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Dirichlet Integrals on General Harmonic Spaces

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Introduction

In the paper [8], the author introduced the notion of Dirichlet integrals, or rather that of gradient measures, on self-adjoint Brelot's harmonic spaces. The purpose of the present paper is to give the definition and basic properties of gradient measures on more general harmonic spaces which are not necessarily self-adjoint.

We shall consider a harmonic space X in the sense of C. Constantinescu and A. Cornea [3], and only assume that X has a countable base and that there is a sheaf homomorphism σ of $\mathscr{R}$ into the sheaf of Radon measures on X, where $\mathscr{R}$ is the sheaf of germs of differences of continuous superharmonic functions (cf. [12], [5]), such that a non-negative measure corresponds to a superharmonic function. We shall see that such a sheaf homomorphism exists for a large class of harmonic spaces.

The definition of gradient measures is given in terms of σ, in almost the same way as in [8]; however, due to the lack of symmetric Green functions, a different approach is necessary to obtain basic properties, i.e., Theorems 1 and 2. It turns out that our new approach considerably simplifies the proofs of the corresponding theorems in [8] and [9].

As an application of the general theory, we can associate a differential equation (whose coefficients are measures) to a harmonic structure given on a euclidean domain (Theorem 4). Problems of this type were considered first by G. L. Tautz [10], and later by J.-M. Bony [1]. Our result contains that in [10]-I; but not directly related to the results in [10]-II and [1].

1. The sheaves $\mathscr{R}$ and $\mathscr{R}_0$

Let $(X, \mathscr{H})$ be a harmonic space in the sense of [3] with a sheaf of "harmonic functions" $\mathscr{H}=\{\mathscr{H}(U)\}$ (U: open$\subseteq X$). We shall assume in this paper that X has a countable base. For various notions in the theory of harmonic spaces, we refer to [3].

As in [12] and [5], we consider the sheaf $\mathscr{R}=\{\mathscr{R}(U)\}$, where for each open set U in X, $\mathscr{R}(U)$ is the set of functions on U which are locally expressed as differences of continuous superharmonic functions. (In [8], this space $\mathscr{R}(U)$ was denoted by $\mathscr{B}_{C,\mathrm{loc}}(U)$.) Each $\mathscr{R}(U)$ is a linear space of continuous real-valued functions on U. The proof of [5, Proposition 3.3] applies to our case and yields

PROPOSITION 1. *If f, g, $h \in \mathscr{R}(U)$ and $h>0$ on U, then $fg/h \in \mathscr{R}(U)$.*

By using [3, Theorem 2.3.1], we can easily show that there is $h_0 \in \mathscr{R}(X)$ such that $h_0>0$ on X. We introduce another sheaf $\mathscr{R}_0=\{\mathscr{R}_0(U)\}$ which is defined by

$$\mathscr{R}_0(U) = \{f/h; f,\ h \in \mathscr{R}(U),\ h > 0 \text{ on } U\}$$

for each open set U. By Proposition 1, we see that each $\mathscr{R}_0(U)$ is an algebra containing 1, $\mathscr{R}_0(U)=\{f/h_0; f \in \mathscr{R}(U)\}$ with the above h_0 and $g \in \mathscr{R}_0(U)$ implies $gh \in \mathscr{R}(U)$ for any $h \in \mathscr{R}(U)$. Obviously, $\mathscr{R}_0(U)=\mathscr{R}(U)$ if and only if $1 \in \mathscr{R}(U)$.

Let W be an open set in X and h be a strictly positive continuous function on W. For each open set $U \subseteq W$, we write

$$\mathscr{H}^{(h)}(U) = \{u/h;\ u \in \mathscr{H}(U)\}.$$

Then $(W, \mathscr{H}_W^{(h)})$ with $\mathscr{H}_W^{(h)}=\{\mathscr{H}^{(h)}(U)\}_{U\subset W}$ is again a harmonic space. The spaces $\mathscr{R}(U)$ and $\mathscr{R}_0(U)$ for $U \subseteq W$ considered with respect to this harmonic space will be denoted by $\mathscr{R}^{(h)}(U)$ and $\mathscr{R}_0^{(h)}(U)$, respectively. Obviously, they are given by

$$\mathscr{R}^{(h)}(U) = \{f/h; f \in \mathscr{R}(U)\} \quad \text{and} \quad \mathscr{R}_0^{(h)}(U) = \{f/h; f \in \mathscr{R}_0(U)\}.$$

If $h \in \mathscr{R}(W)$ and $h>0$ on W, then by Proposition 1, $\mathscr{R}^{(h)}(U)=\mathscr{R}_0^{(h)}(U)=\mathscr{R}_0(U)$ for any open set $U \subseteq W$.

2. Measure representations

By a sheaf homomorphism σ of $\mathscr{R}$ into the sheaf $\mathscr{M}$ of Radon measures on X, we mean a set of mappings $\{\sigma_U\}$ such that

(i) for each open set U, σ_U is a linear mapping of $\mathscr{R}(U)$ into the linear space $\mathscr{M}(U)$ of (signed) Radon measures on U, and

(ii) for open sets U and V with $V \subseteq U$, $f \in \mathscr{R}(U)$ implies

$$\sigma_U(f)|V = \sigma_V(f|V).$$

By virtue of property (ii), there will be no ambiguity in writing $\sigma(f)$ instead of $\sigma_U(f)$.

A sheaf homomorphism $\sigma: \mathscr{R} \to \mathscr{M}$ will be called a *measure representation* of $\mathscr{R}$ if it satisfies the following condition:

(iii) For each open set U and $f \in \mathscr{R}(U)$, $\sigma(f) \geqq 0$ on U if and only if f is superharmonic on U.

