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A Semigroup Treatment of the Mixed Problem for the Hamilton-Jacobi Equation in One Space Variable

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1. Introduction

This paper deals with the mixed problem in the domain $D=\{(t, x); 0\leq t<+\infty, 0\leq x\leq 1\}$ for the Hamilton-Jacobi equation (hereafter called (MP))

$$(1.1)\qquad u_t+f(u_x)=0, \qquad t>0, \quad 0<x<1,$$

$$(1.2)\qquad u(0, x)=u_0(x), \qquad 0\leq x\leq 1,$$

$$(1.3)\qquad u(t, 0)=u(t, 1)=0, \qquad t\geq 0.$$

This investigation was strongly motivated by a recent paper [1] by S. Aizawa, in which the Cauchy problem for the Hamilton-Jacobi equation

$$(*)\qquad u_t+f(u_x)=0, \qquad t>0, \quad -\infty<x<\infty,$$

is treated from the viewpoint of the nonlinear semigroup theory. Aizawa succeeded in constructing a global solution of the Cauchy problem for $(*)$ in the sense of Kružkov when the initial value lies in $W_1^{\infty}(R)$ and its derivative is continuous in $L^1(R)$ under the assumption that f is merely continuous.

Mixed problems for the Hamilton-Jacobi equation have been treated by several authors. See, E. D. Conway and E. Hopf [7], S. Aizawa and N. Kikuchi [3] and S. H. Benton [5, 6]. These authors proved the existence of generalized solutions of mixed problems by using the variational method under the assumption that f is convex.

The purpose of this paper is to prove the existence of a generalized solution of (MP) by using the nonlinear semigroup theory. Our treatment needs no convexity condition on f. Here we define a generalized solution of (MP) in the sense of Kružkov [13].

DEFINITION 1.1. *A Lipschitz continuous function $u(t, x)$ on $D=[0, \infty)\times[0, 1]$ is called a generalized solution of* (MP) *if*

(i) *u satisfies* (1.1) *a.e. on D as well as* (1.2) *and* (1.3),

(ii) *u satisfies the integral inequality*

$$\int_0^T\int_0^1\{|u_x-k|\varphi_t+\operatorname{sign}_0(u_x-k)[f(u_x)-f(k)]\varphi_x\}dxdt \geq 0$$

for every $T>0$, *every* $\varphi(t, x)\in C_0^\infty((0, T)\times(0, 1))$ *such that* $\varphi\geq 0$ *and every* $k\in R$.

We shall, as in [1], choose $L^\infty(I)$ as the Banach space associated with (MP), where I is the interval $[0, 1]$. Let $f: R\to R$ be a continuous function and satisfy the following condition:

(f. 1) (i) $f(0)\leq 0$.

(ii) *There exists a nonnegative constant* p_0 *such that* $f(p)>0$ *for any* p: $|p|\geq p_0$.

Then, as we shall see, the semigroup approach enables us to prove the existence of a nonnegative generalized solution of (MP). Moreover, as an intermediate step in the development, the existence and uniqueness of nonnegative generalized solutions of the boundary value problem for any $\lambda>0$ (hereafter called (BVP))

$$\text{(BVP)}\qquad \begin{aligned} &u(x)+\lambda f(u'(x)) = h(x), \qquad 0<x<1,\\ &u(0) = u(1) = 0, \end{aligned}$$

are established, where $h(x)$ is a given nonnegative and Lipschitz continuous function defined on I with $h(0)=h(1)=0$. In order to solve (BVP), we introduce the regularized boundary value problem

$$(\text{BVP}_\varepsilon)\qquad \begin{aligned} &u(x)+\lambda f(u'(x))-\varepsilon u''(x) = h(x), \qquad 0\leq x\leq 1,\\ &u(0) = u(1) = 0, \end{aligned}$$

for any $\varepsilon>0$.

In Section 2, we give a fundamental lemma concerning the generation of nonlinear semigroups and define the operator A_0: $v\mapsto f(v_x)$ in $L^\infty(I)$ that may be associated with (MP). In Section 3, we shall study the existence and uniqueness of generalized solutions of (BVP). Here the solutions are obtained as the limits of solutions of (BVP_ε). Various results concerning (BVP_ε) are developed as needed. Section 4 deals with the existence of generalized solutions of (MP). Here the solution of (MP) is obtained under the assumption that $f\in C^1$ satisfies Condition (f. 1) and the initial value u_0 is nonnegative and Lipschitz continuous such that $u_0(0)=u_0(1)=0$.

The question of uniqueness of our solution will be taken up in a later paper.

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2. Preliminaries and definition of the operator A_0

In this paper we shall work in the Banach space $L^\infty(I)$ of all real-valued, bounded and measurable functions v on the compact interval I with norm $\|v\|_\infty = \text{ess-sup}\{|v(x)|;\, x \in I\}$. Before proceeding to the proof of an existence theorem for a generalized solution of (MP) in the sense of Definition 1.1, we begin with a basic lemma which is essentially included in the generation theorem of Crandall and Liggett [9]. In order to construct a generalized solution of (MP) under the assumption on f prescribed in this paper, it seems simpler to apply this lemma than to apply the generation theorem.

Let X be a Banach space and A be an operator in X. As in [9], we interpret A as a subset of $X \times X$. For real λ, J_λ will denote the set $(I+\lambda A)^{-1}$ and $D_\lambda = D(J_\lambda)$ its domain.

