HIROSHIMA MATH. J.
7 (1977), 9–31

On the Asymptotic Behavior of Nonoscillatory Solutions of Differential Equations with Deviating Arguments

V. A. STAIKOS and Ch. G. PHILOS

(Received March 24, 1976)

1. Introduction

Let r_i, $i=0, 1,\dots, n$ be positive continuous real-valued functions on the interval $[t_0, \infty)$. For a real-valued function h on $[T, \infty)$, $T \geqq t_0$, and any $k=0, 1,\dots, n$ we define the *k-th r-derivative* of h by the formula

$$D_r^{(k)}h = r_k(r_{k-1}(r_{k-2}(\cdots(r_1(r_0h)')'\cdots)')')'$$

when obviously we have

$$D_r^{(0)}h = r_0h$$

and

$$D_r^{(k)}h = r_k(D_r^{(k-1)}h)' \qquad (k = 1, 2,\dots, n).$$

Moreover, if $D_r^{(k)}h$ is defined as a continuous function on $[T, \infty)$, then h is said to be k-times *continuously* r-differentiable. We note that in the case where

$$r_0 = r_1 = \cdots = r_n = 1$$

the above notion of r-differentiability specializes to the usual one.

Now, we consider the n-th order $(n>1)$ differential equation with deviating arguments of the form

$$(E_m) \qquad (D_r^{(n)}x)(t) + a(t)F(x[\sigma_1(t)],\dots, x[\sigma_m(t)]) = b(t), \qquad t \geqq t_0,$$

where $r_n=1$. The continuity of the functions involved in the above equation (E_m) as well as sufficient smoothness to guarantee the existence of solutions of (E_m) on an infinite subinterval of $[t_0, \infty)$ will be assumed without mention. In what follows the term "solution" is always used only for such solutions $x(t)$ of (E_m) which are defined for all large t. The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function which is defined on an interval of the form $[T, \infty)$ is called *oscillatory* if it has no last zero, and otherwise it is called *nonoscillatory*.

Furthermore, the conditions (i) and (ii) below are assumed to hold throughout the paper:

(i) *For every* $i=1, 2,\ldots, m$

$$\lim_{t\to\infty} \sigma_i(t) = \infty.$$

(ii) *The function F has the following sign property*

$$(\forall i = 1, 2,\ldots, m) y_i > 0 \Rightarrow F(y_1, y_2,\ldots, y_m) > 0$$

and

$$(\forall i = 1, 2,\ldots, m) y_i < 0 \Rightarrow F(y_1, y_2,\ldots, y_m) < 0$$

2. Preliminaries

Let ρ be a real-valued function which is defined and positive at least on the interval (t_0, ∞) and let R_i, $i=0, 1,\ldots, n$ be the functions defined as follows:

$$R_n = \rho$$

and for every $j=n-1, n-2,\ldots, 0$

$$R_j = r_j R'_{j+1}.$$

The function ρ is said to be *of the type* $r[k]$, $0\leqq k\leqq n-1$, if:

(α) the functions R_j, $j=k+1,\ldots, n$ are defined at least on (t_0, ∞),

(β) R_{k+1} is a constant nonzero function on (t_0, ∞),

(γ) if $k < n-1$, then for every $j = k+2,\ldots, n$

$$\lim_{t\to\infty} R_j(t) \quad \text{exists in} \quad \{0, -\infty, +\infty\}$$

and

(δ) if $k < n-2$, then for every $j = k+2,\ldots, n-1$

$$R_j(t) \neq 0 \qquad \text{for all} \quad t > t_0.$$

We give below some interesting examples of functions of the above type which will be used later.

1. Let i, j be integers with $0\leqq i\leqq j\leqq n-1$ and let

$$\varphi_{ij}(t) = \begin{cases} 1, & \text{if} \quad i = j \\ \displaystyle\int_{t_0}^{t} \frac{1}{r_j(s_j)} \int_{t_0}^{s_j} \frac{1}{r_{j-1}(s_{j-1})} \cdots \int_{t_0}^{s_{i+2}} \frac{1}{r_{i+1}(s_{i+1})} ds_{i+1} \cdots ds_{j-1} ds_j, & \text{if} \quad i < j. \end{cases}$$

For each k, $0 \leqq k \leqq n-1$, the function $\varphi_{k,n-1}$ is of the type $r[k]$, provided that in the case where $k<n-1$,

$$\lim_{t\to\infty} \varphi_{k,j}(t) = \infty \qquad \text{for every} \quad j = k+1,\dots, n-1 .$$

We note that the last requirement is satisfied, if

$$\int^{\infty} \frac{dt}{r_j(t)} = \infty \qquad \text{for every} \quad j = k+1,\dots, n-1.$$

2. Let the integer k, $0 \leqq k \leqq n-1$ be such that in the case where $k<n-1$,

$$\int^{\infty} \frac{dt}{r_j(t)} < \infty \qquad \text{for every} \quad j = k+1,\dots, n-1.$$

Then the function χ_k,

$$\chi_k(t) = \begin{cases} 1, & \text{if} \quad k = n-1 \\ \displaystyle\int_t^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} \frac{1}{r_{n-2}(s_{n-2})} \cdots \int_{s_{k+2}}^{\infty} \frac{1}{r_{k+1}(s_{k+1})} ds_{k+1} \cdots ds_{n-2} ds_{n-1}, & \\ & \text{if} \quad k < n-1 \end{cases}$$

is of the type $r[k]$.

3. Let $r_j \equiv 1$ for $j \neq 0$, $n-1$ and $r_{n-1} \equiv r$. If for some integer k, $0 \leqq k \leqq n-2$,

$$\int^{\infty} \frac{t^{n-2-k}}{r(t)} dt < \infty,$$

then the function ψ_k,

$$\psi_k(t) = \int_t^{\infty} \frac{(s-t_0)^{n-2-k}}{r(s)} ds$$

is of the type $r[k]$.

