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Volterra Integral Equations as Functional Differential Equations on Infinite Intervals

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1. Introduction

Our objective is to treat the nonlinear functional differential equation

$$(1.1)\qquad \dot{x}(t) = f(t, x(t)) + F(t, x_t), \qquad t \geq s \geq 0$$
$$x(s) = h \in H, \qquad x_s = \phi \in L^p(-r, 0; H)$$

by means of the theory of nonlinear evolutions in Banach spaces. The notation of (1.1) means that H is a Hilbert space, $1 \leq p < \infty$, $0 < r \leq \infty$, $x: (s-r, \infty) \to H$, $f: [0, \infty) \times H \to H$, $F: [0, \infty) \times L^p(-r, 0; H) \to H$, and $x_t \in L^p(-r, 0; H)$ is defined by $x_t(\theta) = x(t+\theta)$ for $-r < \theta < 0$. The equation (1.1) may be formulated as the abstract ordinary differential equation

$$(1.2)\qquad du(t)/dt = -Au(t) - B(t, u(t)), \qquad t \geq s \geq 0$$
$$u(s) = x \in X$$

in the Banach space $X \overset{\mathrm{def}}{=} L^p(-r, 0; H) \times H$. The notation of (1.2) means u: $[s, \infty) \to X$, $A: X \to X$ such that $-A$ is the infinitesimal generator of a strongly continuous semigroup of linear operators in X, and $B: [0, \infty) \times X \to X$ such that $B(t, \cdot)$ is nonlinear. By converting (1.1) to the form (1.2), we will be able to take advantage of the extensive theory which has been developed in recent years for nonlinear evolution equations of the form (1.2). The special semi-linear form of (1.2) will allow us to state explicitly the relationship between the solutions of (1.1) and (1.2).

There is a growing literature associated with the treatment of functional differential equations as abstract ordinary differential equations in function spaces, and some recent papers on this subject are listed in our references. Our work continues the investigations of [17], [18], and [19], where (1.1) was treated with F independent of t and $r < \infty$. In the present study we will allow the case that $r = \infty$, so that we may treat nonlinear Volterra integral equations of the form

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$$x(t) = y(t) + \int_0^t g(t-\tau, x(\tau))d\tau, \qquad t \geq 0. \tag{1.3}$$

The semigroup approach to linear Volterra equations has been explored in [1], [3], and [13]. As an application of our theory we will obtain new stability information for the equation (1.3).

2. An abstract theorem

For a Banach space X the duality mapping $J\colon X \to 2^{X^*}$ is defined by

$$j \in J(x) \quad \text{iff} \quad <x, j> = \|x\|^2 = \|j\|^2 \qquad \text{for} \quad x \in X. \tag{2.1}$$

The (nonlinear) operator $A\colon X \to X$ is accretive iff

$$\text{for all} \quad x, y \in D(A) \qquad \text{and some} \quad j \in J(x-y), \tag{2.2}$$
$$<Ax - Ay, j> \geq 0.$$

THEOREM 2.1. *Let X be a Banach space and let ω be a real constant. Let $A\colon X \to X$ such that $-A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$, $t \geq 0$ in X. Let $B\colon [0, \infty) \times X \to X$ such that B is continuous, everywhere defined, and maps bounded sets into bounded sets. Let $A + B(t, \cdot) + \omega I$ be accretive for each $t \geq 0$. Then for each $x \in X$ and $s \geq 0$, there exists a unique continuous function $u\colon [s, \infty) \to X$ satisfying*

$$u(t) = T(t-s)x - \int_s^t T(t-\tau)B(\tau, u(\tau))d\tau, \qquad t \geq s. \tag{2.3}$$

If we define $U(t, s)x = u(t)$ as in (2.3), then the family of (nonlinear) operators $U(t, s)$, $t \geq s \geq 0$ satisfies

(2.4) $U(s, s)x = x$;

(2.5) $U(t, s) = U(t, r)U(r, s), \qquad t \geq r \geq s$;

(2.6) *$U(t, s)x$ is continuous as a function in t for x and s fixed and $t \geq s$;*

(2.7) $\|U(t, s)x - U(t, s)y\| \leq e^{\omega(t-s)}\|x - y\| \qquad$ *for* $\quad t \geq s \geq 0 \quad$ *and* $\quad x, y \in X$.