FUNDAMENTAL LEMMA. *Suppose that $\hat{R}$ is a convex set satisfying the following conditions*:

(a) $D(A) \subset \hat{R} \subset R(I+\lambda A)$ *for each* $\lambda > 0$,

(b) *if we restrict the domain of $J_\lambda = (I+\lambda A)^{-1}$ to $\hat{R}$, then J_λ is a contraction mapping (we denote J_λ restricted on $\hat{R}$ by $\hat{J}_\lambda$),*

(c) *for each $h \in \hat{R}$ there is a constant C such that for any $\lambda > 0$, $\|y_\lambda\| \leq C$, where y_λ satisfies $h = u_\lambda + \lambda y_\lambda$, $y_\lambda \in Au_\lambda$ and $u_\lambda = \hat{J}_\lambda h$.*

Then

$$\lim_{n\to\infty} \hat{J}^n_{t/n} u \tag{2.1}$$

exists for $u \in \overline{\hat{R}}$ and $t \geq 0$. Moreover, if $S(t)u$ is defined as the limit (2.1), *then $S(t)$ is a contraction semigroup on $\overline{\hat{R}}$:*

(1) *We have $S(t)\colon \overline{\hat{R}} \to \overline{\hat{R}}$ for $t \geq 0$; $S(t)S(\tau) = S(t+\tau)$ for $t, \tau \geq 0$; $\|S(t)v - S(t)w\| \leq \|v-w\|$ for $v, w \in \overline{\hat{R}}$ and $t \geq 0$; $S(0) = I$ and $S(t)v$ is continuous in the pair (t, v).*

(2) *If $v \in \hat{R}$, then $S(t)v$ is Lipschitz continuous in t on every compact interval.*

(3) *If we denote $u^\varepsilon(t) = \hat{J}_\varepsilon^{[t/\varepsilon]} u_0$ for $u_0 \in \overline{\hat{R}}$, then*

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(t) = S(t)u_0.$$

PROOF. It is sufficient to prove only the convergence of (2.1). To prove this, it suffices to check the properties corresponding to Lemmas 1.2 and 1.3 in [9]. The statements (i) and (iii) of Lemma 1.2 hold under our conditions. Conditions (b) and (c) imply that

$$\|\hat{J}_\lambda x - x\| \leq C\lambda \quad \text{for} \quad \lambda > 0 \quad \text{and} \quad x \in \hat{R}. \tag{2.2}$$

In fact, for $x \in \hat{R}$, by (b), there exists uniquely $x_\lambda = \hat{J}_\lambda x$ such that $x_\lambda + \lambda y_\lambda = x$ for some $y_\lambda \in Ax_\lambda$. By (c), we have $\|\hat{J}_\lambda x - x\| = \|x_\lambda - x\| = \lambda \|y_\lambda\| \leq C\lambda$.

The estimate (2.2) corresponds to (ii) of Lemma 1.2 in [8]. For $x \in \hat{R}$, $\lambda > 0$ and real μ, we have

$$\frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} \hat{J}_\lambda x \in D_\mu \quad \text{and} \quad \hat{J}_\lambda x \in J_\mu\Big(\frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} \hat{J}_\lambda x\Big).$$

In particular, for $\lambda \geq \mu > 0$, these give

$$\hat{J}_\lambda x = \hat{J}_\mu\Big(\frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} \hat{J}_\lambda x\Big) \tag{2.3}$$

by the convexity of $\hat{R}$ and (b). From this it follows that Lemma 1.3 in [9] is valid with $\omega = 0$. Thus the proof is complete.

By applying the above lemma, we shall prove the existence of a generalized solution of (MP). Assume that the function f satisfies Condition (f. 1). Let us define the operator A_0 associated with (MP) in $L^\infty(I)$. $\mathscr{L}(I)$ denotes the subspace of $L^\infty(I)$ consisting of all Lipschitz continuous functions. Let

$$\mathscr{L}_0(I) = \{u \in \mathscr{L}(I);\ u(0) = u(1) = 0\}$$

and

$$\hat{\mathscr{L}}_0(I) = \{u \in \mathscr{L}_0(I);\ u(x) \geq 0 \quad \text{on} \quad I\}.$$

DEFINITION 2.1. *A_0 is the operator in $L^\infty(I)$ defined by: $v \in D(A_0)$ and $w = A_0 v$ if $v \in \hat{\mathscr{L}}_0(I)$, $w \in \mathscr{L}_0(I)$, $w = f(v_x)$ and*

$$\int_0^1 \operatorname{sign}_0(v_x - k)\{[f(v_x) - f(k)]\varphi_x + w_x \varphi\} dx \geq 0 \tag{2.4}$$

for every $\varphi \in C_0^\infty(\mathring{I})$ such that $\varphi \geq 0$ and every $k \in R$, where $\mathring{I} = (0, 1)$.

The following remark will clarify our definition of A_0.

REMARK 2.1. Let $f \in C^1$ satisfy Condition (f. 1). Then there are $\alpha \leq 0$ and $\beta \geq 0$ such that $f(\alpha) = f(\beta) = 0$. It is easy to verify that if $v \in C^2(I)$ satisfies $v(x) \geq 0$ on I, $v(0) = v(1) = 0$, $v'(0) = \beta$ and $v'(1) = \alpha$, then $v \in D(A_0)$ and $A_0 v = f(v_x)$.

3. The boundary value problem (BVP)

In this section we want to discuss the existence and uniqueness of generalized solutions of (BVP) for given $h \in \hat{\mathscr{L}}_0(I)$ under the assumption that f satisfies Condition (f. 1). Throughout this section we can assume $\lambda = 1$ in (BVP) without loss of generality. We start with the definition of a generalized solution of (BVP).

Definition 3.1. *For each Lipschitz continuous function h, a Lipschitz continuous function u will be called a generalized solution of* (BVP) *if u satisfies*

(i) $u(x)+f(u'(x)) = h(x)$ *a.e. on* I,

(ii) $u(0) = u(1) = 0$,

(iii) $\int_0^1 \mathrm{sign}_0(u'-k)\{[f(u')-f(k)]\varphi' + [h'-u']\varphi\}dx \geq 0$

for every $\varphi \in C_0^\infty(\mathring{I})$ such that $\varphi \geq 0$ and every $k \in R$.

