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A Note on Generalized Factorial Series Expansions

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§ 1. Introduction

As is well known, actual solutions of linear ordinary differential equations in a neighborhood of an irregular singular point are characterized and asymptotically represented by formal power series solutions which are in general divergent. It may be more desirable to obtain the convergent representation of solutions in a neighborhood of an irregular singular point in computing the exact value of a solution. For this objective, J. Horn [1, 2], W. J. Trjitzinsky [12] and H. L. Turrittin [14] attempted to sum formal power series solutions by means of the so-called Borel exponential summation and obtained convergent generalized factorial series expansions of actual solutions near an irregular singular point in some cases. In particular, H. L. Turrittin attacked this problem of summation for systems of linear differential equations believing that all formal power series solutions could be summed in every case. Although a considerable progress was made, he did not succeed in summing formal power series solutions in all cases in his paper [14]. See also [15].

The method of obtaining convergent generalized factorial series expansions of actual solutions near an irregular singular point is due to the decomposition of an original system of linear differential equations into a sum of a certain number of nonhomogeneous systems of linear differential equations whose solutions are expressed in terms of Laplace integrals.

We here consider a system of linear differential equations of the form

$$\tau\frac{dX}{d\tau} = \tau^{g}A(\tau)X, \tag{1.1}$$

where the matrix $A(\tau)$ is holomorphic at $\tau=\infty$, i.e., it permits a convergent power series expansion

$$A(\tau) = \sum_{m=0}^{\infty} A_m\tau^{-m} \tag{1.2}$$

for sufficiently large values of τ.
If we then put

$$X(\tau) = \sum_{l=0}^{q-1} \{C_l\tau^{-l} + \tau^{-l}Z_l(\xi)\}\tau^{-h} \qquad (\xi = \tau^{q}) \tag{1.3}$$

for a certain number h and an appropriately chosen number q, the functions $Z_l(\xi)(l = 0, 1, \ldots, q-1)$ satisfy nonhomogeneous systems of linear differential equations and are expressed in terms of Laplace integrals. Thereupon we apply the following N. E. Nörlund theorem [9] to those Laplace integrals. See W. Wasow [16; Chapter 11].

THEOREM (N. E. Nörlund). *Let $Z(\xi)$ be a function of the form*

$$Z(\xi) = \frac{Z_0}{\xi} + \frac{\lambda(\xi)}{\xi^2}, \tag{1.4}$$

where $\lambda(\xi)$ is holomorphic and bounded in a right half-plane $\mathrm{Re}\,\xi \geqq \kappa > 0$, *and be expressed in terms of the Laplace integral*

$$Z(\xi) = \int_0^\infty e^{-\xi s}\mathscr{Z}(s)ds, \tag{1.5}$$

where $\mathscr{Z}(s)$ is holomorphic in a half-infinite parallel strip containing the non-negative real s-axis and has the property

$$\lim_{s\to+\infty} e^{-\kappa s}\mathscr{Z}(s) = 0 \tag{1.6}$$

in that strip.

Then there exists a constant $\omega_0 \geqq 1$ such that for $\omega > \omega_0$ the function $Z(\xi)$ can be represented by an absolutely and uniformly convergent factorial series

$$Z(\xi) = \sum_{r=0}^{\infty} \frac{Z_r}{\frac{\xi}{\omega}\left(\frac{\xi}{\omega}+1\right)\cdots\left(\frac{\xi}{\omega}+r\right)} \qquad (\mathrm{Re}\,\xi > \kappa). \tag{1.7}$$

For our purpose, we may only determine the number of decomposition q appropriately and investigate whether or not the functions $Z_l(\xi)(l = 0, 1, \ldots, q-1)$ have the properties stated in the above theorem. H. L. Turrittin, following W. J. Trjitzinsky's work [12], determined the number q in each considered problem corresponding to the rank g and then analyzed the functions $Z_l(\xi)$. But the validity of the determination of the number q seems not to be clear. Therefore in this note we shall show that the number q is determined by the growth of coefficients of formal power series solutions as an inevitable consequence and obtain a small extension of H. L. Turrittin's results.

In the system of linear differential equations (1.1), the integer g is called Poincaré's rank of the singular point at $\tau = \infty$. Since it is easily verified that when $g \leqq 0$, i.e., $\tau = \infty$ is a regular or a regular singular point, there exists a fundamental set of convergent power series solutions in the neighborhood of $\tau = \infty$, we assume that $g > 0$. It should however be noticed that even if $g > 0$, $\tau = \infty$ is not necessarily an irregular singular point. See [7, 8]. We may therefore assume that the system of linear differential equations (1.1) has $\tau=\infty$ as an irregular singular point. We can always seek independent formal power series solutions of linear differential equations near an irregular singular point. The precise form of such formal power series solutions of the system of linear differential equations (1.1) was ingeneously given by M. Hukuhara [3, 4] and independently by H. L. Turrittin [13]. As the starting point of our study, we here quote H. L. Turrittin-M. Hukuhara's results from the paper [14] and write them in the following

