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Notes on the Brauer Liftings of Finite Classical Groups

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Introduction

Let k be a finite field with algebraic closure K, let H be a finite group and let $r: H \to GL_m(k)$ be a (modular) representation of H. For $x \in H$, let $u_1(x), \dots, u_m(x)$ denote the eigenvalues of $r(x)$. Define the complex valued function $b_{r,\theta}$ on H by $b_{r,\theta}(x) = \sum_{i=1}^{m} \theta(u_i(x))$, where θ is a character $K^\times \to \mathbf{C}^\times$. J. A. Green [5] proved that $b_{r,\theta}$ is then a generalized character of H, i.e. an integral linear combination of irreducible characters of H. In this paper we call $b_{r,\theta}$ the Brauer lifting of r associated to θ. It seems interesting to know the irreducible constituents of $b_{r,\theta}$ for a finite Chevalley group H, i.e. a finite group of k-rational points of a connected reductive linear algebraic group defined over k. For $H = GL_m(k)$, r the natural representation, J. A. Green [5] decomposed $b_{r,\theta}$, and when θ is in general position he obtained an important irreducible character, a cuspidal character.

We are interested in other classical groups $H = SO_{2n+1}(k)$, $GSp_{2n}(k), \dots$ etc. Let r be the natural representation $H \to GL_m(k)$ and assume that θ is injective. If the number of elements in k is greater than 3, then the inner product on H, $< b_{r,\theta}, b_{r,\theta} >_H$ equals m. This is proved in § 2 by making use of a certain inner product formula, which is the simplest one among those obtained by N. Kawanaka [7]. Next in § 3, using an induction argument, we decompose $b_{r,\theta}$ into an alternating sum of irreducible characters. The same result is announced by G. Lusztig [10] at Vancouver Congress of I. C. M. and when $H = GL_m(k)$, T. A. Springer [12] has decomposed $b_{r,\theta}$ using the similar method to ours.

In the case of the group of symplectic similitudes $H = GSp_{2n}(k)$, we have the following result.

As maximal parabolic subgroups of $GSp_{2n}(k)$, we choose

$$P_o = \left\{ \begin{bmatrix} A & * \\ O & D \end{bmatrix} \in GSp_{2n}(k) \,|\, A,\, D \in GL_n(k) \right\},$$

$$P_i = \left\{ \begin{bmatrix} A & & * \\ & X & \\ O & & D \end{bmatrix} \in GSp_{2n}(k) \,|\, A,\, D \in GL_{n-i}(k),\ X \in GSp_{2i}(k) \right\},\ (i = 1, \dots, n-1).$$

Let χ be a character of $GL_{n-i}(k)$ (resp. $GSp_{2i}(k)$). We extend χ to a character $\tilde{\chi}$ of P_i by the natural homomorphism

$$P_i \ni \begin{bmatrix} A & & * \\ & X & \\ O & & D \end{bmatrix} \longmapsto A \in GL_{n-i}(k) \quad (\text{resp. } P_i \ni \begin{bmatrix} A & & * \\ & X & \\ O & & D \end{bmatrix} \longmapsto X \in GSp_{2i}(k)).$$

In general, for a subgroup $P \subset H$ and a representation (resp. a character) ρ of P, we denote by $\mathrm{Ind}_P^H(\rho)$ the induced representation (resp. the induced character) of ρ from P to H. With these notations we have

THEOREM 3.4. *Let $H = GSp_{2n}(k)$ and write $b_{2n,\theta} = b_{r,\theta}$ for brevity.*

a) *There then exists a unique irreducible cuspidal character $\pi'_{n,\theta}$ of H such that $< b_{2n,\theta}, \pi'_{n,\theta} >_H \neq 0$.*

b) *For $i = 0, 1, 2, \ldots, n-1$, let $\pi_{n-i,\theta}$ be the irreducible cuspidal character of $GL_{n-i}(k)$ obtained by decomposing the Brauer lifting of the natural representation of $GL_{n-i}(k)$ associated to a injective character θ (3.3). Then $\mathrm{Ind}_{P_i}^H(\tilde{\pi}_{n-i,\theta})$ is irreducible for every i.*

c) *For $i = 1, 2, \ldots, n-1$, let $\pi'_{i,\theta}$ be the irreducible cuspidal character of $GSp_{2i}(k)$ defined in a). Then $< \mathrm{Ind}_{P_i}^H(\tilde{\pi}'_{i,\theta}), \mathrm{Ind}_{P_i}^H(\tilde{\pi}'_{i,\theta}) >_H = 2$ and there exists the irreducible constituent m_i of $\mathrm{Ind}_{P_i}^H(\tilde{\pi}'_{i,\theta})$ such that $< b_{2n,\theta}, m_i >_H \neq 0$.*

d) $b_{2n,\theta} = \sum_{i=0}^{n-1} (-1)^{n-i-1} \mathrm{Ind}_{P_i}^H(\tilde{\pi}_{n-i,\theta}) + (-1)^n \pi'_{n,\theta} + \sum_{i=1}^{n-1} (-1)^i m_i$.

Similarly we can decompose the Brauer liftings for other classical groups (§ 3).

In § 4, we further investigate the irreducible constituents of $b_{r,\theta}$ obtained in § 3, making use of Deligne-Lusztig's theory [3]. Let ρ be an irreducible character of a finite Chevalley group H. We assume that the algebraic group whose k-rational points constitute H has a connected center. If the characteristic p of k is good (e.g. if $p \neq 2$ for classical groups), J. A. Green, G. I. Lehrer and G. Lusztig [6] proved that $\rho(u) = 0$, 1 or -1 for any regular unipotent element u in H. When $\rho(u) \neq 0$, we call such ρ a "semisimple character"[1)] (4.1.17). We then have $b_{r,\theta}(u) = m$ for any unipotent u in H. Hence, the irreducible constituents of $b_{r,\theta}$ have to be semisimple characters. Recently, P. Deligne and G. Lusztig [3] have constructed a virtual representation R_T^ω and proved that such semisimple characters are described explicitly in terms of R_T^ω (4.1.15), where T is a maximal torus fixed by the Frobenius map σ and ω is a character of T_σ. We also denote by R_T^ω the (generalized) character defined by R_T^ω. In the case of $H = G_\sigma = GSp_{2n}(k)$, we have

1) This notion "semisimple character" is communicated by Professor N. Kawanaka.

THEOREM. a) (4.2.4, 4.2.5) $\pi'_{n,\theta}=(-1)^n R_{T(n)}^{\theta(n)}$, *where* $T(n)$ *is the Coxter torus of* GSp_{2n} *and* $\theta(n)$ *is an injective character of* $T(n)_\sigma$ *defined by* θ.

b) (4.2.11) *Let* $\mathfrak{T}_{n-i}$ $(i=1,\dots,n-1)$ *be the set of* σ*-stable maximal tori of* $GSp_{2(n-i)}$ *(up to the conjugacy of* $GSp_{2(n-i)}(k)$*). Put*

$$\mathfrak{T}=\left\{\begin{bmatrix} T_{11} & O & T_{12} \\ O & T(i) & O \\ T_{21} & O & T_{22}\end{bmatrix}\subset GSp_{2n}\ \middle|\ \begin{array}{l} T(i) \text{ is the Coxter torus of } GSp_{2i}, \\ \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22}\end{bmatrix}\in\mathfrak{T}_{n-i}\end{array}\right\}$$

(a set of σ*-stable maximal tori of* GSp_{2n}*), and for* $T\in\mathfrak{T}$ *put*

$$\omega\left(\begin{bmatrix} t_{11} & o & t_{12} \\ o & t(i) & o \\ t_{21} & o & t_{22}\end{bmatrix}\right)=\theta(i)(t(i)),\ \begin{bmatrix} t_{11} & o & t_{12} \\ o & t(i) & o \\ t_{11} & o & t_{22}\end{bmatrix}\in T_\sigma,\quad \textit{where}$$

$\theta(i)$ *is an injective character of* $T(i)_\sigma$ *defined by* θ. *Then*

$$m_i=(-1)^i\textstyle\sum_{T\in\mathfrak{T}}\dfrac{1}{<R_T^\omega,\,R_T^\omega>}\,R_T^\omega.$$

Finally we compute the degrees of the irreducible constituents of $b_{r,\theta}$. In the case of $H=GSp_{2n}(k)$, we have

COROLLARY 4.2.13. *Let* $H=GSp_{2n}(k)$. *Then*

a) $\quad \mathrm{Ind}_{P_i}^H(\tilde{\pi}_{n-i,\theta})(1)=\dfrac{(q^{2(i+1)}-1)\cdots(q^{2(n-1)}-1)(q^{2n}-1)}{q^{n-i}-1},$

b) $\quad \pi'_{n,\theta}(1)=\dfrac{(q^2-1\cdots(q^{2(n-1)}-1)\ (q^{2n}-1)}{q^n+1},$

c) $\quad m_i(1)=\dfrac{(q^{2(n-i+1)}-1)\cdots(q^{2(n-1)}-1)(q^{2n}-1)}{q^i+1}.$

For other classical groups, similar results are described in 4.2.14∼4.2.18.

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§1. Notation and Preliminaries

For a set A, $|A|$ denotes the number of its elements. When σ is a transformation of A, x^σ denotes the image by σ of $x\in A$ and A_σ denotes the set of fixed points of σ. For a group H, 1 denotes the identity element of H; when H is the matrix group with size n, we denote by I_n the identity matrix. For a finite group H, $\mathrm{I}(H)$ denotes the space of all complex valued class functions on H with inner product $<f, g>_H=|H|^{-1}\sum_{x\in H}f(x)\overline{g(x)}$ $(x\in H)$, where $\overline{g(x)}$ is the complex conjugate of $g(x)$. For a ring A, $A^{\times}$ denotes the group of units in A.

We recall some facts about algebraic groups over finite fields ([1], [17]). Let k be a finite field of characteristic p with q elements. We denotes its algebraic closure by K. Let G be a connected reductive linear algebraic group defined over K, and σ a surjective endomorphism of G such that $G_\sigma=\{x\in G|x^\sigma=x\}$ is finite. The finite group G_σ obtained in this manner is called a finite Chevalley group. Such Chevalley groups are classified in R. Steinberg [17].

EXAMPLES 1.1 ([2], [4], [16]).

a) Let $A=GL_m$ be the general linear group considered as a linear algebraic group defined over k. It is connected and reductive. The Frobenius map $c\mapsto c^q$ on k extends to an endomorphism σ of A onto itself, i.e. sending the matrix elements to their q-th powers. Then $A_\sigma=GL_m(k)$, the finite general linear groups.

a') Let A and σ be as in *a*). For $x\in A$, put

$$\tau(x)=J_m{}^t(x^\sigma)^{-1}J_m,\quad J_m=\underbrace{\begin{bmatrix} O & & & -1 \\ & & 1 & \\ & -1 & & \\ \cdot^{\cdot^{\cdot}} & & & O \end{bmatrix}}_{m}\Bigg\} m\ .$$

Then τ is an endomorphism of A onto itself. Let k_2 be the unique extention of k of degree 2 in K. Then $A=U_m(k_2)=\{x\in GL_m(k_2)|xJ_m{}^t(x^\sigma)=J_m\}$, the finite unitary group.

b) Assume that char. $k\neq 2$. Let $B=SO_{2n+1}=\{x\in GL_{2n+1}|\det x=1,$ $xJ_{2n+1}{}^tx=J_{2n+1}\}$, $J_{2n+1}=\underbrace{\begin{bmatrix} O & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & O \end{bmatrix}}_{2n+1}\Big\}2n+1.$

We consider B as a linear algebraic group defined over k, which is connected and semisimple. If σ is as in a), then $B_\sigma=SO_{2n+1}(k)=\{x\in SO_{2n+1}(k)|xJ_{2n+1}{}^tx=J_{2n+1}\}$, the finite special orthogonal group.

c) Let $C=GSp_{2n}=\{x\in GL_{2n}|xJ'_n{}^tx=\lambda J'_n,\ \lambda\in K^{\times}\}$,

$$J'_n=\begin{bmatrix} O & J_n \\ -J_n & O \end{bmatrix},\qquad J_n=\underbrace{\begin{bmatrix} O & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & O \end{bmatrix}}_{n}\Big\}n\ .$$

We consider C as a linear algebraic group defined over k. The center $Z(C)$

of C is $\{cI_{2n} | c \in K^{x}\}$ and we have $C = Z(C)Sp_{2n}$ where $Sp_{2n} = \{x \in GL_{2n} | xJ_n'{}^t x = J_n'\}$. It follows that C is connected and reductive. If σ is as in a), then $C_\sigma = GSp_{2n}(k) = \{x \in GL_{2n}(k) | xJ_n'{}^t x = \lambda J_n', \lambda \in k^x\}$, the finite group of symplectic similitudes.

d) Assume that char. $k \neq 2$. Let $D = GO_{2n}(J_{2n}) = \{x \in GL_{2n} | xJ_{2n}{}^t x = \lambda J_{2n}, \lambda \in K^x\}$, $J_{2n} = \underbrace{\begin{bmatrix} O & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & O \end{bmatrix}}_{2n} \Big\} 2n.$

We consider D as a linear algebraic group defined over k. Let D^o be the connected component of 1. Then D^o is connected and reductive. For $x \in D$, $xJ_{2n}{}^t x = \lambda J_{2n}$, we have $x \in D^o$ if and only if $\det x = \lambda^n$. Similarly, if σ is as in a), then $D^o = GO_{2n}^{o+}(k) = \{x \in GL_{2n}(k) | xJ_{2n}{}^t x = \lambda J_{2n}, \det x = \lambda^n, \lambda \in k^x\}$, the finite group of split orthogonal and direct similitudes.

d') Assume that cahr. $k \neq 2$. Let $D' = GO_{2n}(F_n) = \{x \in GL_{2n} | xF_n{}^t x = \lambda F_n, \lambda \in K^x\}$, $F_n = \begin{bmatrix} O & & J_{n-1} \\ & F_o & \\ J_{n-1} & & O \end{bmatrix}$, and $J_{n-1} = \begin{bmatrix} O & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & O \end{bmatrix}$, $F_o = \begin{bmatrix} 1 & o \\ o & -\varepsilon \end{bmatrix}$, where ε is a non-square element in k.

We note that D and D' are isomorphic over k_2 but not over k, where k_2 is the unique extention field of k of degree 2 in K. Let D'^o be the connected component of 1. Similarly, if σ is as in a), then $D_\sigma'^o = GO_{2n}^{o-}(k) = \{x \in GL_{2n}(k) | xF_n{}^t x = \lambda F_n, \det x = \lambda^n, \lambda \in k^x\}$, the finite group of twisted orthogonal and direct similitudes.

The basic tool for carrying out the study of algebraic groups over finite fields is the following extention of a theorem of Lang.

LEMMA 1.2 ([17], 10.1). *Let G be a connected linear algebraic group and σ a surjective endomorphism of G such that G_σ is finite. Then the map $f: x \mapsto x(x^{-1})^\sigma$ of G into G is surjective.*

COROLLARY 1.3 ([17], 10.10). *Assume G and σ are as in 1.2. Then σ fixes a Borel subgroup B and a maximal torus T_o contained in it. Any two such couples are conjugate by an element of G_σ.*

Until the end of this section, we assume that G is connected and reductive. Let P be a σ-stable parabolic subgroup of G. Its unipotent radical U is then also σ-stable. There exists a Levi subgroup M of P, which is a connected reductive subgroup of P. Note that M can be chosen to be σ-stable. In the Levi decomposition $P = MU$, we have

LEMMA 1.4 ([2], 8.5). a) $P_\sigma = N_{G_\sigma}(U_\sigma)$, *the normalizer of U_σ in G_σ.*
b) $P_\sigma = M_\sigma U_\sigma$ *and* $M_\sigma \cap U_\sigma = \{1\}$.

In particular, if P is a σ-stable Borel subgroup $B=MU$ of G, then M is a σ-stable maximal torus and U_σ is a p-Sylow subgroup of G_σ.