REMARK. If we consider the specific order in $\mathscr{R}(U)$ (see [3, Chapter 8]) and the natural order in $\mathscr{M}(U)$, then $\mathscr{R}(U)$ and $\mathscr{M}(U)$ are vector lattices and condition (iii) above means that σ gives a vector lattice homomorphism of $\mathscr{R}(U)$ into $\mathscr{M}(U)$ whose kernel is $\mathscr{H}(U)$.

The existence of measure representations will be discussed in § 6.

If h is a strictly positive continuous function on an open set W, then

$$\sigma^{(h)}(g) = \sigma(hg) \qquad \text{for} \quad g \in \mathscr{R}^{(h)}(U), \qquad U \subseteq W$$

defines a measure representation $\sigma^{(h)}$ of $\{\mathscr{R}^{(h)}(U)\}_{U \subset W}$ with respect to the harmonic space $(W, \mathscr{H}_W^{(h)})$.

LEMMA 1. *Let U be an open set in X and suppose $1 \in \mathscr{H}(U)$. Let s_1 and s_2 be bounded continuous superharmonic functions on U and put $f = s_1 - s_2$, $\alpha = \sup_U f$ and $\beta = \inf_U f$. Then*

$$v_p = -(\alpha - f)^p + p(\alpha - \beta)^{p-1} s_2$$

is superharmonic on U for any $p > 1$.

PROOF. Let V be any relatively compact resolutive set (see [3]) such that $\bar{V} \subseteq U$. For a continuous function φ on the boundary of V, let H_φ denote the Dirichlet solution for V with boundary values φ. Put $w_i = s_i - H_{s_i}$ on V $(i = 1, 2)$. Then $w_i \geqq 0$, $i = 1, 2$. Since $1 \in \mathscr{H}(U)$ and $\alpha - f \geqq 0$ on U, we see that $H_{(\alpha - f)^p} \geqq (H_{\alpha - f})^p$. Hence

$$\begin{aligned} H_{v_p} &= -H_{(\alpha - f)^p} + p(\alpha - \beta)^{p-1} H_{s_2} \\ &\leqq -(H_{\alpha - f})^p + p(\alpha - \beta)^{p-1} H_{s_2} \\ &= -(\alpha - f + w_1 - w_2)^p + p(\alpha - \beta)^{p-1}(s_2 - w_2) \end{aligned}$$

on V. It is easy to see that

$$(\alpha - f + w_1 - w_2)^p \geqq (\alpha - f)^p - p(w_2 - w_1)(\alpha - f)^{p-1}$$

on V. Hence

$$H_{v_p} \leqq v_p - p(\alpha - f)^{p-1} w_1 \leqq v_p$$

on V. Therefore v_p is superharmonic on U.

PROPOSITION 2. *Let U be an open set in X and p be an integer > 1. If $1 \in \mathscr{H}(U)$ and if σ is a measure representation of $\mathscr{R}$, then for any $f \in \mathscr{R}(U)$,*

(1) $$\sum_{k=1}^{p}(-1)^{k+1}\binom{p}{k}f^{p-k}\sigma(f^k) \geqq 0$$

on U.

PROOF. Let V be any open subset of U for which $f|V=s_1-s_2$ with bounded continuous superharmonic functions s_1 and s_2 on V. Put $M=\sup_V|f|$. Let $\varepsilon>0$ ($\varepsilon<1$) be given. Since f is continuous, for each $x\in V$ there is an open subset W of V containing x such that $\sup_W f-\inf_W f<\varepsilon$. Put $\alpha=\sup_W f$ and $\beta=\inf_W f$. By the previous lemma, we have

$$-\sigma[(\alpha-f)^p]+p(\alpha-\beta)^{p-1}\sigma(s_2) \geqq 0$$

on W. Hence

$$\sum_{k=1}^{p}(-1)^{k+1}\binom{p}{k}\alpha^{p-k}\sigma(f^k) \geqq -p\varepsilon^{p-1}\sigma(s_2) \geqq -p\varepsilon\sigma(s_2)$$

on W. (Note that $\sigma(1)=0$ since $1\in\mathscr{H}(U)$.) Since

$$|\alpha^{p-k}-f^{p-k}| \leqq (p-k)M^{p-k-1}\varepsilon$$

on W ($k=1,\dots, p$), we have

(2) $$\sum_{k=1}^{p}(-1)^{k+1}\binom{p}{k}f^{p-k}\sigma(f^k) \geqq -\varepsilon\mu$$

on W, where

$$\mu = \sum_{k=1}^{p}\binom{p}{k}(p-k)M^{p-k-1}|\sigma(f^k)|+p\sigma(s_2)$$

is a non-negative measure defined on V. Since both sides of (2) are independent of W, (2) holds on V. Therefore, $\varepsilon>0$ being arbitrary, (1) holds on V; and since such V's cover U, (1) holds on U.

COROLLARY. *Let U be an open set in X and suppose $1\in\mathscr{H}(U)$. If σ is a measure representation of $\mathscr{R}$, then*

(3) $$\sigma(fgh)-f\sigma(gh)-g\sigma(fh)-h\sigma(fg)+fg\sigma(h)+fh\sigma(g)+gh\sigma(f) = 0$$

on U for any $f, g, h\in\mathscr{R}(U)$.

PROOF. In case $p=3$, the inequality (1) in the above proposition becomes

$$\sigma(f^3)-3f\sigma(f^2)+3f^2\sigma(f) \geqq 0.$$

Applying this to $-f$, we obtain the converse inequality. Hence

$$\sigma(f^3)-3f\sigma(f^2)+3f^2\sigma(f)=0. \tag{3)'$$

Then (3) follows from (3)′ by the linearity of σ. (For instance, substitute $f+tg+sh$ with real numbers t and s for f in (3)′ and compare the coefficient of ts.)

