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Divisorial Objects in Abelian Categories

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Introduction

Recently, in [3], the notion of divisorial modules was introduced in Mod(R), the category of R-modules, where R is a completely integrally closed domain. In [3], the class of all pseudo-null modules is a Serre subcategory, more precisely, a localizing subcategory of Mod(R). This fact is meaningful. In fact, if R is a commutative ring with unit, then some closure operations on the lattice of ideals of R which have the same characters as the divisorial envelope of ideals, correspond to localizing subcategories of Mod(R). Another important fact is the following: If R is noetherian, then there is a one-to-one correspondence between the class of localizing subcategories of Mod(R) and the class of subsets of Spec(R) which are stable under specialization. And if Z is a subset of Spec(R), stable under specialization, then we can define the local cohomology modules with supports in Z. Therefore, there must be some relationship between the divisorial envelopes (more generally, *$\mathscr{C}$-divisorial envelopes*, defined in § 2) of modules and the local cohomology modules. In this paper, we shall study the above problem, mostly in § 2. Since both the divisorial envelopes of R-modules and the local cohomology modules are defined functorially, we shall deal with all things in an abelian category and its localizing subcategories.

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§ 1. Weak $\mathscr{C}$-envelopes

Let $\mathscr{A}$ be an abelian category, $\mathscr{C}$ a Serre subcategory of $\mathscr{A}$. For the definitions of $\mathscr{C}$-closed objects, $\mathscr{C}$-isomorphisms and $\mathscr{C}$-envelopes, we shall refer to [1]. Also, we shall assume basic properties of them (see [1] or [2]). For the purpose of convenience, we say that an object L is *$\mathscr{C}$-pure* if L has no $\mathscr{C}$-subobjects. The following lemma is an easy consequence of this definition.

LEMMA 1.1. *An object L is $\mathscr{C}$-pure if and only if, for every $\mathscr{C}$-isomorphism α: $M \to N$, $\mathrm{Hom}(N, L) \to \mathrm{Hom}(M, L)$ is injective.*

PROOF. (Necessity) Let f: $N \to L$ be a morphism. Suppose $\alpha f = 0$, then there is an epimorphism $\mathrm{Coker}(\alpha) \to \mathrm{Im}(f)$. Hence $\mathrm{Im}(f)$ is a $\mathscr{C}$-subobject of

L. Since $\mathscr{C}$-subobjects of L are null, $\mathrm{Im}(f)=0$. Therefore $f=0$.

(Sufficiency) Let C be a $\mathscr{C}$-subobject of L. Since $0\to C$ is a $\mathscr{C}$-isomorphism, $\mathrm{Hom}(C, L)\to\mathrm{Hom}(0, L)$ is injective. Hence $\mathrm{Hom}(C, L)=0$. Therefore $C=0$.

DEFINITION 1.2. *We say that a morphism ρ: $A\to L$ is a strict $\mathscr{C}$-isomorphism if ρ and L satisfy the following conditions*:

(1) *L is $\mathscr{C}$-pure.*

(2) *ρ is a $\mathscr{C}$-isomorphism.*

(3) *Let α: $M\to N$ be a monomorphism such that* $\mathrm{Coker}(\alpha)$ *is an object in $\mathscr{C}$. Then, for every morphism f: $M\to A$, there is a morphism (necessarily unique by Lemma* 1.1) *g: $N\to L$ such that $g\alpha=\rho f$.*

For example, if ρ: $A\to L$ is a $\mathscr{C}$-envelope of A, then ρ is a strict $\mathscr{C}$-isomorphism.

DEFINITION 1.3. *We say that a morphism ρ: $A\to L$ is a weak $\mathscr{C}$-envelope of A, if ρ and L satisfy the following conditions*:

(1) *(Strictness) ρ is a strict $\mathscr{C}$-isomorphism.*

(2) *(Universality) If ρ': $A'\to L'$ is a strict $\mathscr{C}$-isomorphism, then, for every morphism f: $A\to A'$, there is a morphism (necessarily unique by Lemma* 1.1, *since ρ is a $\mathscr{C}$-isomorphism) g: $L\to L'$ such that $g\rho=\rho' f$.*

A weak $\mathscr{C}$-envelope of A, if it exists, is unique up to isomorphisms because of its universality. It is easy to see that if a strict $\mathscr{C}$-isomorphism is a monomorphism, then it is a weak $\mathscr{C}$-envelope. On the other hand, if u: $A\to L$ is a $\mathscr{C}$-isomorphism with L $\mathscr{C}$-pure, then $\mathrm{Ker}(u)$ is a largest $\mathscr{C}$-subobject of A. Therefore, a $\mathscr{C}$-envelope of a $\mathscr{C}$-pure object A is a weak $\mathscr{C}$-envelope of A. More precisely

PROPOSITION 1.4. *Let A be a $\mathscr{C}$-pure object. Then a morphism ρ: $A\to L$ is a $\mathscr{C}$-envelope of A if and only if it is a weak $\mathscr{C}$-envelope of A.*

PROOF. It is sufficient to show the "if" part. All that remains to be proved is that L is $\mathscr{C}$-closed. Let $0\to L\xrightarrow{\alpha}X\to C\to 0$ be an exact sequence and C an object in $\mathscr{C}$. Then $\alpha\rho$ is a $\mathscr{C}$-isomorphism, and also, is a monomorphism. Hence there is a morphism β: $X\to L$ such that $\beta\alpha\rho=\rho$, by strictness of ρ. Then the universality of ρ implies $\beta\alpha=1_L$. Thus the above exact sequence is splitable. Therefore L is $\mathscr{C}$-closed.

It is also easy to see that if a strict $\mathscr{C}$-isomorphism is an epimorphism, then it is a weak $\mathscr{C}$-envelope (since $\mathrm{Hom}(A/N, L)\to\mathrm{Hom}(A, N)$ is bijective for every $\mathscr{C}$-pure object L and $\mathscr{C}$-subobject N of A).

PROPOSITION 1.5. *Let E be an injective object. Suppose that E has a* *largest $\mathscr{C}$-subobject E_0. Then the canonical morphism $\rho: E \to E/E_0$ is a weak $\mathscr{C}$-envelope of E.*

PROOF. Clearly, E/E_0 is $\mathscr{C}$-pure. Let $\alpha: M \to N$ be a monomorphism such that Coker(α) is an object in $\mathscr{C}$, and let $f: M \to E$. Since E is injective, there is a morphism $\delta: N \to E$ such that $\delta\alpha = f$. Then $g = \rho\delta: N \to E/E_0$ has the property $g\alpha = \rho f$. Thus, ρ is a strict $\mathscr{C}$-isomorphism and epimorphism. Therefore ρ is a weak $\mathscr{C}$-envelope of E.

PROPOSITION 1.6. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence, and let $u: B \to Y$ (resp. $v: C \to Z$) be a weak $\mathscr{C}$-envelope of B (resp. of C). By the universality of v, there is a unique morphism $g': Y \to Z$ such that $g'u = vg$. Now, let $f': X \to Y$ be a kernel of g', and let $\rho: A \to X$ be a unique morphism such that $uf = f'$. Suppose that u is an epimorphism. Then ρ is a weak $\mathscr{C}$-envelope of A.*

PROOF. (Strictness) Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & \mathrm{Im}(g) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & & \end{array}$$

By snake lemma, we have an exact sequence: $0 \to \mathrm{Ker}(\rho) \to \mathrm{Ker}(u) \to \mathrm{Ker}(\psi) \to \mathrm{Coker}(\rho) \to \mathrm{Coker}(u)$. Since Ker$(u)$, Coker$(u)$ and $\mathrm{Ker}(\psi) = \mathrm{Ker}(v) \cap \mathrm{Im}(g)$ are objects in $\mathscr{C}$, Ker(ρ) and Coker(ρ) are objects in $\mathscr{C}$. Hence ρ is a $\mathscr{C}$-isomorphism. On the other hand, it is obvious that X is $\mathscr{C}$-pure. Now, let $\alpha: M \to N$ be a monomorphism such that Coker(α) is an object in $\mathscr{C}$, and let $\beta: M \to A$ be a morphism. Then there is a morphism $\gamma': N \to A$ such that $uf\beta = \gamma'\alpha$, since u is a weak $\mathscr{C}$-envelope of B. Note that $g'\gamma'\alpha = 0$. Then by Lemma 1.1, we have $g'\gamma' = 0$. This means that there is a unique morphism $\gamma: N \to X$ such that $f'\gamma = \gamma'$, and so $f'\gamma\alpha = f'\rho\beta$. Since f' is a monomorphism, we have $\gamma\alpha = \rho\beta$.