Let T_0 be a σ-stable maximal torus of G as in 1.3 and N be the normalizer of T_0 in G. Then N is also σ-stable. Let $W=N/T_0$ the Weyl group of G with respect to T_0. Then σ induces an automorphism of the finite group W. Let $T=x^{-1}T_0x$ $(x\in G)$ be a maximal torus of G. It is clear that T is σ-stable if and only if $x(x^{-1})^\sigma\in N$.

LEMMA 1.5 ([14], II, 1.2). a) *The classes of σ-stable maximal tori under conjugation by G_σ are in one-one correspondence with the elements of $H^1(\sigma, W)$ by*

$$x^{-1}T_0x \longrightarrow x(x^{-1})^\sigma T_0\in W\,,$$

where $H^1(\sigma, W)$ denotes W modulo the equivalence relation; $w_1\sim w_2$ $(w_1, w_2\in W)$ if $w_1=ww_2(w^{-1})^\sigma$ for some $w\in W$.

b) *If σ fixes each element of W, i.e., commutes with the action of W on T, then the classes in* a) *correspond to the conjugacy classes of W.*

LEMMA 1.6 (B. Srinivasan [15], Lemma 4). *If $c_T\in H^1(\sigma, W)$ corresponds to a σ-stable maximal torus T, then*

$$N_G(T)_\sigma/T_\sigma\simeq\{w\in W|ww'(w^\sigma)^{-1}=w'\},\qquad (w'\in c_T)\,.$$

Let G, σ and T be as above, and set $W(T)=N_G(T)/T$. Each element of $W(T)_\sigma$ is then represented in $N_G(T)_\sigma$ and $W(T)_\sigma$ is always isomorphic to $N_{G_\sigma}(T)/T_\sigma$ where $N_{G_\sigma}(T)$ is the normalizer of T in G_σ.

LEMMA 1.7 ([14], II, 1.8). *Let G, σ, T and $W(T)$ be as above. If no root relative to T vanishes on T_σ, then $N_G(T)_\sigma$ is the normalizer of T_σ in G_σ and $N_G(T)_\sigma/T_\sigma\simeq W(T)_\sigma$.*

By Lemma 1.6, we have

COROLLARY 1.8. *If no root relative to T vanishes on T_σ, then $W(T)_\sigma\simeq\{w\in W|ww'(w^\sigma)^{-1}=w'\}$, $(w'\in c_T)$.*

We recall the basic facts about the Jordan decomposition $x=x_sx_u=x_ux_s$ of an element of G into its semisimple and unipotent parts. The components x_s and x_u are uniquely determined by x. Further the decomposition is preserved by σ; thus if $x\in G_\sigma$, then x_s, $x_u\in G_\sigma$. The next lemma due to R. Steinberg is basic in the next section.

LEMMA 1.9 ([17], 14.14, 15.1). *Let G, σ, G_σ be as above and* Q *denote the order of a p-Sylow subgroup of G_σ. Then*

a) *The number of maximal tori of* G *fixed by* σ *is* Q^2.
b) *The number of unipotent elements of* G_σ *is* Q^2.

DEFINITION 1.10. An element x of G is said to be a *regular element* if $Z_G(x)$, the centralizer of x in G, has the minimum dimension among the centralizers of elements of G or, equivalently, if $C(x)$, the conjugacy class of x, has the maximum dimension.

Clearly, regular elements exist for dimension reasons.

LEMMA 1.11 ([14], III, § 1). *Let* G, σ *and* G_σ *be as above.*
a) *A unipotent element is regular if and only if it is contained in a unique Borel subgroup.*
b) *Assume that the center of* G *is connected and the characteristic* p *of* k *is good (see* [14], I, §4., *e.g. for type* A_r: *all, for type* B_r, C_r, *and* D_r: $p \neq 2$), *then the set of regular unipotent elements of* G_σ *forms a single conjugacy class.*

Let G, σ, G_σ, B, T, P and M be as above. By abuse of language, we shall call B_σ, T_σ, P_σ and M_σ a Borel subgroup of G_σ, a torus of G_σ, a parabolic subgroup of G_σ and a Levi subgroup of P_σ.

§2. The Brauer lifting and an inner product formula

An integral linear combination of characters of complex representations of a finite group is called a virtual character or a generalized character.

The following Lemma was introduced by J. A. Green, in his well-known work on the characters of $GL_n(k)$, whose proof is based upon Brauer's characterization of generalized characters.

LEMMA 2.1 (J. A. Green [5], Theorem 1). *Let* H *be a finite group,* $r: H \to GL_m(k)$ *be a (modular) representation of* H *and let* $\theta: K^\times \to \mathbf{C}^\times$ *be a character. For* $x \in H$, $u_1(x), \dots, u_m(x)$ *denote the eigenvalues of* $r(x)$. *Then the function* $b_{r,\theta}$ *on* H *defined by* $b_{r,\theta}(x) = \sum_{i=1}^{m} \theta(u_i(x))$, $(x \in H)$, *is a generalized character of* H.

In this paper we call $b_{r,\theta}$ the *Brauer lifting of* r associated to θ.

PROPOSITION 2.2 (T. A. Springer [12], 1.2). *Let* G_σ *be as in* § 1 *and* f *be an element of* $\mathrm{I}(G_\sigma)$. *Assume* $f(x) = f(y)$ *for any elements* $x, y \in G_\sigma$ *such that* $x_s = y_s$. *Then*

$$\sum_{x \in G_\sigma} f(x) = \sum_T \sum_{t \in T_\sigma} f(t),$$

where $\sum_T$ *means sum over all* σ*-stable maximal tori of* G.

In the proof, Lemma 1.9 plays an important role.

Let $T_1, \ldots, T_h$ be the set of representatives of G_σ-conjugacy classes of σ-stable maximal tori of G. Put $W_i = N_{G_\sigma}(T_i)/T_{i\sigma}$ for $i = 1, \ldots, h$. The next corollary is the simplest one among those obtained by N. Kawanaka [7].

COROLLARY 2.3 ([7], Theorem 3.1). *Let f and g be elements of $\mathrm{I}(G_\sigma)$ which satisfy the condition of Proposition 2.2. Then*

$$<f, g>_{G_\sigma} = \sum_{i=1}^{h} |W_i|^{-1} <f, g>_{T_{i\sigma}}.$$

PROOF. By Proposition 2.2,

$$<f, g>_{G_\sigma} = |G_\sigma|^{-1} \sum_T \sum_{t \in T_\sigma} f(t)\overline{g(t)} = \sum_T |T_\sigma|\,|G_\sigma|^{-1} <f, g>_{T_\sigma}$$
$$= \sum_{i=1}^{h} |T_{i\sigma}|\,|N_{G_\sigma}(T_i)|^{-1} <f, g>_{T_{i\sigma}} = \sum_{i=1}^{h} |W_i|^{-1} <f, g>_{T_{i\sigma}}.$$

q. e. d.

PROPOSITION 2.4. *Let G and G_σ be as in §1. Let $r: H = G_\sigma \to GL_m(k)$, u_i $(i = 1, \ldots, m)$ and θ be as in Lemma 2.1. Assume that θ is injective and that for any σ-stable maximal torus T of G, no root relative to T vanishes on T_σ and that there exists $t \in T_\sigma$ such that $u_i(t) \neq u_j(t)$ $(i \neq j)$. Then*

$$<b_{r,\theta}, b_{r,\theta}>_{G_\sigma} = m.$$

PROOF. For any σ-stable maximal torus T of G,

$$<b_{r,\theta}, b_{r,\theta}>_{T_\sigma} = |T_\sigma|^{-1} \sum_{t \in T_\sigma} \left(\sum_{i=1}^{m} \theta(u_i(t))\right)\left(\sum_{j=1}^{m} \overline{\theta(u_j(t))}\right)$$
$$= \sum_{i,j} |T_\sigma|^{-1} \sum_{t \in T_\sigma} \theta(u_i(t))\overline{\theta(u_j(t))} = \sum_{i,j} <\theta u_i, \theta u_j>_{T_\sigma} = \sum_{i,j} \delta_{ij} = m.$$

Let $T_1, \ldots, T_h$ be the set of representatives of G_σ-conjugacy classes of σ-stable maximal tori of G and $W_i = N_{G_\sigma}(T_i)/T_{i\sigma}$ for $i = 1, \ldots, h$. By Corollary 2.3,

$$<b_{r,\theta}, b_{r,\theta}>_{G_\sigma} = m \sum_{i=1}^{h} |W_i|^{-1}.$$

If we denote by c_i the element of $H^1(\sigma, W)$ corresponding to the maximal torus T_i (Lemma 1.5), then we have by Corollary 1.8,

$$|W|\,|W_i|^{-1} = |c_i|,$$

where W is the Weyl group of G. Hence

$$\sum_{i=1}^{h} |W_i|^{-1} = |W|^{-1} \sum_{i=1}^{h} |c_i| = 1. \qquad \text{q. e. d.}$$

COROLLARY 2.5. *Let H be the finite classical group mentioned in 1.1, r be the natural representation of H and $\theta: K^\times \to \mathbf{C}^\times$ be injective. Then we have:*

a) *If $H = GL_m(k)$ and $|k| > 2$, then $<b_{r,\theta}, b_{r,\theta}>_{GL_m(k)} = m$.*

a') *If* $H=U_m(k_2)$ *and* $|k|>2$, *then* $<b_{r,\theta}, b_{r,\theta}>_{U_m(k_2)}=m$.
b) *If* $H=SO_{2n+1}(k)$ *and* $|k|>3$, *then* $<b_{r,\theta}, b_{r,\theta}>_{SO_{2n+1}(k)}=2n+1$.
c) *If* $H=GSp_{2n}(k)$ *and* $|k|>3$, *then* $<b_{r,\theta}, b_{r,\theta}>_{GSp_{2n}(k)}=2n$.
d) *If* $H=GO^{\circ+}_{2n}(k)$ *and* $|k|>3$, *then* $<b_{r,\theta}, b_{r,\theta}>_{GO^{\circ-}_{2n}(k)}=2n$.
d') *If* $H=GO^{\circ-}_{2n}(k)$ *and* $|k|>3$, *then* $<b_{r,\theta}, b_{r,\theta}>_{GO^{\circ+}_{2n}(k)}=2n$.

PROOF. The condition of Proposition 2.4 is satisfied if $|k|>2$ when $G=A$, if $|k|>3$ when $G=B$, C, D and D', which can be easily checked.

§3. Decomposition of the Brauer lifting $b_{r,\theta}$

The aim of this section is to describe $b_{r,\theta}$ as an alternating sum of irreducible characters of G_σ when G, σ and G_σ are as in 1.1, r is the natural representation and θ is injective. Our proof is based upon the same method as that of T. A. Springer [12].

We begin with the following notion and some consequences due to Harish-Chandra for the characters of finite Chevalley groups G_σ.

DEFINITION 3.1. A complex valued function f on G_σ is called a *cusp form* if

$$\sum_{y \in U_\sigma} f(xy) = 0$$

for all elements x of G_σ and all parabolic subgroups $P_\sigma = M_\sigma U_\sigma \neq G_\sigma$. A character of G_σ which is a cusp form is called a *cuspidal character*. An ordinary representation of G_σ is called *cuspidal* if its character is a cusp form.

Let $P_\sigma = M_\sigma U_\sigma$ be a parabolic subgroup of G_σ and $\chi \in \mathrm{I}(M_\sigma)$. We can extend χ to an element $\tilde{\chi}$ of $\mathrm{I}(P_\sigma)$ by putting $\tilde{\chi}(mu)=\chi(m)$, $(m \in M_\sigma,\ u \in U_\sigma)$. Now define the two-sided ideal $\mathrm{I}(P_\sigma)$ of the group algebra $\mathbf{C}[G_\sigma]$ as the smallest two-sided ideal of $\mathbf{C}[G_\sigma]$ which contains all the irreducible constituents of $\mathrm{Ind}_{P_\sigma}^{G_\sigma}(\tilde{\chi})$ as χ runs over all the cuspidal characters of M_σ.

This definition depends only on P_σ and not on the Levi decomposition $M_\sigma U_\sigma$.

Two parabolic subgroups P_σ and Q_σ of G_σ are said to be *associated* if there exists $g \in G_\sigma$ such that P_σ and $gQ_\sigma g^{-1}$ have a common Levi subgroup. This is an equivalent relation over the set of all parabolic subgroups of G_σ. The importance of Definition 3.1 is the following

PROPOSITION 3.2 ([13], §6). a) *Let* P_σ *and* Q_σ *be parabolic subgroups of* G_σ. *Then* $\mathrm{I}(P_\sigma)$ *and* $\mathrm{I}(Q_\sigma)$ *either coincide or annihilate each other, according as* P_σ *and* Q_σ *are, or are not, associated.*

b) *Let* $\mathfrak{C}$ *be the set of representatives for equivalence classes of associated parabolic subgroups of* G_σ. *Then* $\mathbf{C}[G_\sigma]$ *is the direct sum of the two-sided ideals* $\mathrm{I}(P_\sigma)$, $(P_\sigma \in \mathfrak{C})$.

With these preparations on character theory for finite Chevalley groups, we study $b_{r,\theta}$ for finite classical groups mentioned in 1.1, using a case by case analysis.

Case 1. $GL_n(k)$. We assume that $q=|k|>2$. As in Lemma 2.1, we assume θ to be injective and r to be the natural representation $GL_n(k)\to GL_n(k)$, and we write $b_{n,\theta}$ for $b_{r,\theta}$. As maximal parabolic subgroups in $\mathfrak{C}$ of $GL_n(k)$, we choose

$$P_i = \left\{\begin{bmatrix} A & * \\ O & D \end{bmatrix} \middle| A \in GL_i(k),\ D \in GL_{n-i}(k)\right\}, \qquad i=1,\dots, n-1.$$

For a character χ of $GL_i(k)$, we extend χ to the character $\tilde{\chi}$ of P_i by the natural homomorphism $P_i\to GL_i(k)$. The following theorem is proved in T. A. Springer [12].

THEOREM 3.3. a) *There exists a unique irreducible cuspidal character* $\pi_{n,\theta}$ *of* $GL_n(k)$ *such that* $<b_{n,\theta}, \pi_{n,\theta}>_{GL_n(k)}\neq 0$.

b) *Let* $\pi_{i,\theta}$ *be the irreducible cuspidal character of* $GL_i(k)$ *defined in* a). *Then* $\mathrm{Ind}\,_{P_i}^{GL_n(k)}(\tilde{\pi}_{i,\theta})$ *is irreducible for* $i=1,\dots, n-1$.

c) $b_{n,\theta}=\sum_{i=1}^{n-1}(-1)^{i-1}\mathrm{Ind}_{P_i}^{GL_n(k)}(\tilde{\pi}_{i,\theta})+(-1)^{n-1}\pi_{n,\theta}$.

Case 2. $GSp_{2n}(k)=\{x\in GL_{2n}(k) | xJ_n'{}^t x=\lambda J_n',\ \lambda\in k^{\times}\}$,

$$J_n' = \begin{bmatrix} O & J_n \\ -J_n & O \end{bmatrix}, \qquad J_n = \underbrace{\begin{bmatrix} o & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & o \end{bmatrix}}_{n}\Big\}\, n\ .$$

We assume that $q=|k|>3$. As in Lemma 2.1, we assume θ to be injective and r to be the natural representation $GSp_{2n}(k)\to GL_{2n}(k)$, and we write $b_{2n,\theta}$ for $b_{r,\theta}$. As parabolic subgroups in $\mathfrak{C}$ of $GSp_{2n}(k)$, we choose $P=P_{j_1,j_2,\dots;i}=$

$$\left\{\begin{bmatrix} A & & & & & * \\ & B & & & & \\ & & \ddots & & & \\ & & & X & & \\ & & & & \ddots & \\ & & & & B' & \\ O & & & & & A' \end{bmatrix} \middle| \begin{array}{l} A,\ A' \in GL_{j_1}(k),\ AJ_{j_1}{}^tA' = \lambda J_{j_1};\ B,\ B' \in GL_{j_2}(k), \\ BJ_{j_2}{}^tB' = \lambda J_{j_2};\dots; \\ \qquad X\in GSp_{2i}(k),\quad XJ_i'{}^tX = \lambda J_i',\ \lambda\in k^{\times} \end{array}\right\},$$

$(j_1 \geq 0, j_2 \geq 0,\dots; i\geq 0, j_1+j_2+\cdots+i = n)$.