If, in addition, B is of the form $B(t, x) = C(x) + c(t)$ and $C(0) = 0$, then

(2.8) $\|U(t, s)x\| \leq e^{\omega(t-s)}\|x\| + \int_s^t e^{\omega(t-\tau)}|c(\tau)|d\tau, \qquad t \geq s \geq 0, \quad x \in X.$

The equation (2.3) is the integrated form of equation (1.2). Several authors have treated the existence problem for equation (2.3), among them [6], [11], [12], [14], and [16]. The version of Theorem 2.1 which we have stated here is slightly different from those found in the above references, but may be established readily using the methods of these references.

3. Functional differential equations as abstract ordinary differential equations

Let H be a Hilbert space with norm $|\ |$ and inner-product $(\ ,\ )$. Let $1 \leq p < \infty$, $0 < r \leq \infty$, and let $\alpha \in \mathbf{R}$. We will formulate equation (1.1) as equation (2.3) in the Banach space

(3.1) $$X \stackrel{\text{def}}{=} L^p(-r, 0; H; \mu) \times H, \quad \text{where} \quad d\mu(\theta) \stackrel{\text{def}}{=} e^{-p\alpha\theta} d\theta, \quad \text{with norm}$$

$$\|\{\phi, h\}\| \stackrel{\text{def}}{=} \left(\int_{-r}^{0} |\phi(\theta)|^p e^{-p\alpha\theta} d\theta + |h|^p \right)^{1/p}.$$

The introduction of the weight function $e^{-p\alpha\theta}$ in the norm of X will allow us to obtain stability information about the solutions of (1.1). The duality mapping for X is given by

(3.2) if $\{\phi, h\} \in X$, then $j \in J(\{\phi, h\})$, where j is defined at $\{\psi, k\} \in X$

$$\text{by } \langle \{\psi, k\}, j \rangle = \|\{\phi, h\}\|^{2-p} \left(\int_{-r}^{0} (\psi(\theta), \phi(\theta)) |\phi(\theta)|^{p-2} e^{-p\alpha\theta} d\theta + (k, h)|h|^{p-2} \right)$$

(see [17], Proposition 2.1). We will let π_1 and π_2 be the projections on X defined by $\pi_1\{\phi, h\} = \phi$ and $\pi_2\{\phi, h\} = h$.

We require the following hypotheses on f and F in (1.1):

(3.3) $f\colon [0, \infty) \times H \to H$ such that f is continuous, maps bounded sets into bounded sets, and for each $t \geq 0$, $-f(t, \cdot) + \gamma I$ is accretive in H, where γ is a real constant;

(3.4) $F\colon [0, \infty) \times L^p(-r, 0; H; \mu) \to H$, where $p \geq 1$, $\alpha \stackrel{\text{def}}{=} \gamma + 1/p$, $d\mu(\theta) \stackrel{\text{def}}{=} e^{-p\alpha\theta} d\theta$, F is continuous, F maps bounded sets into bounded sets, and for each $t \geq 0$, $F(t, \cdot)$ is Lipschitz continuous with Lipschitz constant $\leq \beta$, β some positive constant.