If we define a generalized solution without requiring (iii), we can construct infinitely many generalized solutions. The following example has been used in [2]. We consider the boundary value problem:

$$u(x)+(u'(x))^2/2 = 0, \qquad x \in I, \tag{3.1}$$

$$u(0) = u(1) = 0, \tag{3.2}$$

in the case of $h=0$. For any couple α, β: $0<\alpha<\beta<1$, we define

$$u_{\alpha\beta}(x) = \begin{cases} 0 & 0 \leq x \leq \alpha \quad \text{or} \quad \beta \leq x \leq 1, \\ -(x-\alpha)^2/2 & \alpha \leq x \leq (\alpha+\beta)/2, \\ -(x-\beta)^2/2 & (\alpha+\beta)/2 \leq x \leq \beta. \end{cases}$$

Then $u_{\alpha\beta}(x)$ satisfies both (3.1) and (3.2). By the arbitrariness of α and β, we can construct infinitely many Lipschitz continuous functions that satisfy (i) and (ii).

First we prove the uniqueness.

Theorem 3.1. *There is at most one generalized solution of* (BVP).

In order to prove this, we need the next Lemmas. Let $u: I \to R$ be measurable. We shall denote by $\mathrm{sign}\, u$ the set of all measurable α: $I \to R$ such that $|\alpha(x)| \leq 1$ a.e. on I and $\alpha(x)u(x) = |u(x)|$ a.e. on I.

Lemma 3.1. *Let $\{u^n\}$ and u be measurable functions defined on I such that $u^n \to u$ in $L^1(S)$, where S is a measurable subset of I. If $\alpha^n \in \mathrm{sign}\, u^n$, then there is a subsequence $\{\alpha^{n(i)}\}$ and $\alpha \in \mathrm{sign}\, u$ (depending, perhaps, on S) such that $\{\alpha^{n(i)}\}$ converges to α in the weak-star topology on $L^\infty(S)$.*

Proof is clear from the fact that $L^\infty(S) = (L^1(S))^*$.

LEMMA 3.2. *Let h, g be Lipschitz continuous functions. If u, v are generalized solutions of* (BVP) *corresponding to h and g, respectively, then there is an $\alpha \in \operatorname{sign}(u'-v')$ (depending, perhaps, on φ) such that*

$$\int_0^1 \alpha\{[f(u')-f(v')]\varphi' + [w'-z']\varphi\}dx \geq 0$$

for each given $\varphi \in C_0^\infty(\mathring{I})$, where $w=f(u')$ and $z=f(v')$.

PROOF. We can prove this in the same way as in the proof of [8; Prop. 2.1].

PROOF OF THEOREM 3.1. In proving the uniqueness, we may assume, without loss of generality, that h and g are Lipschitz continuous functions such that $h(0)=h(1)=0$ and $g(0)=g(1)=0$.

Let $\{\varphi_m\}$ be a sequence of piecewise $C^1(I)$ functions:

$$\varphi_m(x) = \begin{cases} 0 & 0 \leq x \leq 1/m, \\ m(x-1/m) & 1/m \leq x \leq 2/m, \\ 1 & 2/m \leq x \leq 1-2/m, \\ -m(x-1+1/m) & 1-2/m \leq x \leq 1-1/m, \\ 0 & 1-1/m \leq x \leq 1. \end{cases} \tag{3.3}$$

By virtue of Lemma 3.2, we can choose $\{\alpha_m\} \subset \operatorname{sign}(u'-v')$ such that

$$\int_0^1 \alpha_m\{[f(u')-f(v')]\varphi_m' + [w'-z']\varphi_m\}dx \geq 0.$$

We show that

$$\int_0^1 \alpha_m[f(u')-f(v')]\varphi_m' dx = \int_0^1 \alpha_m[(h-u)-(g-v)]\varphi_m' dx$$

tends to 0 as $m \to \infty$. Since the integrand of the above converges to 0 as $m \to \infty$ for every $x \in (0, 1)$, and $h(x)-u(x)-g(x)+v(x)$ is Lipschitz continuous on I and vanishes at both of the end points $x=0$ and $x=1$, we have

$$\lim_{m\to\infty}\int_0^1 \alpha_m[f(u')-f(v')]\varphi_m' dx = 0.$$

Using Lemma 3.1, we can find an $\alpha \in \operatorname{sign}(u'-v')$ such that

$$\int_0^1 \alpha[h'-u'-g'+v']dx \geq 0.$$

From this, it follows

$$\int_0^1 |u'-v'|\,dx \le \int_0^1 |h'-g'|\,dx. \tag{3.4}$$

Here, assuming $h=g$, we have $u=v$. Thus the proof of Theorem 3.1 has been completed.

REMARK 3.1. The estimate (3.4) will be used essentially in Proposition 4.4.

We now proceed to the construction of a generalized solution of (BVP). As was stated in the introduction, the generalized solution of (BVP) will be obtained as a limit of solutions of (BVP_ε).

We shall first establish a priori bounds for the solutions of (BVP_ε), and then prove the existence of solutions of (BVP_ε) by using the a priori estimates and existence theorems for $C^2(I)$-solutions of the boundary value problems for the second order ordinary differential equations.

It is obvious that if f is Lipschitz continuous on R and $h\in C(I)$, then there is a solution $u_\varepsilon\in C^2(I)$ of (BVP_ε). Indeed, we have only to apply, for instance, [4; Theorem 6.3] or [14; Theorem 3.1]. It should be noted that the bound for $\|u_\varepsilon'\|_\infty$ depends, in general, on ε.