4. Let $r_j \equiv 1$ for $j \neq 0$, $n-N$ and $r_{n-N} \equiv r$, where N is an integer with $1<N \leqq n-1$. If for some integer k, $0 \leqq k \leqq n-N-1$,

$$\int^{\infty} \frac{t^{n-N-1-k}}{r(t)} dt < \infty,$$

and for $j=1, 2,\dots, N-1$,

$$\omega_{kj}(t) = \int_{t_0}^{t} (t-s)^{j-1} \int_s^{\infty} \frac{(u-t_0)^{n-N-1-k}}{r(u)} du ds,$$

then the function $\omega_{k,N-1}$ is of the type $r[k]$ provided that for every $j=1,\dots, N-1$

$$\lim_{t\to\infty} \omega_{kj}(t) \quad \text{exists in} \quad \{0, \infty\}.$$

LEMMA 1. *Consider the linear differential equation*

$$(L) \qquad z' - \frac{R'(t)}{R(t)} z + \frac{R'(t)}{R(t)} H(t) = 0,$$

where H is continuous on the interval $[T, \infty)$, R is continuously differentiable on $[T, \infty)$ and such that:

$$R(t) \neq 0 \quad \textit{and} \quad R'(t) \neq 0 \quad \textit{for every} \quad t \geqq T$$

and

$$\lim_{t\to\infty} R(t) \quad \textit{belongs in} \quad \{0, -\infty, +\infty\}.$$

If $\lim_{t\to\infty} H(t)$ exists in the extended real line $\mathbf{R}^*$, then so does the $\lim_{t\to\infty} u(t)$, where u is the solution of (L) with $u(T)=0$. Moreover,

$$\lim_{t\to\infty} |H(t)| = \infty \quad \textit{implies} \quad \lim_{t\to\infty} |u(t)| = \infty.$$

PROOF. The solution u is given by the formula

$$u(t) = -R(t)\int_T^t \frac{R'(s)H(s)}{R^2(s)} ds.$$

If $\lim_{t\to\infty} u(t)$ does not exist in $\mathbf{R}^*$, then following the arguments used by Kusano and Onose [3, Lemma 2] we conclude that $\lim_{t\to\infty} [(-R(t)/R'(t))u'(t)+u(t)]$ does not exist in $\mathbf{R}^*$ and so does the $\lim_{t\to\infty} H(t)$, since from (L)

$$H(t) = -\frac{R(t)}{R'(t)} u'(t) + u(t).$$

To complete the proof of the lemma we suppose that $\lim_{t\to\infty} |H(t)| = \infty$ and we consider the following two cases.

Case 1. $\lim_{t\to\infty} R(t) = 0$.

In this case it is easy to see that

$$\int_T^\infty \frac{R'(t)H(t)}{R^2(t)} dt = \pm\infty$$

when we obtain

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} \frac{\left[-\int_T^t \frac{R'(s)H(s)}{R^2(s)} ds\right]'}{[1/R(t)]'} = \lim_{t\to\infty} H(t).$$

Case 2. $\lim_{t\to\infty} R(t) = \pm\infty$.

The improper integral

$$\int_T^{\infty} \frac{R'(t)H(t)}{R^2(t)}\,dt$$

exists in $\mathbf{R}^*$, since the integrand is of constant sign for all large t. So, if this integral is different from zero, obviously

$$\lim_{t\to\infty} |u(t)| = \infty,$$

otherwise,

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} \frac{\left[-\int_T^t \frac{R'(s)H(s)}{R^2(s)}\,ds\right]'}{[1/R(t)]'} = \lim_{t\to\infty} H(t).$$

Lemma 2. *Let ρ be a function of the type $r[k]$, $0 \leqq k \leqq n-1$, and h an n-times continuously r-differentiable function on $[T, \infty)$, $T > t_0$.*

If the improper integral

$$\int_T^{\infty} \rho(t)(D_r^{(n)}h)(t)\,dt$$

exists in $\mathbf{R}^*$, then so does the $\lim_{t\to\infty} (D_r^{(k)}h)(t)$. *Moreover,*

$$\int_T^{\infty} \rho(t)(D_r^{(n)}h)(t)\,dt = \pm\infty \quad \textit{implies} \quad \lim_{t\to\infty} (D_r^{(k)}h)(t) = \pm\infty.$$

Proof. For $k = n-1$ the function ρ must be constant, say $\rho = c_0$, where c_0 is a positive constant. In this case the lemma is obvious, since we have

$$\lim_{t\to\infty} (D_r^{(n-1)}h)(t) = (D_r^{(n-1)}h)(T) + \frac{1}{c_0}\int_T^{\infty} \rho(t)(D_r^{(n)}h)(t)\,dt.$$

Thus, we assume that $k < n-1$ and we consider the functions R_j, $j = n, n-1, \dots, k+1$ assigned to ρ by the definition as a function of the type $r[k]$.

If

$$q_j(t) = \int_T^t R_j(s) \frac{(D_r^{(j)}h)(s)}{r_j(s)}\,ds \qquad (j = k+1, \dots, n),$$

then, integrating by parts, for any $\mu = n, n-1, \dots, k+2$ we obtain

$$\begin{aligned} q_\mu(t) &= \int_T^t R_\mu(s) \frac{(D_r^{(\mu)}h)(s)}{r_\mu(s)}\,ds = \int_T^t R_\mu(s)(D_r^{(\mu-1)}h)'(s)\,ds \\ &= R_\mu(t)(D_r^{(\mu-1)}h)(t) - R_\mu(T)(D_r^{(\mu-1)}h)(T) - \int_T^t R'_\mu(s)(D_r^{(\mu-1)}h)(s)\,ds \end{aligned}$$

$$= \frac{R_\mu(t)}{R_{\mu-1}(t)/r_{\mu-1}(t)}\left[R_{\mu-1}(t)\frac{(D_r^{(\mu-1)}h)(t)}{r_{\mu-1}(t)}\right] - R_\mu(T)(D_r^{(\mu-1)}h)(T) -$$

$$-\int_T^t R_{\mu-1}(s)\frac{(D_r^{(\mu-1)}h)(s)}{r_{\mu-1}(s)}ds$$

$$= \frac{R_\mu(t)}{R'_\mu(t)}q'_{\mu-1}(t) - R_\mu(T)(D_r^{(\mu-1)}h)(T) - q_{\mu-1}(t).$$

Hence $q_{\mu-1}$ is a solution of the differential equation

$$(L_\mu) \qquad z'(t) - \frac{R'_\mu(t)}{R_\mu(t)}z(t) + \frac{R'_\mu(t)}{R_\mu(t)}H_\mu(t) = 0,$$

where $H_\mu(t) = -R_\mu(T)(D_r^{(\mu-1)}h)(T) - q_\mu(t)$. Obviously, this solution satisfies the initial condition $q_{\mu-1}(T)=0$.