3. Gradient measures

In this and the next sections, we assume that there exists a measure representation σ of $\mathscr{R}$ and fix it. Let U be an open set in X. We shall define gradient measures relative to this σ for functions in $\mathscr{R}_0(U)$. First we prepare

LEMMA 2. *Let $f, g\in\mathscr{R}_0(U)$, $h\in\mathscr{R}(U)$ and $h>0$ on U. Then the signed measure*

$$\frac{1}{2h}\{f\sigma(gh)+g\sigma(fh)-\sigma(fgh)-fg\sigma(h)\} \tag{4}$$

does not depend on h.

PROOF. Let V be any open subset of U for which there is $u\in\mathscr{H}(V)$ such that $u>0$. Since $1\in\mathscr{H}^{(u)}(V)$, the corollary to Proposition 2 implies

$$\begin{aligned}&\sigma^{(u)}(fgh/u)-f\sigma^{(u)}(gh/u)-g\sigma^{(u)}(fh/u)+fg\sigma^{(u)}(h/u)\\&=\frac{h}{u}\{\sigma^{(u)}(fg)-f\sigma^{(u)}(g)-g\sigma^{(u)}(f)\}\end{aligned}$$

on V. Since $\sigma^{(u)}(f)=\sigma(uf)$ for any $f\in\mathscr{R}_0(V)=\mathscr{R}^{(u)}(V)$, it follows that

$$\begin{aligned}&\frac{1}{2h}\{f\sigma(gh)+g\sigma(fh)-\sigma(fgh)-fg\sigma(h)\}\\&=\frac{1}{2u}\{f\sigma(gu)+g\sigma(fu)-\sigma(fgu)\},\end{aligned}$$

the right hand side of which is independent of h. Since any point in U is contained in such V (Axiom of positivity; see [3, §2.1]), we obtain the lemma.

We now define the mutual gradient measure $\delta_{[f,g]}$ of $f, g\in\mathscr{R}_0(U)$ (relative to σ) by (4), i.e.,

$$\delta_{[f,g]}=\frac{1}{2h}\{f\sigma(gh)+g\sigma(fh)-\sigma(fgh)-fg\sigma(h)\}$$

for some strictly positive $h\in\mathscr{R}(U)$. By the above lemma, $\delta_{[f,g]}$ is well-defined. Obviously, the mapping $(f, g)\to\delta_{[f,g]}$ is symmetric and bilinear on $\mathscr{R}_0(U)\times\mathscr{R}_0(U)$. The measure $\delta_f=\delta_{[f,f]}$ is called the gradient measure of $f\in\mathscr{R}_0(U)$.

REMARK. In case $1\in\mathscr{R}(X)$, we can choose $h=1$ in the above definition and have the same form as in [8].

THEOREM 1. *$\delta_f \geqq 0$ on U for any $f \in \mathscr{R}_0(U)$.*

PROOF. Let V be an open subset of U for which there is a strictly positive $u \in \mathscr{H}(V)$. Applying Proposition 2 with $p=2$ to the harmonic space $(V, \mathscr{H}_V^{(u)})$ and the measure representation $\sigma^{(u)}$, we obtain

$$2f\sigma^{(u)}(f) - \sigma^{(u)}(f^2) \geqq 0$$

on V for $f \in \mathscr{R}^{(u)}(V) = \mathscr{R}_0(V)$. Hence

$$\delta_f = \frac{1}{2u}\{2f\sigma(fu) - \sigma(f^2u)\} = \frac{1}{2u}\{2f\sigma^{(u)}(f) - \sigma^{(u)}(f^2)\} \geqq 0$$

on V. Then, by the Axiom of positivity, we see that $\delta_f \geqq 0$ on U.

COROLLARY. *For any $f, g \in \mathscr{R}_0(U)$ and for any Borel set A in U,*

$$\{|\delta_{[f,g]}|(A)\}^2 \leqq \delta_f(A) \cdot \delta_g(A);$$

in particular, $\delta_f = 0$ on U implies $\delta_{[f,g]} = 0$ on U for any $g \in \mathscr{R}_0(U)$.

PROPOSITION 3. *Let W be an open set in X. If $h \in \mathscr{R}(W)$ and $h > 0$ on W, then the mutual gradient measure $\delta_{[f,g]}^{(h)}$ of $f, g \in \mathscr{R}^{(h)}(U) = \mathscr{R}_0(U)$ $(U \subseteq W)$ with respect to the harmonic space $(W, \mathscr{H}_W^{(h)})$ and relative to $\sigma^{(h)}$ is given by*

$$\delta_{[f,g]}^{(h)} = h\delta_{[f,g]}.$$

PROOF. Since $1 \in \mathscr{R}^{(h)}(U)$, we have

$$\begin{aligned}\delta_{[f,g]}^{(h)} &= \frac{1}{2}\{f\sigma^{(h)}(g) + g\sigma^{(h)}(f) - \sigma^{(h)}(fg) - fg\sigma^{(h)}(1)\} \\ &= \frac{1}{2}\{f\sigma(gh) + g\sigma(fh) - \sigma(fgh) - fg\sigma(h)\} \\ &= h\delta_{[f,g]}.\end{aligned}$$