LEMMA 3.3. *Let $f\in C(R)$ and u, $v\in C^2(I)$ satisfy*

$$\begin{aligned} &u+f(u')-\varepsilon u'' = h \quad on \quad I, \qquad u(0)=u(1)=0,\\ &v+f(v')-\varepsilon v'' = g \quad on \quad I, \qquad v(0)=v(1)=0, \end{aligned} \tag{3.5}$$

where $\varepsilon>0$. Then

$$\|u-v\|_\infty \le \|h-g\|_\infty. \tag{3.6}$$

PROOF. We first assume that $f\in C^1(R)$. Putting $w_1(x)=u(x)-v(x)$, we have

$$\begin{aligned} &w_1(x)+a(x)w_1'(x)-\varepsilon w_1''(x) = h(x)-g(x), \qquad x\in I,\\ &w_1(0)=w_1(1)=0, \end{aligned}$$

where $a(x)=f'(v'+\theta(x)(u'-v'))$, $0<\theta(x)<1$. Putting $w_2(x)=\|h-g\|_\infty$ and $\tilde{w}(x)=w_1(x)-w_2(x)$, we have

$$\tilde{w}+a(x)\tilde{w}'-\varepsilon\tilde{w}'' = h-g-\|h-g\|_\infty \le 0, \qquad x\in I,$$

and hence $\tilde{w}(x)$ can not have a positive maximum in (0, 1). Since $\tilde{w}(0)=\tilde{w}(1)=-\|h-g\|_\infty\le 0$, we obtain

$$\tilde{w}(x) = u(x)-v(x)-\|h-g\|_\infty \le 0 \qquad \text{for} \quad x\in I.$$

On the other hand, putting $\bar{w}(x)=w_1(x)+w_2(x)$ and proceeding in a similar way to the above, we have

$$\bar{w}(x) = u(x)-v(x)+\|h-g\|_\infty \geq 0 \qquad \text{for} \quad x \in I.$$

Consequently we have $|u(x)-v(x)| \leq \|h-g\|_\infty$ for $x \in I$.

For $f \in C(R)$, if we proceed in the same way as in the second step in the proof of [1; Prop. 3.1], then we see (3.6).

COROLLARY 3.1. *Let $f \in C(R)$. Then there is at most one solution $u \in C^2(I)$ of* (BVP_ε) *for any $h \in C(I)$.*

LEMMA 3.4. *Let $f \in C(R)$ and $f(0) \leq 0$. Suppose that $h \in C(I)$ is nonnegative on I. Then the solution $u \in C^2(I)$ of* (BVP_ε) *is nonnegative on I.*

PROOF. It is easily verified that u can not have a negative minimum in (0, 1) so that $u(x) \geq 0$ for $x \in I$ by $u(0)=u(1)=0$.

LEMMA 3.5. *Suppose that $f \in C^1(R)$ satisfies Condition* (f. 1) *and that $h \in C^1(I)$ is a given nonnegative function with $h(0)=h(1)=0$. Assume that $u \in C^2(I)$ is a solution of* (BVP_ε). *Then we have*

$$\|u'\|_\infty \leq L \quad \textit{and} \quad \|u\|_\infty \leq L, \tag{3.7}$$

where $L=\max\{\|h'\|_\infty, p_0\}$.

PROOF. Differentiation of the equation of (BVP_ε) yields

$$u'(x)+f'(u'(x))u''(x)-\varepsilon u'''(x) = h'(x), \qquad x \in I.$$

Putting $v_1(x)=u'(x)$, $v_2(x)=L$ and $v=v_1-v_2$, we have

$$v(x)+f'(v_1(x))v'(x)-\varepsilon v''(x) = h'(x)-L \leq 0.$$

Thus $v(x)$ can not have a positive maximum in (0, 1). Note that $u'(0) \geq 0$ and $u'(1) \leq 0$ by Lemma 3.4 and $u(0)=u(1)=0$. Then we have $v(1)=u'(1)-L \leq 0$. We shall show that $v(0)=u'(0)-L \leq 0$. If $u'(0)-L>0$, then $u''(0)=\varepsilon^{-1}f(u'(0)) >0$ by Condition (f. 1). Since $v(1)=u'(1)-L \leq 0$, $v(x)$ has a positive maximum in (0, 1). This leads to a contradiction. Therefore $v(x)$ is nonpositive on I. In a similar way we have $u'(x)+L \geq 0$ for $x \in I$. Consequently we have $|u'(x)| \leq L$ for $x \in I$. Thus the proof is complete.

Now we shall prove the existence of a $C^2(I)$-solution of (BVP_ε).

LEMMA 3.6. *Let $f \in C(R)$ satisfy Condition* (f. 1). *Then for each $h \in \hat{\mathscr{L}}_0(I)$ there exists a unique solution $u \in C^2(I)$ of* (BVP_ε) *such that u is nonnegative on I and*

$$\|u\|_\infty \leq L \quad \textit{and} \quad \|u'\|_\infty \leq L, \tag{3.8}$$

where $L = \max\{\|h'\|_\infty, p_0\}$.

PROOF. Choose a sequence $\{f_n\}_{n=1}^\infty$ of functions in $C^1(R) \cap W_1^\infty(R)$ such that $f_n(0) \leq 0$ and $\{f_n(p)\}$ converges to $f(p)$ uniformly on the compact set $\{p;\ |p| \leq L\}$. Further we approximate $h(x)$ by $\{h_n(x)\}_{n=1}^\infty$ of functions in $C^1(I)$ such that $h_n(0) = h_n(1) = 0$, $h_n(x) \geq 0$ on I, $\{h_n\}$ converges to h uniformly on I and $\{h'_n\}$ converges to h' a.e. on I as $n \to \infty$.

By [4; Theorem 6.3], for each n there exists a unique solution $u_n \in C^2(I)$ of the boundary value problem:

$$(\mathrm{BVP}_\varepsilon^n) \qquad \begin{aligned} &u(x) + f_n(u'(x)) - \varepsilon u''(x) = h_n(x), \qquad x \in I, \\ &u(0) = u(1) = 0. \end{aligned}$$

From Corollary 3.1 and Lemmas 3.4 and 3.5, it follows that each u_n is non-negative and satisfies $\|u_n\|_\infty \leq L$ and $\|u'_n\|_\infty \leq L$. Therefore the relation

$$u''_n(x) = \varepsilon^{-1}\{u_n(x) + f_n(u'_n(x)) - h_n(x)\}$$

implies that $\{u''_n(x)\}$ is uniformly bounded on I. Hence we can choose a subsequence $\{u_{n(i)}\}$ of $\{u_n\}$ such that $\{u_{n(i)}\}$ converges to u in $C^1(I)$. Since $\{u''_{n(i)}\}$ converges to $\varepsilon^{-1}\{u + f(u') - h\}$ as $n(i) \to \infty$ uniformly on I, the limit $u \in C^2(I)$ is a solution of $(\mathrm{BVP}_\varepsilon)$. The proof is complete.