In particular, as maximal parabolic subgroups in $\mathfrak{C}$ of $GSp_{2n}(k)$, we choose

$$P_o = \left\{ \begin{bmatrix} A & * \\ O & D \end{bmatrix} \middle| A, D \in GL_n(k),\ AJ_n{}^tD = \lambda J_n,\ \lambda \in k^\times \right\},$$

$$P_i = \left\{ \begin{bmatrix} A & & * \\ & X & \\ O & & D \end{bmatrix} \middle| \begin{array}{l} A,\ D \in GL_{n-i}(k),\ AJ_{n-i}{}^tD = \lambda J_{n-i} \\ X \in GSp_{2i}(k),\ XJ_i'{}^tX = \lambda J_i',\ \lambda \in k^\times \end{array} \right\}, \quad (i=1,\dots, n-1).$$

For a character χ of $GL_{n-i}(k)$ (resp. $GSp_{2i}(k)$), we extend χ to the character $\tilde{\chi}$ of P_i by the natural homomorphism

$$P_i \ni \begin{bmatrix} A & & * \\ & X & \\ O & & D \end{bmatrix} \longmapsto A \in GL_{n-i}(k) \quad (\text{resp. } P_i \ni \begin{bmatrix} A & & * \\ & X & \\ O & & D \end{bmatrix} \longmapsto X \in GSp_{2i}(k)).$$

With these notations, we have the following theorem.

THEOREM 3.4. a) *There exists a unique irreducible cuspidal character* $\pi'_{n,\theta}$ *of* $GSp_{2n}(k)$ *such that* $<b_{2n,\theta}, \pi'_{n,\theta}>_{GSp_{2n}(k)} \neq 0$.

b) *For* $i=0, 1, 2,\dots, n-1$, *let* $\pi_{n-i,\theta}$ *be the irreducible cuspidal character of* $GL_{n-i}(k)$ *obtained in Theorem* 3.3. *Then* $\mathrm{Ind}_{P_i}^{GSp_{2n}(k)}(\tilde{\pi}_{n-1,\theta})$ *is irreducible for every* i.

c) *For* $i=1, 2,\dots, n-1$, *let* $\pi'_{i,\theta}$ *be the irreducible cuspidal character of* $GSp_{2i}(k)$ *defined in* a). *Then*

$$<\mathrm{Ind}_{P_i}^{GSp_{2n}(k)}(\tilde{\pi}'_{i,\theta}),\ \mathrm{Ind}_{P_i}^{GSp_{2n}(k)}(\tilde{\pi}'_{i,\theta})>_{GSp_{2n}(k)} = 2$$

and there exists a unique irreducible constituent m_i *of* $\mathrm{Ind}_{P_i}^{GSp_{2n}(k)}(\tilde{\pi}'_{i,\theta})$ *such that* $<b_{2n,\theta}, m_i>_{GSp_{2n}(k)} \neq 0$.

d) $b_{2n,\theta} = \sum_{i=o}^{n-1} (-1)^{n-1} \mathrm{Ind}_{P_i}^{GSp_{2n}(k)}(\tilde{\pi}_{n-i,\theta}) + (-1)^n \pi'_{n,\theta} + \sum_{i=1}^{n-1} (-1)^i m_i$.

PROOF. a) We prove the theorem by induction on n. For $n=1$, $GSp_2(k) = GL_2(k)$. This reduces the proof to the case of Theorem 3.3. We assume that the statement a) is true for $GSp_{2i}(k)$ $(i<n)$. Let P be a parabolic subgroup of $GSp_{2n}(k)$, M be a Levi subgroup of P and χ be an irreducible cuspidal character of M. By Frobenius reciprocity theorem,

$$\begin{aligned} <b_{2n,\theta}, \mathrm{Ind}_P^{GSp_{2n}(k)}(\tilde{\chi})>_{GSp_{2n}(k)} &= <\mathrm{Res}_P^{GSp_{2n}(k)}(b_{2n,\theta}), \tilde{\chi}>_P \\ &= <\mathrm{Res}_M^{GSp_{2n}(k)}(b_{2n,\theta}), \chi>_M. \end{aligned}$$

If $P = P_{j_1, j_2,\dots; i}$ $(\neq GSp_{2n}(k))$, then

$$M=\left\{\begin{bmatrix} A & & & & & & O \\ & B & & & & & \\ & & \ddots & & & & \\ & & & X & & & \\ & & & & \ddots & & \\ & & & & & B' & \\ O & & & & & & A' \end{bmatrix} \left| \begin{array}{l} A,\ A' \in GL_{j_1}(k),\ AJ_{j_1}{}^tA' = \lambda J_{j_1};\ B,\ B' \in GL_{j_2}(k), \\ BJ_{j_2}{}^tB' = \lambda J_{j_2};\ldots; \\ \qquad X \in GSp_{2i}(k),\ XJ_i'{}^tX = \lambda \in J_i',\ \lambda \in k^{\times} \end{array} \right.\right\}.$$

For $m=\mathrm{diag}(A, B,\ldots, X,\ldots, B', A')\in M$, we have

$$b_{2n,\theta}(m) = b_{j_1,\theta}(A)+b_{j_2,\theta}(B)+\cdots+b_{2i,\theta}(X)+\cdots+b_{j_2,\theta}(B')+b_{j_1,\theta}(A').$$

Following this decomposition we have

$$b_{2n,\theta} = b^p_{j_1,\theta}+b^p_{j_2,\theta}+\cdots+b^o_{2i,\theta}+\cdots+b^n_{j_2,\theta}+b^n_{j_1,\theta},$$

where $b^p_{j_1,\theta}(m) = b_{j_1,\theta}(A),\ldots, b^o_{2i,\theta}(m) = b_{2i,\theta}(X),\ldots, b^n_{j_1,\theta}(m) = b_{j_1,\theta}(A')$.

Then

$$<\mathrm{Res}_M^{GSp_{2n}(k)}(b_{2n,\theta}),\ \chi>_M \,=\, <b^p_{j_1,\theta},\ \chi>_M + <b^p_{j_2,\theta},\ \chi>_M+\cdots$$
$$+<b^o_{2i,\theta},\ \chi>_M+\cdots+<b^n_{j_2,\theta},\ \chi>_M+<b^n_{j_1,\theta},\ \chi>_M.$$

Let

$$M_1^h = \{\mathrm{diag}(A_1,\ldots, A_{h-1}, I_{j_h}, A_{h+1},\ldots, A_r, X, A_r',\ldots, A_{h+1}', \lambda I_{j_h}, A_{h-1}',\ldots, A_1')$$
$$\in M | A_1,\ A_1' \in GL_{j_1}(k),\ A_1J_{j_1}{}^tA_1' = \lambda J_{j_1},\ldots;\ X\in GSp_{2i}(k),\ XJ_i'{}^tX = \lambda J_i',$$
$$\lambda\in k^{\times}\}$$

and

$$M_2^h = \{\mathrm{diag}(I_{j_1},\ldots, I_{j_{h-1}}, D, I_{j_{h+1}},\ldots, I_{j_r}, I_{2i}, I_{j_r},\ldots, I_{j_{h+1}}, D', I_{j_{h-1}},\ldots, I_{j_1})$$
$$\in M | D,\ D'\in GL_{j_h}(k),\ DJ_{j_h}{}^tD' = J_{j_h}\},\qquad (h = 1,\ldots, r).$$

Then M_1^h and M_2^h are subgroups of M and

$$M = M_1^hM_2^h \qquad \text{(direct product)}.$$

We can write $\chi=\chi_1^h\chi_2^h$, where χ_i^h is an irreducible cuspidal character of M_i^h ($i=1$, 2). Let

$$N_1 = \{\mathrm{diag}(1,\ldots, 1, X, \lambda,\ldots, \lambda)\in M | X\in GSp_{2i}(k),\ XJ_i'{}^tX = \lambda J_i,\ \lambda\in k^{\times}\}$$

and

$$N_2 = \{\mathrm{diag}(A, B,\ldots, I_{2i},\ldots, B', A')\in M | A,\ A'\in GL_{j_1}(k),\ AJ_{j_1}{}^tA' = J_{j_1},\ldots\}.$$

Then N_1 and N_2 are subgroups of M and

$$M = N_1 N_2 \qquad \text{(direct product)}.$$

We can write $\chi=\chi_1\chi_2$, where χ_i is an irreducible cuspidal character of N_i ($i=1$, 2). We have

$$\begin{aligned}&<\mathrm{Res}_M^{GSp_{2n}(k)}(b_{2n,\theta}),\, \chi>_M \;=\; <\mathbf{1}_{M_1^1},\, \chi_1^1>_{M_1^1}<b_{j_1,\theta}^p,\, \chi_2^1>_{M_2^1}+\cdots+\\ &<\mathbf{1}_{M_1^h},\, \chi_1^h>_{M_1^h}<b_{j_h,\theta}^p,\, \chi_2^h>_{M_2^h}+\cdots+<b_{2i}^o,\, \chi_1>_{N_1}<\mathbf{1}_{N_2},\, \chi_2>_{N_2}+\cdots+\\ &<\mathbf{1}_{M_1^h},\, \chi_1^h>_{M_1^h}<b_{j_h,\theta}^p,\, \chi_2^h>_{M_2^h}+\cdots+<\mathbf{1}_{M_1^1},\, \chi_1^1>_{M_1^1}<b_{j_1,\theta}^n,\, \chi_2^1>_{M_2^1},\end{aligned}$$

where $\mathbf{1}_H$ denotes the trivial character of a group H. Hence

$<b_{2n,\theta},\, \mathrm{Ind}_P^{GSp_{2n}(k)}(\tilde{\chi})>_{GSp_{2n}(k)}=0$ unless P associates with

$$Q_j = \left\{ \begin{bmatrix} A & & & & & & * \\ & b & & & & & \\ & & \ddots & & & & \\ & & & d & & & \\ & & & & d' & & \\ & & & & & \ddots & \\ & & & & & & b' \\ O & & & & & & & A' \end{bmatrix} \;\middle|\; \begin{array}{l} A,\, A' \in GL_j(k),\, AJ_j{}^tA' = J_j \\ b,\, b',\ldots,\, d,\, d',\, \lambda \in k^{x}, \\ bb' = \cdots = dd' = \lambda \end{array} \right\}, \qquad (j = 1,\ldots, n),$$

or

$$Q_i' = \left\{ \begin{bmatrix} a & & & & & & * \\ & \ddots & & & & & \\ & & d & & & & \\ & & & X & & & \\ & & & & d' & & \\ & & & & & \ddots & \\ O & & & & & & a' \end{bmatrix} \;\middle|\; \begin{array}{l} X \in GSp_{2i}(k),\, XJ_i'{}^tX = \lambda J_i' \\ a,\, a',\ldots,\, d,\, d',\, \lambda \in k^{x}, \\ aa' = \cdots = dd' = \lambda \end{array} \right\}, \qquad (i = 1,\ldots, n-1).$$

If $P=Q_j$ ($j=1,\ldots, n$), then

$$<b_{2n,}\;,\, \mathrm{Ind}_P^{GSp_{2n}(k)}(\tilde{\chi})>_{GSp_{2n}(k)} = \begin{cases} (-1)^{j-1} & \text{if } \tilde{\chi} = \pi_{j,\theta}^p \text{ or } \pi_{j,\theta}^n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\pi_{j,\theta}$ is the irreducible cuspidal character of $GL_j(k)$ obtained in 3.3 and π_j^p, (resp. π_j^n) is the character of Q_j obtained from $\pi_{j,\theta}$ by the natural homomorphism

$$Q_j \ni \begin{bmatrix} A & & & & & * \\ & b & & & & \\ & & \ddots & & & \\ & & & d & & \\ & & & & d' & \\ & & & & & \ddots \\ & & & & & b' \\ O & & & & & A' \end{bmatrix} \longmapsto A \in GL_j(k) \qquad (\text{resp.}$$

$$Q_j \ni \begin{bmatrix} A & & & & & * \\ & b & & & & \\ & & \ddots & & & \\ & & & d & & \\ & & & & d' & \\ & & & & & \ddots \\ & & & & & b' \\ O & & & & & A' \end{bmatrix} \longmapsto A' \in GL_j(k) \quad).$$

If $P=Q_i'$ $(i=1,\dots, n-1)$, then by induction hypothesis

$$< b_{2n,}\ , \mathrm{Ind}_P^{GSp_{2n}(k)}(\tilde{\chi}) >_{GSp_{2n}(k)} = \begin{cases} (-1)^i & \text{if } \tilde{\chi} = \pi_{i,\theta}'^{o}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\pi_{i,\theta}'^{o}$ is the character of Q_i' obtained from $\pi_{i,\theta}'$ by the natural homomorphism

$$Q_i' \in \begin{bmatrix} a & & & & & * \\ & \ddots & & & & \\ & & d & & & \\ & & & X & & \\ & & & & d' & \\ & & & & & \ddots \\ O & & & & & a' \end{bmatrix} \longmapsto X \in GSp_{2i}(k)\,,$$

(Induction hypothesis defines $\pi_{i,\theta}'$ for $i<n$). On the other hand, by [13], 4.8, 4.9, we see

$$\mathrm{Ind}_{Q_j}^{GSp_{2n}(k)}(\pi_{j,\theta}^{n}) = \mathrm{Ind}_{Q_j}^{GSp_{2n}(k)}(\pi_{j,\theta}^{p})$$

and

$$< \mathrm{Ind}_{Q_j}^{GSp_{2n}(k)}(\pi_{j,\theta}^{p}),\ \mathrm{Ind}_{Q_j}^{GSp_{2n}(k)}(\pi_{i,\theta}^{o}) >_{GSp_{2n}(k)} = 0.$$

Hence, there are $2n-1$ irreducible characters which intertwines with $b_{2n,\theta}$ by ± 1 and induced from irreducible cuspidal characters of proper parabolic subgroups. From 2.5, a) is now proved.

b) The maximal parabolic subgroups P_i of $GSp_{2n}(k)$ contains Q_{n-i} $(i=0, 1,\dots, n-1)$. We have

$$\mathrm{Ind}_{Q_{n-i}}^{GSp_{2n}(k)}(\pi_{n-i,\theta}^{p}) = \mathrm{Ind}_{P_i}^{GSp_{2n}(k)}\,\mathrm{Ind}_{Q_{n-i}}^{P_i}(\pi_{n-i,\theta}^{p}),$$

and by Frobenius theorem,

$$<\mathrm{Ind}_{Q_{n-i}}^{P_i}(\pi_{n-i,\theta}^{p}),\ \tilde{\pi}_{n-i,\theta}>_{P_i} = 1.$$

On the other hand, by the same computation as in the proof of part a),

$$<b_{2n,\theta},\ \mathrm{Ind}_{P_i}^{GSp_{2n}(k)}(\tilde{\pi}_{i,\theta})>_{GSp_{2n}(k)} = (-1)^{n-i-1}.$$

The irreducibility of $\mathrm{Ind}_{P_i}^{GSp_{2n}(k)}(\tilde{\pi}_{i,\theta})$ follows from [13], 4.12.