Let X be defined as in (3.1) and define $A\colon X \to X$ by

(3.5) $D(A) = \{\{\phi, h\} \in X\colon \phi$ is absolutely continuous on bounded intervals of $(-r, 0]$, $\phi' \in L^p(-r, 0; H; \mu)$, and $\phi(0) = h\}$

$$A\{\phi, h\} = \{-\phi', 0\} \qquad \text{for} \quad \{\phi, h\} \in D(A).$$

Define B: $[0, \infty)\times X\to X$ for all $t\geq 0$, $\{\phi, h\}\in X$ by

$$B(t, \{\phi, h\}) = \{0, -f(t, h)-F(t, \phi)\}. \tag{3.6}$$

The proposition which follows establishes that A and B as defined above satisfy the hypothesis of Theorem 2.1. In the proof we will use the following elementary inequality:

(3.7) $ab^{p-1} \leq w(p)(a^p+b^p)$, $a \geq 0$, $b \geq 0$, $p \geq 1$, where $w(p) = (p-1)^{1/q}/p$ if $p > 1$, $1/p+1/q=1$, and $w(p) = 1$ if $p = 1$.

PROPOSITION 3.1. *Let* (3.3) *and* (3.4) *hold, let X be defined as in* (3.1) *with $\alpha=\gamma+1/p$, let A be defined as in* (3.5), *let B be defined as in* (3.6), *and let $\omega=\alpha+w(p)\beta$.*

Then the hypothesis of Theorem 2.1 *is satisfied.*

PROOF. For $t\geq 0$ define the linear operator $T(t)$ in X as follows:

$$(3.8)\qquad \begin{aligned} &(\pi_1 T(t)\{\phi, h\})(\theta) = \phi(t+\theta) && \text{for a.e. } \theta\in(-r, -t)\\ &(\pi_1 T(t)\{\phi, h\})(\theta) = h && \text{for a.e. } \theta\in(-t, 0)\\ &\pi_2 T(t)\{\phi, h\} = h. \end{aligned}$$

It is easily verified that $T(t)$, $t\geq 0$ is a strongly continuous semigroup of bounded linear operators in X. We will show that $-A$ is the infinitesimal generator of $T(t)$, $t\geq 0$. Let A_1 be the infinitesimal generator of $T(t)$, $t\geq 0$. If λ is real and sufficiently large, then for all $\{\phi, h\}\in X$

$$(\lambda I-A_1)^{-1}\{\phi, h\} = \int_0^\infty e^{-\lambda s}\, T(s)\{\phi, h\}\, ds \tag{3.9}$$

(see [4], Theorem 11, p. 622). From (3.8) we have for a.e. $\theta\in(-r, 0)$,

$$(3.10)\qquad \begin{aligned} &\left(\pi_1\int_0^\infty e^{-\lambda s}T(s)\{\phi, h\}ds\right)(\theta)\\ &\quad = \int_0^{-\theta} e^{-\lambda s}\phi(s+\theta)ds + \int_{-\theta}^\infty e^{-\lambda s}h\, ds\\ &\quad = \int_\theta^0 e^{\lambda(\theta-s)}\phi(s)ds + he^{\lambda\theta}/\lambda, \quad \text{and}\\ &\pi_2\int_0^\infty e^{-\lambda s}T(s)\{\phi, h\}ds = h/\lambda. \end{aligned}$$

Define K: $X\to X$ by $K\{\phi, h\}=\{\lambda\phi-\phi', \lambda h\}$ with $D(K)=D(A)$. Using (3.9) and (3.10) we see that $(\lambda I-A_1)^{-1}K=I$ on $D(A)$ and $K(\lambda I-A_1)^{-1}=I$ on X.

Thus, $K=\lambda I-A_1$, which implies that $A_1=-A$.

