0}}\cdot\mu_1^{-\rho_1-\cdots-\rho_{m_0}}f(t,\ \mu_1\mathbf{1},\ \mu_2^2\mathbf{1},\ldots,\ \mu_2^2\mathbf{1}) \\ &= f(t,\ \mu_1\mathbf{1},\ \mu_2^2\mathbf{1},\ldots,\ \mu_2^2\mathbf{1}). \end{aligned}$$

Therefore, by (11) and (16), we see that $y=\Phi x$ satisfies the following inequalities for $t\geqq T$:

$$|y(t)-c_0| \leqq \frac{1}{(n-1)!}\int_T^{\infty}(s-T)^{n-1}f(s,\ \mu_1\mathbf{1},\ \mu_2^2\mathbf{1},\ldots,\ \mu_2^2\mathbf{1})ds \leqq \delta,$$

$$|y^{(i)}(t)| \leqq \frac{(n-1)(n-2)\cdots(n-i)}{(n-1)!}\int_t^{\infty}(s-t)^{n-1-i}f(s,\ \mu_1\mathbf{1},\ \mu_2^2\mathbf{1},\ldots,\ \mu_2^2\mathbf{1})ds$$

$$\leqq \frac{1}{(n-1-i)!}\int_T^{\infty}(s-T)^{n-1-i}f(s,\,\mu_1\mathbf{1},\,\mu_2^2\mathbf{1},\ldots,\,\mu_2^2\mathbf{1})ds \leqq \delta.$$

ii) *$\overline{\Phi X}$ is compact.* Let $y_\nu=\Phi x_\nu$, $x_\nu \in X$, $\nu=1, 2,\ldots$, be any sequence of elements of ΦX. In view of (16) we have

$$|y_\nu^{(n-1)}(t_2)-y_\nu^{(n-1)}(t_1)| \leqq \left|\int_{t_1}^{t_2} f(s,\,\mu_1\mathbf{1},\,\mu_2^2\mathbf{1},\ldots,\,\mu_2^2\mathbf{1})ds\right|,$$

so that $\{y_\nu^{(n-1)}\}$ is equicontinuous at each point of $[T, \infty)$. Since, moreover, $\{y_\nu^{(n-1)}\}$ is uniformly bounded, there exists a subsequence $\{z_\nu\}$ of $\{y_\nu\}$ for which $\{z_\nu^{(n-1)}\}$ converges uniformly on every compact subinterval of $[T, \infty)$. Since the sequences $\{z_\nu^{(i)}(T)\}$, $i=0, 1,\ldots, n-2$, are bounded, there exists a subsequence $\{w_\nu\}$ of $\{z_\nu\}$ such that each $\{w_\nu^{(i)}(T)\}$, $i=0, 1,\ldots, n-2$, is convergent. It follows that the sequence $\{w_\nu\}$ converges to an element of X in the topology of Y. This shows that $\overline{\Phi X}$ is a compact subset of X.

iii) *Φ is continuous.* Let $\{x_\nu\}$ be a sequence of elements of X such that $x_\nu \to x \in X$ in the topology of Y. Put $y_\nu=\Phi x_\nu$ and $y=\Phi x$. It is clear that $x_\nu(t) \to x(t)$ at every point $t \in [T, \infty)$ and that by (16) the function

$$(s-t)^{n-1}f(s,\,x_\nu<\mathbf{g}_0(s)>,\,[x_\nu']^2<\mathbf{g}_1(s)>,\ldots,\,[x_\nu^{(n-1)}]^2<\mathbf{g}_{n-1}(s)>)$$

is bounded above by $(s-T)^{n-1}f(s,\,\mu_1\mathbf{1},\,\mu_2^2\mathbf{1},\ldots,\,\mu_2^2\mathbf{1})$ which is integrable on $[T, \infty)$. Hence, by the Lebesgue dominated convergence theorem, we see that $y_\nu(t) \to y(t)$ at every point $t \in [T, \infty)$.

Let $\{u_\nu\}$ be an arbitrary subsequence of $\{y_\nu\}$. Since $\overline{\Phi X}$ is compact, there is a subsequence $\{v_\nu\}$ of $\{u_\nu\}$ such that $v_\nu \to v \in \overline{\Phi X}$ in the topology of Y. The convergence in Y implies the pointwise convergence on $[T, \infty)$, and so we have $v=y$. It follows that $y_\nu \to y$ in the topology of Y.