Denote by $u_\varepsilon(x)$ the solution of $(\mathrm{BVP}_\varepsilon)$. We shall prove that $\{u_\varepsilon(x)\}_{\varepsilon>0}$ is a precompact set in a certain topology and that the limit function as $\varepsilon \downarrow 0$ is our desired (generalized) solution of (BVP).

The following lemma is a slight modification of [8; Prop. 2.3] (see also [1; Prop. 3.3]).

LEMMA 3.7. *Let* $f \in C^1$. *Suppose that* $u, v \in C^2(I_1)$ *satisfy*

$$u(x) + f(u'(x)) - \varepsilon u''(x) = h(x),$$

$$v(x) + f(v'(x)) - \varepsilon v''(x) = g(x)$$

for $x \in I_1$, *where* I_1 *is an arbitrary compact interval* $I_1 \subset\subset I$. *If* h *and* g *are Lipschitz continuous, then*

$$\|u' - v'\|_{L^1(I_1)} \leq \|h' - g'\|_{L^1(I_1)} + 2\|h - g\|_\infty + 2\|u - v\|_\infty. \tag{3.9}$$

PROOF. Let $I_1 = [a, b]$. Define $\Phi_l(s)$ as

$$\Phi_l(s) = \begin{cases} -s & \text{if} \quad s \leq -1/l, \\ (l/2)s^2 + 1/2l & \text{if} \quad |s| \leq 1/l, \\ s & \text{if} \quad s \geq 1/l, \end{cases} \tag{3.10}$$

and define $\alpha_l(s)$ by $\alpha_l(s) = \Phi_l'(s)$. Let $\{\varphi_m\}_{m=m_0}^{\infty}$ of $C(I)$-functions, where m_0 is sufficiently large, be defined by

$$\varphi_m(x) = \begin{cases} 1 & \text{on} \quad I_m = [a+1/m,\ b-1/m], \\ 0 & \text{on outside} \quad I_1, \\ \text{linear} & \text{elsewhere.} \end{cases}$$

It is clear that $w = u' - v'$ satisfies

$$w + (f(u') - f(v'))_x - \varepsilon w'' = h' - g' \quad \text{a.e.} \quad \text{on} \quad I_1.$$

Multiplying the above by $\alpha_l(w)\varphi_m$ and integrating over I_1, we have

$$\begin{aligned} &\int_{I_1} \{w\alpha_l(w)\varphi_m + (f(u') - f(v'))_x \alpha_l(w)\varphi_m - \varepsilon w''\alpha_l(w)\varphi_m\} dx \\ &\quad = \int_{I_1} (h' - g')\alpha_l(w)\varphi_m dx \leq \|h' - g'\|_{L^1(I_1)}. \end{aligned} \tag{3.11}$$

We will estimate each term on the left hand side of (3.11) from below. Evidently we have

$$\int_{I_1} w\alpha_l(w)\varphi_m dx \geq \int_{I_m} w\alpha_l(w)\, dx \to \int_{I_m} |w| dx = \|w\|_{L^1(I_m)} \tag{3.12}$$

as $l \to \infty$ by using the oddness of α_l. Integration by parts yields

$$\begin{aligned} &\int_{I_1} (f(u') - f(v'))_x \alpha_l(w)\varphi_m dx \\ &\quad = -\int_{\Omega_l} (f(u') - f(v'))\alpha_l'(w) w' \varphi_m dx - \int_{I_1} (f(u') - f(v'))\alpha_l(w)\varphi_m' dx, \end{aligned}$$

where $\Omega_l = \{x \in I_1;\ |w(x)| = |u'(x) - v'(x)| \leq 1/l\}$. We estimate the first term of the above. By the bounded convergence theorem, we have

$$\overline{\lim_{l \to \infty}} \left| \int_{\Omega_l} (f(u') - f(v'))\alpha_l'(w) w' \varphi_m dx \right| \leq \|f'\|_{\infty} \int_{\Omega} |w'| dx, \tag{3.13}$$

where $\Omega = \bigcap_{l=1}^{\infty} \Omega_l$. Since $u' = v'$ on Ω, $w' = 0$ a.e. on Ω. Thus $\int_{\Omega} |w'| dx = 0$. For the second term,

$$-\int_{I_1}(f(u')-f(v'))\alpha_l(w)\varphi'_m dx$$

$$\geq -\int_{I_1}\|u-v\|_\infty|\varphi'_m|dx-\int_{I_1}\|h-g\|_\infty|\varphi'_m|dx-\varepsilon\int_{I_1}w'\alpha_l(w)\varphi'_m dx \tag{3.14}$$

$$= -2\|u-v\|_\infty-2\|h-g\|_\infty-\varepsilon\int_{I_1}w'\alpha_l(w)\varphi'_m dx.$$

For the last term on the left hand side of (3.11), by the integration by parts and $\alpha'_l \geq 0$, we have

$$-\varepsilon\int_{I_1}w''\alpha_l(w)\varphi_m dx \geq \varepsilon\int_{I_1}w'\alpha_l(w)\varphi'_m dx. \tag{3.15}$$

Consequently, using (3.11)–(3.15) and letting $l, m\to\infty$, we have (3.9). Thus the proof is complete.

An immediate consequence of Lemma 3.7 is:

LEMMA 3.8. *Let u_ε be a solution of* (BVP_ε). *Then for any compact interval $I_1\subset\subset I$, we have*

$$\|u'_\varepsilon(x+y)-u'_\varepsilon(x)\|_{L^1(I_1)} \leq \|h'(x+y)-h'(x)\|_{L^1(I_1)}+4L|y| \tag{3.16}$$

for any small y so that $I_1\subset(|y|, 1-|y|)$, where L is a constant appearing in Lemma 3.6.