From (3.3) and (3.4) it is clear that B is continuous, everywhere defined, and maps bounded sets into bounded sets. It remains to show that $A+B(t,\cdot)+\omega I$ is accretive for each $t\geq 0$. Let $\{\phi_1, h_1\}, \{\phi_2, h_2\}\in D(A)$, let $\phi=\phi_1-\phi_2$, let $h=h_1-h_2$, and let $j\in J(\{\phi, h\})$ as in (3.2). Then, using an integration by parts and (3.7), we obtain

$$
\begin{aligned}
&<(A+B(t,\cdot))\{\phi_1, h_1\}-(A+B(t,\cdot))\{\phi_2, h_2\}, j> \\
&= \|\{\phi, h\}\|^{2-p}\Big(\int_{-r}^{0}(-\phi'(\theta), \phi(\theta))\,|\phi(\theta)|^{p-2}e^{-p\alpha\theta}d\theta \\
&\quad +(-f(t, h_1)-F(t, \phi_1)+f(t, h_2)+F(t, \phi_2), h)\,|h|^{p-2}\Big) \\
&= \|\{\phi, h\}\|^{2-p}\Big(\int_{-r}^{0}-(1/p)d/d\theta|\phi(\theta)|^{p}e^{-p\alpha\theta}d\theta \\
&\quad +(-f(t, h_1)-F(t, \phi_1)+f(t, h_2)+F(t, \phi_2), h)\,|h|^{p-2}\Big) \\
&\geq \|\{\phi, h\}\|^{2-p}\Big(\int_{-r}^{0}(1/p)\,|\phi(\theta)|^{p}(-p\alpha e^{-p\alpha\theta})d\theta \\
&\quad -(1/p)\,|\phi(0)|^{p}-(\gamma|h|^{2}+\beta\|\phi\|_{L^{p}(-r,0;H;\mu)}|h|)\,|h|^{p-2}\Big) \\
&= \|\{\phi, h\}\|^{2-p}(-\alpha\|\phi\|^{p}_{L^{p}(-r,0;H;\mu)}-\alpha|h|^{p}-\beta\|\phi\|_{L^{p}(-r,0;H;\mu)}|h|^{p-1}) \\
&\geq \|\{\phi, h\}\|^{2-p}(-\alpha-w(p)\beta)\,\|\{\phi, h\}\|^{p} \\
&= -\omega\|\{\phi, h\}\|^{2},
\end{aligned}
$$

and the proof is complete.

By virtue of Theorem 2.1 there is a unique solution $u(t)=U(t, s)\{\phi, h\}$ to (2.3). Our goal will be to show that the solution $x(t)$ to (1.1) is given by

$$
\begin{aligned}
(3.11)\qquad x(t) &= \phi(t-s) \quad \text{for a.e. } t\in(s-r, s) \\
x(t) &= \pi_2 U(t, s)\{\phi, h\} \quad \text{for } t\geq s.
\end{aligned}
$$

We must first establish a certain "translation property" of the function $x(t)$.

Proposition 3.2. *For $\{\phi, h\}\in X$, $t\geq s\geq 0$, and $x\colon(s-r, \infty)\to X$ as in (3.11),*

$$
(3.12)\qquad x_t = \pi_1 U(t, s)\{\phi, h\}.
$$

Proof. If we can establish

$$
(3.13)\qquad (\pi_1 U(t, s)\{\phi, h\})(\theta) = \phi(t-s+\theta) \quad \text{for a.e. } \theta\in(-r, s-t)
$$

$$(3.14)\qquad (\pi_1 U(t, s)\{\phi, h\})(\theta) = \pi_2 U(t+\theta, s)\{\phi, h\} \quad \text{for a.e.}\quad \theta \in (s-t, 0),$$

then (3.12) will follow immediately. Let k: $[s, \infty)\to H$ be the continuous function $k(t)=f(t, \pi_2 U(t, s)\{\phi, h\})+F(t, \pi_1 U(t, s)\{\phi, h\})$. Using (3.8) we observe that

$$(3.15)\qquad \int_s^t \pi_1 T(t-\tau)\{0, k(\tau)\}d\tau$$

$$= \int_s^t (\pi_1 T(t-\tau)\{0, k(\tau)\}\chi_{(-r,\tau-t)}+\pi_1 T(t-\tau)\{0, k(\tau)\}\chi_{(\tau-t,0)})d\tau$$