THEOREM 2. *For $f, g, h \in \mathscr{R}_0(U)$,*

$$\delta_{[fg,h]} = f\delta_{[g,h]} + g\delta_{[f,h]}. \tag{5}$$

PROOF. Let V be an open subset of U for which there is a strictly positive $u \in \mathscr{H}(V)$. By the corollary to Proposition 2, we have

$$\begin{aligned}&2\{\delta_{[fg,h]}^{(u)} - f\delta_{[g,h]}^{(u)} - g\delta_{[f,h]}^{(u)}\} \\ &\quad = fg\sigma^{(u)}(h) + h\sigma^{(u)}(fg) - \sigma^{(u)}(fgh)\end{aligned}$$

$$\begin{aligned}
&-f\{g\sigma^{(u)}(h)+h\sigma^{(u)}(g)-\sigma^{(u)}(gh)\}\\
&-g\{f\sigma^{(u)}(h)+h\sigma^{(u)}(f)-\sigma^{(u)}(fh)\}\\
=\ &-\{\sigma^{(u)}(fgh)-f\sigma^{(u)}(gh)-g\sigma^{(u)}(fh)-h\sigma^{(u)}(fg)\\
&\quad+fg\sigma^{(u)}(h)+fh\sigma^{(u)}(g)+gh\sigma^{(u)}(f)\}\\
=\ &0.
\end{aligned}$$

Thus, (5) holds on V by virtue of the above proposition. Since such V's cover U, (5) holds on U.

PROPOSITION 4. *$\mathscr{R}(U)$ and $\mathscr{R}_0(U)$ are vector lattices with respect to the max. and min. operations and $\delta_{|f|}=\delta_f$ for $f\in\mathscr{R}_0(U)$.*

This proposition can be proved in the same way as in [8, §4.2].

4. Composition of functions in $\mathscr{R}_0(U)$ with C^2-functions

First we prepare

LEMMA 3. *Let g_j $(j=1,\dots,m)$ and f_n $(n=1,2,\dots)$ be functions in $\mathscr{R}(U)$ and let $\varphi_{j,n}$ $(j=1,\dots,m;\ n=1,2,\dots)$ be continuous functions on U. Suppose f_n (resp. $\varphi_{j,n}$) converges to f (resp. φ_j) locally uniformly on U as $n\to\infty$ and*

$$\sigma(f_n)=\sum_{j=1}^{m}\varphi_{j,n}\sigma(g_j),\qquad n=1,2,\dots.$$

Then $f\in\mathscr{R}(U)$ and

$$\sigma(f)=\sum_{j=1}^{m}\varphi_j\sigma(g_j).$$

PROOF. Let V be a relatively compact open set such that $\overline{V}\subseteq U$ and $g_1,\dots,g_m$ can be expressed as differences of continuous superharmonic functions on V. Then there is a continuous superharmonic function s on V such that $|\sigma(g_j)|\leqq\sigma(s)$ for all j. Put

$$\gamma_n=\sum_{j=1}^{m}\sup_V|\varphi_{j,n}-\varphi_j|,\qquad n=1,2,\dots.$$

Then $\gamma_n\to 0$ $(n\to\infty)$ by assumption and

$$|\sigma(f_n)-\sum_{j=1}^{m}\varphi_j\sigma(g_j)|=|\sum_{j=1}^{m}(\varphi_{j,n}-\varphi_j)\sigma(g_j)|\leqq\gamma_n\sigma(s)\tag{6}$$

on V for all n. Let $\varepsilon>0$ be given. Since $\varphi_1,\dots,\varphi_m$ are continuous, for any $x\in V$ there is an open set W such that $x\in W\subseteq\overline{W}\subseteq V$ and

$$\sup_W \varphi_j - \inf_W \varphi_j < \varepsilon, \qquad j = 1,\dots, m.$$

Put $\alpha_j = \sup_W \varphi_j$, $\beta_j = \inf_W \varphi_j$, $j = 1,\dots, m$ and

$$v_1 = \sum_{j=1}^m \{\beta_j s - \alpha_j(s - g_j)\}, \quad v_2 = \sum_{j=1}^m \{\alpha_j s - \beta_j(s - g_j)\}.$$

Since $\sigma(s) \geqq 0$ and $\sigma(s - g_j) \geqq 0$,

$$\sigma(v_1) \leqq \sum_{j=1}^m \varphi_j \sigma(g_j) \leqq \sigma(v_2) \tag{7}$$

on W. Hence, it follows from (6) that $f_n - v_1 + \gamma_n s$ and $-f_n + v_2 + \gamma_n s$ are superharmonic on W, so that their uniform limits $f - v_1$ and $-f + v_2$ are superharmonic on W. Since this is true for any $x \in V$, we conclude that $f \in \mathscr{R}(V)$. Furthermore, for the above W, $\sigma(v_1) \leqq \sigma(f) \leqq \sigma(v_2)$ on W. Hence, in view of (7) we obtain

$$|\sigma(f) - \sum_{j=1}^m \varphi_j \sigma(g_j)| \leqq \sigma(v_2 - v_1) \leqq 3\varepsilon m \sigma(s)$$

on W. Noting that the first and the last terms are independent of W, we see that

$$\sigma(f) = \sum_{j=1}^m \varphi_j \sigma(g_j)$$

on V. Since such V's cover U, we obtain the lemma.