PROOF. By the uniqueness of solutions of (BVP_ε), $u_\varepsilon(x+y)$ is a unique solution of the boundary value problem:

$$u(x)+f(u'(x))-\varepsilon u''(x) = h(x+y), \qquad -y\leq x\leq 1-y,$$

$$u(-y) = u(1-y) = 0,$$

for each $y\in R$. Using Lemma 3.7 and (3.9), we get (3.16) for each sufficiently small y. Note that we can apply Lemma 3.7 under our assumptions, since $u_\varepsilon(x)$ is obtained as the limit of solutions of ($\mathrm{BVP}^n_\varepsilon$). Thus the proof is complete.

We now establish the existence of a generalized solution of (BVP).

THEOREM 3.2. *Let f satisfy Condition* (f. 1). *Then for each $h\in\hat{\mathscr{L}}_0(I)$ there exists a unique generalized solution u of* (BVP) *such that*

$$\|u\|_\infty \leq L \quad \text{and} \quad \|u'\|_\infty \leq L \tag{3.17}$$

$$u(x)\geq 0 \quad \text{for} \quad x\in I. \tag{3.18}$$

PROOF. It is sufficient to prove the existence. We proceed in a similar way as in the proof of [1; Theorem 3.1]. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of C^1 functions given in the proof of Lemma 3.6. Given $h \in \hat{\mathscr{L}}_0(I)$, let $u_n \in C^2(I)$ be the unique solution of $(\mathrm{BVP}''_{1/n})$ guaranteed by Lemma 3.6. By (3.8) and Lemma 3.8, there is a subsequence $\{u_{n(i)}\}$ and a nonnegative Lipschitz continuous function u such that $\{u_{n(i)}\}$ converges to u uniformly on I, $\{u'_{n(i)}\}$ converges to u' a.e. on I and u vanishes at both of the end points $x=0$ and $x=1$.

Multiplying the equation of $(\mathrm{BVP}''_{1/n})$ by any $\varphi \in C_0^{\infty}(I)$ and integrating over I, we have

$$\int_0^1 \left\{(u_n + f_n(u'_n))\varphi + \frac{1}{n} u'_n \varphi'\right\} dx = \int_0^1 h\varphi dx.$$

Letting n tend to infinity through the subsequence $\{n(i)\}$, we have

$$\int_0^1 (u + f(u'))\varphi dx = \int_0^1 h\varphi dx.$$

Since φ is arbitrary, $u + f(u') = h$ a.e. on I.

Next we shall prove that u satisfies (iii) of Definition 3.1. Let $\varphi \in C_0^{\infty}(\mathring{I})$ and Φ have a piecewise continuous second derivative. Multiplying the equation

$$u'_n + f'_n(u'_n)u''_n - \frac{1}{n} u'''_n = h' \text{ a.e. on } I$$

by $\Phi'(u'_n)\varphi$ and integrating over I, we have

$$\begin{aligned}\int_0^1 \Big\{ & u'_n \Phi'(u'_n)\varphi - [\Phi'(u'_n)f_n(u'_n) - \Phi'(k)f_n(k)]\varphi' \\ & + \left[\int_k^{u'_n(x)} \Phi''(s)f_n(s)ds\right]\varphi' + \frac{1}{n}[\Phi''(u'_n)u''^2_n\varphi - \Phi(u'_n)\varphi'']\Big\} dx \\ = & \int_0^1 h'\Phi'(u'_n)\varphi dx\end{aligned}$$

for every $k \in R$. Assuming, in addition, $\Phi'' \geq 0$ and $\varphi \geq 0$, we see $\int_0^1 \Phi''(u'_n)u''^2_n\varphi dx \geq 0$. Letting $n \to \infty$ through the subsequence $\{n(i)\}$, we obtain

$$\begin{aligned}\int_0^1 & \left\{u'\Phi'(u')\varphi - \Phi'(u')f(u')\varphi' + \left[\int_k^{u'(x)} \Phi''(s)f(s)ds\right]\varphi'\right\} dx \\ & \leq \int_0^1 h'\Phi'(u')\varphi dx\end{aligned}$$

for $\varphi \in C_0^{\infty}(\mathring{I})$ such that $\varphi \geq 0$ and $k \in R$. Hence, choosing $\Phi(s) = \Phi_l(s-k)$, where Φ_l is given by (3.10), and letting $l \to \infty$, we have

$$\int_0^1 \mathrm{sign}_0(u'-k)\{[f(u')-f(k)]\varphi'+[h'-u']\varphi\}dx \geq 0$$

for every $\varphi \in C_0^\infty(\mathring{I})$ such that $\varphi \geq 0$ and every $k \in R$. This completes the proof.

4. Existence of generalized solutions of (MP)

The aim in this section is to establish our main result, that is, the existence of a generalized solution of (MP). We shall prove the existence via the theory of semigroups of nonlinear transformations. To this end, we choose $L^\infty(I)$ as the Banach space and $\hat{\mathscr{L}}_0(I)$ as the convex set $\hat{R}$ in Fundamental Lemma associated with (MP). We have to verify the conditions (a), (b) and (c) of Fundamental Lemma for the operator A_0 of Definition 2.1.

By the definition of A_0 and Definition 3.1, we have

PROPOSITION 4.1. *Let $f \in C$ satisfy Condition* (f. 1). *Then for any $\lambda > 0$ and every $h \in \hat{\mathscr{L}}_0(I)$, u is a generalized solution of*

$$\text{(BVP)} \qquad u + \lambda f(u') = h, \qquad u(0) = u(1) = 0$$

if and only if $u \in D(A_0)$ and $u + \lambda A_0 u = h$.

PROOF. The "if" part is evident by Definition 2.1 and the "only if" part is clear by Theorems 3.1 and 3.2.