Now, if $\lim_{t\to\infty} q_\mu(t)$ exists in $\mathbf{R}^*$, then so does the $\lim_{t\to\infty} H_\mu(t)$. Hence, applying Lemma 1 for the differential equation (L_μ) we obtain that $\lim_{t\to\infty} q_{\mu-1}(t)$ also exists in $\mathbf{R}^*$ and moreover that

$$\lim_{t\to\infty} |q_\mu(t)| = \infty \quad \text{implies} \quad \lim_{t\to\infty} |q_{\mu-1}(t)| = \infty.$$

Next, we assume that the improper integral $\int_T^\infty \rho(t)(D_r^{(n)}h)(t)dt$ exists in $\mathbf{R}^*$. Since

$$\lim_{t\to\infty} q_n(t) = \int_T^\infty \rho(t)(D_r^{(n)}h)(t)dt,$$

we can consecutively apply the above procedure for $\mu=n, n-1,\dots, k+2$ to obtain that $\lim_{t\to\infty} q_{k+1}(t)$ exists in $\mathbf{R}^*$ and it is infinite in the case where

$$\int_T^\infty \rho(t)(D_r^{(n)}h)(t)dt = \pm\infty.$$

Finally, since R_{k+1} is a constant nonzero function c, we have

$$q_{k+1}(t) = c\int_T^t \frac{(D_r^{(k+1)}h)(s)}{r_{k+1}(s)}ds = c[(D_r^{(k)}h)(t) - (D_r^{(k)}h)(T)]$$

and consequently

$$\lim_{t\to\infty}(D_r^{(k)}h)(t) = (D_r^{(k)}h)(T) - (1/c)\lim_{t\to\infty} q_{k+1}(t)$$

which proves the lemma.

3. Main results

In this section we give sufficient conditions which guarantee that

$$\liminf_{t\to\infty}|x(t)| = 0 \quad \text{or} \quad \lim_{t\to\infty} x(t) = 0$$

for the nonoscillatory solutions x of the differential equation (E_m).

THEOREM 1. *Consider the differential equation (E_m) subject to the conditions* (i), (ii) *and*

(iii) $$\limsup_{t\to\infty} r_0(t) < \infty.$$

Moreover, let there exist a function ρ of the type $r[k]$, $0 \leqq k \leqq n-1$, so that the conditions (C_1) and (C_2) below are satisfied:

(C_1) *If $k>0$, then for every $i=1, 2,\dots, k$*

$$\int^{\infty} \frac{dt}{r_i(t)} = \infty.$$

(C_2) *For every μ_1, μ_2 with $0<\mu_1 \leqq 1$ and $\mu_2>0$,*

$$\int^{\infty} \rho(t)\,[\mu_1 a^+(t) - a^-(t) - \mu_2|b(t)|]dt = \infty$$

or

$$\int^{\infty} \rho(t)\,[\mu_1 a^-(t) - a^+(t) - \mu_2|b(t)|]dt = \infty,$$

where $a^+(t)=\max\{a(t), 0\}$ and $a^-(t)=\max\{-a(t), 0\}$.

Then for all bounded solutions x of the differential equation (E_m),

$$\liminf_{t\to\infty}|x(t)| = 0.$$

PROOF. We assume that the conclusion of the theorem is not valid, when there exists a bounded nonoscillatory solution x of (E_m) with $\liminf_{t\to\infty}|x(t)|>0$. Without loss of generality, we suppose that x is a solution on the whole interval $[t_0, \infty)$. Moreover, this solution is supposed positive on $[t_0, \infty)$, since the substitution $u=-x$ transforms (E_m) into an equation of the same form satisfying the assumptions of the theorem.

For the solution x, by taking into account (i), it is easy to see that there exist positive constants M_1 and M_2 such that for some $T>t_0$ and every $t \geqq T$,

$$M_1 \leqq x[\sigma_i(t)] \leqq M_2 \qquad (i = 1, 2,\dots, m).$$

Thus, we have that for every $t \geqq T$,

$$c_1 \leqq F(x[\sigma_1(t)], x[\sigma_2(t)], \dots, x[\sigma_m(t)]) \leqq c_2,$$

where the constants c_1 and c_2 are chosen to be positive, because of (ii).

Now, from equation (E_m) after some manipulations, we obtain that for every $t \geqq T$,

$$\rho(t)(D_r^{(n)}x)(t) \leqq -c_2\rho(t)[\mu_1 a^+(t) - a^-(t) - \mu_2 b(t)]$$

and

$$\rho(t)(D_r^{(n)}x)(t) \geqq c_2\rho(t)[\mu_1 a^-(t) - a^+(t) + \mu_2 b(t)],$$

where $\mu_1 = c_1/c_2$ and $\mu_2 = 1/c_2$. Obviously, $0 < \mu_1 \leqq 1$ and $\mu_2 > 0$, when, by condition (C_2), we have

$$\int_T^\infty \rho(t)(D_r^{(n)}x)(t)dt = \pm\infty.$$

Next, by applying Lemma 2, we get

$$\lim_{t\to\infty}(D_r^{(k)}x)(t) = \pm\infty$$

which implies

$$\lim_{t\to\infty}(D_r^{(0)}x)(t) = \pm\infty.$$

Indeed, we assume that $k > 0$ and we observe that for some $T_1 \geqq T$ we have

$$(D_r^{(k)}x)(t) \geqq 1, \text{ i.e. } (D_r^{(k-1)}x)'(t) \geqq \frac{1}{r_k(t)} \quad \text{for every} \quad t \geqq T_1$$

or

$$(D_r^{(k)}x)(t) \leqq -1, \text{ i.e. } (D_r^{(k-1)}x)'(t) \leqq -\frac{1}{r_k(t)} \quad \text{for every} \quad t \geqq T_1.$$

Hence, by integrating from T_1 to t, it follows respectively

$$(D_r^{(k-1)}x)(t) \geqq (D_r^{(k-1)}x)(T_1) + \int_{T_1}^t \frac{ds}{r_k(s)}$$

or

$$(D_r^{(k-1)}x)(t) \leqq (D_r^{(k-1)}x)(T_1) - \int_{T_1}^t \frac{ds}{r_k(s)}$$

and consequently, by condition (C_1),

$$\lim_{t\to\infty}(D_r^{(k-1)}x)(t) = \pm\infty.$$

Thus, following the same procedure, we finally obtain our assertion.