THEOREM 3. *Let $f_1,\dots, f_k \in \mathscr{R}_0(U)$ and put $\mathbf{f} = (f_1,\dots, f_k)$. Let Ω be an open set in $\mathbf{R}^k$ containing $\mathbf{f}(U)$. If $\varphi \in \mathbf{C}^2(\Omega)$, then $\varphi \circ \mathbf{f} \in \mathscr{R}_0(U)$ and the following equalities hold*:

$$\sum_{i,j=1}^k \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j} \circ \mathbf{f}\right) \delta_{[f_i, f_j]} = \frac{1}{h}\left\{\sum_{j=1}^k \left(\frac{\partial \varphi}{\partial x_j} \circ \mathbf{f}\right)\{\sigma(f_j h) - f_j \sigma(h)\} + (\varphi \circ \mathbf{f})\sigma(h) - \sigma[(\varphi \circ \mathbf{f})h]\right\} \tag{8}$$

for any $h \in \mathscr{R}(U)$ with $h > 0$ on U;

$$\delta_{[\varphi \circ \mathbf{f}, g]} = \sum_{j=1}^k \left(\frac{\partial \varphi}{\partial x_j} \circ \mathbf{f}\right) \delta_{[f_j, g]} \tag{9}$$

for any $g \in \mathscr{R}_0(U)$, in particular

$$\delta_{[\varphi \circ \mathbf{f}, \psi \circ \mathbf{f}]} = \sum_{i,j=1}^k \left(\frac{\partial \varphi}{\partial x_i} \circ \mathbf{f}\right)\left(\frac{\partial \psi}{\partial x_j} \circ \mathbf{f}\right) \delta_{[f_i, f_j]} \tag{9$'$}$$

for $\varphi, \psi \in \mathbf{C}^2(\Omega)$.

Proof. (I) First we consider the case where φ is a polynomial of k variables. Since $\mathscr{R}_0(U)$ is an algebra with 1, $\varphi\circ\mathbf{f}\in\mathscr{R}_0(U)$. If $\varphi\equiv 1$, then both sides of (8) and (9) are reduced to 0. If $\varphi(x_1,\dots,x_k)=x_j$, then both sides of (8) (resp. (9)) are reduced to 0 (resp. $\delta_{[f_j,g]}$). Since both sides of (8) and (9) are linear in φ, it follows that these equalities hold if the degree of φ is 1.

Now suppose (8) and (9) are valid for polynomials φ_1 and φ_2 and let $\varphi=\varphi_1\varphi_2$. Then, using Theorem 2, we have

$$\begin{aligned}\delta_{[\varphi\circ\mathbf{f},g]} &= \delta_{[(\varphi_1\circ\mathbf{f})(\varphi_2\circ\mathbf{f}),g]}\\ &= (\varphi_1\circ\mathbf{f})\delta_{[\varphi_2\circ\mathbf{f},g]}+(\varphi_2\circ\mathbf{f})\delta_{[\varphi_1\circ\mathbf{f},g]}\\ &= (\varphi_1\circ\mathbf{f})\sum_{j=1}^{k}\left(\frac{\partial\varphi_2}{\partial x_j}\circ\mathbf{f}\right)\delta_{[f_j,g]}+(\varphi_2\circ\mathbf{f})\sum_{j=1}^{k}\left(\frac{\partial\varphi_1}{\partial x_j}\circ\mathbf{f}\right)\delta_{[f_j,g]}\\ &= \sum_{j=1}^{k}\left(\frac{\partial\varphi}{\partial x_j}\circ\mathbf{f}\right)\delta_{[f_j,g]}\end{aligned}$$

and

$$\begin{aligned}&\sum_{i,j=1}^{k}\left(\frac{\partial^2\varphi}{\partial x_i\,\partial x_j}\circ\mathbf{f}\right)\delta_{[f_i,f_j]}\\ &= \sum_{i,j=1}^{k}(\varphi_1\circ\mathbf{f})\left(\frac{\partial^2\varphi_2}{\partial x_i\partial x_j}\circ\mathbf{f}\right)\delta_{[f_i,f_j]}+\sum_{i,j=1}^{k}(\varphi_2\circ\mathbf{f})\left(\frac{\partial^2\varphi_1}{\partial x_i\partial x_j}\circ\mathbf{f}\right)\delta_{[f_i,f_j]}\\ &\quad +2\sum_{i,j=1}^{k}\left(\frac{\partial\varphi_1}{\partial x_i}\circ\mathbf{f}\right)\left(\frac{\partial\varphi_2}{\partial x_j}\circ\mathbf{f}\right)\delta_{[f_i,f_j]}\\ &= \frac{\varphi_1\circ\mathbf{f}}{h}\left\{\sum_{j=1}^{k}\left(\frac{\partial\varphi_2}{\partial x_j}\circ\mathbf{f}\right)\{\sigma(f_jh)-f_j\sigma(h)\}+(\varphi_2\circ\mathbf{f})\sigma(h)-\sigma[(\varphi_2\circ\mathbf{f})h]\right\}\\ &\quad +\frac{\varphi_2\circ\mathbf{f}}{h}\left\{\sum_{j=1}^{k}\left(\frac{\partial\varphi_1}{\partial x_j}\circ\mathbf{f}\right)\{\sigma(f_jh)-f_j\sigma(h)\}+(\varphi_1\circ\mathbf{f})\sigma(h)-\sigma[(\varphi_1\circ\mathbf{f})h]\right\}\\ &\quad +2\delta_{[\varphi_1\circ\mathbf{f},\varphi_2\circ\mathbf{f}]}\\ &= \frac{1}{h}\left\{\sum_{j=1}^{k}\left(\frac{\partial\varphi}{\partial x_j}\circ\mathbf{f}\right)\{\sigma(f_jh)-f_j\sigma(h)\}+2(\varphi\circ\mathbf{f})\sigma(h)\right.\\ &\qquad \left. -(\varphi_1\circ\mathbf{f})\sigma[(\varphi_2\circ\mathbf{f})h]-(\varphi_2\circ\mathbf{f})\sigma[(\varphi_1\circ\mathbf{f})h]+2h\delta_{[\varphi_1\circ\mathbf{f},\varphi_2\circ\mathbf{f}]}\right\}\\ &= \frac{1}{h}\left\{\sum_{j=1}^{k}\left(\frac{\partial\varphi}{\partial x_j}\circ\mathbf{f}\right)\{\sigma(f_jh)-f_j\sigma(h)\}+(\varphi\circ\mathbf{f})\sigma(h)-\sigma[(\varphi\circ\mathbf{f})h]\right\}.\end{aligned}$$

Thus, (8) and (9) hold for φ. Then by induction we see that (8) and (9) hold for any polynomial φ.