(Universality) Consider the following pull back diagram:

$$\begin{array}{ccc} K & \xrightarrow{j} & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f'} \\ B & \xrightarrow{u} & Y \end{array}$$

From the universality of the pull back diagram, we have a morphism $i: A \to K$ such that $pi = f$ and $ji = \rho$. Since f and f' are monomorphisms, p and i are also monomorphisms. Hence Ker(j) is a subobject of Ker(u), so that Ker(j) is an object in $\mathscr{C}$. On the other hand, j is an epimorphism; hence j is a $\mathscr{C}$-isomorphism. Therefore i is also a $\mathscr{C}$-isomorphism. Now let $\rho': A' \to X'$ be a strict $\mathscr{C}$-isomorphism, and let $\sigma: A \to A'$ be a morphism. Then there is a morphism $\tau': K \to X'$ such that $\tau' i = \rho'\sigma$. However $\tau'(\mathrm{Ker}(j)) = 0$ since X' is $\mathscr{C}$-pure and

Ker(j) is an object in $\mathscr{C}$. Therefore we have a morphism $\tau: X \to X'$ such that $\tau j = \tau'$. It is clear that $\tau\rho = \rho'\sigma$.

COROLLARY 1.7. *Assume that $\mathscr{A}$ has enough injectives. If every object of $\mathscr{A}$ has a largest $\mathscr{C}$-subobject, then every object has a weak $\mathscr{C}$-envelope.*

PROOF. Let A be an object, and let $0 \to A \to E_0 \to E_1$ be an injective resolution of A. By Prop. 1.5, E_i has a weak $\mathscr{C}$-envelope $\rho_i: E_i \to L_i$ which is an epimorphism ($i=0, 1$). The assertion follows from Prop. 1.6.

COROLLARY 1.8. *Assume that $\mathscr{A}$ has enough injectives. Then a Serre subcategory $\mathscr{C}'$ of $\mathscr{A}$ is a localizing subcategory if and only if every object in $\mathscr{A}$ has a largest $\mathscr{C}'$-subobject.*

PROOF. If $\mathscr{C}'$ is a localizing subcategory of $\mathscr{A}$, then it is clear that every object in $\mathscr{A}$ has a largest $\mathscr{C}'$-subobject. Conversely, if every object in $\mathscr{A}$ has a largest $\mathscr{C}'$-subobject, then by Coroll. 1.7 every object has a weak $\mathscr{C}'$-envelope; hence by Prop. 1.4 every $\mathscr{C}'$-pure object has a $\mathscr{C}'$-envelope. Therefore, by [2], Prop. 4, Chap. III, $\mathscr{C}'$ is a localizing subcategory of $\mathscr{A}$.

For the rest of this section, we assume that $\mathscr{A}$ has enough injectives and $\mathscr{C}$ is a localizing subcategory of $\mathscr{A}$. For each object A in $\mathscr{A}$, choose a weak $\mathscr{C}$-envelope $\rho(A): A \to \mathrm{T}(A)$. If B is another object in $\mathscr{A}$, then we have the map $\mathrm{Hom}(A, B) \to \mathrm{Hom}(\mathrm{T}(A), \mathrm{T}(B))$ from the universality of $\rho(A)$, which is a group homomorphism. Therefore we have a covariant additive functor $\mathrm{T}: \mathscr{A} \to \mathscr{A}$ and a morphism $\rho: 1_{\mathscr{A}} \to \mathrm{T}$ of functors such that for each A, $\rho(A): A \to \mathrm{T}(A)$ is a weak $\mathscr{C}$-envelope of A. On the other hand, we also have a functor $\mathrm{L}_{\mathscr{C}}: \mathscr{A} \to \mathscr{A}$ and a morphism $\mathrm{L}_{\mathscr{C}} \to 1_{\mathscr{A}}$ of functors such that $\mathrm{L}_{\mathscr{C}}(A) \to A$ is a largest $\mathscr{C}$-subobject of A, for each A. It is easy to see that $\mathrm{L}_{\mathscr{C}}$ is a left exact, additive, covariant functor.

Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathscr{A}$. We can construct an injective resolution $E_1.$ (resp. $E_2.$, $E_3.$) of A (resp. B, C) and a splitable exact sequence of complexes $0 \to E_1. \to E_2. \to E_3. \to 0$. Since $\mathrm{L}_{\mathscr{C}}$ is left exact, $0 \to \mathrm{L}_{\mathscr{C}}(E_1.) \to \mathrm{L}_{\mathscr{C}}(E_2.) \to \mathrm{L}_{\mathscr{C}}(E_3.) \to 0$ is exact, and hence $0 \to E_1./\mathrm{L}_{\mathscr{C}}(E_1.) \to E_2./\mathrm{L}_{\mathscr{C}}(E_2.) \to E_3./\mathrm{L}_{\mathscr{C}}(E_3.) \to 0$ is exact. Therefore Prop. 1.5 and 1.6 imply that $0 \to \mathrm{T}(A) \to \mathrm{T}(B) \to \mathrm{T}(C)$ is exact. We can thus see that T is left exact, and also it is the zero-th right derived functor of $\mathrm{Coker}(\mathrm{L}_{\mathscr{C}} \to 1_{\mathscr{A}})$. Summarizing the properties of T and ρ, we have

(1) T *is left exact, covariant and additive.*
(2) *For an injective object E, $\rho(E)$ is an epimorphism.*
(3) $\mathrm{T}(\rho(A)) = \rho(\mathrm{T}(A))$, *and it is a monomorphism.*

Now, let $\mathrm{R}^p\mathrm{T}$ (resp. $\mathrm{R}^p\mathrm{L}_{\mathscr{C}}$) be the p-th right derived functor of T (resp. $\mathrm{L}_{\mathscr{C}}$). For every object A, choose an injective resolution $E.$ of A. Since the sequence $0 \to \mathrm{L}_{\mathscr{C}}(E.) \to E. \to \mathrm{T}(E.) \to 0$ of complexes is exact, we have an exact sequence and isomorphisms

$$(4)\quad 0\to L_{\mathscr{C}}(A)\to A\to T(A)\to R^1L_{\mathscr{C}}(A)\to 0$$

$$R^pT(A)\cong R^{p+1}L_{\mathscr{C}}A) \qquad (p>0).$$

REMARK. If $\mathscr{A}$ is the category of R-modules, where R is a noetherian ring, and if $\mathscr{C}$ is its localizing subcategory consisting of R-modules whose supports are contained in a closed subset Z of $\mathrm{Spec}(R)$, then the exact sequence (4) above is the fundamental exact sequence connecting the local cohomology modules with supports in Z and the global cohomologies on $\mathrm{Spec}(R)-Z$.

PROPOSITION 1.9. *There is a one-to-one correspondence between the class of localizing subcategories of* $\mathscr{A}$ *and the class of pairs* (T, ρ) *consisting of a functor* $T\colon \mathscr{A}\to\mathscr{A}$ *and a morphism* $\rho\colon 1_{\mathscr{A}}\to T$ *of functors satisfying the conditions* (1), (2) *and* (3) *above.*

PROOF. Let (T, ρ) be a pair of a functor and a morphism of functors satisfying (1), (2) and (3). Let $\mathscr{C}$ be the class of all objects A such that $T(A)=0$. We regard $\mathscr{C}$ as a full subcategory of $\mathscr{A}$. We shall prove that $\mathscr{C}$ is a localizing subcategory of A and $\rho(A)\colon A\to T(A)$ is a weak $\mathscr{C}$-envelope of A for each A.