But this assertion is exactly a contradiction, since, by (iii), the function $D_r^{(0)}x = r_0 x$ is bounded.

THEOREM 2. *Consider the differential equation* (E_m) *subject to the conditions* (i), (ii) *and* (iii). *Moreover, let there exist a function* ρ *of the type* $r[0]$ so that the conditions (C_3) and (C_4) below are satisfied:

$$(C_3) \qquad \int^\infty \rho(t)|b(t)|dt < \infty,$$

$$(C_4) \qquad \int^\infty \rho(t)a^+(t)dt = \infty \quad and \quad \int^\infty \rho(t)a^-(t)dt < \infty$$

or

$$\int^\infty \rho(t)a^+(t)dt < \infty \quad and \quad \int^\infty \rho(t)a^-(t)dt = \infty.$$

Then for all bounded nonoscillatory solutions x *of the differential equation* (E_m),

$$\lim_{t\to\infty} r_0(t)x(t) = 0.$$

Note. Obviously, under the additional condition

$$\text{(iv)} \qquad \liminf_{t\to\infty} r_0(t) > 0$$

we obtain that

$$\lim_{t\to\infty} x(t) = 0$$

for all bounded nonoscillatory solutions x of (E_m).

PROOF. Let x be a bounded nonoscillatory solution of (E_m). As in the proof of Theorem 1, we suppose that x is a solution on the whole interval $[t_0, \infty)$ and positive on $[t_0, \infty)$.

If, by (i), $T > t_0$ is chosen so that for every $t \geqq T$,

$$\sigma_i(t) \geqq t_0 \qquad (i = 1, 2,\dots, m),$$

then from equation (E_m) we obtain

$$\int_T^t \rho(s)(D_r^{(n)}x)(s)ds = \int_T^t \rho(s)b(s)ds - \int_T^t \rho(s)a^+(s)F(x[\sigma_1(s)],\dots, x[\sigma_m(s)])ds$$

$$+\int_T^t \rho(s)a^-(s)F(x[\sigma_1(s)],\dots, x[\sigma_m(s)])ds$$

for every $t \geqq T$. Because of (ii) and the boundedness of x, the function $F\circ(x\circ\sigma_1,\dots, x\circ\sigma_m)$ is positive and bounded on $[T, \infty)$. Thus, by conditions (C_3) and (C_4), the improper integral

$$\int_T^\infty \rho(t)(D_r^{(n)}x)(t)dt$$

exists in $\mathbf{R}^*$ and consequently, by virtue of Lemma 2, the $\lim_{t\to\infty}(D_r^{(0)}x)(t)$ also exists in $\mathbf{R}^*$.

Now, we observe that conditions (C_3) and (C_4) imply condition (C_2), when, by applying Theorem 1 for $k=0$, we obtain

$$\liminf_{t\to\infty} x(t) = 0$$

which, by (iii), gives

$$\liminf_{t\to\infty}(D_r^{(0)}x)(t) = 0$$

and finally $\lim_{t\to\infty}(D_r^{(0)}x)(t)=0$, i.e.

$$\lim_{t\to\infty} r_0(t)x(t) = 0.$$

THEOREM 3. *Consider the differential equation* (E_1),

(E_1) $$(D_r^{(n)}x)(t)+a(t)F(x[\sigma(t)]) = b(t), \quad \textit{where} \quad \sigma \equiv \sigma_1$$

subject to the conditions (i), (ii), (iii) *and*

(v) $$\int^\infty \frac{dt}{r_1(t)} = \infty,$$

(vi) $$\liminf_{t\to\infty} r_1(t) > 0.$$

Moreover, let there exist a function ρ *of the type* $r[1]$ *which satisfies condition* (C_3) *and the following one*:

(C_5) *For some* $\delta>0$,

$$\liminf_{t\to\infty}\int_t^{t+\delta}\rho(s)a^+(s)ds > 0 \quad \textit{and} \quad \int^\infty \rho(t)a^-(t)dt < \infty$$

or

$$\liminf_{t\to\infty}\int_t^{t+\delta}\rho(s)a^-(s)ds > 0 \quad \textit{and} \quad \int^\infty \rho(t)a^+(t)dt < \infty.$$

If the function σ is differentiable with bounded derivative on $[t_0, \infty)$, then for all bounded nonoscillatory solutions x of the differential equation (E_1),

$$\lim_{t\to\infty} r_0(t)x(t) = 0 = \lim_{t\to\infty} r_1(t)(r_0(t)x(t))'.$$

PROOF. Let x be a bounded nonoscillatory solution on $[t_0, \infty)$ of the differential equation (E_1) which is supposed positive on $[t_0, \infty)$.

From equation (E_1) we obtain

$$\text{(1)} \qquad \int_T^t \rho(s)(D_r^{(n)}x)(s)ds = \int_T^t \rho(s)b(s)ds - \int_T^t \rho(s)a^+(s)F(x[\sigma(s)])ds + \int_T^t \rho(s)a^-(s)F(x[\sigma(s)])\,ds$$

for every $t \geqq T > t_0$, where T, as in the proof of Theorem 2, is chosen so that for every $t \geqq T$,

$$\sigma(t) \geqq t_0.$$

Thus, since the function $F\circ(x\circ\sigma)$ is positive and bounded on $[T, \infty)$, by conditions (C_3) and (C_5), the improper integral

$$\int_T^\infty \rho(t)(D_r^{(n)}x)(t)dt$$

exists in $\mathbf{R}^*$ and consequently, by virtue of Lemma 2, the $\lim_{t\to\infty}(D_r^{(1)}x)(t)$ also exists in $\mathbf{R}^*$. Moreover,

$$\text{(2)} \qquad \lim_{t\to\infty}(D_r^{(1)}x)(t) = 0.$$

Indeed, in the opposite case there exists a positive constant M such that for some $T_1 \geqq T$ we have

$$(D_r^{(1)}x)(t) \geqq M, \text{ i.e. } (D_r^{(0)}x)'(t) \geqq \frac{M}{r_1(t)} \qquad \text{for every} \quad t \geqq T_1$$

or

$$(D_r^{(1)}x)(t) \leqq -M, \text{ i.e. } (D_r^{(0)}x)'(t) \leqq -\frac{M}{r_1(t)} \qquad \text{for every} \quad t \geqq T_1.$$

Hence, by integrating from T_1 to t and taking into account condition (v), we conclude that

$$\lim_{t\to\infty}(D_r^{(0)}x)(t) = \pm\infty,$$

which is a contradiction, since, by (iii), the function $D_r^{(0)}x = r_0 x$ is bounded.