(II) Next, let $\varphi\in\mathbf{C}^2(\Omega)$. We can find a sequence $\{\varphi_n\}$ of polynomials of

k variables which converges to φ in the topology of $\mathbf{C}^2(\Omega)$. By the result in (I), we have

$$\sigma[(\varphi_n\circ\mathbf{f})h] = -h\sum_{i,j=1}^{k}\left(\frac{\partial^2\varphi_n}{\partial x_i\partial x_j}\circ\mathbf{f}\right)\delta_{[f_i,f_j]}$$

$$+\sum_{j=1}^{k}\left(\frac{\partial\varphi_n}{\partial x_j}\circ\mathbf{f}\right)\{\sigma(f_jh)-f_j\sigma(h)\}+(\varphi_n\circ\mathbf{f})\sigma(h).$$

Since $\varphi_n\circ\mathbf{f}\to\varphi\circ\mathbf{f}$, $(\partial\varphi_n/\partial x_j)\circ\mathbf{f}\to(\partial\varphi/\partial x_j)\circ\mathbf{f}$ and $(\partial^2\varphi_n/\partial x_i\partial x_j)\circ\mathbf{f}\to(\partial^2\varphi/\partial x_i\partial x_j)\circ\mathbf{f}$ all locally uniformly on U as $n\to\infty$, the previous lemma implies that $(\varphi\circ\mathbf{f})h\in\mathscr{R}(U)$ and

$$\sigma[(\varphi\circ\mathbf{f})h] = -h\sum_{i,j=1}^{k}\left(\frac{\partial^2\varphi}{\partial x_i\partial x_j}\circ\mathbf{f}\right)\delta_{[f_i,f_j]}$$

$$+\sum_{j=1}^{k}\left(\frac{\partial\varphi}{\partial x_j}\circ\mathbf{f}\right)\{\sigma(f_jh)-f_j\sigma(h)\}+(\varphi\circ\mathbf{f})\sigma(h).$$

Hence $\varphi\circ\mathbf{f}\in\mathscr{R}_0(U)$ and (8) holds.

Next, consider the function

$$\Phi(x_1,\dots,x_k,x_{k+1}) = x_{k+1}\varphi(x_1,\dots,x_k).$$

Then $\Phi\in\mathbf{C}^2(\Omega\times\mathbf{R})$. Given $g\in\mathscr{R}_0(U)$, applying (8) to $\mathbf{f}^*=(f_1,\dots,f_k,g)$ and Φ, we obtain

$$g\sum_{i,j=1}^{k}\left(\frac{\partial^2\varphi}{\partial x_i\partial x_j}\circ\mathbf{f}\right)\delta_{[f_i,f_j]}+2\sum_{j=1}^{k}\left(\frac{\partial\varphi}{\partial x_j}\circ\mathbf{f}\right)\delta_{[f_j,g]}$$

$$=\frac{1}{h}\left\{g\sum_{j=1}^{k}\left(\frac{\partial\varphi}{\partial x_j}\circ\mathbf{f}\right)\{\sigma(f_jh)-f_j\sigma(h)\}+(\varphi\circ\mathbf{f})\{\sigma(gh)-g\sigma(h)\}\right.$$

$$\left.+(\varphi\circ\mathbf{f})g\sigma(h)-\sigma[(\varphi\circ\mathbf{f})gh]\right\}.$$

Hence, again using (8), we have

$$2\sum_{j=1}^{k}\left(\frac{\partial\varphi}{\partial x_j}\circ\mathbf{f}\right)\delta_{[f_j,g]}$$

$$=\frac{1}{h}\{(\varphi\circ\mathbf{f})\sigma(gh)+g\sigma[(\varphi\circ\mathbf{f})h]-\sigma[(\varphi\circ\mathbf{f})gh]-(\varphi\circ\mathbf{f})g\sigma(h)\}$$

$$=2\delta_{[\varphi\circ\mathbf{f},g]},$$

which is the required equality (9).

REMARK. Given $f\in\mathscr{R}_0(U)$, if $\varphi\in\mathbf{C}^2(\Omega)$ with $\Omega\supseteq f(U)$ and $\varphi''\neq 0$ on $f(U)$, then from (8) we deduce

$$(10)\quad \delta_f = \frac{1}{(\varphi''\circ f)h}[(\varphi'\circ f)\{\sigma(fh)-f\sigma(h)\} - \{\sigma[(\varphi\circ f)h]-(\varphi\circ f)\sigma(h)\}].$$

In case $\varphi(t)=t^2$, (10) is nothing but the definition of δ_f. Thus we may define δ_f by (10) using any φ as above.

5. The case where X is a euclidean domain

THEOREM 4 (cf. [9, §5]). *Suppose that the base space X of the harmonic space is an open set in the k-dimensional euclidean space $\mathbf{R}^k$ and that a measure representation σ of $\mathscr{R}$ is given. If $\mathbf{C}^2(X)\cap\mathscr{R}_0(X)$ is dense in $\mathbf{C}^1(X)$ (or, if the coordinate functions x_j all belong to $\mathscr{R}_0(X)$), then $\mathbf{C}^2(U)\subseteq\mathscr{R}_0(U)$ for any open set U in X, and writing*

$$\alpha_{ij} = \delta_{[x_i,x_j]}, \qquad i, j = 1,\dots, k,$$

we have

$$(11)\qquad \delta_{[f,g]} = \sum_{i,j=1}^{k} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\alpha_{ij}$$

for $f, g\in\mathbf{C}^2(U)$. The matrix (α_{ij}) is symmetric and positive semi-definite in the sense that $\mu_\xi=\sum_{i,j}\xi_i\xi_j\alpha_{ij}$ is a non-negative measure on X for each $\xi\in\mathbf{R}^k$. If, in addition, $1\in\mathscr{R}(X)$, then with $\beta_j=-\sigma(x_j)+x_j\sigma(1)$, $j=1,\dots, k$, and $\gamma=-\sigma(1)$, every $f\in\mathbf{C}^2(U)$ satisfies the equation

$$(12)\qquad \sum_{i,j=1}^{k}\frac{\partial^2 f}{\partial x_i\partial x_j}\alpha_{ij}+\sum_{j=1}^{k}\frac{\partial f}{\partial x_j}\beta_j+f\gamma = -\sigma(f)$$

on U.