The maximal parabolic subgroups P_i of $GSp_{2n}(k)$ contains Q'_i $(i=1,\dots,n-1)$. Similarly to b), we have

$$\mathrm{Ind}_{Q'_i}^{GSp_{2n}(k)}(\pi'^{o}_{i,\theta}) = \mathrm{Ind}_{P_i}^{GSp_{2n}(k)}\,\mathrm{Ind}_{Q'_i}^{P_i}(\pi'^{o}_{i,\theta}),$$

$<\mathrm{Ind}_{Q'_i}^{P_i}(\pi'^{o}_{i,\theta}),\ \tilde{\pi}'_{i,\theta}>_{P_i} = 1$ and $<b_{2n,\theta},\ \mathrm{Ind}_{P_i}^{GSp_{2n}(k)}(\tilde{\pi}'_{i,\theta})>_{GSp_{2n}(k)} = (-1)^i$.
From Mackey's formula,

$$\begin{aligned}&<\mathrm{Ind}_{P_i}^{GSp_{2n}(k)}(\tilde{\pi}'_{i,\theta}),\ \mathrm{Ind}_{P_i}^{GSp_{2n}(k)}(\tilde{\pi}'_{i,\theta})>_{GSp_{2n}(k)}\\ &= \textstyle\sum_{g\in P_i\backslash GSp_{2n}(k)/P_i}<\mathrm{Res}_{P_i\cap {}^gP_i}^{P_i}(\tilde{\pi}'_{i,\theta}),\ \mathrm{Res}_{P_i\cap {}^gP_i}^{{}^gP_i}({}^g\tilde{\pi}'_{i,\theta})>_{P_i\cap {}^gP_i}\end{aligned}$$

We can make an identification $P_i\backslash GSp_{2n}(k)/P_i=\{1, w\}$, $(w\in W)$, where W is the Weyl group of GSp_{2n}. We easily see that

$${}^w\tilde{\pi}'_{i,\theta}(x) = \tilde{\pi}'_{i,\theta}(x) \qquad \text{for} \quad x\in P_i\cap {}^wP_i.$$

Thus

$$<\mathrm{Ind}_{P_i}^{GSp_{2n}(k)}(\tilde{\pi}'_{i,\theta}),\ \mathrm{Ind}_{P_i}^{GSp_{2n}(k)}(\tilde{\pi}'_{i,\theta})>_{GSp_{2n}(k)} = 2.$$

d) follows from a), b), c). q. e. d.

Similarly one can decompose the Brauer lifting for other classical groups. Since main parts of proofs are quite similar to that of Theorem 3.4, we shall only note some points for each case.

Case 3. $SO_{2n+1}(k)=\{x\in SL_{2n+1}(k)|xJ_{2n+1}{}^tx=J_{2n+1}\}$,

$$J_{2n+1} = \underbrace{\begin{bmatrix} O & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ & 1 & \\ 1 & & O\end{bmatrix}}_{2n+1}\left.\vphantom{\begin{bmatrix} O\\ \\ \\ 1\end{bmatrix}}\right\}2n+1\,.$$

We assume that $q=|k|>3$. As in Lemma 2.1, we assume θ to be injective and r to be the natural representation $SO_{2n+1}(k)\to GL_{2n+1}(k)$, and we write

$b_{2n+1,\theta}$ for $b_{r,\theta}$. As maximal parabolic subgroups in $\mathfrak{C}$ of $SO_{2n+1}(k)$, we choose

$$P_i = \left\{ \begin{bmatrix} A & & * \\ & X & \\ O & & D \end{bmatrix} \middle| \begin{matrix} A,\ D \in GL_{n-i}(k),\ AJ_{n-i}{}^tD = J_{n-i} \\ X \in SO_{2i+1}(k) \end{matrix} \right\}, \qquad (i = 0, 1, \dots, n-1).$$

THEOREM 3.5. a) *There exists a unique irreducible cuspidal character* $\pi'_{n,\theta}$ *of* $SO_{2n+1}(k)$ *such that* $<b_{2n+1,}\ ,\ \pi'_{n,\theta}>_{SO_{2n+1}(k)} \neq 0$.

b) *For* $i=0, 1, 2, \dots, n-1$, *let* $\pi_{n-i,\theta}$ *be the irreducible cuspidal character of* $GL_{n-i}(k)$ *obtained in* 3.3. *Then* $\mathrm{Ind}_{P_i}^{SO_{2n+1}(k)}(\tilde{\pi}_{n-i,\theta})$ *are all irreducible.*

c) *For* $i=1, 2, \dots, n-1$, *let* $\pi'_{i,\theta}$ *be the irreducible cuspidal character of* $SO_{2i+1}(k)$ *defined in* a). *Then*

$$<\mathrm{Ind}_{P_i}^{SO_{2n+1}(k)}(\tilde{\pi}'_{i,\theta}),\ \mathrm{Ind}_{P_i}^{SO_{2n+1}(k)}(\tilde{\pi}'_{i,\theta})>_{SO_{2n+1}(k)} = 2$$

and there exists a unique irreducible constituent m_i *of* $\mathrm{Ind}_{P_i}^{SO_{2n+1}(k)}(\tilde{\pi}'_{i,\theta})$ *such that* $<b_{2n+1,\theta}, m_i>_{SO_{2n+1}(k)} \neq 0$.

d) $b_{2n+1,\theta} = \sum_{i=0}^{n-1}(-1)^{n-i-1}\mathrm{Ind}_{P_i}^{SO_{2n+1}(k)}(\tilde{\pi}_{n-i,\theta}) + (-1)^n\pi'_{n,\theta} + \sum_{i=1}^{n-1}(-1)^i m_i$

$+\mathbf{1}$,

where $\mathbf{1}$ *is the trivial character of* $SO_{2n+1}(k)$.

PROOF. We only check that $\mathbf{1}$ is a constituent of $b_{2n+1,\theta}$ with multiplicity one. By 2.3,

$$<b_{2n+1,\theta}, \mathbf{1}>_{SO_{2n+1}(k)} = \sum_{i=1}^{h}|W_i|^{-1}<b_{2n+1,}\ ,\ \mathbf{1}>_{T_{i\sigma}} = 1,$$

because any element of $T_{i\sigma}$ is conjugate to $\mathrm{diag}(a_1, \dots, a_n, 1, a_n^{-1}, \dots, a_1^{-1})$, $a_i \in K^\times$ $(i=1, \dots, n)$. q.e.d.

Case 4. $GO_{2n}^{\circ +}(k) = \{x \in GL_{2n}(k) | xJ_{2n}{}^tx = \lambda J_{2n},\ det\, x = \lambda^n,\ \lambda \in k^\times\}$,

$$J_{2n} = \underbrace{\begin{bmatrix} O & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ & 1 & \\ 1 & & O \end{bmatrix}}_{2n} \Big\} 2n\,.$$

We assume that $q=|k|>3$ and $char.\,k \neq 2$. As in Lemma 2.1, we assume θ to be injective and r to be the natural representation $GO_{2n}^{\circ +}(k) \to GL_{2n}(k)$, and we write $b_{2n,\theta}$ for $b_{r,\theta}$. As maximal parabolic subgroups in $\mathfrak{C}$ of $GO_{2n}^{\circ +}(k)$, we choose

$$P_O = \left\{\begin{bmatrix} A & * \\ O & D \end{bmatrix} \middle| A,\ D \in GL_n(k),\ AJ_n{}^tD = \lambda J_n,\ \lambda \in k^x \right\},$$

$$P_i = \left\{\begin{bmatrix} A & & * \\ & X & \\ O & & D \end{bmatrix} \middle| \begin{array}{l} A,\ D \in GL_{n-i}(k),\ AJ_{n-i}{}^tD = \lambda J_{n-i} \\ X \in GO_{2i}^{o+}(k),\ XJ_{2i}{}^tX = \lambda J_{2i},\ \lambda \in k^x \end{array} \right\}, \quad (i=1,\dots,n-1).$$

THEOREM 3.6. *Assume that* $n \geq 2$. a) *There exists a unique irreducible cuspidal character* $\pi'_{n,\theta}$ *of* $GO_{2n}^{o+}(k)$ *such that* $<b_{2n,\theta}, \pi_{n,\theta}>_{GO_{2n}^{o+}(k)} \neq 0$.

b) *For* $i=0, 1, 2,\dots, n-1$, *let* $\pi_{n-1,\theta}$ *be the irreducible cuspidal character of* $GL_{n-i}(k)$ *obtained in* 3.3. *Then* $\mathrm{Ind}_{P_i}^{GO_{2n}^{o+}(k)}(\tilde{\pi}_{n-i,\theta})$ *are all irreducible.*

c) *For* $i=2,\dots, n-1$, *let* $\pi'_{i,\theta}$ *be the irreducible cuspidal character of* $GO_{2i}^{o+}(k)$ *defined in* a) *and let*

$$\pi'_{1,\theta}\left(\begin{bmatrix} a & o \\ o & d \end{bmatrix}\right) = \theta(a) \qquad \textit{for} \ \begin{bmatrix} a & o \\ o & d \end{bmatrix} \in GO_2^{o+}(k). \quad \textit{Then}$$

$$<\mathrm{Ind}_{P_i}^{GO_{2n}^{o+}(k)}(\tilde{\pi}'_{i,\theta}),\ \mathrm{Ind}_{P_i}^{GO_{2n}^{o+}(k)}(\tilde{\pi}'_{i,\theta})>_{GO_{2n}^{o+}(k)} = 2 \qquad \textit{for} \quad i = 1, 2,\dots, n-1,$$

and there exists a unique irreducible constituent m_i *of* $\mathrm{Ind}_{P_i}^{GO_{2n}^{o+}(k)}(\tilde{\pi}'_{i,\theta})$ *such that* $<b_{2n,\theta}, m_i>_{GO_{2n}^{o+}(k)} \neq 0$ *for* $i=1,\dots, n-1$.

d) $b_{2n,\theta} = \sum_{i=0}^{n-1}(-1)^{n-i-1}\mathrm{Ind}_{P_i}^{GO_{2n}^{o+}(k)}(\tilde{\pi}_{n-1,\theta}) + (-1)^{n-1}\pi'_{n,\theta} + \sum_{i=1}^{n-1}(-1)^{i-1} m_i$.

PROOF. We only note here the following. For $GO_2^{o+}(k) = \left\{\begin{bmatrix} a & o \\ o & d \end{bmatrix} \middle| a,\ d \in k^x\right\}$, we have $b_{2,\theta} = \pi + \pi'$, where $\pi\left(\begin{bmatrix} a & o \\ o & d \end{bmatrix}\right) = \theta(a)$, $\pi'\left(\begin{bmatrix} a & o \\ o & d \end{bmatrix}\right) = \theta(d)$. Let

$$Q_1 = \left\{\begin{bmatrix} a & & & & & * \\ & \ddots & & & & \\ & & d & & & \\ & & & X & & \\ & & & & d' & \\ & & & & & \ddots \\ O & & & & & a' \end{bmatrix} \middle| \begin{array}{l} X \in GO_2^{o+}(k),\ XJ_2{}^tX = \lambda J_2,\ a, a',\dots, d, d',\ \lambda \in k^x \\ aa' = \cdots = dd' = \lambda \end{array} \right\}.$$

It is a Borel subgroup contained in P_1. Let π^p and π'^n be the character of Q_1 obtained from π and π' by the natural homomorphism

$$Q_1 \ni \begin{bmatrix} a & & & & * \\ & \ddots & & & \\ & & d & & \\ & & & X & & \\ & & & & d' & \\ & & & & & \ddots \\ O & & & & & a' \end{bmatrix} \longmapsto X \in GO_2^{\circ+}(k).$$

Then by [13], 4.9, we have $\mathrm{Ind}_{Q_1}^{GO_{2n}^{\circ\pm}(k)}(\pi^p)=\mathrm{Ind}_{Q_1}^{GO_{2n}^{\circ\pm}(k)}(\pi'^n)$. q.e.d.

Case 5. $GO_{2n}^{\circ-}(k)=\{x\in GL_{2n}(k)|xF_n{}^tx=\lambda F_n,\ \det x=\lambda^n,\ \lambda\in k^\times\}$,

$$F_n=\begin{bmatrix} O & & J_{n-1} \\ & F_o & \\ J_{n-1} & & O \end{bmatrix},\ J_{n-1}=\underbrace{\begin{bmatrix} O & & 1 \\ & 1^{\cdot^{\cdot^{\cdot}}} & \\ 1 & & O \end{bmatrix}}_{n-1}\Big\}\, n-1 \quad \text{and} \quad F_o=\begin{bmatrix} 1 & o \\ o & -\varepsilon \end{bmatrix},$$

where ε is a non-square element in k.

We assume that $q=|k|>3$ and *char.* $k\neq 2$. As in Lemma 2.1, we assume θ to be injective and r to be the natural representation $GO_{2n}^{\circ-}(k)\to GL_{2n}(k)$, and we write $b_{2n,\theta}$ for $b_{r,\theta}$. As maximal parabolic subgroups in $\mathfrak{C}$ of $GO_{2n}^{\circ-}(k)$, we choose

$$P_i=\left\{\begin{bmatrix} A & & * \\ & X & \\ O & & D \end{bmatrix}\ \middle|\ \begin{matrix} A,\ D\in GL_{n-i}(k),\ AJ_{n-i}{}^tD=\lambda J_{n-i} \\ X\in GO_{2i}^{\circ-}(k),\ XF_i{}^tX=\lambda F_i,\ \lambda\in k^\times \end{matrix}\right\},\quad (i=1,\dots,n-1).$$

THEOREM 3.7. *Assume that* $n\geq 2$. a) *There exists a unique irreducible cuspidal character* $\pi'_{n,\theta}$ *of* $GO_{2n}^{\circ-}(k)$ *such that* $<b_{2n,\theta},\pi_{n,\theta}>_{GO_{2n}^{\circ-}(k)}\neq 0$.

b) *For* $i=1,2,\dots,n-1$, *let* $\pi_{n-i,\theta}$ *be the irreducible cuspidal character of* $GL_{n-i}(k)$ *obtained in* 3.3. *Then* $\mathrm{Ind}_{P_i}^{GO_{2n}^{\circ-}(k)}(\tilde{\pi}_{n-i,\theta})$ *are all irreducible.*

c) *For* $i=2,\dots,n-1$, *let* $\pi'_{i,\theta}$ *be the irreducible cuspidal character of* $GO_{2n}^{\circ-}(k)$ *defined in* a). *Then*

$$<\mathrm{Ind}_{P_i}^{GO_{2n}^{\circ-}(k)}(\tilde{\pi}'_{i,\theta}),\ \mathrm{Ind}_{P_i}^{GO_{2n}^{\circ-}(k)}(\tilde{\pi}'_{i,\theta})>_{GO_{2n}^{\circ-}(k)}=2$$

and there exists a unique irreducible constituent m_i *of* $\mathrm{Ind}_{P_i}^{GO_{2n}^{\circ-}(k)}(\tilde{\pi}'_{i,\theta})$ *such that* $<b_{2n,\theta},m_i>_{GO_{2n}^{\circ-}(k)}\neq 0$. *For* $x\in GO_2^{\circ-}(k)$, *let* $u_1(x)$ *and* $u_2(x)$ *be the eigenvalues of* x. *Then* θu_1 *and* θu_2 *are characters of* $GO_2^{\circ-}(k)$. *There exists a unique irreducible constituent* m_1 *(resp.* m'_1*) of* $\mathrm{Ind}_{P_1}^{GO_{2n}^{\circ-}(k)}(\widetilde{\theta u_1})$ (resp. $\mathrm{Ind}_{P_1}^{GO_{2n}^{\circ-}(k)}(\widetilde{\theta u_2})$) *such that* $<b_{2n,\theta},m_1>_{GO_{2n}^{\circ-}(k)}\neq 0$ *(resp.* $<b_{2n,\theta},m'_1>_{GO_{2n}^{\circ-}(k)}\neq 0$*).*

d) $b_{2n,\theta}=\sum_{i=1}^{n-1}(-1)^{n-i-1}\mathrm{Ind}_{P_i}^{GO_{2n}^{\circ-}(k)}(\tilde{\pi}_{n-i,\theta})+(-1)^{n-1}\pi'_{n,\theta}+m_1+m'_1+\sum_{i=2}^{n-1}$

$(-1)^{i-1}m_i$.