Also, it is noteworthy that

$$\lim_{t\to\infty} (D_r^{(0)}x)'(t) = 0, \tag{3}$$

which immediately follows from

$$(D_r^{(0)}x)' = \frac{D_r^{(1)}x}{r_1},$$

by virtue of (2) and condition (vi).

Now, it remains to prove that

$$\lim_{t\to\infty} (D_r^{(0)}x)(t) = 0. \tag{4}$$

To do this, we observe that (v) implies condition (C_1) for $k=1$, as well as (C_3) and (C_5) imply condition (C_2). Thus, applying Theorem 1 for $k=1$, we obtain

$$\liminf_{t\to\infty} x(t) = 0$$

and consequently, because of (iii),

$$\liminf_{t\to\infty} (D_r^{(0)}x)(t) = 0.$$

To complete the proof of (4), we have to verify that

$$\limsup_{t\to\infty} (D_r^{(0)}x)(t) = 0.$$

Indeed, in the opposite case we have

$$\liminf_{t\to\infty} (D_r^{(0)}x)[\sigma(t)] = 0 \quad \text{and} \quad \limsup_{t\to\infty} (D_r^{(0)}x)[\sigma(t)] \geqq K$$

for some positive constant K. Hence, based on the arguments used by Hammett [1] (cf. also Singh [5, 6]), we derive that there exist three sequences (α_ν), (β_ν) and (γ_ν) with $\lim \alpha_\nu = \infty$ and such that for every $\nu = 1, 2, \ldots$

$$T \leqq \alpha_\nu < \gamma_\nu < \beta_\nu \leqq \alpha_{\nu+1},$$

$$(D_r^{(0)}x)[\sigma(\alpha_\nu)] = \tfrac{1}{2}K = (D_r^{(0)}x)[\sigma(\beta_\nu)],$$

$$(D_r^{(0)}x)[\sigma(\gamma_\nu)] > K,$$

$$(D_r^{(0)}x)[\sigma(t)] > \tfrac{1}{2}K \qquad \text{for every} \quad t \in (\alpha_\nu, \beta_\nu).$$

By mean-value theorem, we have

$$\frac{(D_r^{(0)}x)[\sigma(\gamma_\nu)]-(D_r^{(0)}x)[\sigma(\alpha_\nu)]}{\gamma_\nu-\alpha_\nu}=\sigma'(\xi_\nu)(D_r^{(0)}x)'[\sigma(\xi_\nu)]$$

and consequently

$$\frac{K}{2(\beta_\nu-\alpha_\nu)}<\sigma'(\xi_\nu)(D_r^{(0)}x)'[\sigma(\xi_\nu)],$$

where obviously $\lim \xi_\nu=\infty$. Thus, because of (3) and the boundedness of σ' we obtain

$$\lim(\beta_\nu-\alpha_\nu)=\infty. \tag{5}$$

Next, we observe that for every $\nu=1, 2,\dots$

$$(D_r^{(0)}x)[\sigma(t)]\geqq \tfrac{1}{2}K \qquad \text{for every} \quad t\in[\alpha_\nu, \beta_\nu]$$

and consequently, by (iii),

$$x[\sigma(t)]\geqq\frac{K}{2r_0[\sigma(t)]}\geqq\frac{K}{2\sup\limits_{t\geqq t_0} r_0(t)}>0 \qquad \text{for every} \quad t\in[\alpha_\nu, \beta_\nu].$$

We have thus proved that the bounded function $x\circ\sigma$ has a positive lower bound on the set $\cup_{\nu=1}^{\infty}[\alpha_\nu, \beta_\nu]$. Hence, because of (ii), we have

$$F(x[\sigma(t)])\geqq M \qquad \text{for every} \quad t\in\cup_{\nu=1}^{\infty}[\alpha_\nu, \beta_\nu],$$

where the constant M is positive.

Obviously,

$$\int_T^\infty \rho(t)a^{\pm}(t)F(x[\sigma(t)])dt\geqq\sum_{\nu=1}^{\infty}\int_{\alpha_\nu}^{\beta_\nu}\rho(t)a^{\pm}(t)F(x[\sigma(t)])dt$$

$$\geqq M\sum_{\nu=1}^{\infty}\int_{\alpha_\nu}^{\beta_\nu}\rho(t)a^{\pm}(t)dt.$$

But, by virtue of (5) and condition (C_5), we have

$$\sum_{\nu=1}^{\infty}\int_{\alpha_\nu}^{\beta_\nu}\rho(t)a^{+}(t)dt=\infty \quad \text{or} \quad \sum_{\nu=1}^{\infty}\int_{\alpha_\nu}^{\beta_\nu}\rho(t)a^{-}(t)dt=\infty$$

and consequently

$$\int_T^\infty \rho(t)a^{+}(t)F(x[\sigma(t)])dt=\infty \quad \text{or} \quad \int_T^\infty \rho(t)a^{-}(t)F(x[\sigma(t)])dt=\infty$$

Thus, from (1) it follows that

$$\int_T^\infty \rho(t)(D_r^{(n)}x)(t)dt=\pm\infty.$$

Finally, by Lemma 2, we obtain

$$\lim_{t\to\infty}(D_r^{(1)}x)(t) = \pm\infty$$

which contradicts (2).

Finally, we shall restrict our attention to the case where a is nonnegative. For nonnegative a, Kusano and Onose [2, 3] have considered the cases

(vii) $$\int^{\infty}\frac{dt}{r_i(t)} = \infty \qquad (i=1, 2,\dots, n-1)$$

and

(viii) $$\int^{\infty}\frac{dt}{r_i(t)} < \infty \qquad (i=1, 2,\dots, n-1)$$

and they established conditions under which for all nonoscillatory solutions x of the differential equation (E_1) with $r_0\equiv 1$,

$$\lim_{t\to\infty}x(t) = 0.$$

The results of Kusano and Onose [2, Theorem 3; 3, Theorem 3] can easily generalized for the differential equation (E_m). More precisely, we have the following two theorems.