By Theorem 3.2 and Proposition 4.1, we have

PROPOSITION 4.2. *Under Condition* (f. 1) *on f, we obtain*

$$D(A_0) \subset \hat{\mathscr{L}}_0(I) \subset R(I+\lambda A_0)$$

for any $\lambda > 0$, that is, for every $h \in \hat{\mathscr{L}}_0(I)$ and any $\lambda > 0$, there is a unique generalized solution u of (BVP) *such that $\|u\|_\infty \leq L$ and $\|u_x\|_\infty \leq L$, where $L = \max\{\|h_x\|_\infty, p_0\}$.*

The next proposition is an immediate consequence of Lemma 3.3.

PROPOSITION 4.3. *Let f satisfy Condition* (f. 1). *If $u, v \in D(A_0)$ satisfy $u + \lambda A_0 u = h \in \hat{\mathscr{L}}_0(I)$ and $v + \lambda A_0 v = g \in \hat{\mathscr{L}}_0(I)$, then we have*

$$\text{(4.1)} \qquad \|u - v\|_\infty \leq \|h - g\|_\infty.$$

PROOF. By Proposition 4.1, Theorems 3.1 and 3.2, we can regard u, v as the limits of $\{u_n\}$ and $\{v_n\}$, respectively, that are $C^2(I)$-solutions of

$$u + \lambda f(u_x) - \frac{1}{n} u_{xx} = h \quad \text{on} \quad I, \quad u(0) = u(1) = 0,$$

$$v+\lambda f(v_x)-\frac{1}{n}v_{xx}=g \quad \text{on} \quad I, \quad v(0)=v(1)=0.$$

Applying Lemma 3.3, we have $\|u_n-v_n\|_\infty \le \|h-g\|_\infty$ and hence (4.1). The proof is complete.

According to Propositions 4.2 and 4.3, we can verify that A_0 satisfies (a) and (b) in Fundamental Lemma. Next we will observe (c). If $v \in \hat{\mathscr{L}}_0(I)$, then $u=\hat{J}_\lambda v$ satisfies $u+\lambda f(u_x)=v$ and hence, by (3.17), we have $\|u_x\|_\infty \le L=\max\{\|v_x\|_\infty, p_0\}$. Consequently we have

$$\|A_0 u\|_\infty = \|f(u_x)\|_\infty \le \sup_{|p|\le L}|f(p)|. \tag{4.2}$$

Thus we have

THEOREM 4.1. *Let $f \in C(R)$ satisfy Condition* (f. 1). *Then the semigroup $S(t)$ obtained from A_0 through Fundamental Lemma is the contraction semigroup on $\overline{\hat{\mathscr{L}}_0(I)}$.*

Let $u_0 \in \hat{\mathscr{L}}_0(I)$ and $u^\varepsilon(t)$ be a unique solution of

$$\begin{cases} \varepsilon^{-1}(u(t)-u(t-\varepsilon))+A_0u(t)=0 & t\ge 0, \\ u(t)=u_0 & t<0. \end{cases} \tag{4.3}$$

Then

$$u^\varepsilon(t)=(I+\varepsilon A_0)^{-[t/\varepsilon]-1}u_0=\hat{J}_\varepsilon^{[t/\varepsilon]+1}\,u_0$$

for $t\ge 0$ and by Proposition 4.2,

$$\|u^\varepsilon(t)\|_\infty \le L \quad \text{and} \quad \|(u^\varepsilon(t))_x\|_\infty \le L \tag{4.4}$$

hold for $t\ge 0$, where $L=\max\{\|u_0'\|_\infty, p_0\}$. The following proposition is a core in our proof of the existence theorem for (MP).

PROPOSITION 4.4. *Suppose that $f\in C$ satisfies Condition* (f. 1). *Let u_0 be an element of $\hat{\mathscr{L}}_0(I)$ such that $(u_0)_x$ is regarded as the limit of $\{(u_n)_x\}$ in $L^1(I)$, where $u_n \in D(A_0)$. Then for all $t\ge 0$*

$$(u^\varepsilon(t))_x \longrightarrow (S(t)u_0)_x \quad \text{in} \quad L^1(I)$$

as $\varepsilon\downarrow 0$. Assume, in addition, that $f\in C^1$. Then this convergence holds for every $u_0\in\hat{\mathscr{L}}_0(I)$ and all $t\ge 0$.

PROOF. Let $u\in D(A_0)$ and $A_0u=v$. Then $u_x, v_x \in L^\infty(I)\subset L^1(I)$. Define the operator B_0: $L^1(I)\to L^1(I)$ by $B_0u_x=v_x$ if $u\in D(A_0)$ and $A_0u=v$. We shall show that $D(B_0)\subseteq R(I+\lambda B_0)$ for $\lambda>0$. For each $h_x\in D(B_0)$ we have some u in $D(A_0)$ such that $h=u+\lambda A_0 u$ by Proposition 4.2. From this, it follows that $u_x\in D(B_0)$ and $B_0 u_x=\lambda^{-1}(h_x-u_x)$, and hence $h_x=u_x+\lambda B_0 u_x\in R(I+\lambda B_0)$. From the inequality (3.4) proved in the proof of Theorem 3.1, we find that B_0 is an accretive operator in $L^1(I)$. We define the operator B as the closed extension of B_0 in $L^1(I)$. Then B is accretive in $L^1(I)$. For any $\tilde{h}\in D(B)$ we have a sequence $\{h_x^n\}\subset D(B_0)$ such that $\{h_x^n\}$ converges to $\tilde{h}$ in $L^1(I)$. Let $\{u_x^n\}$ be a sequence in $L^1(I)$ such that $u_x^n+\lambda B_0 u_x^n=h_x^n$. Then by the accretiveness of B_0, $\{u_x^n\}$ is a Cauchy sequence in $L^1(I)$. Hence $\{u_x^n\}$ converges to some $\tilde{u}$ in $L^1(I)$. Then $B_0 u_x^n=\lambda^{-1}(h_x^n-u_x^n)\to\lambda^{-1}(\tilde{h}-\tilde{u})$ in $L^1(I)$. Thus we get $\tilde{u}\in D(B)$ and $h\in\tilde{u}+\lambda B\tilde{u}\subset R(I+\lambda B)$. Therefore $\overline{D(B)}\subset\overline{R(I+\lambda B)}=R(I+\lambda B)$ for $\lambda>0$, since B is accretive and closed.