THEOREM (H. L. Turrittin-M. Hukuhara). *An appropriate nonsingular transformation*

$$X = P(\tau)Y = \left(\sum_{k=0}^{N} P_k \tau^{-\frac{k}{p}}\right)Y, \tag{1.8}$$

where N and p are suitably chosen positive integers and the change of variables $t=\tau^{1/p}$ reduce the system of linear differential equations (1.1) *to the canonical form*

$$t\frac{dY}{dt} = (\delta_{ij}(\rho_i(t)I_i + J_i) + \sum_{m=1}^{\infty} B_{ij}(m)t^{-m})Y \tag{1.9}$$

$$(i, j = 1, 2, \dots, \mu),$$

where δ_{ij} is the Kronecker delta, I_i $(i=1, 2,\dots, \mu)$ are n_i by n_i identity matrices, J_i $(i=1, 2,\dots, \mu)$ are n_i by n_i matrices of the form

$$J_i = \begin{pmatrix} 0 & & & & \\ \varepsilon_i^{(1)} & \ddots & & \text{\Large 0} & \\ & \ddots & \ddots & & \\ & \varepsilon_i^{(2)} & & \ddots & \\ & \text{0} & \ddots & \varepsilon_i^{(n_i-1)} & 0 \end{pmatrix}, \quad \varepsilon_i^{(k)} = 0 \text{ or } 1 \qquad (k = 1, 2,\dots, n_i - 1) \tag{1.10}$$

and the infinite series of n_i by n_j matrices

$$\sum_{m=1}^{\infty} B_{ij}(m)t^{-m} \qquad (i, j = 1, 2,\dots, \mu) \tag{1.11}$$

are convergent for sufficiently large $|t|$, say $|t| > t_0 > 0$. Moreover, $\rho_i(t)$ $(i=1, 2,\dots, \mu)$ are polynomials of the form

$$\rho_i(t) = \alpha_i^h t^h + \alpha_i^{h-1}t^{h-1} + \dots + \alpha_i^0, \tag{1.12}$$

h being a positive integer, and no two of the polynomials (1.12) *are identical, i.e., if $i \neq j (i, j=1, 2,\dots, \mu)$, for some k $(0 \leqq k \leqq h)$*

$$\alpha_i^k \neq \alpha_j^k \tag{1.13}$$

holds. In particular, if

$$\alpha_i^k = \alpha_j^k \qquad (1 \leqq k \leqq h)$$

for $i \neq j$, *then the difference* $\alpha_i^0 - \alpha_j^0$ *is not only not zero, but it also is not an integer.*

We here remark that in (1.12) h cannot be zero from the assumption that the system of linear differential equations (1.1) has not a regular singularity at $\tau = \infty$.

We can then find independent formal power series solutions of the canonical system of linear differential equations (1.9) of the form

$$Y(t) = (U_{ij}(t))(\delta_{ij} \exp(p_i(t) I_i + J_i \log t)) \qquad (i, j = 1, 2, \ldots, \mu), \tag{1.14}$$

where

$$p_i(t) = \frac{\alpha_i^h}{h} t^h + \frac{\alpha_i^{h-1}}{h-1} t^{h-1} + \cdots + \alpha_i^1 t + \alpha_i^0 \log t \tag{1.15}$$

$$(i = 1, 2, \ldots, \mu)$$

and the matrices of formal power series $U_{ij}(t)(i, j = 1, 2, \ldots, \mu)$ are of the form

$$U_{jj}(t) = \sum_{m=0}^{\infty} U_{jj}(m) t^{-m}, \tag{1.16}$$

$$U_{ij}(t) = t^{-h_{ij}-1} \sum_{m=0}^{\infty} U_{ij}(m) t^{-m} \qquad (i \neq j;\, i, j = 1, 2, \ldots, \mu). \tag{1.17}$$

In the above expression the nonnegative integers h_{ij} ($i \neq j$; $i, j = 1, 2, \ldots, \mu$) are determined as the largest integers such that

$$\begin{aligned} \gamma_{ij}(t) &\equiv \rho_i(t) - \rho_j(t) \\ &= \Gamma_{ij}(h_{ij}) t^{h_{ij}} + \Gamma_{ij}(h_{ij} - 1) t^{h_{ij}-1} + \cdots + \Gamma_{ij}(0), \end{aligned} \tag{1.18}$$

where $\Gamma_{ij}(h_{ij}) \neq 0$. In particular, if $h_{ij} = 0$, $\Gamma_{ij}(0)$ is not an integer. Evidently, the relations

$$\begin{cases} h_{ij} = h_{ji}, \\ \Gamma_{ij}(h_{ij}) = -\Gamma_{ji}(h_{ji}) \qquad (i \neq j;\, i, j = 1, 2, \ldots, \mu) \end{cases} \tag{1.19}$$

hold.