$$= \int_s^t (0\chi_{(-r,\tau-t)}+k(\tau)\chi_{(\tau-t,0)})d\tau$$

$$= \int_s^t k(\tau)\chi_{(\tau-t,0)}d\tau.$$

Then, for a.e. $\theta\in(-r, s-t)$, (3.13) follows from (2.3), (3.8), (3.15), and

$$(\pi_1 U(t, s)\{\phi, h\})(\theta)$$

$$= (\pi_1 T(t-s)\{\phi, h\})(\theta)+\left(\int_s^t \pi_1 T(t-\tau)\{0, k(\tau)\}d\tau\right)(\theta)$$

$$= \phi(t-s+\theta)+\left(\int_s^t k(\tau)\chi_{(\tau-t,0)}d\tau\right)(\theta)$$

$$= \phi(t-s+\theta)+0.$$

Also, for a.e. $\theta\in(s-t, 0)$, (3.14) follows from (2.3), (3.8), (3.15), and

$$(\pi_1 U(t, s)\{\phi, h\})(\theta)$$

$$= (\pi_1 T(t-s)\{\phi, h\})(\theta)+\left(\int_s^t \pi_1 T(t-\tau)\{0, k(\tau)\}d\tau\right)(\theta)$$

$$= h+\left(\int_s^t k(\tau)\chi_{(\tau-t,0)}d\tau\right)(\theta)$$

$$= h+\int_s^{t+\theta} k(\tau)d\tau$$

$$= \pi_2 T(t+\theta-s)\{\phi, h\}+\int_s^{t+\theta} \pi_2 T(t+\theta-\tau)\{0, k(\tau)\}d\tau$$

$$= \pi_2 U(t+\theta, s)\{\phi, h\}.$$

PROPOSITION 3.3. *Let f satisfy* (3.3), *let $p\geq 1$, let $\alpha=\gamma+1/p$, let X be* *defined as in* (3.1), *let* F *satisfy* (3.4), *and let* $\omega=\alpha+w(p)\beta$. *If* $\{\phi, h\}\in X$ *and* $s\geq 0$, *then the unique solution of* (1.1) *is given by* (3.11). *Moreover, if* $\{\phi, h\}$, $\{\hat{\phi}, \hat{h}\}\in X$, *and* $x(t)$, $\hat{x}(t)$ *satisfy* (1.1) *for* $\{\phi, h\}$, $\{\hat{\phi}, \hat{h}\}$, *respectively, then for* $t\geq s$

$$|x(t)-\hat{x}(t)|\leq e^{\omega(t-s)}\|\{\phi, h\}-\{\hat{\phi}, \hat{h}\}\|, \tag{3.16}$$

$$\|x_t-\hat{x}_t\|_{L^p(-r,0;H;\mu)}\leq e^{\omega(t-s)}\|\{\phi, h\}-\{\hat{\phi}, \hat{h}\}\|. \tag{3.17}$$

Finally, if f, F *are of the form* $f(t, h)+F(t, \phi)=\hat{f}(h)+\hat{F}(\phi)+c(t)$, *where* $\hat{f}(0)=0$ *and* $\hat{F}(0)=0$, *then the solution* $x(t)$ *of* (1.1) *satisfies*

$$|x(t)|\leq e^{\omega(t-s)}\|\{\phi, h\}\|+\int_s^t e^{\omega(t-\tau)}|c(\tau)|d\tau, \qquad t\geq s, \tag{3.18}$$

$$\|x_t\|_{L^p(-r,0;H;\mu)}\leq e^{\omega(t-s)}\|\{\phi, h\}\|+\int_s^t e^{\omega(t-\tau)}|c(\tau)|d\tau, \qquad t\geq s. \tag{3.19}$$