PROOF. We only note here the following. For $GO_2^{o-}(k)$, $b_{2,\theta}=\theta u_1+\theta u_2$.

$$<\mathrm{Ind}_{Q_1}^{GO_{2n}^{o-}(k)}(\widetilde{\theta u_1}),\ \mathrm{Ind}_{Q_1}^{GO_{2n}^{o-}(k)}(\widetilde{\theta u_2})>_{GO_{2n}^{o-}(k)} = 0\,,$$

where

$$Q_1 = \left\{ \begin{bmatrix} a & & & & & * \\ & \ddots & & & & \\ & & d & & & \\ & & & X & & \\ & & & & d' & \\ & & & & & \ddots \\ O & & & & & a' \end{bmatrix} \middle| \begin{array}{l} X \in GO_2^{o-}(k),\ XF_o{}^tX = \lambda F_o \\ a, a' \dots, d, d', \lambda \in k^\times, \\ aa' = \cdots = dd' = \lambda \end{array} \right\},$$

a Borel subgroup of $GO_{2n}^{o-}(k)$. For $i=1, 2$,

$$<\mathrm{Ind}_{P_1}^{GO_{2n}^{o-}(k)}(\widetilde{\theta u_i}),\ \mathrm{Ind}_{P_1}^{GO_{2n}^{o-}(k)}(\widetilde{\theta u_i})>_{GO_{2n}^{o-}(k)}=2. \qquad \text{q. e. d.}$$

Case 6. $U_{2n}(k_2)=\{x\in GL_{2n}(k_2)|xJ_{2n}{}^t(\sigma(x))=J_{2n}\}$,

$$k = \mathbf{F}_q, \quad k_2 = \mathbf{F}_{q^2} \quad \text{and} \quad J_{2n} = \begin{bmatrix} O & & & & 1 \\ & & & -1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & 1 & & & \\ -1 & & & & O \end{bmatrix}.$$

We assume that $q=|k|>2$. As in Lemma 2.1, we assume θ to be injective and r to be the natural representation $U_{2n}(k_2)\to GL_{2n}(k_2)$, and we write $b_{2n,\theta}$ for $b_{r,\theta}$. As maximal parabolic subgroups in $\mathfrak{C}$ of $U_{2n}(k_2)$, we choose

$$P_o = \left\{ \begin{bmatrix} A & * \\ O & D \end{bmatrix} \middle| A, D\in GL_n(k_2),\ AJ_nD^* = J_n \right\},$$

$$P_i = \left\{ \begin{bmatrix} A & & * \\ & X & \\ O & & D \end{bmatrix} \middle| \begin{array}{l} X\in U_{2i}(k_2),\ A, D\in GL_{n-i}(k) \\ AJ_{n-1}D^* = J_{n-i} \end{array} \right\}, \quad (i = 1,\dots, n-1),$$

$D^* = {}^t(\sigma(D))$.

THEOREM 3.8. a) *There exists a unique irreducible cuspidal character* $\pi'_{n,\theta}$ *of* $U_{2n}(k_2)$ *such that* $<b_{2n,\theta}, \pi'_{n,\theta}>_{U_{2n}(k_2)}\neq 0$.

b) *For* $i=0, 1, 2,\dots, n-1$, *let* $\pi_{n-i,\theta}$ *be the irreducible cuspidal character of* $GL_{n-i}(k_2)$ *obtained in* 3.3. *Then* $\mathrm{Ind}_{P_i}^{U_{2n}(k)}(\tilde{\pi}_{n-i,\theta})$ *are all irreducible.*

c) *For* $i=1, 2,\dots, n-1$, *let* $\pi'_{i,\theta}$ *be the irreducible cuspidal character of*

$U_{2i}(k_2)$ *defined in* a). *Then* $<\mathrm{Ind}_{P_i}^{U_{2n}(k_2)}(\tilde{\pi}'_{i,\theta}), \mathrm{Ind}_{P_i}^{U_{2n}(k_2)}(\tilde{\pi}'_{i,\theta})>_{U_{2n}(k_2)}=2$ *and there exists a unique irreducible constituent* m_i *of* $\mathrm{Ind}_{P_i}^{U_{2n}(k_2)}(\tilde{\pi}'_{i,\theta})$ *such that* $<b_{2n,\theta}, m_i>_{U_{2n}(k_2)}\neq 0$.

d) $b_{2n,\theta}=\sum_{i=0}^{n-1}(-1)^{n-i-1}\mathrm{Ind}_{P_i}^{U_{2n}(k_2)}(\tilde{\pi}_{n-i,\theta})+(-1)^{n-1}\pi'_{n,\theta}+\sum_{i=1}^{n-1}(-1)^{i-1} m_i$.

Case 7. $U_{2n+1}(k_2)=\{x\in GL_{2n+1}(k_2)|xJ_{2n+1}{}^t(\sigma(x))=J_{2n+1}\}$,

$$J_{2n+1}=\underbrace{\begin{bmatrix} O & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ \cdot^{\cdot^{\cdot}} & & & & O \end{bmatrix}}_{2n+1}\left.\vphantom{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}}\right\}2n+1\,.$$

We assume that $q=|k|>2$. As in Lemma 2.1, we assume θ to be injective and r to be the natural representation $U_{2n+1}(k_2)\to GL_{2n+1}(k_2)$, and we write $b_{2n+1,\theta}$ for $b_{r,\theta}$. As maximal parabolic subgroups in $\mathfrak{C}$ of $U_{2n+1}(k_2)$, we choose

$$P_i=\left\{\begin{bmatrix} A & & * \\ & X & \\ O & & D \end{bmatrix}\middle| X\in U_{2i+1}(k_2),\ A,\ D\in GL_{n-i}(k_2),\ AJ_{n-i}D^*=J_{n-i}\right\},$$

$$(i=0, 1,\ldots, n-1),\ D^*={}^t(\sigma(D)).$$

THEOREM 3.9. a) *There exists a unique irreducible cuspidal character* $\pi'_{n,\theta}$ *of* $U_{2n+1}(k_2)$ *such that* $<b_{2n+1,\theta}, \pi'_{n,\theta}>_{U_{2n+1}(k_2)}\neq 0$.

b) *For* $i=0, 1, 2,\ldots, n-1$, *let* $\pi_{n-i,\theta}$ *be the irreducible cuspidal character of* $GL_{n-i}(k_2)$ *obtained in* 3.3. *Then* $\mathrm{Ind}_{P_i}^{U_{2n}(k_2)}(\tilde{\pi}_{n-i,\theta})$ *are all irreducible.*

c) *Let* $\pi'_{o,\theta}$ *be the character of* $U_1(k_2)=\{x\in k_2|xx^q=1\}$ *defined by* $\pi'_{o,\theta}(x)=\theta(x)$ $(x\in U_1(k_2))$, *and for* $i=1, 2,\ldots, n-1$, *let* $\pi'_{i,\theta}$ *be the irreducible character of* $U_{2i+1}(k_2)$ *defined in* a). *Then*

$$<\mathrm{Ind}_{P_i}^{U_{2n+1}(k_2)}(\tilde{\pi}'_{i,\theta}), \mathrm{Ind}_{P_i}^{U_{2n+1}(k_2)}(\tilde{\pi}'_{i,\theta})>_{U_{2n+1}(k_2)}=2$$

and there exists a unique irreducible constituent m_i *of* $\mathrm{Ind}_{P_i}^{U_{2n+1}(k_2)}(\tilde{\pi}'_{i,\theta})$ *such that* $<b_{2n+1,\theta}, m_i>_{U_{2n+1}(k_2)}\neq 0$.

d) $b_{2n+1,\theta}=\sum_{i=0}^{n-1}(-1)^{n-i-1}\mathrm{Ind}_{P_i}^{U_{2n+1}(k_2)}(\tilde{\pi}_{n-i,\theta})+(-1)^n\pi'_{n,\theta}+\sum_{i=0}^{n-1}(-1)^i m_i$.

REMARK 3.10. Similarly one can decompose the Brauer liftings for $SL_n(k)$, $Sp_{2n}(k)$, $SO_{2n}(k)$ and $SU_m(k_2)$, as alternating sums of induced and cuspidal characters.

§4. Constituents in the Brauer lifting $b_{r,\theta}$

Our purpose in this section is to investigate the irreducible constituents of $b_{r,\theta}$ obtained in § 3. These are given in 4.2. We first summarize some results obtained by P. Deligne and G. Lusztig [3].

4.1. Recently P. Deligne and G. Lusztig [3] have constructed certain virtual representation of finite Chevalley groups and have proved Macdonald's conjecture (see [13], 6.7). More precisely, let G be a connected reductive algebraic group defined over a finite field k, let σ be the Frobenius map of G with respect to k, let T be a σ-stable maximal torus of G and let ω be an arbitrary character of T_σ. They have then introduced the virtual representation R_T^ω of G_σ as the alternating sum of the l-adic cohomologies with compact supports of the variety consisting of Borel subgroups of G which are in a fixed relative position with their σ-transform, with coefficients in a certain G_σ-equivariant locally constant l-adic sheaf of rank one (see [3], Chapter 1). If ω is in general position (for definition, see below Def. 4.1.2), then R_T^ω is irreducible. The vanishing theorem (see [3], Chapter 9) provides an explicit model for it provided that $q=|k|\geq 30$ (if G is a classical group or G_2-type, then any q will do).

Let G, σ and T be as above. The *Green function* $Q_{T,G}$ is the restriction to the unipotent elements of the character of the virtual representation $R_T^{\mathbf{1}}$, where $\mathbf{1}$ is the trivial character of T_σ.

The character of R_T^ω, denoted also by R_T^ω, can be expressed in terms of ω and of the Green functions.

THEOREM 4.1.1. a) ([3], 4.2). *Let $x=su$ be the Jordan decomposition of $x\in G$. Then*

$$R_T^\omega(x) = \frac{1}{|Z^o(s)_\sigma|}\sum_{g\in G_\sigma, gTg^{-1}\subset Z^o(s)}\omega(g^{-1}sg)Q_{gTg^{-1},Z^o(s)}(u),$$

where $Z^o(s)$ is the identity component of $Z_G(s)$, the centralizer of s in G.

b) ([3], 7.1). *Let $s(G)$ (resp. $s(T)$) be the k-rank of G (resp. T). Then*

$$Q_{T,G}(1) = (-1)^{s(G)-s(T)}\frac{|G_\sigma|}{|T_\sigma|St_G(1)},$$

where St_G is the Steinberg character of G_σ (see [15]).

c) ([3], 7.2). *For any semisimple element $s\in G_\sigma$,*

$$R_T^\omega(s) = (-1)^{s(Z^o(s))-s(T)}\frac{1}{St_{Z^o(s)}(1)|T_\sigma|}\sum_{g\in G, g^{-1}sg\in T_\sigma}\omega(g^{-1}sg).$$

The values of $Q_{T,G}(u)$ seem to be polynomials in q ($q=|k|$) but it is not known in general.

DEFINITION 4.1.2. The character ω of T_σ is said to be *in general position* if it is not kept fixed by any non-trivial element of $W(T)_\sigma$.

PROPOSITION 4.1.3 ([3], 7.4). *If ω is a character in general position of T_σ, then $(-1)^{s(G)-s(T)}R_T^\omega$ is proper irreducible.*

Let k_m be the unique extension of k of degree m in K and σ^m be the Frobenius map of G with respect to k_m. Let T be a σ-stable maximal torus in G. We define the norm map $N\colon T_{\sigma^m}\to T_\sigma$ by $N(x)=xx^\sigma x^{\sigma^{m-2}}x^{\sigma^{m-1}}$, $(x\in T_{\sigma^m})$. It is known that N is a surjective homomorphism.

DEFINITION 4.1.4. *Let T and T' be two σ-stable maximal tori of G, and let ω, ω' be characters of T_σ, T'_σ. The pairs (T, ω), (T', ω') are said to be geometrically conjugate if for some m the pairs $(T, \omega N)$, $(T', \omega' N)$, where N is the norm from T_{σ^m} to T_σ (resp. T'_{σ^m} to T'_σ), are G_{σ^m}-conjugate.*

THEOREM 4.1.5. a) ([3], 6.3). *If (T, ω) and (T', ω') are not geometrically conjugate, then no irreducible representation of G_σ can occur in both virtual representations R_T^ω and $R_{T'}^{\omega'}$.*

b) ([3], 6.8).

$$<R_{T'}^{\omega'}, R_T^\omega>_{G_\sigma} = |\{w\in W(T, T')_\sigma | {}^w\omega' = \omega\}|,$$

where $W(T, T')=\{g\in G| Tg=gT'\}/T'$.

PROPOSITION 4.1.6 ([3], 7.7). *For any irreducible representation ρ of G_σ, there exist a σ-stable maximal torus T and a character ω of T_σ such that $<\rho, R_T^\omega>_{G_\sigma}\neq 0$.*

We have considered the Brauer lifting $b_{r,\theta}$ in § 2 and in § 3. This generalized character satisfies the condition $b_{r,\theta}(su)=b_{r,\theta}(s)$. Such kinds of characters are considered in [3], Chapter 7.

PROPOSITION 4.1.7. a) ([3], 7.11). *Let ρ be a generalized character of G_σ such that $\rho(su)=\rho(s)$ for any $su\in G_\sigma$. Let T be a σ-stable maximal torus and ω be a character of T_σ. Then $<\rho, R_T^\omega>_{G_\sigma}=<\rho, \omega>_{T_\sigma}$.*

b) ([3], 7.12). *Let ρ be as in* a). *Then*

$$\rho=\sum_T\frac{1}{|W(T)_\sigma|}\sum_{\omega\in T_\sigma^\vee}<\rho, \omega>_{T_\sigma}R_T^\omega,$$

where $\sum_T$ means the summation over all G_σ-conjugacy classes of σ-stable maximal tori T of G and $T_\sigma^\vee$ is the character group of the finite abelian group T_σ.

By 4.1.5, b) and 4.1.7, b), we have:

COROLLARY 4.1.8 (N. Kawanaka [7]; [3], 7.13). *Let ρ be as in* 4.1.7. *Then*

$$<\rho, \rho>_{G_\sigma} = \sum_T \frac{1}{|W(T)_\sigma|} <\rho, \rho>_{T_\sigma}.$$

Let P be a σ-stable parabolic subgroup of G and let $T\subset P$ be a σ-stable maximal torus. Let M be a σ-stable Levi subgroup of P and let $\pi\colon P\to M$ be the canonical projection. This π induces an isomorphism $T\xrightarrow{\sim}\pi(T)$; hence also an isomorphism $T_\sigma\xrightarrow{\sim}\pi(T)_\sigma$. Let ω be a character of T_σ and $\bar{\omega}$ be the corresponding character of $\pi(T)_\sigma$.

DEFINITION 4.1.9. We denote by $R^\omega_{T,P}$ the pull-back of the virtual representation $R^{\bar{\omega}}_{\pi(T)}$ of M_σ under $\pi\colon P\to M$.

THEOREM 4.1.10. a) ([3], 8.2). $R^\omega_T=\mathrm{Ind}^{G_\sigma}_{P_\sigma}(R^\omega_{T,P})$

b) ([3], 8.3). *Let T be a minisotropic σ-stable maximal torus in G, i.e., T is not contained in any σ-stable proper parabolic subgroup of G. Let ω be a character in general position of T_σ. Then $(-1)^{s(G)-s(T)}R^\omega_T$ can be represented by a cuspidal G_σ-module.*

DEFINITION 4.1.10. Let G (resp. G^*) be a reductive algebraic group defined over k and let T (resp. T^*) be a σ-stable maximal torus contained in a σ-stable Borel subgroup of G (resp. G^*). We call G^* a *dual* to G if there is an isomorphism from T^* to a dual of T, this isomorphism carrying simple roots to simple coroots.