THEOREM 4. *Consider the differential equation (E_m) subject to the conditions* (i), (ii), (iii), (iv), (vii) *and*:

(C_6) *The function a is nonnegative and such that*

$$\int^{\infty}a(t)dt = \infty.$$

(C_7) $$\liminf_{\substack{y_i\to\infty\\ i=1,2,\dots,m}} F(y_1, y_2,\dots, y_m) > 0 \quad \textit{and} \quad \limsup_{\substack{y_i\to-\infty\\ i=1,2,\dots,m}} F(y_1, y_2,\dots, y_m) < 0.$$

(C_8) $$\int^{\infty}\rho(t)|b(t)|dt < \infty,$$

where

$$\rho(t) = \int_{t_0}^{t}\frac{1}{r_{n-1}(s_{n-1})}\int_{t_0}^{s_{n-1}}\frac{1}{r_{n-2}(s_{n-2})}\cdots\int_{t_0}^{s_2}\frac{1}{r_1(s_1)}ds_1\cdots ds_{n-2}ds_{n-1}.$$

Then for all nonoscillatory solutions x of the differential equation (E_m),

$$\lim_{t\to\infty}x(t) = 0.$$

PROOF. Let x be a nonoscillatory solution on $[t_0, \infty)$ of the equation (E_m) which is supposed positive on $[t_0, \infty)$.

From equation (E_m) we obtain

$$\int_T^t \rho(s)(D_r^{(n)}x)(s)ds = \int_T^t \rho(s)b(s)ds - \int_T^t \rho(s)a(s)F(x[\sigma_1(s)],\dots, x[\sigma_m(s)])ds$$

for every $t \geqq T > t_0$, where T is chosen as in the proof of Theorem 2. Thus, because of (ii) and (C_8), the improper integral

$$\int_T^\infty \rho(t)(D_r^{(n)}x)(t)dt$$

exists in $\mathbf{R}^*$ and consequently, by Lemma 2, so does also the $\lim_{t\to\infty}(D_r^{(0)}x)(t)$.

Now, we assume that $\lim_{t\to\infty}(D_r^{(0)}x)(t) = \infty$, when, by (iii),

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{(D_r^{(0)}x)(t)}{r_0(t)} = \infty.$$

So, by (ii) and condition (C_7), the function $F\circ(x\circ\sigma_1,\dots, x\circ\sigma_m)$ has a positive lower bound and consequently, by condition (C_6),

$$\int_T^\infty a(t)F(x[\sigma_1(t)],\dots, x[\sigma_m(t)])dt = \infty.$$

From equation (E_m) we have

$$(D_r^{(n-1)}x)(t) = (D_r^{(n-1)}x)(T) - \int_T^t a(s)F(x[\sigma_1(s)],\dots, x[\sigma_m(s)])ds + \int_T^t b(s)ds.$$

But obviously, because of (C_8),

$$\int^\infty |b(t)|dt < \infty.$$

Hence,

$$\lim_{t\to\infty}(D_r^{(n-1)}x)(t) = -\infty$$

from which, by taking into account (vii) and following the same arguments as in the proof of Theorem 1, we derive the contradiction

$$\lim_{t\to\infty}(D_r^{(0)}x)(t) = -\infty.$$

Thus, we have just proved that $\lim_{t\leftarrow\infty}(D_r^{(0)}x)(t)$ is finite and consequently, by (iv), the solution $x = (D_r^{(0)}x/r_0)$ must be bounded.

Finally, we note that $\rho=\varphi_{0,n-1}$, where the function $\varphi_{0,n-1}$ is given in the first example of section 2. So, ρ is of the type $r[0]$ and hence the theorem follows now immediately from Theorem 2.

THEOREM 5. *Consider the differential equation* (E_m) *subject to the conditions* (i), (ii), (iii), (iv), (viii) *and*:

$$(C_9) \qquad \int^{\infty}|b(t)|\,dt < \infty.$$

(C_{10}) *The function a is nonnegative and such that*

$$\int^{\infty}\rho(t)a(t)dt = \infty,$$

where

$$\rho(t) = \int_t^{\infty}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{\infty}\frac{1}{r_{n-2}(s_{n-1})}\cdots\int_{s_2}^{\infty}\frac{1}{r_1(s_1)}ds_1\cdots ds_{n-2}ds_{n-1}.$$

Then for all nonoscillatory solutions x of the differential equation (E_m),

$$\lim_{t\to\infty}x(t) = 0.$$

PROOF. Let x be again a nonoscillatory solution on $[t_0, \infty)$ of the equation (E_m) which is supposed positive on $[t_0, \infty)$.

From equation (E_m) we have

$$(D_r^{(n-1)}x)(t) = (D_r^{(n-1)}x)(T) - \int_T^t a(s)F(x[\sigma_1(s)],\dots, x[\sigma_m(s)])ds + \int_T^t b(s)ds$$

for every $t \geqq T > t_0$, where T is chosen as in the proof of Theorem 2. Thus, because of (ii) and (C_9), for some constant M_{n-1} and every $t \geqq T$,

$$(D_r^{(n-1)}x)(t) \leqq M_{n-1}.$$

By integrating from T to t and taking into account (viii), we obtain that for every $t \geqq T$,

$$(D_r^{(n-2)}x)(t) \leqq (D_r^{(n-2)}x)(T) + M_{n-1}\int_T^t\frac{ds}{r_{n-1}(s)} \leqq M_{n-2}.$$

Thus, following this procedure we finally obtain that $D_r^{(0)}x = r_0x$ is bounded and because of (iv), the solution x is also bounded.

Furthermore we note that $\rho=\chi_0$, where the function χ_0 is given in the second example of section 2. So, ρ is of the type $r[0]$ and hence the theorem follows from Theorem 2.

4. Applications

We shall now clarify the importance of the main results by applying them in the particular case where for some integer N, $1 \leqq N \leqq n-1$, we have

$$r_j \equiv 1 \quad \text{for} \quad j \neq n-N \quad \text{and} \quad r_{n-N} = r.$$

More precisely, we shall give below some corollaries concerning the differential equation

$$(D_{mN}) \qquad [r(t)x^{(n-N)}(t)]^{(N)} + a(t)F(x[\sigma_1(t)],\dots, x[\sigma_m(t)]) = b(t), \qquad t \geqq t_0.$$

All these corollaries are new, except Corollary 1 which slightly improves a recent result due to Staikos and Sficas [7].