(a) For every object A, $\mathrm{Ker}(\rho(A))$ is an object in $\mathscr{A}$. In particular, A is an object in $\mathscr{C}$ if and only if $\rho(A)=0$. In fact, consider the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Ker}(\rho(A)) & \longrightarrow & A & \xrightarrow{\rho(A)} & T(A) \\
 & & \downarrow & & \downarrow{\scriptstyle \rho(A)} & & \downarrow{\scriptstyle \rho(T(A))} \\
0 & \longrightarrow & T(\mathrm{Ker}(\rho(A))) & \longrightarrow & T(A) & \xrightarrow{T(\rho(A))} & TT(A).
\end{array}$$

Then $T(\mathrm{Ker}(\rho(A)))=0$, since $T(\rho(A))$ is a monomorphism. Therefore $\mathrm{Ker}(\rho(A))$ is an object in $\mathscr{C}$.

(b) $\mathscr{C}$ is a Serre subcategory of $\mathscr{A}$. In fact, let $0\to A\to B\to C\to 0$ be an exact sequence. By (a) above, $B\in\mathscr{C}\Leftrightarrow\rho(B)=0\Leftrightarrow\rho(A)=0=\rho(C)\Leftrightarrow A,\ C\in\mathscr{C}$.

(c) $T(A)$ is $\mathscr{C}$-pure for each A. In fact, let $i\colon N\to T(A)$ be a non-trivial subobject of $T(A)$. Since $\rho(T(A))i=T(i)\rho(N)$ is a monomorphism, $\rho(N)$ is also a monomorphism. Therefore $T(A)$ has no $\mathscr{C}$-subobjects.

(d) $\mathrm{Ker}(\rho(A))$ is a largest $\mathscr{C}$-subobject of A, for each A. In fact, if $i\colon N\to A$ is a $\mathscr{C}$-subobject of A, then $\rho(A)i=0$. Therefore i factors through $\mathrm{Ker}(\rho(A))$.

(e) We have thus proved that $\mathscr{C}$ is a Serre subcategory such that every object A has a largest $\mathscr{C}$-subobject $\mathrm{Ker}(\rho(A))$. Hence, by Prop. 1.7, for every object A, a weak $\mathscr{C}$-envelope of A exists. From the condition (2) for ρ and Prop. 1.5, $\rho(E)\colon E\to T(E)$ is a weak $\mathscr{C}$-envelope of E for every injective object E. Therefore by left exactness of T and Prop. 1.5, $\rho(A)$ is a weak $\mathscr{C}$-envelope of A for every object A.

REMARK. Let $\mathscr{C}$ be a localizing subcategory of $\mathscr{A}$, where $\mathscr{A}$ may not have enough injectives. Then we have a functor $\mathrm{F}: \mathscr{A} \to \mathscr{A}$ and a morphism $\sigma: 1_{\mathscr{A}} \to F$ of functors such that, for every object A, $\sigma(A): A \to \mathrm{F}(A)$ is a $\mathscr{C}$-envelope of A (cf. [1] or [2]). F and σ have the following properties:

(5) F *is left exact, additive and covariant.*

(6) $\mathrm{F}(\sigma(A)) = \sigma(\mathrm{F}(A))$, *and it is an isomorphism.*

An analogus theorem for F and σ, corresponding to Prop. 1.9, is also true. In fact we have

PROPOSITION. *There is a one-to-one correspondence between the class of localizing subcategories of $\mathscr{A}$ and the class of pairs* (F, σ) *of a functor* $\mathrm{F}: \mathscr{A} \to \mathscr{A}$ *and a morphism* $\sigma: 1_{\mathscr{A}} \to \mathrm{F}$ *of functors satisfying* (5) *and* (6) *above.*

PROOF. For a pair (F, σ) satisfying (5) and (6), we put $\mathscr{C} = \{A \in \mathrm{Obj}(\mathscr{A}): \mathrm{F}(A) = 0\}$, which is regarded as a full subcategory of $\mathscr{A}$. We shall prove that $\mathscr{C}$ is a localizing subcategory and $\sigma(A): A \to \mathrm{F}(A)$ is a $\mathscr{C}$-envelope of A for each A. Steps (a) to (d) in the proof of Prop. 1.9 are also valid in this case. Therefore it is sufficient to show that $\mathrm{F}(A)$ is $\mathscr{C}$-closed and $\sigma(A)$ is a $\mathscr{C}$-isomorphism for each A. Let $0 \to \mathrm{F}(A) \xrightarrow{\alpha} X \to C \to 0$ be an exact sequence such that C is an object in $\mathscr{C}$. Since $\mathrm{F}(C) = 0$, $\mathrm{F}(\alpha)$ is an isomorphism. Hence $\sigma(\mathrm{F}(A))^{-1}\mathrm{F}(\alpha)^{-1}\sigma(X)\alpha = 1_{\mathrm{F}(A)}$. Therefore the above exact sequence is splitable, and so $\mathrm{F}(A)$ is $\mathscr{C}$-closed. On the other hand, $\sigma(A)$ is factored as follows:

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathrm{Ker}(\sigma(A)) \longrightarrow A \longrightarrow & \mathrm{Im}(\sigma(A)) & \longrightarrow 0 & & \\ & \| & & & \\ 0 \longrightarrow & \mathrm{Im}(\sigma(A)) & \longrightarrow \mathrm{F}(A) \longrightarrow \mathrm{Coker}(\sigma(A)) \longrightarrow 0, & & \end{array}$$

so that we have two exact sequences:

$$\begin{array}{ccc} 0 \longrightarrow \mathrm{F}(A) \xrightarrow{i} & \mathrm{F}(\mathrm{Im}(\sigma(A))) & \\ & \| & \\ 0 \longrightarrow & \mathrm{F}(\mathrm{Im}(\sigma(A))) & \xrightarrow{j} \mathrm{FF}(A) \xrightarrow{u} \mathrm{F}(\mathrm{Coker}(\sigma(A))). \end{array}$$

Since $ji = \mathrm{F}(\sigma(A))$ is an isomorphism, and since i and j are monomorphism, i and j are isomorphism. Hence $u = 0$, and so we have $\sigma(\mathrm{Coker}(\sigma(A))) = 0$. Therefore $\mathrm{Coker}(\sigma(A))$ is an object in $\mathscr{C}$. This shows that $\sigma(A)$ is a $\mathscr{C}$-isomorphism.

§2. $\mathscr{C}$-divisorial envelopes

A characterization of divisorial modules in the category of R-modules, where R is a completely integrally closed domain, is given in [3], Prop. 8. However, the corresponding characterization of divisorial envelopes has not been given yet.

We, here, state this characterization and adopt it as the definition of "*$\mathscr{C}$-divisorial envelopes*".

Let $\mathscr{A}$ be an abelian category, $\mathscr{C}$ a localizing subcategory of $\mathscr{A}$.

DEFINITION 2.1. *A $\mathscr{C}$-divisorial envelope of an object A is a morphism $d: A \to D$ satisfying the following conditions:*

(1) *d is a monomorphism and* Coker(d) *is an object in $\mathscr{C}$.*

(2) *If a monomorphism $\alpha: M \to N$ is a $\mathscr{C}$-isomorphism, then, for every morphism $f: M \to A$, there is a morphism $g: N \to D$ such that $g\alpha = df$.*

(3) *If $\delta: D \to D$ satisfies $\delta d = d$, then δ is an isomorphism.*

It is easy to see that a $\mathscr{C}$-divisorial envelope of A, if it exists, is unique up to isomorphisms.

COROLLARY 2.2. *If $d: A \to D$ is a $\mathscr{C}$-divisorial envelope of A, then d is an essential morphism.*

PROOF. Let N be a subobject of D such that $N \cap A = 0$, and let β be the canonical projection $D \to D/N$. Then βd is a monomorphism and Coker(βd) is an object in $\mathscr{C}$. Hence there is a morphism $g: D/N \to D$ such that $g\beta d = d$. Then, by (3) above, $g\beta$ is an isomorphism, and so β is a monomorphism. Therefore $N = 0$.