It is important to note that for $u\in D(A_0)$ and $\lambda>0$,

$$u_x+\lambda B_0 u_x = h_x \quad \text{iff} \quad u+\lambda A_0 u = h.$$

Then we find that for $u_0\in\hat{\mathscr{L}}_0(I)$,

$$u^\varepsilon(t) = (I+\varepsilon A_0)^{-[t/\varepsilon]-1}u_0$$

is equivalent to

$$(u^\varepsilon(t))_x = (I+\varepsilon B_0)^{-[t/\varepsilon]-1}(u_0)_x.$$

Applying the generation theorem of Crandall and Liggett, we can see that for every $u_0\in\hat{\mathscr{L}}_0(I)$ such that $(u_0)_x\in D(B)$ and for every $t\geq 0$, $(u^\varepsilon(t))_x$ converges to some $v(t, x)$ in $L^1(I)$. Hence we have

$$(S(t)u_0)(x) = \int_0^x v(t, y)dy$$

for $x\in I$. Consequently $(S(t)u_0)_x=v(t, x)$ a.e. on I.

Assuming, in addition, that $f\in C^1$, we can see easily that $h_x\in\overline{D(B)}$ in $L^1(I)$ for every $h\in\hat{\mathscr{L}}_0(I)$ by using Remark 2.1 and an approximation argument. Thus the proof of Proposition 4.4 is complete.

We now prove that $(S(t)u_0)(x)=u(t, x)$ is a generalized solution of (MP). By the definition of A_0, $u^\varepsilon(t)$ satisfies the equation

$$\varepsilon^{-1}(u^\varepsilon(t)-u^\varepsilon(t-\varepsilon))+f((u^\varepsilon(t))_x) = 0 \tag{4.5}$$

for $t\geq 0$. Let $T>0$. Since $S(t)u_0$ is Lipschitz continuous for $0\leq t\leq T$ and $\|(S(t)u_0)_x\|_\infty\leq L$ for every $t\geq 0$, $u(t, x)$ is (totally) differentiable a.e. on $[0, T]\times I$. Moreover, by Proposition 4.4 and the bounded convergence theorem, we find that $(u^\varepsilon(t))_x$ converges to $(S(t)u_0)_x$ in $L^1([0, T]\times I)$ as $\varepsilon\downarrow 0$. Therefore we find a subsequence $\{\varepsilon(i)\}$ such that $\{(u^{\varepsilon(i)}(t))_x\}$ converges to $(S(t)u_0)_x$ a.e. on $[0, T]$

$\times I$.

Multiply (4.5) by $\varphi \in C_0^\infty((0, T)\times \mathring{I})$ and integrate over $[0, T]\times I$. Integrating by parts and letting $\varepsilon \downarrow 0$ through the subsequence $\{\varepsilon(i)\}$, we have

$$\int_0^T\int_0^1\{-(S(t)u_0)\varphi_t+f((S(t)u_0)_x)\varphi\}dxdt = 0,$$

which implies that $u(t, x)$ satisfies (i) of Definition 1.1. To prove that $u(t, x)$ satisfies (ii) of Definition 1.1, we proceed as in the case of the Cauchy problem [1; Theorem 4.3]. Let $u^\varepsilon(t, x)=u^\varepsilon(t)(x)$. By Definition 2.1, we have

$$\int_0^1\{\mathrm{sign}_0(u_x^\varepsilon(t, x)-k)[f(u_x^\varepsilon(t, x))-f(k)]\varphi_x + \varepsilon^{-1}(u_x^\varepsilon(t-\varepsilon, x)-u_x^\varepsilon(t, x))\,\mathrm{sign}_0(u_x^\varepsilon(t, x)-k)\varphi\}dx \geq 0 \tag{4.6}$$

for every $\varphi=\varphi(t, x)\in C_0^\infty((0, T)\times \mathring{I})$: $\varphi\geq 0$ and every $k\in R$.

Integrating (4.6) over $[0, T]$, we have

$$\int_0^T\int_0^1\{\mathrm{sign}_0(u_x^\varepsilon(t, x)-k)[f(u_x^\varepsilon(t, x))-f(k)]\varphi_x + \varepsilon^{-1}[h^\varepsilon(t-\varepsilon, x)-h^\varepsilon(t, x)]\varphi\}dxdt \geq 0, \tag{4.7}$$

where $h^\varepsilon(t, x)=(u_x^\varepsilon(t, x)-k)\,\mathrm{sign}_0(u_x^\varepsilon(t, x)-k)$. Since $u_x^\varepsilon(t, x)$ converges to $u_x(t, x)$ in $L^1([0, T]\times I)$, the second term converges to

$$\int_0^T\int_0^1|u_x(t, x)-k|\varphi_t(t, x)dxdt$$

as $\varepsilon \downarrow 0$. Letting $\varepsilon\downarrow 0$ through the subsequence $\{\varepsilon(i)\}$ in (4.7), we see that $u(t, x)$ satisfies (ii).

Thus we conclude

THEOREM 4.2. *Suppose that $f\in C$ satisfies Condition* (f.1). *Then for $u_0 \in \hat{\mathscr{L}}_0(I)$ such that $(u_0)_x\in \overline{D(B_0)}$ in $L^1(I)$, $u(t, x)=(S(t)u_0)(x)$ is a generalized solution of* (MP). *Assume, in addition, that $f\in C^1$. Then $u(t, x)$ is a generalized solution of* (MP) *for $u_0\in \hat{\mathscr{L}}_0(I)$.*

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