The three first corollaries follow from Theorem 1.

COROLLARY 1. *Consider the differential equation* (D_{mN}) *subject to the conditions* (i), (ii) *and*

$$\int^{\infty} \frac{dt}{r(t)} = \infty.$$

Moreover, let there exist a real number T *and an integer* k, $0 \leqq k \leqq N-1$, *such that for every* μ_1, μ_2 *with* $0 < \mu_1 \leqq 1$ *and* $\mu_2 > 0$,

$$\int^{\infty} (t-T)^k [\mu_1 a^+(t) - a^-(t) - \mu_2 |b(t)|] dt = \infty$$

or

$$\int^{\infty} (t-T)^k [\mu_1 a^-(t) - a^+(t) - \mu_2 |b(t)|] dt = \infty.$$

Then for all bounded solutions x *of the differential equation* (D_{mN}),

$$\liminf_{t \to \infty} |x(t)| = 0.$$

PROOF. Without loss of generality, we assume that $t_0 \geqq T$. It is easy to see then that the function ρ,

$$\rho(t) = (t-T)^k$$

is of the type $r[n-1-k]$. Thus, the corollary follows by applying Theorem 1 with $n-1-k$ in place of k.

COROLLARY 2. *Consider the differential equation* (D_{mN}) *subject to the con-*

ditions (i) *and* (ii). *Moreover, let there exist an integer* k, $0 \leqq k \leqq n-N-1$, *such that*

$$\int^{\infty} \frac{t^{n-N-1-k}}{r(t)} dt = \infty$$

and for every μ_1, μ_2 *with* $0<\mu_1 \leqq 1$ *and* $\mu_2>0$,

$$\int^{\infty} \rho_k(t)\left[\mu_1 a^+(t) - a^-(t) - \mu_2 |b(t)|\right] dt = \infty$$

or

$$\int^{\infty} \rho_k(t)\left[\mu_1 a^-(t) - a^+(t) - \mu_2 |b(t)|\right] dt = \infty,$$

where

$$\rho_k(t) = \int_{t_0}^{t} \frac{(t-s)^{N-1}(s-t_0)^{n-N-1-k}}{r(s)} ds.$$

Then for all bounded solutions x *of the differential equation* (D_{mN}),

$$\liminf_{t\to\infty} |x(t)| = 0.$$

PROOF. By Theorem 1, it suffices to verify that ρ_k is of the type $r[k]$. Indeed, in the case considered here the function $\varphi_{k,n-1}$, given in the first example of Section 2, is of the form $c\rho_k$, where c is a positive constant.

COROLLARY 3. *Consider the differential equation* (D_{mN}) *subject to the conditions* (i) *and* (ii). *Moreover, let there exist an integer* k, $0 \leqq k \leqq n-N-1$, *such that*:

(α) $$\int^{\infty} \frac{t^{n-N-1-k}}{r(t)} dt < \infty;$$

(β) *If* $N>1$, *then for every* $j=1, 2, \dots, N-1$

$$\lim_{t\to\infty} \int_{t_0}^{t} (t-s)^{j-1} \int_{s}^{\infty} \frac{(u-t_0)^{n-N-1-k}}{r(u)} du ds \text{ exists in } \{0, \infty\};$$

and

(γ) *For every* μ_1, μ_2 *with* $0<\mu_1 \leqq 1$ *and* $\mu_2>0$,

$$\int^{\infty} \rho_k(t)\left[\mu_1 a^+(t) - a^-(t) - \mu_2 |b(t)|\right] dt = \infty$$

or

$$\int^{\infty}\rho_k(t)\left[\mu_1 a^-(t)-a^+(t)-\mu_2|b(t)|\right]dt=\infty,$$

where

$$\rho_k(t)=\begin{cases}\displaystyle\int_t^{\infty}\frac{(s-t_0)^{n-2-k}}{r(s)}ds, & \textit{if}\quad N=1\\[2ex] \displaystyle\int_{t_0}^{t}(t-s)^{N-2}\int_s^{\infty}\frac{(u-t_0)^{n-N-1-k}}{r(u)}duds, & \textit{if}\quad N>1.\end{cases}$$

Then for all bounded solutions x of the differential equation (D_{mN}),

$$\liminf_{t\to\infty}|x(t)|=0.$$

PROOF. Obviously,

$$\rho_k=\begin{cases}\psi_k, & \text{if}\quad N=1\\ \omega_{k,N-1}, & \text{if}\quad N>1,\end{cases}$$

where ψ_k and $\omega_{k,N-1}$ are the functions given in the third and fourth example of Section 2. Thus, the function ρ_k is of the type $r[k]$ and consequently the corollary follows immediately from Theorem 1.

Kusano and Onose have recently obtained some results [2, Theorems 1 and 2; 3, Theorems 1 and 2] included in Theorem 2. This is exactly the case $m=1$ and $\rho=\varphi_{0,n-1}$ or $\rho=\chi_0$, where the functions $\varphi_{0,n-1}$ and χ_0 are given in the first and second example of Section 2. From the above mentioned results of Kusano and Onose only those in [2] can be applied for the equation (D_{1N}), when the condition

$$\int^{\infty}\frac{t^{n-N-1}}{r(t)}dt=\infty;$$

must be valid. In the opposite case, however, our Theorem 2 can applied successfully for the differential equation (D_{mN}). More precisely we have the following corollary.

COROLLARY 4. *Consider the differential equation (D_{mN}) subject to the conditions* (i), (ii) *and*:

(α)
$$\int^{\infty}\frac{t^{n-N-1}}{r(t)}dt<\infty;$$

(β) *If $N>1$, then for every $j=1, 2,\dots, N-1$*

$$\lim_{t\to\infty}\int_{t_0}^{t}(t-s)^{j-1}\int_s^{\infty}\frac{(u-t_0)^{n-N-1}}{r(u)}duds\qquad \textit{exists in } \{0, \infty\};$$

and

$$(\gamma) \qquad \int^{\infty} \rho_0(t)|b(t)|dt < \infty$$

and either

$$\int^{\infty} \rho_0(t)a^+(t)dt = \infty \quad \textit{and} \quad \int^{\infty} \rho_0(t)a^-(t)dt < \infty$$

or

$$\int^{\infty} \rho_0(t)a^+(t)dt < \infty \quad \textit{and} \quad \int^{\infty} \rho_0(t)a^-(t)dt = \infty,$$

where

$$\rho_0(t) = \begin{cases} \displaystyle\int_t^{\infty} \frac{(s-t_0)^{n-2}}{r(s)} ds, & \textit{if} \quad N=1 \\ \displaystyle\int_{t_0}^{t} (t-s)^{N-2} \int_s^{\infty} \frac{(u-t_0)^{n-N-1}}{r(u)} du ds, & \textit{if} \quad N>1. \end{cases}$$

Then for all bounded nonoscillatory solutions x of the differential equation (D_{mN}),

$$\lim_{t\to\infty} x(t) = 0.$$

PROOF. It is obvious, since the function

$$\rho_0 = \begin{cases} \psi_0, & \text{if} \quad N=1 \\ \omega_{0,N-1}, & \text{if} \quad N>1 \end{cases}$$

is of the type $r[0]$.