About some properties of this duality, see [3], 5.21. Let G and G^* be dual. By the definition, G and G^* have the same Weyl group. Let T be a σ-stable maximal torus in G, and let ω be a character of T_σ. Fix T', a corresponding maximal torus in G^*. The character ω defines a $(N(T')/T')_\sigma$-conjugacy class of elements ω' of T'. In that way, we get a bijection between G_σ-conjugacy classes of pairs (T, ω) as above, and G^*_σ-conjugacy classes of pairs (T', ω'), T' a σ-stable maximal torus of G^* and ω' an element of T'_σ. By forgetting T', we see that each G_σ-conjugacy class of pairs (T, ω) defines an element $\omega'\in G^*_\sigma$ well defined up to G^*_σ-conjugacy.

PROPOSITION 4.1.11. a) ([3], 5.22). *Two pairs (T_1, ω_1) and (T_2, ω_2) are geometrically conjugate if and only if ω'_1 and ω'_2 are geometrically conjugate.*

b) ([3], 5.24). *If the center of G is connected, two pairs (T_1, ω_1) and (T_2, ω_2) are geometrically conjugate if and only if ω'_1 and ω'_2 are G^*_σ-conjugate.*

c) ([3], 5.7). *The number of geometric conjugacy classes of pairs* (T, ω) *is* $|Z_\sigma^o|q^l$, *where* $q=|k|$, Z^o *is the identity component of the center of* G *and* l *is the semisimple rank of* G.

Let $G_\sigma^\vee$ be the set of equivalent classes of ordinary irreducible representations of G_σ, and let $\mathfrak{S}$ be the set of all geometric conjugacy classes of pairs (T, ω). For any $\rho \in G_\sigma^\vee$, there exist a σ-stable maximal torus T and $\omega \in T_\sigma$ such that $<\rho, R_T^\omega> \neq 0$ (4.1.6); moreover the geometric conjugacy class $[\omega]$ of (T, ω) is uniquely determined by ρ (4.1.5, a)). We thus get the well-defined surjective map

$$(1) \qquad G_\sigma^\vee \longrightarrow \mathfrak{S}.$$

In the rest of this part 4.1, we shall assume that *the center* Z *of* G *is connected.* Let T and B be a σ-stable maximal torus and a σ-stable Borel subgroup containing T. Let U be the unipotent radical of B and U^1 be the subgroup of U generated by the root subgroups corresponding to non-simple roots. The quotient U/U^1 is commutative and is a direct product over the simple roots α: $\prod_\alpha U_\alpha$, with U_α one-dimensional. Let I be the set of orbits of σ on the simple roots. For any $i \in \mathrm{I}$, let $U_i=\prod_{\alpha\in i} U_\alpha$. Then $U/U^1=\prod_{i\in \mathrm{I}} U_i$. This decomposition is σ-stable; hence we have also $U_\alpha/U_\alpha^1=\prod_{i\in\mathrm{I}} U_{i\sigma}$.

DEFINITION 4.1.12. The representation $\Gamma_G=\mathrm{Ind}_{U_\sigma}^{G_\sigma}(\chi)$, where χ is a linear character of U_α which is trivial on U_σ^1 and defines a non-trivial linear character of $U_{i\sigma}$ for all $i \in I$, is called the *Gelfand-Graev representation.*

It is known that Γ_G is well-defined up to isomorphism ([8], 6.9).

DEFINITION 4.1.13. Let Δ_G be the class function on G_σ such that, for $x \in G_\sigma$,

$$\Delta_G(x) = \begin{cases} |Z_\sigma|q^l & \text{if } x \text{ is regular unipotent,} \\ 0 & \text{otherwise,} \end{cases}$$

where l is the semisimple rank of G.

The following result shows that this is the character of a virtual representation of G_σ (which will be also denoted by Δ_G).

PROPOSITION 4.1.14 ([3], 10.3; [8], 6.10). *For any subset* $J \subset I$, *let* $P(J) \supset B$ *be the parabolic subgroup generated by* B *and by the root subgroups corresponding to minus the simple roots in* σ*-orbits in* J. *Let* $L(J)$ *be a Levi subgroup of* $P(J)$. *Then*

a) $\Delta_G=\sum_{J\subset I}(-1)^{|J|}\,\mathrm{Ind}_{P(P)_\sigma}^{G_\sigma}(\Gamma_{L(J)})$, *(Note that the center of $L(J)$ is connected, since so is that of G; hence $\Gamma_{L(J)}$ is welldefined).*

b) $\Gamma_G = \sum_{J\subset I}(-1)^{|J|}\operatorname{Ind}^{G_\sigma}_{P(J)_\sigma}(\Delta_{L(J)})$.

THEOREM 4.1.15 ([3], 10.7). *Let $x\in\mathfrak{S}$, and x' be a representative of the corresponding semisimple conjugacy class in the dual group G^*. We define δ_x to be the k-rank of the centralizer of x'.*

a) *The formulas*

$$\rho_X = \sum_{\substack{(T,\omega)\, mod.\, G_\sigma \\ [\omega]=x}} \frac{(-1)^{s(G)-s(T)}}{<R_T^\omega, R_T^\omega>} R_T^\omega$$

and

$$\rho'_x = (-1)^{s(G)-\delta_x}\sum_{\substack{(T,\omega)\, mod.\, G_\sigma \\ [\omega]=x}} \frac{1}{<R_T^\omega, R_T^\omega>} R_T^\omega$$

define irreducible representations of $G_\sigma^\vee$. The $|Z_\sigma|q^l$ elements $\rho_x\in G_\sigma^\vee$, $x\in\mathfrak{S}$ (resp. $\rho'_x\in G_\sigma^\vee$, $x\in\mathfrak{S}$) are distinct. The maps $x\mapsto\rho_x$, $x\mapsto\rho'_x$ are two cross-sections of the map (1).

b) *One has*

$$\Gamma_G = \sum_{x\in S}\rho_x, \quad \Delta_G = \sum_{x\in S}(-1)^{s(G)-\delta_x}\rho'_x.$$

c) *Let S^* be the centralizer of x'. We have*

$$\dim \rho_X = \frac{|G_\sigma|/St_G(1)}{|S_\sigma^*|/St_{S^*}(1)} St_{S^*}(1), \qquad \dim \rho'_x = \frac{|G_\sigma|/St_G(1)}{|S_\sigma^*|/St_{S^*}(1)}.$$

CORLLARY 4.1.16 ([3], 10.8, 10.9; [6]; [8], 6.12). a) *For any $\rho\in G_\sigma$, the average value of the character of ρ on the regular unipotents in G_σ equals* 0, 1 *or* -1. *This value is ± 1 if and only if $\rho=\rho'_x$ for some $x\in\mathfrak{S}$.*

b) *If we assume that p is a good prime for G, then the character of ρ at any regular unipotent in G_σ equals* 0, 1 *or* -1.

PROOF. This average value is just $<\rho, \Delta_G>$, and b) follows from 1.11, b).

DEFINITION 4.1.17. When G has a connected center, we shall call the representation ρ'_x ($x\in\mathfrak{S}$) a *semisimple* representation and ρ_x ($x\in\mathfrak{S}$) a *regular* representation.

4.2 We consider classical groups H treated in § 3. Let r be the natural representation of H and $\theta: K^\times\to\mathbf{C}^\times$ be a character (not necessary injective). Let $b_{r,\theta}$ be the Brauer lifting of r associated to θ (2.1).

DEFINITION 4.2.1. Let G, σ, G_σ and T_o be as in 1.3. Put $W=N_G(T_o)/T_o$. We call an element w of W a *Coxeter element* (*in the twisted sense*) if w is the product in some order of the simple reflections, one in each orbit of σ. Let T be a σ-stable maximal torus of G. When T corresponds to $c_T \in H^1(\sigma, W)$ (1.5), and c_T contains a Coxeter element (in the twisted sense), we shall say that T_σ is a *Coxeter torus* of G_σ.

First we determine the Coxeter tori of classical groups treated in § 3. These are given in the next table.

4.2.2. *Table of the Coxeter tori.*

G_σ	T_σ (Coxeter torus)	$W(T)_\sigma$
$GL_n(k)$	$\{\mathrm{diag}(a, a^q, \dots, a^{q^{n-1}}) \mid a \in k_n^x\}$ $\lvert T_\sigma\rvert = q^n - 1$	generated by $\mathrm{diag}(a, a^q, \dots, a^{q^{n-1}}) \mapsto$ $\mathrm{diag}(a^q, a^{q^2}, \dots, a)$, cyclic of order n
$SO_{2n+1}(k)$	$\{\mathrm{diag}(a, a^q, \dots, a^{q^{n-1}}, 1, a^{q^n},$ $\dots, a^{q^{2n-1}}) \mid a \in k_{2n}, aa^{q^n} = 1\}$ $\lvert T_\sigma\rvert = q^n + 1$	generated by $\mathrm{diag}(a, \dots, a^{q^{n-1}}, 1, a^{q^n},$ $\dots, a^{q^{2n-1}})$ $\mapsto \mathrm{diag}(a^q, \dots, a^{q^n}, 1, a^{q^{n+1}},$ $\dots, a)$, cyclic of order $2n$
$GSp_{2n}(k)$ $GO^{o-}_{2n}(k)$	$\{\mathrm{diag}(a, a^q, \dots, a^{q^{n-1}}, a^{q^n},$ $\dots, a^{q^{2n-1}}) \mid a \in k_{2n}, aa^{q^n} \in k^x\}$ $\lvert T_\sigma\rvert = (q^n+1)(q-1)$	generated by $\mathrm{diag}(a, a^q, \dots, a^{q^{2n-1}}) \mapsto$ $\mathrm{diag}(a^q, a^{q^2}, \dots, a)$, cyclic of order $2n$
$GO^{o+}_{2n}(k)$	$\{\mathrm{diag}(a, b, b^q, \dots, b^{q^{n-2}}, b^{q^{n-1}},$ $\dots, b^{q^{2n-3}}, a^q) \mid a \in k_2,$ $b \in k_{2n-2}, bb^{q^{n-1}} = aa^q\}$ $\lvert T_\sigma\rvert = (q+1)(q^{n-1}+1)(q-1)$	generated by $\mathrm{diag}(a, b, \dots, b^{q^{n-1}},$ $\dots, b^{q^{2n-3}}, a^q) \mapsto$ $\mathrm{diag}(a^q, b^q, \dots, b^{q^n}, \dots, b, a)$, cyclic of order $2n-2$
$U_{2n}(k_2)$ $(\sigma=\tau)$ (1.1)	$\{\mathrm{diag}(a, a^{q^2}, \dots, a^{q^{2(2n-2)}},$ $b) \mid a \in k_{4n-2}, aa^{q^{2n-1}} = 1,$ $b \in k_2, bb^q = 1\}$ $\lvert T_\tau\rvert = (q+1)(q^{2n-1}+1)$	generated by $\mathrm{diag}(a, a^{q^2}, \dots, a^{q^{2(2n-2)}}, b)$ $\mapsto \mathrm{diag}(a^{q^2}, a^{q^4}, \dots, a, b)$, cyclic of order $2n-1$

$U_{2n+1}(k_2)$ $(\sigma=\tau)$ (1.1)	$\{\mathrm{diag}(a, a^{q^2},\dots, a^{q^{2(2n-1)}}, a^{q^{4n}})$ $\mid a\in k_{4n+2},\ aa^{q^{2n+1}}=1\}$ $\lvert T_\tau\rvert=(q^{2n+1}+1)$	generated by $\mathrm{diag}(a, a^{q^2},\dots, a^{q^{4n}})\mapsto$ $\mathrm{diag}(a^{q^2}, a^{q^4},\dots, a)$, cyclic of order $2n+1$

a) We denote by $\mathrm{diag}(a, b,\dots, c)$ the diagonal matrix

$$\begin{bmatrix} a & & O \\ & b & \\ & & \ddots \\ O & & & c \end{bmatrix}.$$

b) The elements in T_σ which we use in this table are not necessarily elements of G_σ, but their canonical forms in an extention field of k.

By 4.1.10. b), for a minisotropic σ-stable maximal torus T and for a character in general position ω of T_σ, we have an irreducible cuspidal character R_T^ω. Note that the Coxeter torus is minisotropic ([14], II, 1.14). For the Coxeter torus T_σ of a finite classical group, we now define a character in general position of T_σ.

4.2.3. For each positive integer m, let k_m be the unique extention of k of degree m in K. We write

$$(k_m)_{2,1}=\{x\in k_{2m}\mid xx^{q^m}=1\},\quad (k_m)_{2,k^\times}=\{x\in k_{2m}\mid xx^{q^m}\in k^\times\}.$$

Case 1. $G_\sigma=GL_n(k)$. For $i=0, 1,\dots, n-1$, let

$$\lambda_i\colon T_\sigma\ni \mathrm{diag}(a, a^q,\dots, a^{q^{n-1}})\longmapsto a^{q^i}\in k_n^\times.$$

If the restriction of θ to $k_n^\times$ satisfies the condition: $\theta(x)\neq\theta(x^q)$ for some $x\in k_n^\times$, then $\theta\lambda_i$ is a character in general position of T_σ.

Case 2. $G_\sigma=SO_{2n+1}(k)$. For $i=0, 1,\dots, 2n-1$, let

$$\lambda_i\colon T_\sigma\ni \mathrm{diag}(a,\dots, a^{q^{n-1}}, 1, a^{q^n},\dots, a^{q^{2n-1}})\longmapsto a^{q^i}\in (k_n)_{2,1}.$$

If the restriction of θ to $(k_n)_{2,1}$ satisfies the condition: $\theta(x)\neq\theta(x^q)$ for some $x\in(k_n)_{2,1}$ then $\theta\lambda_i$ is a character in general position of T_σ.

Case 3. $G_\sigma=GS_{2n}(k)$ or $GO_{2n}^{0-}(k)$. For $i=0, 1,\dots, 2n-1$, let

$$\lambda_i\colon T_\sigma\ni \mathrm{diag}(a,\dots, a^{q^{n-1}}, a^{q^n},\dots, a^{q^{2n-1}})\longmapsto a^{q^i}\in (k_n)_{2,k^\times}.$$

If θ satisfies the condition: $\theta(x)\neq\theta(x^q)$ for some $x\in(k_n)_{2,k^\times}$ then $\theta\lambda_i$ is a character in general position of T_σ.

Case 4. $G_\sigma = GO_{2n}^{0+}(k)$. For $i=0, 1,\dots, 2n-3$, let

$$\lambda_i\colon T_\sigma \ni \operatorname{diag}(a, b,\dots, b^{q^{n-1}},\dots, b^{q^{2n-3}}, a^q) \longmapsto b^{q^i} \in (k_{n-1})_{2,k^\times}.$$

and

$$\mu_j\colon T_\sigma \ni \operatorname{diag}(a, b,\dots, b^{q^{n-1}},\dots, b^{q^{2n-3}}, a^q) \longmapsto a^{q^j} \in k_2^\times \quad (j = 1, 2).$$

If θ satisfies the condition: $\theta(x) \neq \theta(x^q)$ for some $x \in (k_{n-1})_{2,k^\times}$, then $\theta\lambda_i$ is a character in general position of T_σ. If θ satisfies the condition: $\theta(x) \neq \theta(x^q)$ for some $x \in (k_{n-1})_{2,k^\times}$ and for some $x \in k_2^\times$, then $\theta(\lambda_i\mu_j)$ is a character in general position of T_σ, where $(\lambda_i\mu_j)$ is defined by

$$(\lambda_i\mu_j)(\operatorname{diag}(a, b,\dots, b^{q^{n-1}} b,^{q^{n-2}},\dots, b^{q^{2n-3}}, a^q)) = b^{q^i} \cdot a^{q^j} \in k_{2n-2}^\times.$$

Case 5. $G_\sigma = U_{2n}(k_2)$. For $i=0, 1,\dots, 2n-2$, let

$$\lambda_i\colon T_\sigma \ni \operatorname{diag}(a, a^{q^2},\dots, a^{q^{2(2n-2)}}, b) \longmapsto a^{q^{2i}} \in (k_{2n-1})_{2,1}$$

and

$$\mu\colon T_\sigma \ni \operatorname{diag}(a, a^{q^2},\dots, a^{q^{2(2n-2)}}, b) \longmapsto b \in k_{2,1}.$$

If θ satisfies the condition: $\theta(x) \neq \theta(x^{q^2})$ for some $x \in (k_{2n-1})_{2,1}$, then $\theta\lambda_i$ and $\theta(\lambda_i\mu)$ are characters in general position of T_σ, where $(\lambda_i\mu)$ is defined by

$$(\lambda_i\mu)(\operatorname{diag}(a, a^{q^2},\dots, a^{q^{2(2n-2)}}, b)) = a^{q^{2i}} \cdot b \in k_{4n-2}^\times.$$

Case 6. $G_\sigma = U_{2n+1}(k_2)$. For $i=0, 1,\dots, 2n$, let

$$\lambda_i\colon T_\sigma \ni \operatorname{diag}(a, a^{q^2},\dots, a^{q^{4n}}) \longmapsto a^{q^{2i}} \in (k_{2n+1})_{2,1}.$$

If θ satisfies the condition: $\theta(x) \neq \theta(x^{q^2})$ for some $x \in (k_{2n+1})_{2,1}$, then $\theta\lambda_i$ is a character in general position of T_σ.