PROOF. Let V be any bounded convex open set such that $\bar{V}\subseteq X$. Since coordinate functions belong to $\mathbf{C}^1(X)$, by assumption we can find $\psi_1,\dots,\psi_k \in\mathbf{C}^2(X)\cap\mathscr{R}_0(X)$ such that $|(\partial\psi_i/\partial x_j)(x)-\delta_{ij}|<1/4k$ for all $x\in V$ and $i, j=1,\dots, k$. Put $\Psi=(\psi_1|V,\dots,\psi_k|V)$. Then, by the convexity of V, we see that $\Psi\colon V\to\Psi(V)$ is a bijection and if we put $\Psi^{-1}=(\varphi_1,\dots,\varphi_k)$, then $\varphi_j\in\mathbf{C}^2(\Psi(V))$, $j=1,\dots, k$. Hence $f\circ\Psi^{-1}\in\mathbf{C}^2(\Psi(U\cap V))$ for $f\in\mathbf{C}^2(U)$. Since $\psi_j|U\cap V\in\mathscr{R}_0(U\cap V)$, Theorem 3 implies that $f=(f\circ\Psi^{-1})\circ\Psi\in\mathscr{R}_0(U\cap V)$; and since such V's cover X, $f\in\mathscr{R}_0(U)$. Hence $\mathbf{C}^2(U)\subseteq\mathscr{R}_0(U)$. Now we obtain (11) and (12) from (8) and (9) in Theorem 3 by considering $f_j(x)=x_j$, $j=1,\dots, k$. Obviously, (α_{ij}) is symmetric and $\mu_\xi=\delta_{\Sigma\xi_jx_j}\geqq 0$ for $\xi\in\mathbf{R}^k$ by Theorem 1.

COROLLARY. *Under the same assumptions as in the above theorem (including the assumption $1\in\mathscr{R}(X)$), $u\in\mathbf{C}^2(U)$ belongs to $\mathscr{H}(U)$ if and only if*

$$\sum_{i,j=1}^{k} \frac{\partial^2 u}{\partial x_i \partial x_j}\alpha_{ij} + \sum_{j=1}^{k} \frac{\partial u}{\partial x_j}\beta_j + u\gamma = 0$$

on U, *for any open set* U *in* X.

6. Existence of a measure representation

As we have seen in [8] (also, cf. [11]), if integral representation of potentials in terms of Green functions is possible, then we can define a measure representation.

Let V be a $\mathfrak{P}$-set (see [3, § 2.3]) in X. By a Green function on V, we shall mean a lower semicontinuous function G: $V \times V \to [0, +\infty]$ which is finite continuous off the diagonal such that for each $y \in V$, the function $x \to G(x, y)$ is a non-zero potential on V and is harmonic on $V - \{y\}$.

The existence of such a Green function and the possibility of integral representation were investigated by R.-M. Hervé [6] for Brelot's harmonic spaces and by K. Janssen [7] for more general harmonic spaces. To apply Janssen's result, we need the following assumptions:

(K_D) The harmonic sheaf $\mathscr{H}$ possesses the Doob convergence property (see [3, p. 9]);

(A) Each $x \in X$ has a neighborhood W such that X is the smallest absorbent set (see [3, § 6.1]) containing $X - W$;

(P) (Proportionality condition) For each $\mathfrak{P}$-set V, any two non-zero potentials on V having the same point support are proportional.

Note that Brelot's harmonic spaces satisfy (K_D) and (A). The harmonic space determined by the heat equation on a euclidean domain satisfies all of these assumptions (see [3, § 3.3] and [4]).

Under the assumptions (K_D), (A) and (P), each $\mathfrak{P}$-set V in X possesses a Green function G and any potential p on V is expressed as $p(x) = \int_V G(x, y) d\mu(y)$ by a uniquely determined non-negative measure μ on V ([7, Theorem 3.4]). Thus, by the same reasoning as in [11, Lemma 1.5 and Theorem 1.6], we can show that there exists a "consistent" system of Green functions, that is, a system $\{G_V\}_{V:\mathfrak{P}-\mathrm{set}}$ such that each G_V is a Green function on V and for any $\mathfrak{P}$-sets V and V' with $V' \subseteq V$ and for any $y \in V'$, there is $u_y \in \mathscr{H}(V')$ satisfying

$$G_{V'}(x, y) = G_V(x, y) + u_y(x) \qquad (x \in V').$$

With respect to such a system we obtain a sheaf homomorphism σ of $\mathscr{R}$ into $\mathscr{M}$ as follows: for each $f \in \mathscr{R}(U)$, there is a unique $\sigma(f) \in \mathscr{M}(U)$ such that

$$f | V = \int_V G_V(\cdot, y) d\sigma(f)(y) + u_f^V$$

with $u_f^V \in \mathscr{H}(V)$ for every relatively compact $\mathfrak{P}$-set V whose closure is contained in another $\mathfrak{P}$-set $\subseteq U$. It is easy to see that this σ is a measure representation of $\mathscr{R}$. Thus we have shown