COROLLARY 2.3. *Let A be a $\mathscr{C}$-pure object, and let $\rho: A \to L$ be a morphism. Then the following conditions for ρ are equivalent:*

(i) *ρ is a weak $\mathscr{C}$-envelope of A.*

(ii) *ρ is a $\mathscr{C}$-envelope of A.*

(iii) *ρ is a $\mathscr{C}$-divisorial envelope of A.*

PROOF. By Prop. 1.4, the conditions (i) and (ii) are equivalent to each other. The implication (ii)$\Rightarrow$(iii) follows easily from the definition of $\mathscr{C}$-envelopes. Now suppose that ρ is a $\mathscr{C}$-divisorial envelope of A. Since A is $\mathscr{C}$-pure, it follows from Cor. 2.2 that D is $\mathscr{C}$-pure. Hence a monomorphism ρ is a strict $\mathscr{C}$-isomorphism; whence ρ is a weak $\mathscr{C}$-envelope of A. Therefore we have the implication (iii)$\Rightarrow$(i).

LEMMA 2.4. *Let $\rho: A \to B$ be a monomorphism such that* Coker(ρ) *is $\mathscr{C}$-pure; let $\alpha: M \to N$ be a morphism such that* Coker(α) *is an object in $\mathscr{C}$. Let $f: M \to A$ be a morphism. If $g: N \to B$ is a morphism such that $g\alpha = \rho f$, then g factors through ρ.*

PROOF. Let ψ be the canonical projection $B \to$ Coker(ρ). Then, by Lemma 1.1, $\psi g = 0$, since Coker(ρ) is $\mathscr{C}$-pure and $\psi g\alpha = 0$. Therefore g factors through

ρ.

THEOREM 2.5. *Let $d: A \to D$ be a $\mathscr{C}$-divisorial envelope of A. Then ρ: $A \to D/\mathrm{L}_{\mathscr{C}}(D)$ (the composition of d and the canonical projection $\psi: D \to D/\mathrm{L}_{\mathscr{C}}(D)$) is a weak $\mathscr{C}$-envelope of A.*

PROOF. (Strictness) Clearly, ρ is a $\mathscr{C}$-isomorphism and $D/\mathrm{L}_{\mathscr{C}}(D)$ is $\mathscr{C}$-pure. Let $\alpha: M \to N$ be a monomorphism such that Coker(α) is an object in $\mathscr{C}$, and let $f: M \to A$ be a morphism. By Def. 2.1, there is a morphism $g': N \to D$ such that $g'\alpha = df$. Then $\psi g'\alpha = \rho f$. Therefore $\psi g'$ is a morphism we wanted.

(Universality) Let $\rho': A' \to L'$ be a strict $\mathscr{C}$-isomorphism, and let $f: A \to A'$ be a morphism. By the strictness of ρ', there is a morphism $h: D \to L'$ such that $hd = \rho' f$. Then $h(\mathrm{L}_{\mathscr{C}}(D)) = 0$, since L' is $\mathscr{C}$-pure. Hence there is a morphism $g: D/\mathrm{L}_{\mathscr{C}}(D)) \to L'$ such that $h = g\psi$, which implies $g\rho = \rho' f$.

The following proposition gives a sufficient condition for the existence of $\mathscr{C}$-divisorial envelopes, and also it shows that our definition of $\mathscr{C}$-divisorial envelopes is a generalization of divisorial envelopes in [3].

PROPOSITION 2.6. *If $\mathscr{A}$ is an abelian category with injective envelopes, then, for every object A, a $\mathscr{C}$-divisorial envelope of A exists.*

PROOF. Let A be an object. Choose an injective envelope $u: A \to E$ of A. Let ψ be the canonical projection $E \to E/A$. Consider the following pull back diagram:

$$\begin{array}{ccc} L & \xrightarrow{\psi'} & L_{\mathscr{C}}(E/A) \\ \downarrow{\scriptstyle j} & & \downarrow \\ E & \xrightarrow{\psi} & E/A \end{array} \quad .$$

Then u factors through j i.e., there is a unique morphism $d: A \to L$ such that $jd = u$. We shall prove that d is a $\mathscr{C}$-divisorial envelope of A. In fact, it is obvious that d is a monomorphism and Coker(d) is an object in $\mathscr{C}$. Let $\alpha: M \to N$ be a monomorphism such that Coker(α) is an object in $\mathscr{C}$, and let $f: M \to A$ be a morphism. Since E is injective, there is a morphism $g': N \to E$ such that $uf = g'\alpha$. Then, by Lemma 2.4, there is a morphism $g: N \to L$ such that $jg = g'$, and so $g\alpha = df$. Therefore d satisfies the condition (2) of Def. 2.1. If $\delta: L \to L$ satisfies $\delta d = d$, then it is clear that δ is a monomorphism. Since E is injective, there is a morphism δ': $L \to E$ such that $\delta'\delta = j$; δ' is also a monomorphism. Hence we have an endomorphism δ'' of $\mathrm{L}_{\mathscr{C}}(E/A)$ which is a monomorphism and commutes with the inclusion map $\mathrm{L}_{\mathscr{C}}(E/A) \to E/A$, so that δ'' is an isomorphism by the definition of $\mathrm{L}_{\mathscr{C}}(E/A)$; this shows that δ is an isomorphism. Therefore d is a $\mathscr{C}$-divisorial envelope of A.

We now give the definition of $\mathscr{C}$-divisorial objects.

DEFINITION 2.7. *We say that an object D is $\mathscr{C}$-divisorial if $1_D: D \to D$ is a $\mathscr{C}$-divisorial envelope of D, equivalently, for every monomorphism $\alpha: M \to N$ such that* $\mathrm{Coker}(\alpha)$ *is an object in $\mathscr{C}$,* $\mathrm{Hom}(N, D) \to \mathrm{Hom}(M, D)$ *is surjective.*

REMARK 2.8. Injective objects are obviously $\mathscr{C}$-divisorial. Also, it is obvious that $A \oplus B$ is $\mathscr{C}$-divisorial if and only if both A and B are $\mathscr{C}$-divisorial. Moreover, if D is $\mathscr{C}$-divisorial, then $\mathrm{L}_{\mathscr{C}}(D)$ is $\mathscr{C}$-divisorial. In fact, let $\alpha: M \to N$ be a monomorphism such that $\mathrm{Coker}(\alpha)$ is an object in $\mathscr{C}$, and let $f: M \to \mathrm{L}_{\mathscr{C}}(D)$ be a morphism. Since D is $\mathscr{C}$-divisorial, there is a morphism $g': N \to D$ such that $if = g'\alpha$ where i is the inclusion map $\mathrm{L}_{\mathscr{C}}(D) \to D$. Then by Lemma 2.4, there is a morphism $g: N \to \mathrm{L}_{\mathscr{C}}(D)$ such that $g' = ig$. Therefore $g\alpha = f$; this shows that $\mathrm{L}_{\mathscr{C}}(D)$ is $\mathscr{C}$-divisorial.

PROPOSITION 2.9. *Assume that $\mathscr{A}$ has enough injectives. If $d: A \to D$ is a $\mathscr{C}$-divisorial envelope of A, then D is $\mathscr{C}$-divisorial.*

PROOF. Let $u: A \to E$ be an embedding of A into an injective object E, and let ψ be the canonical projection $E \to E/A$. Consider the following pull back diagram:

$$\begin{array}{ccc} L & \longrightarrow & L_{\mathscr{C}}(E/A) \\ \big\downarrow{\scriptstyle j} & & \big\downarrow \\ E & \xrightarrow{\ \psi\ } & E/A \end{array}$$

Let $u': A \to L$ be the canonical monomorphism such that $u = ju'$. Since E is injective, there is a morphism $v: D \to E$ such that $vd = u$ $(= ju')$. Then, by Lemma 2.4, v factors through j i.e. there is a morphism $i: D \to L$ such that $v = ji$. Since j is a monomorphism, $jid = ju'$ implies $id = u'$. On the other hand, by the property of d, we have a morphism $\delta: L \to D$ such that $d = \delta u'$ $(= \delta id)$, which implies that δi is an isomorphism. Now let $\alpha: M \to N$ be a monomorphism such that $\mathrm{Coker}(\alpha)$ is an object in $\mathscr{C}$, and let $f: M \to D$ be a morphism. Then there is a morphism $g': N \to E$ with $g'\alpha = jif$. By Lemma 2.4, g' factors through j i.e., there is a morphism $g'': N \to L$ with $g' = jg''$. Since j is a monomorphism, we have $g''\alpha = if$. Therefore $((\delta i)^{-1}\delta g'')\alpha = f$; this shows that D is $\mathscr{C}$-divisorial.