The following corollaries 5, 6 and 7 follow from Theorem 3.

COROLLARY 5. *Consider the differential equation $(D_{1,n-1})$,*

$$(D_{1,n-1}) \quad [r(t)x'(t)]^{(n-1)} + a(t)F(x[\sigma(t)]) = b(t), \quad \textit{where} \quad \sigma \equiv \sigma_1$$

subject to the conditions (i), (ii) *and*:

$$(\alpha) \qquad \int^{\infty} \frac{dt}{r(t)} = \infty \quad \textit{and} \quad \liminf_{t\to\infty} r(t) > 0;$$

$$(\beta) \qquad \int t^{n-2}|b(t)|dt < \infty;$$

and

(γ) *For some* $\delta>0$,

$$\liminf_{t\to\infty}\int_t^{t+\delta} s^{n-2}a^+(s)ds > 0 \quad \textit{and} \quad \int^\infty t^{n-2}a^-(t)dt < \infty$$

or

$$\liminf_{t\to\infty}\int_t^{t+\delta} s^{n-2}a^-(s)ds > 0 \quad \textit{and} \quad \int^\infty t^{n-2}a^+(t)dt < \infty.$$

If the function σ *is differentiable with bounded derivative on* $[t_0, \infty)$, *then for all bounded nonoscillatory solutions* x *of the differential equation* $(D_{1,n-1})$,

$$\lim_{t\to\infty} x(t) = 0 = \lim_{t\to\infty}(r(t)x(t))'.$$

PROOF. It follows immediately from Theorem 3 applied for the function ρ,

$$\rho(t) = (t-t_0)^{n-2},$$

which is of the type $r[1]$.

COROLLARY 6. *Consider the differential equation* (D_{1N}) *with* $N<n-1$,

(D_{1N}) $\qquad [r(t)x^{(n-N)}(t)]^{(N)}+a(t)F(x[\sigma(t)]) = b(t),$ *where* $\sigma = \sigma_1$

subject to the conditions (i), (ii) *and*:

(α)
$$\int^\infty \frac{t^{n-N-2}}{r(t)}dt = \infty,$$

(β)
$$\int^\infty \rho_1(t)|b(t)|dt < \infty$$

and for some $\delta>0$ *either*

$$\liminf_{t\to\infty}\int_t^{t+\delta} \rho_1(s)a^+(s)ds > 0 \quad \textit{and} \quad \int^\infty \rho_1(t)a^-(t)dt < \infty$$

or

$$\liminf_{t\to\infty}\int_t^{t+\delta} \rho_1(s)a^-(s)ds > 0 \quad \textit{and} \quad \int^\infty \rho_1(t)a^+(t)dt < \infty,$$

where

$$\rho_1(t) = \int_{t_0}^t \frac{(t-s)^{N-1}(s-t_0)^{n-N-2}}{r(s)}ds.$$

If the function σ is differentiable with bounded derivative on $[t_0, \infty)$, *then for all bounded nonoscillatory solutions x of the differential equation* (D_{1N}),

$$\lim_{t\to\infty} x(t) = 0 = \lim_{t\to\infty} x'(t).$$

PROOF. As it is remarked in the proof of Corollary 2, the function ρ_1 is of the type $r[1]$, and so the corollary follows from Theorem 3.

COROLLARY 7. *Consider the differential equation* (D_{1N}), $N<n-1$, *subject to the conditions* (i), (ii) *and*:

$$(\alpha) \qquad \int^{\infty} \frac{t^{n-N-2}}{r(t)} dt < \infty;$$

(β) *If $N>1$, then for every $j=1, 2,\ldots, N-1$*

$$\lim_{t\to\infty} \int_{t_0}^{t} (t-s)^{j-1} \int_{s}^{\infty} \frac{(u-t_0)^{n-N-2}}{r(u)} duds \qquad \textit{exists in } \{0, \infty\};$$

and

$$(\gamma) \qquad \int^{\infty} \rho_1(t)|b(t)|dt < \infty$$

and for some $\delta>0$ either

$$\liminf_{t\to\infty} \int_{t}^{t+\delta} \rho_1(s)a^{+}(s)ds > 0 \quad \textit{and} \quad \int^{\infty} \rho_1(t)a^{-}(t)dt < \infty$$

or

$$\liminf_{t\to\infty} \int_{t}^{t+\delta} \rho_1(s)a^{-}(s)ds > 0 \quad \textit{and} \quad \int^{\infty} \rho_1(t)a^{+}(t)dt < \infty,$$

where

$$\rho_1(t) = \begin{cases} \displaystyle\int_{t}^{\infty} \frac{(s-t_0)^{n-3}}{r(s)} ds, & \textit{if} \quad N=1 \\ \displaystyle\int_{t_0}^{t} (t-s)^{N-2} \int_{s}^{\infty} \frac{(u-t_0)^{n-N-2}}{r(u)} duds, & \textit{if} \quad N>1. \end{cases}$$

If the function σ is differentiable with bounded derivative on $[t_0, \infty)$, *then for all bounded nonoscillatory solutions x of the differential equation* (D_{1N}),

$$\lim_{t\to\infty} x(t) = 0 = \lim_{t\to\infty} x'(t).$$

PROOF. It is obvious, since the function

$$\rho_1 = \begin{cases} \psi_1, & \text{if } N = 1 \\ \omega_{1,N-1}, & \text{if } N > 1 \end{cases}$$

is of the type $r[1]$.

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Department of Mathematics,
University of Ioannina,
Ioannina, Greece