PROOF. Using (2.3), (3.8), and (3.12), we have that for $t\geq s$

$$\begin{aligned} x(t) &= \pi_2 U(t, s)\{\phi, h\} \\ &= \pi_2 T(t-s)\{\phi, h\}-\int_s^t \pi_2 T(t-\tau)B(\tau, U(\tau, s)\{\phi, h\})d\tau \\ &= h+\int_s^t (f(\tau, \pi_2 U(\tau, s)\{\phi, h\})+F(\tau, \pi_1 U(\tau, s)\{\phi, h\}))d\tau \\ &= h+\int_s^t (f(\tau, x(\tau))+F(\tau, x_\tau))d\tau. \end{aligned}$$

Since f and F are continuous, $\dot{x}(t)=f(t, x(t))+F(t, x_t)$ for $t\geq s$. Further, $x(s)=h$ and $x_s=\pi_1 U(s, s)\{\phi, h\}=\phi$ by (2.4). The estimates (3.16), (3.17), (3.18), and (3.19) follow directly from (2.7) and (2.8).

REMARK 3.1. The family of evolution operators $U(t, s)$, $t\geq s\geq 0$, describes the evolution of the solution $x(t)$ of (1.1) in the sense that $\{x_t, x(t)\}=U(t, s)\{\phi, h\}$. The estimates (3.16) and (3.17) yield the continuous dependence of the solutions of (1.1) upon the initial data $\{\phi, h\}$.

4. An application to nonlinear Volterra integral equations

As an application of the preceding development we will treat the nonlinear Volterra integral equation (1.3). We make the following assumptions on the nonlinear kernel g: $[0, \infty)\times H\to H$:

(4.1) $g(0, \cdot)$ is continuous, maps bounded sets into bounded sets, and $-g(0, \cdot)$

$+\gamma I$ is accretive in H, where γ is some real constant;

(4.2) g is continuous, g is differentiable with respect to its first variable, g_1 is continuous, and there is a real-valued function b on $[0, \infty)$ such that $|g_1(t, h)-g_1(t, k)| \leq b(t)|h-k|$ for $t \geq 0$, $h, k \in H$;

(4.3) for some $p \geq 1$ and $\alpha \stackrel{\text{def}}{=} \gamma + 1/p$, either $\left(\int_0^\infty (b(s)e^{-\alpha s})^q ds\right)^{1/q} \stackrel{\text{def}}{=} \beta < \infty$ if $p>1$, or $\operatorname{ess\,sup}_{s \geq 0} b(s)e^{-\alpha s} \stackrel{\text{def}}{=} \beta < \infty$ if $p=1$ (where $1/p+1/q=1$).

PROPOSITION 4.1. *Let $g\colon [0, \infty) \times H \to H$ satisfy* (4.1)–(4.3) *and let* $\omega \stackrel{\text{def}}{=} \alpha + w(p)\beta$. *If $g_1(t, 0)=0$ for $t \geq 0$ and $y\colon [0, \infty) \to H$ is continuously differentiable, then there is a unique solution $x(t)$ to equation* (1.3). *If, in addition,* $g(0, 0)=0$, $\omega \leq 0$, *and*

$$\lim_{t \to \infty} \int_0^t e^{\omega(t-\tau)}|\dot{y}(\tau)|d\tau < \infty, \quad \textit{then} \quad x(t) \quad \textit{is bounded on} \quad [0, \infty).$$