PROPOSITION 2.10. *Assume that $\mathscr{A}$ has enough injectives. Then the following statements concerning an object D are equivalent:*

(i) *D is $\mathscr{C}$-divisorial.*

(ii) $\mathrm{Ext}^1(N, D) = 0$ *for every object N in $\mathscr{C}$.*

(iii) $\mathrm{R}^1\mathrm{L}_{\mathscr{C}}(D) = 0$ *and* $\mathrm{L}_{\mathscr{C}}(D)$ *is $\mathscr{C}$-divisorial.*

PROOF. (ii)⇒(i): If a monomorphism $\alpha: M \to N$ is a $\mathscr{C}$-isomorphism, then $\mathrm{Hom}(N, D) \to \mathrm{Hom}(M, D) \to \mathrm{Ext}^1(N/M, D)$ is exact, and $\mathrm{Ext}^1(N/M, D)=0$ since N/M is an object in $\mathscr{C}$. Therefore D is $\mathscr{C}$-divisorial.

(i)⇒(iii): By Th. 2.5, the canonical morphism $d: D \to D/\mathrm{L}_{\mathscr{C}}(D)$ is a weak $\mathscr{C}$-envelope of D; hence $\mathrm{R}^1\mathrm{L}_{\mathscr{C}}(D)=\mathrm{Coker}(d)=0$. The last assertion follows from Remark 2.8.

(iii)⇒(ii): Let $0 \to D \to E_0 \to E_1 \to E_2$ be an injective resolution of D. Since $\mathrm{R}^1\mathrm{L}_{\mathscr{C}}(D)=0$, the complex $0 \to \mathrm{L}_{\mathscr{C}}(D) \xrightarrow{\alpha} \mathrm{L}_{\mathscr{C}}(E_0) \xrightarrow{\beta} \mathrm{L}_{\mathscr{C}}(E_1) \to \mathrm{L}_{\mathscr{C}}(E_2)$ is exact. Since $\mathrm{L}_{\mathscr{C}}(D)$ is $\mathscr{C}$-divisorial, α is splitable; hence $\beta(\mathrm{L}_{\mathscr{C}}(E_0))$ is isomorphic to a direct sumand of $\mathrm{L}_{\mathscr{C}}(E_0)$, so that it is also $\mathscr{C}$-divisorial by Remark 2.8. The same argument shows that $\beta(\mathrm{L}_{\mathscr{C}}(E_0))$ is a direct sumand of $\mathrm{L}_{\mathscr{C}}(E_1)$. Then the complex $\mathrm{Hom}(A, \mathrm{L}_{\mathscr{C}}(E.))$ is exact for every A. Therefore, for every object N in $\mathscr{C}$, $\mathrm{Ext}^1(N, D)=\mathrm{H}^1(\mathrm{Hom}(N, E.))=\mathrm{H}^1(\mathrm{Hom}(N, \mathrm{L}_{\mathscr{C}}(E.)))=0$.

COROLLARY 2.11. *Assume that $\mathscr{A}$ has enough injectives.*

(1) *An object A is $\mathscr{C}$-pure if and only if* $\mathrm{Ext}^0(N, A)=0$ *for every object N in $\mathscr{C}$.*

(2) *An object A is $\mathscr{C}$-closed if and only if* $\mathrm{Ext}^0(N, A)=\mathrm{Ext}^1(N, A)=0$ *for every object N in $\mathscr{C}$.*

PROOF. The assertion (1) follows from the definition of $\mathscr{C}$-pure objects. As for the last one, we may assume that A is $\mathscr{C}$-pure. Then, by Cor. 2.3, A is $\mathscr{C}$-closed if and only if A is $\mathscr{C}$-divisorial. Therefore the assertion (2) follows from Prop. 2.10.

REMARK 2.12. Let N be a subobject of M. Then we can introduce the notion of the $\mathscr{C}$-divisorial envelope of N in M, naturally extending the definition of divisorial envelopes in a module (cf. [3], 1 n°7), i.e., the subobject D of M containing N such that $D/N=\mathrm{L}_{\mathscr{C}}(M/N)$. We also say that N is $\mathscr{C}$-divisorial in M if M/N is $\mathscr{C}$-pure. It is easy to see that if M is $\mathscr{C}$-divisorial and N is $\mathscr{C}$-divisorial in M, then N is $\mathscr{C}$-divisorial.

For an example, we shall show a property of functors $\mathrm{R}^p\mathrm{L}_{\mathscr{C}}$, which is a well-known theorem in the local cohomology theory. Before stating this, we introduce a definition for complexes; we say that a complex $C_0 \xrightarrow{d_0} C_1 \xrightarrow{d_1} C_2 \to \cdots \xrightarrow{d_n} C_{n+1}$ is a *splitting exact complex* if $0 \to \mathrm{Im}(d_{i-1}) \to C_i \to \mathrm{Im}(d_i) \to 0$ is a splitting exact sequence for each $i=1,\dots, n$, and $\mathrm{Ker}(d_0)$ is a direct summand of C_0.

PROPOSITION 2.13. *Assume that $\mathscr{A}$ has enough injectives. Then the following statements, concerning an object A and a positive integer n, are equivalent*:

(i) $\mathrm{Ext}^p(N, A)=0$ *for every object N in $\mathscr{C}$ and $p<n$.*

(ii) $\mathrm{R}^p\mathrm{L}_{\mathscr{C}}(A)=0$ *for $p<n$.*

PROOF. (i)⇒(ii): Let $0 \to A \to E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots$ be an injective resolution of A. If N is an object in $\mathscr{C}$, then $H^p(\mathrm{Hom}(N, L_{\mathscr{C}}(E.))) = H^p(\mathrm{Hom}(N, E.)) = \mathrm{Ext}^p(N, A) = 0$ for $p<n$. Now, let $j_p: N_p \to L_{\mathscr{C}}(E_p)$ be the kernel of $L_{\mathscr{C}}(d_p)$, where $p<n$. Since $H^p(\mathrm{Hom}(N_p, L_{\mathscr{C}}(E.))) = 0$, there is a morphism $\alpha_p: N_p \to L_{\mathscr{C}}(E_{p-1})$ such that $L_{\mathscr{C}}(d_{p-1})\alpha_p = j_p$. Hence $0 \to L_{\mathscr{C}}(E_0) \to \cdots \to L_{\mathscr{C}}(E_n)$ is a splitting exact complex. Therefore $R^pL_{\mathscr{C}}(A) = 0$ for $p<n$.

(ii)⇒(i): Let $0 \to A \to E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots$ be an injective resolution of A. By $R^pL_{\mathscr{C}}(A) = 0$ for $p<n$, the complex $0 \to L_{\mathscr{C}}(E_0) \to \cdots \to L_{\mathscr{C}}(E_n)$ is exact. Since each E_p is injective, $L_{\mathscr{C}}(E_p)$ is $\mathscr{C}$-divisorial by Remark 2.8. In particular, $0 \to L_{\mathscr{C}}(E_0) \to L_{\mathscr{C}}(E_1) \to L_{\mathscr{C}}(E_2)$ is a splitting exact complex. Now suppose that $0 \to L_{\mathscr{C}}(E_0) \to \cdots \to L_{\mathscr{C}}(E_p)$ $(p<n)$ is a splitting exact complex. Then $\mathrm{Im}(L_{\mathscr{C}}(d_{p-1}))$ is a direct summand of $L_{\mathscr{C}}(E_{p-1})$. Hence it is $\mathscr{C}$-divisorial by Remark 2.8, whence the morphism $\mathrm{Im}(L_{\mathscr{C}}(d_{p-1})) \to L_{\mathscr{C}}(E_p)$ is splitable. By induction, $0 \to L_{\mathscr{C}}(E_0) \to \cdots \to L_{\mathscr{C}}(E_n)$ is a splitting exact complex. Therefore $\mathrm{Ext}^p(N, A) = H^p(\mathrm{Hom}(N, E.)) = H^p(\mathrm{Hom}(N, L_{\mathscr{C}}(E.))) = 0$ for every object N in $\mathscr{C}$ and $p<n$.