THEOREM 4.2.4. *Let G_σ be as above and T_σ be the Coxeter torus of G_σ. If θ satisfies the above regularity condition; Case $1\sim 6$ respectively, then*

a) *$\pm R_T^{\theta\lambda_i}$ is an irreducible cuspidal character of G_σ.*

b) *If G_σ is a split orthogonal similitudes group of size $2n$ (resp. a unitary group of size $2n$), then $\pm R_T^{\theta(\lambda_i\mu_j)}$ (resp. $\pm R_T^{\theta(\lambda_i\mu)}$) is an irreducible cuspidal character of G_σ.*

c) *$\langle b_{r,\theta}, R_T^{\theta\lambda_i}\rangle_{G_\sigma} = 1$; if $G_\sigma = GO_{2n}^{0+}(k)$, then $\langle b_{r,\theta}, R_T^{\theta(\lambda_i\mu_j)}\rangle_{G_\sigma} = 0$; if $G_\sigma = U_{2n}(k_2)$ then $\langle b_{r,\theta}, R_T^{\theta(\lambda_i\mu)}\rangle_{G_\sigma} = 0$.*

d) *$R_T^{\theta\lambda_i} = R_T^{\theta\lambda_j}$ for any i, j; if $G_\sigma = GO_{2n}^{0+}(k)$, then $R_T^{\theta(\lambda_i\mu_j)} = R_T^{\theta(\lambda_h\mu_l)}$ for any i, j, h and l $(i-j=h-l)$; if $G_\sigma = U_{2n}(k_2)$, then $R_T^{\theta(\lambda_i\mu)} = R_T^{\theta(\lambda_j\mu)}$ for any i, j.*

PROOF. a), b) follow from 4.1.10. b).

c) If G_σ is the matrix group of size m, then we can write $b_{r,\theta}=\theta u_1+\cdots+\theta u_m$. By 4.1.7. b),

$$<b_{r,\theta},\ R_T^{\theta\lambda_i}>_{G_\sigma} = <b_{r,\theta},\ \theta\lambda_i>_{T_\sigma} = \sum_{j=1}^{n}<\theta u_j,\ \theta\lambda_i>_{T_\sigma}$$

$$=\begin{cases} 1 & \text{if } u_j=\lambda_i \text{ on } T_\sigma, \\ 0 & \text{otherwise.}\end{cases}$$

On the other hand

$$<b_{r,\theta},\ R_T^{\theta(\lambda_i\mu_j)}>_{G^\sigma} = \sum_{h=1}^{m}<\theta u_h,\ \theta(\lambda_i\mu_j)>_{T_\sigma} = 0.$$

d) follows from 4.1.5. b). q.e.d.

COROLLARY 4.2.5. *The irreducible cuspidal character* $\pi_{n,\theta}$ *(if* $G_\sigma=GL_n(k)$*) or* $\pi'_{n,\theta}$ *(otherwise) obtained in* § 3 *coincides with* $\pm R_T^{\theta\lambda_i}$, θ *injective.*

In the rest of this section we shall assume that the center of G is connected and $q=|k|$ satisfies the condition of § 3 respectively for $G=GL_n$, SO_{2n+1}, $GSp_{2n}, \dots$ etc. Moreover we assume that θ is injective. Now the characteristic p of k is a good prime for G; hence any character ρ of G_σ takes the values 0, 1 or -1 on any regular unipotent in G_σ (4.1.16. b)).

PROPOSITION 4.2.6. *Let* $r\colon G_\sigma\to GL_m(k)$ *be as in* § 3. *Then all the irreducible constituents of* $b_{r,\theta}$ *(*θ*; injective) are semisimple* (4.1.17).

PROOF. Let u be a regular unipotent in G_σ. Then $b_{r,\theta}(u)=m$. On the other hand, we have $<b_{r,\theta},\ b_{r,\theta}>_{G_\sigma}=m$ by 2.4. Hence all the irreducible constituents of $b_{r,\theta}$ have value ± 1 on a regular unipotent in G_σ. q.e.d.

Next we characterize m_i which is, defined in § 3, an irreducible constituent both of $\mathrm{Ind}_{P_i}^{G_\sigma}(\tilde{\pi}'_{i,\theta})$ and of $b_{r,\theta}$.

PROPOSITION 4.2.7. *Let* $P_\sigma=L_\sigma V_\sigma$ *be a parabolic subgroup of* G_σ *and* ψ *be an irreducible semisimple cuspidal character of* L_σ *and* $\tilde{\psi}$ *be a character of* P_σ *defined by* $\tilde{\psi}(hv)=\psi(h)$, $(h\in L_\sigma,\ v\in V_\sigma)$. *Then there is a unique irreducible semisimple character* ρ' *of* G_σ *such that* $<\mathrm{Ind}_{P_i}^{G_\sigma}(\tilde{\psi}),\ \rho'>_{G_\sigma}\neq 0$.

PROOF. By 4.1.9, we can write $\tilde{\psi}=\pm R_{T_o,P}^{\xi}$, where T_o is a minisotropic torus of L and ξ is a character in general position of $T_{o\sigma}$. By 4.1.10. a), we have $\mathrm{Ind}_{P_\sigma}^{G_\sigma}(\tilde{\psi})=\pm R_{T_o}^{\xi}$. Put

$$\rho' = (-1)^{s(G)-\delta'}\sum_{(T_o,\xi)}\frac{1}{<R_T^{\omega},\ R_T^{\omega}>}R_T^{\omega}\ \left(\begin{array}{l}\delta' \text{ is the } k\text{-rank of}\\ Z_{G^*}(\xi')\ (4.1.11,\ 4.1.15)\end{array}\right),$$

where $\sum_{(T_o,\xi)}$ means the sum over all (T, ω) which are geometrically conjugate to (T_o, ξ) (mod. G_σ). Then ρ' is semisimple by 4.1.16. a), and we have

$$<\operatorname{Ind}_{P_\sigma}^{G_\sigma}(\tilde{\psi}), \rho'>_{G_\sigma} = 1.$$

On the other hand, if ρ'' is another semisimple character of G_σ, then by 4.1.16. a), we have

$$\rho'' = (-1)^{s(G)-\delta''}\sum_{(T',\omega')} \frac{1}{<R_{T'}^{\omega'}, R_{T'}^{\omega'}>} R_{T'}^{\omega'} \quad \left(\begin{array}{l}\delta'' \text{ is the } k\text{-rank} \\ \text{of } Z_{G^*}(\omega'')\end{array}\right),$$

where $\sum_{(T',\omega')}$ means the sum over some geometric conjugacy class (mod. G_σ) which does not contain (T_o, ξ). By 4.1.5. a),

$$<\rho'', R_{T_o}^{\xi}>_{G_\sigma} = 0. \qquad \text{q.e.d.}$$

By the proof of this proposition, we also have

COROLLARY 4.2.8. *Let the notations be as in 4.2.7. Let $\tilde{\psi} = \pm R_{T_o,P}^{\xi}$, where T_o is a minisotropic torus of L and let ξ be a character in general position of $T_{o\sigma}$. Then*

$$\rho' = (-1)^{s(G)-\delta'}\sum_{(T_o,\xi)} \frac{1}{<R_T^{\omega}, R_T^{\omega}>} R_T^{\omega} \left(\begin{array}{l}\delta' \text{ is the } k\text{-rank of} \\ Z_{G^*}(\xi')\ (4.1.11, 4.1.15)\end{array}\right),$$

where $\sum_{(T_o,\xi)}$ means the sum over all (T, ω) which are geometrically conjugate to (T_o, ξ) (mod. G_σ).

REMARK. In 4.2.7, one can omit the assumption "ψ *is semisimple*", because one can prove that "*If* $\operatorname{Ind}_{P_\sigma}^{G_\sigma}(\tilde{\psi})$ *contains a semisimple character, then* ψ *is semisimple.*" This remark and 4.2.7 are obtained also by J. A. Green, G. I. Lehrer and G. Lusztig [6].

PROOF. A semisimple character can be written as

$$\rho'_x = (-1)^{s(G)-\delta_x}\sum_{\substack{(T,\omega) \bmod. G_\sigma \\ [\omega]=x}} \frac{1}{<R_T^{\omega}, R_T^{\omega}>} R_T^{\omega},$$

($x \in \mathfrak{S}$, where $\mathfrak{S}$ is the set of all geometric conjugacy classes of pairs (T, ω)). Let (T, ω) be a maximal split pair in x ([3], 5.25). There exists a σ-stable parabolic subgroup $Q \subset G$ such that T is contained in some σ-stable Levi subgroup M of Q and (T, ω) is in general position ([3], 5.28). One chooses Q and M so that the T is minisotropic in M. Then

$$<\rho', \operatorname{Ind}_{Q_\sigma}^{G_\sigma}(R_{T,Q}^{\omega})>_{G_\sigma} = \pm 1.$$

If $\operatorname{Ind}_{P_\sigma}^{G_\sigma}(\tilde{\psi})$ and $\operatorname{Ind}_{Q_\sigma}^{G_\sigma}(R_{T,Q}^{\omega})$ have a common irreducible constituent ρ'_x, then by 3.2, Q and P are associated i.e., there exists $g \in G_\sigma$ such that $M_\sigma = gL_\sigma g^{-1}$, and

${}^{\theta}\tilde{\psi}=R^{\omega}_{T,Q}$. Hence ψ is a semisimple character. q.e.d.

In § 3 (e.g. in the proof of Theorem 3.4), we take a parabolic subgroup Q'_i of G_σ which is contained in P_i such that the character $\tilde{\pi}'_{i,\theta}$ is cuspidal on the Levi subgroup of G'_i. Then irreducible constituents of $\mathrm{Ind}^{G_\sigma}_{P_i}(\tilde{\pi}'_{i,\theta})$ are irreducible constituents of $\mathrm{Ind}^{G_\sigma}_{Q'_i}(\tilde{\pi}'_{i,\theta})$. Hence we have:

COROLLARY 4.2.9. *One can characterize m_i as the unique semisimple character in the irreducible constituents of* $\mathrm{Ind}^{G_\sigma}_{P_i}(\tilde{\pi}'_{i,\theta})$.

Next we describe $b_{r,\theta}$ as the alternating sum of semisimple characters when $G=GSp_{2n}$ and θ is injective. For other classical groups, it can be done similarly. By 3.4,

$$b_{r,\theta}=b_{2n,\theta}=\sum_{i=0}^{n-1}(-1)^{n-i-1}\mathrm{Ind}^{GSp_{2n}(k)}_{P_i}(\tilde{\pi}_{n-i,\theta})+(-1)^n\pi'_{n,\theta}$$
$$+\sum_{i=1}^{n-1}(-1)^i m_i\,.$$

Here $\mathrm{Ind}^{GSp_{2n}(k)}_{P_i}(\tilde{\pi}_{n-i,\theta})$ is irreducible and m_i is an irreducible constituent of $\mathrm{Ind}^{GSp_{2n}(k)}_{P_i}(\tilde{\pi}'_{i,\theta})$. By 4.2.4, we have $\pi'_{n,\theta}=(-1)^nR^{\theta(n)}_{T(n)}$, where $T(n)$ is the Coxeter torus of GSp_{2n} and $\theta(n)$ is an injective character of $T(n)_\sigma$. We study here on m_i.

The irreducible cuspidal character $\pi'_{i,\theta}$ of $GSp_{2i}(k)$ can be described as $\pi'_{i,\theta}=(-1)^iR^{\theta(i)}_{T(i)}$. Put

$$Q'_i=\left\{\begin{bmatrix} a & & & & & * \\ & \ddots & & & & \\ & & d & & & \\ & & & X & & \\ & & & & d' & \\ & & & & & \ddots \\ O & & & & & a' \end{bmatrix}\in GSp_{2n}(k)\ \middle|\ \begin{matrix} X\in GSp_{2i}(k)\,, \\ a,a',\dots,d,d'\in k^{\times}\end{matrix}\right\}.$$

Then Q'_i is a parabolic subgroup of $GSp_{2n}(k)$ and contained in P_i. Let M_i be a Levi subgroup of Q'_i. Let p (resp. π) be the canonical projection $p\colon M_i\to GSp_{2i}(k)$ (resp. $\pi\colon Q'_i\to M_i$). We denote by $l_{i,\theta}$ the character of M_i defined by $l_{i,\theta}(x)=\pi'_{i,\theta}(p(x))$, $x\in M_i$. Then $l_{i,\theta}$ is an irreducible semisimple cuspidal character of M_i. Since M_i and $GSp_{2i}(k)$ have the same unipotent sets, we have $l_{i,\theta}=(-1)^iR^{\xi}_{T_o,M_i}$ by the character formula 4.1.1. a), where

$$T_o=\left\{\begin{bmatrix} a & & & & & O \\ & \ddots & & & & \\ & & d & & & \\ & & & T(i) & & \\ & & & & d' & \\ & & & & & \ddots \\ O & & & & & a' \end{bmatrix}\in GSp_{2n}\ \middle|\ \begin{matrix} T(i)\text{ is the Coxeter torus} \\ \text{of } GSp_{2i}, \\ a,a',\dots,d,d'\in GL_1\end{matrix}\right\}$$

and $\xi(t)=\theta(i)(p(t))$, $t\in T_{o\sigma}$.

Let $\tilde{l}_{i,\theta}$ be the character of Q'_i defined by $\tilde{l}_{i,\theta}=l_{i,\theta}(\pi(x))$, $x\in Q'_i$. We then have $\tilde{l}_{i,\theta}=(-1)^i R^{\xi}_{T_o,Q'_i}$. The pairs (T,ω) which are geometrically conjugate to (T_o,ξ) have the forms

$$T=\begin{bmatrix} T_{11} & O & T_{12} \\ O & T(i) & O \\ T_{21} & O & T_{22}\end{bmatrix}\subset GSp_{2n},\ \text{where}\ \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22}\end{bmatrix}$$

is a σ-stable maximal torus of $GSp_{2(n-i)}$ and

$$\text{(4.2.10)}\qquad \omega\left(\begin{bmatrix} t_{11} & O & t_{12} \\ O & t(i) & O \\ t_{21} & O & t_{22}\end{bmatrix}\right)=\theta(i)(t(i)),\quad \begin{bmatrix} t_{11} & O & t_{12} \\ O & t(i) & O \\ t_{21} & O & t_{22}\end{bmatrix}\in T_{\sigma}.$$

Let $\mathfrak{T}_{n-i}$ be the set of σ-stable maximal tori of $GSp_{2(n-i)}$ (mod. $GSp_{2(n-i)}(k)$). Put

$$\mathfrak{T}=\left\{\begin{bmatrix} T_{11} & O & T_{12} \\ O & T(i) & O \\ T_{21} & O & T_{22}\end{bmatrix}\subset GSp_{2n}\,\middle|\,\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22}\end{bmatrix}\in\mathfrak{T}_{n-i}\right\},$$

$$\mathfrak{X}=\{(T,\omega)\,|\,T\in\mathfrak{T}\ \text{and}\ \omega\ \text{satisfies 4.2.10}\}.$$

Then $\mathfrak{X}$ is a geometric conjugacy class (mod. $GSp_{2n}(k)$) which contains (T_o,ξ). The irreducible constituent m_i of $\mathrm{Ind}_{P_i}^{GSp_{2n}(k)}(\tilde{\pi}'_{i,\theta})$ is an irreducible constituent of $\mathrm{Ind}_{Q'_i}^{GSp_{2n}(k)}(\tilde{l}_{i,\theta})$. By 4.2.7 and 4.2.9, m_i is the unique semisimple character in the irreducible constituents of $\mathrm{Ind}_{Q'_i}^{GSp_{2n}(k)}(\tilde{l}_{i,\theta})$. Hence, by 4.2.8, we have:

PROPOSITION 4.2.11. *Let $\mathfrak{X}$ be as above. Then*

$$m_i=(-1)^i\textstyle\sum_{(T,\omega)\in X}\dfrac{1}{<R^{\omega}_T,R^{\omega}_T>}R^{\omega}_T.$$

We compute the degrees of the irreducible constituents of $b_{2n,\theta}$.