PROOF. Define $f(h)=g(0, h)$ for $h \in H$ and define $F(t, \phi)=\dot{y}(t)+\int_{-\infty}^0 g_1(-\tau, \phi(\tau))d\tau$ for $(t, \phi) \in [0, \infty) \times L^p(-\infty, 0; H; \mu)$, where $d\mu(\theta)=e^{-p\alpha\theta}d\theta$. By virtue of (4.1) f satisfies (3.3) and by virtue of (4.2), (4.3), and the continuity of $\dot{y}$, F satisfies (3.4). Thus, we can apply Proposition 3.3 and the unique solution $x(t)$ of (1.1) is given by (3.11). If we take $s=0$ and $\{\phi, h\}=\{0, y(0)\}$, then the hypothesis that $g_1(t, 0)=0$ for $t \geq 0$ yields

$$\begin{aligned} \dot{x}(t) &= f(t, x(t))+F(t, x_t) \\ &= g(0, x(t))+\dot{y}(t)+\int_0^t g_1(t-\tau, x(\tau))d\tau \\ &= d/dt\left(y(t)+\int_0^t g(t-\tau, x(\tau))d\tau\right). \end{aligned}$$

Since $x(0)=h=y(0)$, $x(t)$ is also the unique solution of (1.3). The last statement of the proposition follows from (3.18).

REMARK 4.1. Suppose C is a linear operator from H to H (possibly unbounded) and λ is in the resolvent set of C. Let $\hat{y}\colon [0, \infty) \to H$ and $\hat{g}\colon [0, \infty) \times H \to H$ such that $y \stackrel{\text{def}}{=} (\lambda I - C)^{-1}\hat{y}$ and $g \stackrel{\text{def}}{=} (\lambda I - C)^{-1}\hat{g}$ satisfy the hypothesis of Theorem 4.1. The equation (1.3) becomes

$$\lambda x(t) = Cx(t)+\hat{y}(t)+\int_0^t \hat{g}(t-\tau, x(\tau))d\tau.$$

PROPOSITION 4.2. *Let $g\colon [0, \infty) \times H \to H$ satisfy* (4.1)–(4.3), *let* $\omega \stackrel{\text{def}}{=} \alpha +$

$w(p)\beta$, and let $d\mu(\theta)=e^{-p\alpha\theta}d\theta$. If y: $[0, \infty)\rightarrow H$ is continuously differentiable and for $t\geq 0$

$$(4.4) \qquad \dot{y}(t) = \int_{-\infty}^{0} g_1(t-\tau, \phi(\tau))d\tau \qquad \textit{for some} \quad \phi\in L^p(-\infty, 0; H; \mu),$$

then there is a unique solution to (1.3). If, in addition, $g(0, 0)=0$, $g_1(t, 0)=0$ for $t\geq 0$, and $\omega<0$, then the solution $x(t)$ of (1.3) converges exponentially to 0 as $t\rightarrow\infty$.

PROOF. Define $f(h)=g(0, h)$ for $h\in H$ and define $F(t, \phi)=\int_{-\infty}^{0} g_1(-\tau, \phi(\tau))d\tau$ for $(t, \phi)\in[0, \infty)\times L^p(-\infty, 0; H; \mu)$. As in the proof of Proposition 4.1 f and F satisfy (3.3) and (3.4), respectively. Again we may apply Proposition 3.3 to obtain a unique solution $x(t)$ to (1.1) given by (3.11). If we take $\{\phi, h\} =\{\phi, y(0)\}$, where ϕ is as in (4.4), then the argument of Proposition (4.1) demonstrates that $x(t)$ also satisfies (1.3). The last statement of the proposition follows from (3.16), using the fact that the solution of (1.1) with $\{\phi, h\}=\{0, 0\}$ is $x(t) \equiv 0$, $t>-r$.

REMARK 4.2. *If* (4.1)–(4.4) are satisfied, then the family of evolution operators $U(t, s)$, $t\geq s\geq 0$, is actually a semigroup of operators, $S(t)$, $t\geq 0$, where $U(t, s) =S(t-s)$. This observation follows from the fact that B is autonomous, that is, $B(t, \{\phi, h\})=\{0, -f(h)-F(\phi)\}$ does not depend on t. In this case we have that the evolution of the solutions of (1.3) is given by $\{x_t, x(t)\}=S(t)\{\phi, y(0)\}$, $t\geq 0$.

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