COROLLARY (to the proof). *If* $\mathrm{Ext}^n(N, A) = 0$ *for every* N *in* $\mathscr{C}$, *then* $R^nL_{\mathscr{C}}(A) = 0$.

For the rest of this section, we assume that $\mathscr{A}$ has enough injectives. We have constructed two functors T (defined by weak $\mathscr{C}$-envelopes) and F (defined by $\mathscr{C}$-envelopes). By Prop. 1.4, these are related by $TT \cong F$. Moreover, the left exactness of T and F implies that T and F are isomorphic to each other if and only if, for every injective object E, $E/L_{\mathscr{C}}(E)$ is $\mathscr{C}$-closed. The following result is essentially contained in [3], Th. 2.

PROPOSITION 2.14. *Assume that* $\mathscr{A}$ *has injective envelopes. Then the followings are equivalent to each other*:

(i) *If* E *is injective, then* $L_{\mathscr{C}}(E)$ *is also injective.*

(ii) *If* A *is not an object in* $\mathscr{C}$, *then* A *has a non-zero* $\mathscr{C}$*-pure subobject.*

(iii) *Let* $A \to B$ *be an essential morphism. If* A *is an object in* $\mathscr{C}$, *then so is* B.

(iv) $E(L_{\mathscr{C}}(A)) = L_{\mathscr{C}}(E(A))$ *for every object* A (*where* $E(X)$ *is an injective envelope of* X).

PROOF. (i)⇒(ii): Let A be an object, $u: A \to E$ an injective envelope of A. By assumption, $E = L_{\mathscr{C}}(E) \oplus L$ for some $\mathscr{C}$-pure subobject L of E. Now, if A is not an object in $\mathscr{C}$, then $L \neq 0$; hence $u^{-1}(L)$ is a non-zero $\mathscr{C}$-pure subobject of A.

(ii)⇒(iii): If B is not an object in $\mathscr{C}$, then B has a non-zero $\mathscr{C}$-pure subobject L. Since B is an essential extension of A, $A \cap L$ is a non-zero $\mathscr{C}$-pure subobject of A, which is a contradiction.

(iii)⇒(iv): Let A be an object, $u: A\to E$ an injective envelope of A. It is clear that $L_{\mathscr{C}}(u): L_{\mathscr{C}}(A)\to L_{\mathscr{C}}(E)$ is essential. Let $v: L_{\mathscr{C}}(A)\to E'$ be an injective envelope of $L_{\mathscr{C}}(A)$. Then, by our assumption, E' is an object in $\mathscr{C}$. Since E' is injective, we have a morphism $f: L_{\mathscr{C}}(E)\to E'$ such that $fL_{\mathscr{C}}(u)=v$, which is also an essential monomorphism; hence f is splitable by the $\mathscr{C}$-divisoriality of $L_{\mathscr{C}}(E)$, and is essential. Therefore f is an isomorphism.

(iv)⇒(i): If E is injective, then by our assumption $L_{\mathscr{C}}(E)=E(L_{\mathscr{C}}(E))$. Therefore $L_{\mathscr{C}}(E)$ is injective.

We say that an object A is $\mathscr{C}$-flasque if $A/L_{\mathscr{C}}(A)$ is $\mathscr{C}$-closed. If the conditions in Prop. 2.14 are satisfied, then $E/L_{\mathscr{C}}(E)$ is injective and $\mathscr{C}$-pure for every injective object E; hence it is $\mathscr{C}$-closed. Therefore injective objects are $\mathscr{C}$-flasque. $\mathscr{C}$-flasqueness of injective objects can be also interpreted by the functor $R^2L_{\mathscr{C}}$.

PROPOSITION 2.15. *Every injective object is $\mathscr{C}$-flasque if and only if $R^2L_{\mathscr{C}}(N)=0$ for every object N in $\mathscr{C}$.*

PROOF. (Necessity) Note that two functors T and F are isomorphic to each other. Let N be an object in $\mathscr{C}$, and let $N\to E_0\to E_1\to E_2$ be an injective resolution of N. Then $0\to F(E_0/N)\to F(E_1)\to F(E_2)$ is exact. Hence $0\to F(E_0)\to F(E_1)\to F(E_2)$ is also exact since $F(E_0)\cong F(E_0/N)$. Therefore $R^2L_{\mathscr{C}}(N)\cong R^1T(N)\cong R^1F(N)=0$.

(Sufficiency) Let E be an injective object. Consider the following exact sequence: $0\to L_{\mathscr{C}}(E)\to E\to E/L_{\mathscr{C}}(E)\to 0$. Then we have an exact sequence: $0\to T(E)\to T(E/L_{\mathscr{C}}(E)\to R^1T(L_{\mathscr{C}}(E))$. By our assumption, $R^1T(L_{\mathscr{C}}(E))\cong R^2L_{\mathscr{C}}(L_{\mathscr{C}}(E))=0$; hence $E/L_{\mathscr{C}}(E)\cong T(E)\cong T(E/L_{\mathscr{C}}(E))$. Therefore, by Prop. 1.4, $E/L_{\mathscr{C}}(E)$ is $\mathscr{C}$-closed.

REMARK. If the conditions in Prop. 2.14 are satisfied, then $R^pL_{\mathscr{C}}(N)=0$ for every object N in $\mathscr{C}$ and $p>0$. In fact, if $E.$ is a minimal injective resolution of an object N in $\mathscr{C}$, then each E_p is also an object in $\mathscr{C}$.

§3. The category of Mod (A)

Let A be a commutative ring with 1. Consider a closure operation D on the lattice of ideals of A satisfying the following conditions:

(1) $\mathfrak{a}\subseteq D(\mathfrak{a})$ *for every ideal* $\mathfrak{a}$.
(2) $DD=D$.
(3) $\mathfrak{a}\subseteq\mathfrak{b}\Rightarrow D(\mathfrak{a})\subseteq D(\mathfrak{b})$.
(4) $D(\mathfrak{a}: x)=D(\mathfrak{a}): x$.

Then D defines a family of ideals **I** satisfying the following conditions:

(5) $\mathfrak{a}\subseteq\mathfrak{b}$ *and* $\mathfrak{a}\in\mathbf{I}\Rightarrow\mathfrak{b}\in\mathbf{I}$.

(6) $\mathfrak{a}\subseteq\mathfrak{b}$, $\mathfrak{b}\in\mathbf{I}$ *and* $\mathfrak{a}: b\in\mathbf{I}$ *for every* b *in* $\mathfrak{b}\Rightarrow\mathfrak{a}\in\mathbf{I}$.

In fact, let $\mathbf{I}=\{\mathfrak{a};\ D(\mathfrak{a})=A\}$. We shall show that $\mathbf{I}$ satisfies the conditions (5) and (6) above. The assertion (5) follows from (1) and (3). Now, if $\mathfrak{a}\subseteq\mathfrak{b}$, $\mathfrak{b}\in\mathbf{I}$ and $\mathfrak{a}: b\in\mathbf{I}$ for every b in $\mathfrak{b}$, then $D(\mathfrak{a}: b)=D(\mathfrak{a}): b=A$ for every b in $\mathfrak{b}$; hence $\mathfrak{b}\subseteq D(\mathfrak{a})$. Therefore $D(\mathfrak{a})=A$.

A family $\mathbf{I}$ of ideals satisfying (5) and (6) also satisfies the following condition:

(7) $\mathfrak{a}, \mathfrak{b}\in\mathbf{I}\Rightarrow\mathfrak{a}\cdot\mathfrak{b}$ *and* $\mathfrak{a}\cap\mathfrak{b}\in\mathbf{I}$.