Thus we are able to apply Tychonoff's fixed point theorem to the operator Φ. Let $x \in X$ be a fixed point of Φ: $x=\Phi x$. As we observed earlier, this fixed point $x=x(t)$ provides the required solution of equation (A). This completes the proof.

Corollary. *Consider the equation*

$$x^{(n)}(t)+f(t,\,x<\mathbf{g}_0(t)>) = 0 \tag{B}$$

which is either superlinear or sublinear. Then, a necessary and sufficient condition in order that every bounded solution of (B) *be oscillatory for n even, and be either oscillatory or strongly monotone for n odd is that*

$$\int^{\infty} t^{n-1}|f(t,\,\mu\mathbf{1})|dt = \infty \qquad \textit{for all} \quad \mu \neq 0.$$

3. Oscillation of all solutions

In this section we confine our attention to equation (A) which is either strongly superlinear or strongly sublinear and present conditions under which all of its solutions are oscillatory.

We start with the strongly sublinear case. Let $\tau_1,\dots,\tau_{m_0}$ be the strong sublinearity constants and put

$$h_{0j}(t) = \min\{g_{0j}(t), t\}, \qquad j = 1,\dots, m_0,$$

$$\boldsymbol{g}_0^{n-1}(t) = ([g_{01}(t)]^{n-1},\dots, [g_{0m_0}(t)]^{n-1}),$$

$$\boldsymbol{g}_i^{2(n-1-i)}(t) = ([g_{i1}(t)]^{2(n-1-i)},\dots, [g_{im_i}(t)]^{2(n-1-i)}), \qquad i = 1,\dots, n-1.$$

THEOREM 5. *Let equation* (A) *be strongly sublinear. Suppose that*

$$\int^\infty \prod_{j=1}^{m_0}\left(\frac{h_{0j}(t)}{g_{0j}(t)}\right)^{(n-1)\tau_j}|f(t, \mu\boldsymbol{g}_0^{n-1}(t), \mu^2\boldsymbol{g}_1^{2(n-2)}(t), \mu^2\boldsymbol{g}_2^{2(n-3)}(t),\dots, \mu^2\mathbf{1})|dt = \infty \tag{17}$$

for all $\mu\neq 0$. *Then, every solution of* (A) *is oscillatory when n is even, and every solution is either oscillatory or strongly monotone when n is odd.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (A) such that $\lim_{t\to\infty} x(t)\neq 0$. We may suppose $x(t)$ is eventually positive. Thus from (A) $x^{(n)}(t)<0$ for $t\geqq t_1$, provided t_1 is sufficiently large. According to a lemma of Kiguradze [2, Lemma 2] there is an integer l, $0<l<n$, such that

$$x^{(i)}(t) > 0\ (i = 0, 1,\dots, l), \quad (-1)^{n+i-1}x^{(i)}(t) > 0\ (i = l+1,\dots, n-1), \tag{18}$$

and

$$x(t) \geqq \frac{(t-t_1)^{n-1}x^{(n-1)}(2^{n-l-1}t)}{(n-1)(n-2)\cdots(n-l)} \tag{19}$$

for all $t\geqq t_1$.

By Taylor's formula for $x^{(i)}(t)$ we obtain

$$x^{(i)}(t) \leqq \sum_{k=0}^{n-1-i}\frac{x^{(i+k)}(t_1)}{k!}(t-t_1)^{i+k}, \qquad i=0, 1, \dots, l,$$

for $t\geqq t_1$, which implies that $x^{(i)}(t)/t^{n-1-i}$ $(i=0, 1,\dots, l)$ are eventually bounded. Consequently, there are positive numbers c and $t_2\geqq t_1$ such that

$$0 < x^{(i)}(g_{ij}(t)) \leqq c[g_{ij}(t)]^{n-1-i}, \qquad j = 1,\dots, m_i, \quad i = 0, 1,\dots, l, \tag{20}$$

for $t \geqq t_2$. Since $|x^{(i)}(t)|$ $(i=l+1,\dots, n-1)$ are bounded, we have