PROPOSITION 5. *Under the assumptions* (K_D), (A) *and* (P), *there exists a measure representation of* $\mathscr{R}$, *defined with respect to a consistent system of Green functions. In particular, a Brelot's harmonic space satisfying* (P) *possesses such a measure representation.*

REMARK. If there is a consistent system of Green functions $\{G_V\}$ such that each G_V is symmetric, i.e., $G_V(x, y) = G_V(y, x)$, then we may call $(X, \mathscr{H})$ self-adjoint (cf. [8]). Under the assumptions (K_D), (A) and (P), we can show that if $(X, \mathscr{H})$ is self-adjoint, then it is a Brelot's harmonic space. In fact, Theorem 1 and a result similar to [8, Proposition 2.2], which is also valid in the present case, imply that each G_V is of positive type. Then each G_V satisfies energy principle in view of [3, Theorem 2.3.1] (cf. [2]). By a classical method of H. Cartan (cf. [2]), we see that the domination principle holds on each $\mathfrak{P}$-set. Then, by [3, Propositions 9.2.1 and 3.1.4, Corollary 3.1.2 and Exercise 3.1.3], we conclude that $(X, \mathscr{H})$ is a Brelot's harmonic space.

The following examples show that conditions (K_D), (A) and (P) are by no means necessary for the existence of a measure representation.

EXAMPLE 1. Let $\tilde{\mathscr{H}}$ be the harmonic sheaf on $\mathbf{R}^2$ defined by the solutions of the heat equation

$$Lu \equiv \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad ((x, t) \in \mathbf{R}^2).$$

Let $X = \mathbf{R} \times (-\infty, 0]$ and for an open set U of X, let $U^* = U \cap \{\mathbf{R} \times (-\infty, 0)\}$ and

$$\mathscr{H}(U) = \{u \in \mathbf{C}(U);\ u|U^* \in \tilde{\mathscr{H}}(U^*)\}.$$

Then $(X, \mathscr{H})$ is a harmonic space which does not satisfy (K_D) (see [3, Exercise 11.1.9]). If we define

$$\sigma(f) = L(f|U^*) \qquad \text{for} \quad f \in \mathscr{R}(U),$$

where Lf is taken in the distribution sense, then σ gives a measure representation of $\mathscr{R}$ (cf. [4, p. 263]).

EXAMPLE 2. Let $(X, \mathscr{H})$ be the harmonic space given in [7, Remark (3.1)], i.e.,

$$X = \bigcup_{n=0}^{\infty} X_n \subseteq \mathbf{R}^2$$

with

$$X_0 = (-1, +\infty) \times \{0\} \quad \text{and} \quad X_n = \{1/n\} \times (0, 1/n], \qquad n = 1, 2, \dots,$$

and

$$\mathscr{H}(U) = \left\{ h \in \mathbf{C}(U); \begin{array}{l} h|U \cap X_n \text{ is linear for } n = 0, 1, \dots; h \text{ is} \\ \text{constant on } U \cap X_n \ (n \geqq 1) \text{ if } (1/n, 1/n) \in U \end{array} \right\}.$$

This harmonic space satisfies neither (A) nor (P). Still we can construct a measure representation as follows:

If U is an open set in X and $u(x, y)$ is a continuous superharmonic function on U, then u is a concave function of x (resp. of y) on $X_0 \cap U$ (resp. on each $X_n \cap U$, $n \geqq 1$), and furthermore, monotone increasing in y on $X_n \cap U$ ($n \geqq 1$) in case $(1/n, 1/n) \in U$. It follows that $\mu_0 = -(\partial^2 u/\partial x^2)$ on $X_0 \cap U$ (resp. $\mu_n = -(\partial^2 u/\partial y^2)$ on $X_n \cap U$, $n \geqq 1$) in the distribution sense is a non-negative measure on $X_0 \cap U$ (resp. $X_n \cap U$). We see that if $(1/n, 0) \in U$, then $\int_0 y d\mu_n(y) < +\infty$ and if in addition $X_n \subseteq U$, then

$$\int_0^{1/n} y d\mu_n(y) \leqq u(1/n, 1/n) - u(1/n, 0).$$

It follows that if we define $\sigma(u) = \mu_0$ on $X_0 \cap U$ and $\sigma(u) = (1/n^2) y \mu_n$ on $X_n \cap U$, $n = 1, 2, \dots$, then $\sigma(u)$ is a non-negative Radon measure on U. Now it is easy to see that the mappings $u \to \sigma(u)$ induce a measure representation of $\mathscr{R}$.

7. Open questions

Here, we raise two open questions, whose answers are known to be affirmative in the case of self-adjoint harmonic spaces ([8] and [9]).

(1) In case the harmonic space is elliptic (see [3, p. 66]), does $\delta_f = 0$ imply $f = \text{const.}$ for $f \in \mathscr{R}_0(U)$ when U is connected?

(2) Let U be an open set in X. For each compact set K in U and $f \in \mathscr{R}_0(U)$, put

$$\rho_K(f) = \delta_f(K)^{1/2} + \sup_K |f|.$$

Then $\mathscr{R}_0(U)$ is a Hausdorff locally convex space with respect to the family of seminorms $\{\rho_K\}$. Is it possible to embed the completion of $\mathscr{R}_0(U)$ into $\mathbf{C}(U)$? In other words, if $\{f_n\}$ and $\{g_n\}$ are Cauchy nets in $\mathscr{R}_0(U)$ and if $\lim_n f_n = \lim_n g_n$

pointwise, then do $\{f_n\}$ and $\{g_n\}$ determine the same element in the completion of $\mathscr{R}_0(U)$?

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