In fact, for every b in $\mathfrak{b}$, $\mathfrak{ab}: b$ contains $\mathfrak{a}$. Hence by (5), $\mathfrak{ab}: b\in\mathbf{I}$ for every b in $\mathfrak{b}$. Therefore $\mathfrak{ab}\in\mathbf{I}$ by (6).

The corresponding property for D is

(8) $D(\mathfrak{a}\cap\mathfrak{b})=D(\mathfrak{a})\cap D(\mathfrak{b})$.

In fact, $x\in D(\mathfrak{a})\cap D(\mathfrak{b})\Leftrightarrow D(\mathfrak{a}: x)=A=D(\mathfrak{b}: x)\Leftrightarrow\mathfrak{a}: x$ and $\mathfrak{b}: x\in\mathbf{I}\Leftrightarrow\mathfrak{a}\cap\mathfrak{b}: x=(\mathfrak{a}: x)\cap(\mathfrak{b}: x)\in\mathbf{I}\Leftrightarrow D(\mathfrak{a}\cap\mathfrak{b}): x=A\Leftrightarrow x\in D(\mathfrak{a}\cap\mathfrak{b})$.

PROPOSITION 3.1. *There is a bijective correspondence between the class of closure operations on the lattice of ideals of A, satisfying* (1) (2) (3) *and* (4), *and the class of families of ideals, satisfying* (5) *and* (6).

PROOF. Let $\mathbf{I}$ be a family of ideals satisfying (5) and (6). Then we define an operation D as follow: $D(\mathfrak{a})=\{a\in A;\ \mathfrak{a}: a\in\mathbf{I}\}$ for every ideal $\mathfrak{a}$. We must show that $D(\mathfrak{a})$ is an ideal of A. If $x, y\in D(\mathfrak{a})$, then $\mathfrak{a}: x+y\supseteq(\mathfrak{a}: x)\cap(\mathfrak{a}: y)$; hence $\mathfrak{a}: x+y\in\mathbf{I}$ by (7) and (5); therefore $x+y\in D(\mathfrak{a})$. If $x\in D(\mathfrak{a})$ and $a\in A$, then $\mathfrak{a}: ax\supseteq\mathfrak{a}: x$; hence $\mathfrak{a}: ax\in\mathbf{I}$ by (5); therefore $ax\in D(\mathfrak{a})$. These show that $D(\mathfrak{a})$ is an ideal of A. Obviously, D satisfies (1) and (3). If $x\in DD(\mathfrak{a})$, then $D(\mathfrak{a}): x\in\mathbf{I}$. For every $b\in D(\mathfrak{a}): x$, $(\mathfrak{a}: x): b=\mathfrak{a}: xb\in\mathbf{I}$, since $xb\in D(\mathfrak{a})$; hence $\mathfrak{a}: x\in\mathbf{I}$ by (6). Therefore $x\in D(\mathfrak{a})$, so we have $D(\mathfrak{a})=DD(\mathfrak{a})$. Next, $y\in D(\mathfrak{a}: x)\Leftrightarrow(\mathfrak{a}: x): y=\mathfrak{a}: xy\in\mathbf{I}\Leftrightarrow xy\in D(\mathfrak{a})\Leftrightarrow y\in D(\mathfrak{a}): x$; hence $D(\mathfrak{a}: x)=D(\mathfrak{a}): x$. Finally, we must show that $\mathbf{I}=\{\mathfrak{a};\ D(\mathfrak{a})=A\}$. In fact, if $D(\mathfrak{a})=A$, then $\mathfrak{a}=\mathfrak{a}: 1\in\mathbf{I}$ by definition. Conversely, if $\mathfrak{a}\in\mathbf{I}$, then $\mathfrak{a}: 1=\mathfrak{a}\in\mathbf{I}$; hence $1\in D(\mathfrak{a})$ by definition; therefore $D(\mathfrak{a})=A$.

EXAMPLE 1. If A is a domain, then the operation $A:_K(A:_K*)$ satisfies the conditions (1) (2) (3) and (4), where K is the field of fractions of A and $A:_K N=\{x\in K;\ xN\subseteq A\}$ for a fractional ideal N. The corresponding family of ideals is $\{\mathfrak{a};\ A:_K\mathfrak{a}=A\}$.

EXAMPLE 2. Let $f: A\to B$ be a ring homomorphism. Then the family of ideals $\{\mathfrak{a};\ \mathfrak{a}$ is an ideal of A such that $f(\mathfrak{a})B=B\}$ satisfies the conditions (5) and

(6).

A family of ideals satisfying the conditions (5) and (6) is a Gabriel topology on A (cf. [4], §5, Chap. VI). We shall quote some propositions from [4].

PROPOSITION 3.2. ([4], Prop. 2.1 and Th. 5.1, Chap. VI) *There are bijective correspondences between*

(i) *Gabriel topologies on* A,

(ii) *Left exact radicals*,

(iii) *Classes of* A*-modules closed under quotient, coproduct, extension and subobject (or, equivalently, localizing subcategories of* $\mathrm{Mod}(A)$).

If $\mathbf{I}$ is a Gabriel topology on A, then the corresponding localizing subcategory $\mathscr{C}$ is $\{A\text{-module } M \text{ such that } \mathrm{Ann}(x)\in\mathbf{I} \text{ for every } x \text{ in } M\}$. Conversely, if $\mathscr{C}$ is a localizing subcategory of $\mathrm{Mod}(A)$, then $\mathbf{I}=\{\text{ideal } \mathfrak{a} \text{ of } A \text{ such that } A/\mathfrak{a} \text{ is an object in } \mathscr{C}\}$ is the corresponding Gabriel topology on A. Moreover, the operation D on the class of ideals, corresponding to a localizing subcategory $\mathscr{C}$, has the property: $D(\mathfrak{a})/\mathfrak{a}=\mathrm{L}_{\mathscr{C}}(A/\mathfrak{a})$ i.e. $D(\mathfrak{a})$ is the $\mathscr{C}$-divisorial envelope of $\mathfrak{a}$ in A.

PROPOSITION 3.3. *Let* $\mathscr{C}$ *be a localizing subcategory of* $\mathrm{Mod}(A)$, $\mathbf{I}$ *the corresponding Gabriel topology on* A. *Let* M *be an* A*-module. Then the canonical map* $M\to\varinjlim_{\mathfrak{a}\in\mathbf{I}}\mathrm{Hom}(\mathfrak{a}, M)$ *is a weak* $\mathscr{C}$*-envelope of* M.

PROOF. It is easy to see that $\mathrm{R}^p\mathrm{L}_{\mathscr{C}}(M)=\varinjlim_{\mathfrak{a}\in\mathbf{I}}\mathrm{Ext}^p(A/\mathfrak{a}, M)$. Since $M=\varinjlim_{\mathfrak{a}\in\mathbf{I}}\mathrm{Hom}(A, M)$, the exact sequences $0\to\mathfrak{a}\to A\to A/\mathfrak{a}\to 0$, $\mathfrak{a}\in\mathbf{I}$, induce an exact sequence $0\to\mathrm{L}_{\mathscr{C}}(M)\to M\to\varinjlim_{\mathfrak{a}\in\mathbf{I}}\mathrm{Hom}(\mathfrak{a}, M)\to\mathrm{R}^1\mathrm{L}_{\mathscr{C}}(M)\to 0$. Hence, for every injective A-module E, $E\to\varinjlim_{\mathfrak{a}\in\mathbf{I}}\mathrm{Hom}(\mathfrak{a}, E)$ is a weak $\mathscr{C}$-envelope of E. On the other hand, it is clear that $\varinjlim_{\mathfrak{a}\in\mathbf{I}}\mathrm{Hom}(\mathfrak{a}, *)$ is a left exact functor. Therefore $M\to\varinjlim_{\mathfrak{a}\in\mathbf{I}}\mathrm{Hom}(\mathfrak{a}, M)$ is a weak $\mathscr{C}$-envelope of M by Prop. 1.6.