$$(21)\qquad |x^{(i)}(g_{ij}(t))| \leqq c[g_{ij}(t)]^{n-1-i}, \qquad j=1,\dots, m_i, \quad i=l+1,\dots, n-1,$$

for $t \leqq t_2$. On the other hand, using (19) and the decreasing nature of $x^{(n-1)}(t)$, it is easy to check that there are positive constants d and $t_3 \geqq t_2$ such that

$$(22)\qquad x(h_{0j}(t)) \geqq [dh_{0j}(t)]^{n-1}x^{(n-1)}(t), \qquad j=1,\dots, m_0,$$

for $t \geqq t_3$. Noting that $x(t)$ is increasing and using (20), (21), (22) and the strong sublinearity, we obtain

$$\begin{aligned}(23)\quad & f(t, x\langle \boldsymbol{g}_0(t)\rangle, [x']^2\langle \boldsymbol{g}_1(t)\rangle,\dots, [x^{(n-1)}]^2\langle \boldsymbol{g}_{n-1}(t)\rangle)\\ &\geqq \prod_{j=1}^{m_0}[x(h_{0j}(t))]^{\tau_j}\cdot \prod_{j=1}^{m_0}[x(g_{0j}(t))]^{-\tau_j} f(t, x\langle \boldsymbol{g}_0(t)\rangle,\\ &\qquad\qquad [x']^2\langle \boldsymbol{g}_1(t)\rangle,\dots, [x^{(n-1)}]^2\langle \boldsymbol{g}_{n-1}(t)\rangle)\\ &\geqq (d/c)^{(n-1)\tau}[x^{(n-1)}(t)]^{\tau}\prod_{j=1}^{m_0}[h_{0j}(t)/g_{0j}(t)]^{(n-1)\tau_j}\cdot\\ &\qquad\qquad f(t, c\boldsymbol{g}_0^{n-1}(t), c^2\boldsymbol{g}_1^{2(n-2)}(t),\dots, c^2\mathbf{1}),\end{aligned}$$

where $\tau=\tau_1+\cdots+\tau_{m_0}<1$. From (A) and (23) it follows that

$$\begin{aligned} &x^{(n)}(t)+(d/c)^{(n-1)\tau}[x^{(n-1)}(t)]^{\tau}\prod_{j=1}^{m_0}[h_{0j}(t)/g_{0j}(t)]^{(n-1)\tau_j}\cdot\\ &\qquad\qquad f(t, c\boldsymbol{g}_0^{n-1}(t), c^2\boldsymbol{g}_1^{2(n-2)}(t),\dots, c^2\mathbf{1}) \leqq 0\end{aligned}$$

for $t \geqq t_3$. Dividing the above inequality by $[x^{(n-1)}(t)]^{\tau}$ and integrating, we conclude that

$$\int^{\infty}\prod_{j=1}^{m_0}[h_{0j}(t)/g_{0j}(t)]^{(n-1)\tau_j} f(t, c\boldsymbol{g}_0^{n-1}(t), c^2\boldsymbol{g}_1^{2(n-2)}(t),\dots, c^2\mathbf{1})dt < \infty.$$

But this contradicts (17) and the proof is complete.

REMARK. If in addition $f(t, \boldsymbol{y}_0, \boldsymbol{y}_1,\dots, \boldsymbol{y}_{n-1})$ is assumed to be nondecreasing in $\boldsymbol{y}_0$, then the assertion of Theorem 5 is true under the condition

$$\int^{\infty}|f(t, \mu\boldsymbol{h}_0^{n-1}(t), \mu^2\boldsymbol{g}_1^{2(n-2)}(t),\dots, \mu^2\mathbf{1})|dt = \infty \qquad \text{for all} \quad \mu \neq 0,$$

where $\boldsymbol{h}_0^{n-1}(t)=([h_{01}(t)]^{n-1},\dots, [h_{0m_0}(t)]^{n-1})$, which is weaker than (17).

We now turn to the strongly superlinear equation (A). Let $\sigma_1,\dots, \sigma_{m_0}$ be the strong superlinearity constants with $\sigma=\sigma_1+\cdots+\sigma_{m_0}>1$.