LEMMA 4.2.12 (J. A. Green). *Let χ_n be an irreducible cuspidal character of $GL_n(k)$. Then the degree of χ_n, i.e., $\chi_n(1)$, is $(q-1)(q^2-1)\cdots(q^{n-1}-1)$.*

COROLLARY 4.2.13.

a) $\mathrm{Ind}_{P_i}^{GSp_{2n}(k)}(\tilde{\pi}_{n-i,\theta})(1)=\dfrac{(q^{2(i+1)}-1)\cdots(q^{2(n-1)}-1)(q^{2n}-1)}{q^{n-i}-1}$,

b) $\pi'_{n,\theta}(1) = \dfrac{(q^2-1)\cdots(q^{2(n-1)}-1)(q^{2n}-1)}{q^n+1}$,

c) $m_i(1) = \dfrac{(q^{2(n-i+1)}-1)\cdots(q^{2(n-1)}-1)(q^{2n}-1)}{q^i+1}$.

PROOF. Since $\pi_{n-i,\theta}$ is an irreducible cuspidal character of $GL_{n-i}(k)$, we have

$$\tilde{\pi}_{n-i,\theta}(1) = \frac{(q-1)\cdots(q^{n-i-1}-1)}{q^{n-i}-1}.$$

Here $\mathrm{Ind}_{P_i}^{GSp_{2n}(k)}(\tilde{\pi}_{n-i,\theta})$ is irreducible, so we have a) if we compute $|GS_{2n}(k)|$ and $|P_i|$. Let $f\colon Sp_{2n}\to PSp_{2n}=GS_{2n}/Z$ be the central covering. Let T and T' be the σ-stable Coxeter tori of GSp_{2n} and Sp_{2n} respectively and let P'_i be the parabolic subgroup of Sp_{2n} such that $f(P'_i)=P_i/Z$ i.e., P'_i has the form

$$\left\{\left.\begin{bmatrix} A & & * \\ & X & \\ O & & B \end{bmatrix}\right| A, B\in GL_{n-i},\ AJ\,{}^tB = J,\ X\in Sp_{2i}\right\}.$$

Note that we use the notational identifications $P_{i\sigma}=P_i$ and $P'_{i\sigma}=P'_i$. Then, by [1], (16.8), we have

$$|GSp_{2n}(k)| = |Sp_{2n}(k)|\,|Z_\sigma| = (q^2-1)\cdots(q^{2(n-1)}-1)(q^{2n}-1)q^2(q-1),$$

$$|P_i| = |P'_i|\,|Z_\sigma| = |GL_{n-i}(k)|\,|Sp_{2i}(k)|q^\alpha(q-1), \qquad (\alpha = (n-i)(n+3i+1)/2),$$

and

$$|T_\sigma| = |T'_\sigma|\,|Z_\sigma| = (q^n+1)(q-1).$$

Hence we have a). b) follows from 4.1.1. b). Let $\mathfrak{X}$ be as in 4.2.11. Let S^* be the connected centralizer in the dual group GSp_{2n}^* corresponding to $\mathfrak{X}$. Then the dual group S of S^* have a form

$$S = \left\{\begin{bmatrix} x_{11} & O & x_{12} \\ O & t(i) & O \\ x_{21} & O & x_{22} \end{bmatrix}\in GSp_{2n}\ \middle|\ \begin{array}{l} t(i) \text{ is in the Coxeter torus of} \\ GSp_{2i},\ \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\in GSp_{2(n-i)} \end{array}\right\}.$$

Since $|S^*_\sigma|=|S_\sigma|$, we can compute c) by 4.1.15. c). q.e.d.

In the rest of this section we give the similar results as those of 4.2.11 and 4.2.13 for other classical groups $SO_{2n+1}(k)$, $GO_{2n}^{\eta-}(k)$, $GO_{2n}^{\eta+}(k)$, $U_{2n}(k_2)$ and $U_{2n+1}(k_2)$. Main parts of proofs are quite similar to those of 4.2.11 and 4.2.13. Hence we omit them in most cases.

PROPOSITION 4.2.14 (*Case* $G_\sigma = SO_{2n+1}(k)$, 3.5). a) *Let* $\mathfrak{T}_{n-i}$ *be the set of* σ*-stable maximal tori of* $SO_{2(n-i)+1}$ (*mod.* $SO_{2(n-i)+1}(k)$). *Let* $\mathfrak{T}$ *be the set of all* σ*-stable maximal tori of* SO_{2n+1} *isomorphic to* $T(i)\times T'$, $(T'\in\mathfrak{T}_{n-i})$ *and let* $\theta(i)$ *be an injective character of* $T(i)_\sigma$ *defined by* θ, *where* $T(i)$ *is the Coxeter torus of* SO_{2i+1}. *Put*

$$\mathfrak{X} = \{(T,\omega) \mid T\in\mathfrak{T},\ \omega((t(i),t')) = \theta(i)(t(i)),\ ((t(i),t')\in T(i)_\sigma\times T'_\sigma \simeq T_\sigma)\}.$$

Then $\mathfrak{X}$ *is a geometric conjugacy class* (*mod.* $SO_{2n+1}(k)$) *and we have*

$$m_i = (-1)^i \sum\nolimits_{(T,\omega)\in\mathfrak{X}} \frac{1}{<R_T^\omega, R_T^\omega>} R_T^\omega,$$

b) $\displaystyle \mathrm{Ind}_{P_i}^{SO_{2n+1}(k)}(\tilde{\pi}_{n-i,\theta})(1) = \frac{(q^{2(i+1)}-1\cdots(q^{2(n-1)}-1)(q^{2n}-1)}{q^{n-i}-1},$

c) $\displaystyle \pi'_{n,\theta}(1) = \frac{(q^2-1)\cdots(q^{2(n-1)}-1)(q^{2n}-1)}{q^n+1},$

d) $\displaystyle m_i(1) = \frac{(q^{2(n-i+1)}-1)\cdots(q^{2(n-1)}-1)(q^{2n}-1)}{q^i+1}.$

PROOF. The connected centralizer in the dual group $SO^*_{2n+1} = Sp_{2n}$ which corresponds to $\mathfrak{X}$ is

$$S^* = \left\{ \begin{bmatrix} x_{11} & O & x_{12} \\ O & t(i) & O \\ x_{21} & O & x_{22} \end{bmatrix} \in Sp_{2n} \,\middle|\, \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \in Sp_{2(n-i)},\ t(i)\in T(i) \right\},$$

where $T(i)$ is the Coxeter torus of Sp_{2i}. Then

$$|S^*| = (q^i+1)(q^2-1)\cdots(q^{2(n-i)}-1)q^{(n-i)^2}.$$

Hence we have d). q.e.d.

PROPOSITION 4.2.15 (*Case* $G_\sigma = GO^{o+}_{2n}(k)$, 3.6). a) *Let* $\mathfrak{T}_{n-i}$ *be the set of maximal tori of* $GO^{o+}_{2(n-i)}$ (*mod.* $GO^{o+}_{2(n-i)}(k)$). *Put*

$$\mathfrak{T} = \left\{ \begin{bmatrix} T_{11} & O & T_{12} \\ O & T(i) & O \\ T_{21} & O & T_{22} \end{bmatrix} \subset GO^{o+}_{2n} \,\middle|\, \begin{array}{l} T(i) \text{ is the Coxeter torus of} \\ GO^{o+}_{2i},\ \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathfrak{T}_{n-i} \end{array} \right\}.$$

Let $\theta(i)$ *be an injective character of* $T(i)_\sigma$ *defined by* θ. *Put*

$$\mathfrak{X} = \{(T, \omega) | T \in \mathfrak{T}, \omega\left(\begin{bmatrix} t_{11} & o & t_{12} \\ o & t(i) & o \\ t_{21} & o & t_{22} \end{bmatrix}\right) = \theta(i)(t(i))\}.$$

Then

$$m_i = (-1)^{i-1} \sum_{(T,\omega)\in\mathfrak{X}} \frac{1}{<R_T^\omega, R_T^\omega>} R_T^\omega,$$

a) $\operatorname{Ind}_{P_i}^{GO_{2n}^{o+}(k)}(\tilde{\pi}_{n-i,\theta})(1) = \dfrac{(q^i+1)(q^{2(i+1)}-1)\cdots(q^{2(n-1)}-1)(q^n-1)}{q^{n-i}-1}$,

c) $\pi'_{n,\theta}(1) = \dfrac{(q^2-1)\cdots(q^{2(n-1)}-1)(q^n-1)}{(q+1)(q^{n-1}+1)}$,

d) $m_i(1) = \dfrac{(q^{n-i}+1)(q^{2(n-i+1)}-1)\cdots(q^{2(n-1)}-1)(q^n-1)}{(q+1)(q^{i-1}+1)}$.

PROPOSITION 4.2.16 (*Case* $G_\sigma = GO_{2n}^{o-}(k)$, 3.7). a) *Let* $\mathfrak{T}_{n-i}$ *be the set of maximal tori of* $GO_{2(n-i)}^{o+}$ (*mod.* $GO_{2(n-i)}^{o+}(k)$). *Put*

$$\mathfrak{T} = \left\{\begin{bmatrix} T_{11} & O & T_{12} \\ O & T(i) & O \\ T_{21} & O & T_{22} \end{bmatrix} \subset GO_{2n}^{o-} \;\middle|\; \begin{array}{l} T(i) \text{ is the Coxeter torus of} \\ GO_{2i}^{o-}, \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathfrak{T}_{n-i} \end{array}\right\}.$$

Let $\theta(i)$ *be an injective character of* $T(i)_\sigma$ *defined by* θ. *Put*

$$\mathfrak{X} = \{(T, \omega) | T \in \mathfrak{T}, \omega\left(\begin{bmatrix} t_{11} & o & t_{12} \\ o & t(i) & o \\ t_{21} & o & t_{22} \end{bmatrix}\right) = \theta(i)(t(i))\}.$$

Then

$$m_i = (-1)^{i-1} \sum_{(T,\omega)\in\mathfrak{X}} \frac{1}{<R_T^\omega, R_T^\omega>} R_T^\omega,$$

b) $\operatorname{Ind}_{P_i}^{GO_{2n}^{o-}(k)}(\tilde{\pi}_{n-i,\theta})(1) = \dfrac{(q^i-1)(q^{2(i+1)}-1)\cdots(q^{2(n-1)}-1)(q^n+1)}{q^{n-i}-1}$,

c) $\pi'_{n,\theta}(1) = \dfrac{(q^2-1)\cdots(q^{2(n-1)}-1)(q^n+1)}{q^n+1}$,

d) $m_i(1) = \dfrac{(q^{n-i}-1)(q^{2(n-i+1)}-1)\cdots(q^{2(n-1)}-1(q^n+1)}{q^i+1}$.

PROPOSITION 4.2.17 (*Case* $G_\sigma = U_{2n}(k_2)$, 3.8). a) *Let* $\mathfrak{T}_{n-i}$ *be the set of all maximal tori of* $U_{2(n-i)}(k_2)$ (*mod.* $U_{2(n-i)}(k_2)$). *Put*

$$\mathfrak{T} = \left\{ \begin{bmatrix} T_{11} & O & T_{12} \\ O & T(i) & O \\ T_{21} & O & T_{22} \end{bmatrix} \subset U_{2n}(k_2) \;\middle|\; \begin{array}{l} T(i) \text{ is the Coxeter torus of} \\ U_{2i}(k_2), \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathfrak{T}_{n-i} \end{array} \right\}.$$

Let $\theta(i)$ *be an injective character of* $T(i)$ *defined by* θ. *Put*

$$\mathfrak{X} = \{(T, \omega) | T\text{'s are tori of } GL_{2n} \text{ such that } T_\sigma \in \mathfrak{T},\ \omega\left(\begin{bmatrix} t_{11} & o & t_{12} \\ o & t(i) & o \\ t_{21} & o & t_{22} \end{bmatrix}\right) = \theta(i)(t(i))\}.$$

Then

$$m_i = (-1)^i \sum_{(T,\omega)\in\mathfrak{X}} \frac{1}{<R_T^\omega, R_T^\omega>} R_T^\omega,$$

b) $\operatorname{Ind}_{P_i}^{U_{2n}(k_2)}(\tilde{\pi}_{n-i,\theta})(1) = \dfrac{(q^{2i}+1)(q^{2(i+1)}-1)\cdots(q^{2n-1}+1)\ (q^{2n}-1)}{q^{2(n-i)}-1}$,

c) $\pi'_{n,\theta}(1) = \dfrac{(q+1)(q^2-1)\cdots(q^{2n-1}+1)(q^{2n}-1)}{(q+1)(q^{2n-1}+1)}$,

d) $m_i(1) = \dfrac{(q^{2(n-i)}+1)(q^{2(n-i+1)}-1)\cdots(q^{2n-1}+1)(q^{2n}-1)}{(q+1)(q^{2i-1}+1)}$.

PROPOSITION 4.2.18 (*Case* $G_\sigma = U_{2n+1}(k_2)$, 3.9). a) *Let* $\mathfrak{T}_{n-i}$ *be the set of all maximal tori of* $U_{2(n-i)}(k_2)$ (*mod.* $U_{2(n-i)}(k_2)$). *Let* $\mathfrak{T}$ *be the set of all maximal tori* T *of* GL_{2n+1} *such that* T_σ *is isomorphic to* $T(i)\times T'$ ($T(i)$ *is the Coxeter torus of* $U_{2i+1}(k_2)$ *and* $T' \in \mathfrak{T}_{n-i}$) *and let* $\theta(i)$ *be an injective character of* $T(i)$ *defined by* θ. *Put*

$$\mathfrak{X} = \{(T, \omega) | T \in \mathfrak{T},\ \omega((t(i), t')) = \theta(i)(t(i)),\ ((t(i), t') \in T(i)\times T' \simeq T_\sigma)\}.$$

Then

$$m_i = (-1)^i \sum_{(T,\omega)\in\mathfrak{X}} \frac{1}{<R_T^\omega, R_T^\omega>} R_T^\omega,$$

b) $\operatorname{Ind}_{P_i}^{U_{2n+1}(k_2)}(\tilde{\pi}_{n-i,\theta})(1) = \dfrac{(q^{2(i+1)}-1)\cdots(q^{2n}-1)(q^{2n+1}+1)}{q^{2(n-i)}-1}$,

c) $\pi'_{n,\theta}(1) = \dfrac{(q+1)(q^2-1)\cdots(q^{2n}-1)(q^{2n+1}+1)}{q^{2n+1}+1}$,

d) $m_i(1) = \dfrac{(q^{2(n-i)+1}+1)\cdots(q^{2n}-1)(q^{2n+1}+1)}{q^{2i+1}+1}$.

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