COROLLARY 3.4. $M\to\varinjlim_{\mathfrak{a}\in\mathbf{I}}\mathrm{Hom}(\mathfrak{a}, M/\mathrm{L}_{\mathscr{C}}(M))$ *is a* $\mathscr{C}$*-envelope of* M.

PROPOSITION 3.5. (cf. [4] Prop. 6.13, Chap. VI) *Let* $\mathscr{C}$ *be a localizing subcategory of* $\mathrm{Mod}(A)$, $\mathbf{I}$ *the corresponding Gabriel topology on* A. *Then the following conditions are equivalent*:

(i) *There is a subset* Z *of* $\mathrm{Spec}(A)$, *stable under specialization, such that* $\mathscr{C}=\{A\text{-module } M;\ \mathrm{Supp}(M)\subseteq Z\}$.

(ii) *There is a subset* Z *of* $\mathrm{Spec}(A)$, *stable under specialization, such that* $\mathbf{I}=\{\text{ideal } \mathfrak{a} \text{ of } A;\ \mathrm{V}(\mathfrak{a})\subseteq Z\}$.

(iii) *If* $\mathrm{V}(\mathfrak{a})\subseteq\mathbf{I}$, *then* $\mathfrak{a}\in\mathbf{I}$.

COROLLARY 3.6. ([4], Cor. 6.15, Chap. VI) *If* $\mathbf{I}$ *has a cofinal subfamily consisting of finitely generated ideals, then the conditions in* Prop. 3.5 *are all* *satisfied. In particular, if A is noetherian, then there is a one-to-one correspondence between the class of localizing subcategories of* Mod(A) *and the class of subsets of* Spec(A) *which are stable under specialization.*

PROPOSITION 3.7. *Let Z be a subset of* Spec(A), *stable under specialization, and let $\mathscr{C}$ be the localizing subcategory whose objects are A-modules M such that* $\mathrm{Supp}(M)\subseteq Z$. *Assume that* $\mathrm{Spec}(A)-Z$ *is quasi-compact. Then, for every A-module M, the map* $M\ (=\Gamma(\mathrm{Spec}(A),\ \tilde{M}))\rightarrow \varinjlim_{\mathrm{V}(\mathfrak{a})\subseteq Z} \Gamma(\mathrm{Spec}(A)-\mathrm{V}(\mathfrak{a}),\ \tilde{M})$ *induced by restrictions is a $\mathscr{C}$-envelope of M, where $\tilde{M}$ is the quasi-coherent* $\mathcal{O}_{\mathrm{Spce}(A)}$*-module associated to M. In particular, if* $\mathrm{Spec}(A)-Z$ *is quasi-compact and open, then a $\mathscr{C}$-envelope of M is given by* $M\rightarrow\Gamma(\mathrm{Spec}(A)-Z,\ \tilde{M})$.

PROOF. Let J (resp. J') be the family of open (resp. quasi-compact open) subsets of Spec(A) which contain $\mathrm{Spec}(A)-Z$. Since $\mathrm{Spec}(A)-Z$ is quasi-compact, J' is a cofinal subfamily of J. Let U be an element of J', and let $i\colon U\rightarrow \mathrm{Spec}(A)$ be the inclusion map. Then, for every quasi-coherent $\mathcal{O}_U$-module $\mathscr{F}$, $i_*(\mathscr{F})$ is also a quasi-coherent $\mathcal{O}_{\mathrm{Spec}(A)}$-module. Hence the same argument described in EGA. IV, 5.9 is also valid in our case. Therefore the functor $\varinjlim_{\mathrm{V}(\mathfrak{a})\subseteq Z} \Gamma(\mathrm{Spec}(A)-\mathrm{V}(\mathfrak{a}),\ \tilde{*})$ satisfies the conditions (5) and (6) in the last remark in § 1, and the map $M\rightarrow \varinjlim_{\mathrm{V}(\mathfrak{a})\subseteq Z} \Gamma(\mathrm{Spec}(A)-\mathrm{V}(\mathfrak{a}),\ \tilde{M})$ is a $\mathscr{C}$-isomorphism.

PROPOSITION 3.8. *Let $\mathscr{C}$ be a localizing subcategory of* Mod(A). *Then the conditions in* Prop. 2.14 *are also equivalent to the followings*:

(v) *Let $\mathfrak{a}$ be an ideal of A such that $A/\mathfrak{a}$ does not belong to $\mathscr{C}$. Then $A/\mathfrak{a}: a$ is $\mathscr{C}$-pure for some $a\in A-\mathfrak{a}$.*

(vi) *Let $\mathfrak{a}$ be an ideal of A. Then there are ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ such that $\mathfrak{a}=\mathfrak{a}_1\cap\mathfrak{a}_2$, $A/\mathfrak{a}_1\in\mathscr{C}$ and $A/\mathfrak{a}_2$ is $\mathscr{C}$-pure.*

PROOF. See [3], Th. 2.

REMARK. Let $\mathscr{C}$ be a localizing subcategory of Mod(A), $\mathbf{I}$ the corresponding Gabriel topology on A. Then our definition of $\mathscr{C}$-divisorial envelopes is equal to the definition of $\mathbf{I}$-injective envelopes by Prop. 2.6 and [4], Prop. 2.1, Chap. IX.

Let A be a graded ring. We denote the category of graded A-modules by *Mod(A).

PROPOSITION 3.9. *There are bijective correspondences between*

(i) *operations D on the homogeneous ideals of A, satisfying the following conditions*:

(1) $\mathfrak{a}\subseteq D(\mathfrak{a})$

(2) $DD=D$

(3) $\mathfrak{a}\subseteq\mathfrak{b}\Rightarrow D(\mathfrak{a})\subseteq D(\mathfrak{b})$

(4) $D(\mathfrak{a}: x)=D(\mathfrak{a}): x$, *where* x *is a homogeneous element of* A

(ii) *families* $\mathbf{I}$ *of homogeneous ideals satisfying the following conditions*:

(5) $\mathfrak{a}\subseteq\mathfrak{b}$ *and* $\mathfrak{a}\in\mathbf{I}\Rightarrow\mathfrak{b}\in\mathbf{I}$

(6) $\mathfrak{a}\subseteq\mathfrak{b}$, $\mathfrak{b}\in\mathbf{I}$ *and* $\mathfrak{a}: b\in\mathbf{I}$ *for every homogeneous element* b *of* $\mathfrak{b}\Rightarrow\mathfrak{a}\in\mathbf{I}$

(iii) *localizing subcategories of* $^*\mathrm{Mod}(A)$.

PROOF. The same arguments, which we used in the proofs of Prop. 3.1 and 3.2 are also valid in this case.

PROPOSITION 3.10. *Let* $\mathscr{C}$ *be a localizing subcategory of* $^*\mathrm{Mod}(A)$, $\mathbf{I}$ *the corresponding family of homogeneous ideals. Then the functor* T, *defined by weak* $\mathscr{C}$*-envelopes, is* $\bigoplus_{n\in\mathbf{Z}}(\varinjlim_{\mathfrak{a}\in\mathbf{I}}\mathrm{Hom}(\mathfrak{a}, *(n)))$.

PROOF. Since $M\cong\bigoplus_{n\in\mathbf{Z}}\mathrm{Hom}(A, M(n))$ and $\mathrm{L}_{\mathscr{C}}(M)\cong\bigoplus_{n\in\mathbf{Z}}(\varinjlim_{\mathfrak{a}\in\mathbf{I}}\mathrm{Hom}(A/\mathfrak{a}, M(n)))$ for every graded A-module M, we have an exact sequence $0\to\mathrm{L}_{\mathscr{C}}(M)\to M\xrightarrow{\rho(M)}\bigoplus_{n\in\mathbf{Z}}(\varinjlim_{\mathfrak{a}\in\mathbf{I}}\mathrm{Hom}(\mathfrak{a}, M(n)))$ such that, if M is injective, then $\rho(M)$ is surjective. Therefore $\rho(M)$ is a weak $\mathscr{C}$-envelope of M, since $\bigoplus_{n\in\mathbf{Z}}(\varinjlim_{\mathfrak{a}\in\mathbf{I}}\mathrm{Hom}(\mathfrak{a}, *(n)))$ is left exact.

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