THEOREM 6. *Let equation* (A) *be strongly superlinear. Suppose that there exist differentiable functions* $k_{0j}(t)$, $j=1,\dots, m_0$, *on* $[t_0, \infty)$ *such that*

$$k_{0j}(t) \leqq \min\{g_{0j}(t), t\}, \quad k'_{0j}(t) \geqq 0, \quad \lim_{t\to\infty} k_{0j}(t) = \infty,$$

and

$$\int^{\infty} \frac{\sum_{j=1}^{m_0} [k_{0j}(t)]^{n-1}}{\prod_{j=1}^{m_0} [k_{0j}(t)]^{(n-1)(\sigma-\sigma_j)}} |f(t, \mu\mathbf{1}, \mathbf{0},\dots, \mathbf{0})|\, dt = \infty \tag{24}$$

for all $\mu \neq 0$. *Then, every solution of* (A) *is oscillatory when n is even, and every solution is either oscillatory or strongly monotone when n is odd.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (A) such that $\lim_{t\to\infty} x(t) \neq 0$. We may suppose $x(t)$ is eventually positive. Kiguradze's lemma [2, Lemma 2] implies in particular that $x(t)$ is eventually increasing, so that there are positive numbers c and t_1 such that

$$c \leqq x(h_{0j}(t)) \leqq x(g_{0j}(t)) \qquad \text{for} \quad t \geqq t_1, \quad j = 1,\dots, m_0. \tag{25}$$

In view of (25) and the strong superlinearity we find

$$\begin{aligned} &f(t, x\langle \boldsymbol{g}_0(t)\rangle, [x']^2\langle \boldsymbol{g}_1(t)\rangle,\dots, [x^{(n-1)}]^2\langle \boldsymbol{g}_{n-1}(t)\rangle) \\ &\geqq \prod_{j=1}^{m_0} [x(h_{0j}(t))]^{\sigma_j} \cdot \prod_{j=1}^{m_0} [x(g_{0j}(t))]^{-\sigma_j} f(t, x\langle \boldsymbol{g}_0(t)\rangle, \\ &\qquad\qquad [x']^2\langle \boldsymbol{g}_1(t)\rangle,\dots, [x^{(n-1)}]^2\langle \boldsymbol{g}_{n-1}(t)\rangle) \\ &\geqq c^{-\sigma} f(t, c\mathbf{1}, \mathbf{0},\dots, \mathbf{0}) \prod_{j=1}^{m_0} [x(h_{0j}(t))]^{\sigma_j}. \end{aligned}$$

Combining this with (A), we conclude that $x(t)$ satisfies the *retarded* differential inequality

$$x^{(n)}(t) + c^{-\sigma} f(t, c\mathbf{1}, \mathbf{0},\dots, \mathbf{0}) \prod_{j=1}^{m_0} [x(h_{0j}(t))]^{\sigma_j} \leqq 0 \tag{26}$$

for $t \geqq t_1$. We now apply the theory developed by Sficas and Staikos [6] to the differential inequality (26). Then, because of (24), it can be shown that $x(t)$ is either oscillatory or strongly monotone. This, however, contradicts the hypothesis that $x(t)$ is a nonoscillatory solution not tending to zero as $t\to\infty$. For the details the reader is referred to the paper [6].

REMARK. In the particular case when $g_{0j}(t) \geqq t$ for $t \geqq t_0$, $j=1,\dots, m_0$, we can replace (24) by

$$\int^{\infty} t^{n-1}|f(t, \mu\mathbf{1}, \mathbf{0},\ldots, \mathbf{0})|dt = \infty \qquad \text{for all} \quad \mu \neq 0.$$

From this remark and Theorem 3 we have the following result which characterizes the oscillation situation for strongly superlinear advanced equations of the form

$$\text{(B)} \qquad x^{(n)}(t)+f(t, x<\mathbf{g}_0(t)>) = 0.$$

COROLLARY. *Let* (B) *be strongly superlinear and suppose that*

$$g_{0j}(t) \geqq t \qquad \text{for} \quad t \geqq t_0, \quad j = 1,\ldots, m_0.$$

Then, a necessary and sufficient condition in order that every solution of (B) *be oscillatory when n is even, and be either oscillatory or strongly monotone when n is odd is that*

$$\int^{\infty} t^{n-1}|f(t, \mu\mathbf{1})|dt = \infty \qquad \textit{for all} \quad \mu \neq 0.$$

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