§ 2. Estimates of coefficients of formal power series solutions

In this section we shall analyze the growth order of coefficients $U_{ij}(m)$ of formal power series solutions for sufficiently large values of m. To this end, we first seek recurrence formulas satisfied by the coefficients $U_{ij}(m)$, substituting the formal power series (1.16) and (1.17) into the following systems of linear differential equations respectively:

$$(2.1)\quad tU'_{jj} = U_{jj}J_j - J_jU_{jj} + (\sum_{m=1}^{\infty} B_{jj}(m)t^{-m})U_{jj} + \sum_{\substack{k=1\\k\neq j}}^{\mu}(\sum_{m=1}^{\infty} B_{jk}(m)t^{-m})U_{kj},$$

$$(2.2)\quad tU'_{ij} = \gamma_{ji}(t)U_{ij} + (U_{ij}J_j - J_iU_{ij}) + (\sum_{m=1}^{\infty} B_{ii}(m)t^{-m})U_{ij}$$
$$+ (\sum_{m=1}^{\infty} B_{ij}(m)t^{-m})U_{jj} + \sum_{\substack{k=1\\k\neq i,j}}^{\mu}(\sum_{m=1}^{\infty} B_{ik}(m)t^{-m})U_{kj}.$$

We then obtain

$$(2.3)\quad U_{jj}(m)(-m - J_j) + J_jU_{jj}(m) = \sum_{s=1}^{m} B_{jj}(s)U_{jj}(m-s)$$
$$+ \sum_{\substack{k=1\\k\neq j}}^{\mu}\sum_{s=1}^{m-h_{kj}-1} B_{jk}(s)U_{kj}(m-h_{kj}-1-s)\quad (m\geqq 0),$$

putting $U_{jj}(m)=0$ for $m<0$. We may therefore put

$$(2.4)\qquad U_{jj}(0) = I_j \qquad (j=1, 2,\dots, \mu).$$

If $i\neq j$ and $h_{ij}\neq 0$, we have, considering (1.19),

$$(2.5)\quad \begin{aligned} \Gamma_{ij}(h_{ij})U_{ij}(m+h_{ij}) &= -\sum_{k=1}^{h_{ij}-1}\Gamma_{ij}(k)U_{ij}(m+k) \\ &+ U_{ij}(m)(m+h_{ij}+1-\Gamma_{ij}(0)+J_j) - J_iU_{ij}(m) \\ &+ \sum_{s=1}^{m} B_{ii}(s)U_{ij}(m-s) + \sum_{s=1}^{m+h_{ij}+1} B_{ij}(s)U_{jj}(m+h_{ij}+1-s) \\ &+ \sum_{\substack{k=1\\k\neq i,j}}^{\mu}\sum_{s=1}^{m+h_{ij}-h_{kj}} B_{ik}(s)U_{kj}(m+h_{ij}-h_{kj}-s)\qquad (m\geqq 0), \end{aligned}$$

putting $U_{ij}(m)=0$ for $m<0$ and hence from (2.4)

$$(2.6)\qquad U_{ij}(0) = \frac{B_{ij}(1)}{\Gamma_{ij}(h_{ij})}.$$

In particular, if $i \neq j$ and $h_{ij}=0$, we have

$$(2.7)\qquad U_{ij}(m)(-m-1+\Gamma_{ij}(0)-J_j)+J_iU_{ij}(m)=\sum_{s=1}^{m}B_{ii}(s)U_{ij}(m-s)$$
$$+\sum_{s=1}^{m+1}B_{ij}(s)U_{jj}(m+1-s)+\sum_{\substack{k=1\\k\neq i,j}}^{\mu}\sum_{s=1}^{m-h_{kj}}B_{ik}(s)U_{kj}(m-h_{kj}-s)$$
$$(m \geqq 0),$$

putting $U_{ij}(m)=0$ for $m<0$ again and hence

$$(2.8)\qquad U_{ij}(0)(-1+\Gamma_{ij}(0)-J_j)+J_iU_{ij}(0)=B_{ij}(1).$$

From the above recurrence formulas the coefficients $U_{ij}(m)$ can be uniquely determined in succession.

Now in order to proceed to estimating the coefficient matrices, we rewrite $n_i \times n_j$ elements in the coefficient matrices $U_{ij}(m)(i, j=1, 2,\dots, \mu)$ in the form of $n_i \times n_j$-dimensional column vectors $\mathscr{U}_{ij}(m)(i, j=1, 2,\dots, \mu)$. Then the recurrence formulas (2.3), (2.5) and (2.7), the second subscript j being fixed and hereafter dropped from Γ_{ij}, h_{ij} and the vectors $\mathscr{U}_{ij}(m)$, can be rewritten in the form

$$(2.9)\qquad (-m-\mathscr{B}_{jj}(0))\mathscr{U}_j(m)=\sum_{s=1}^{m}\mathscr{B}_{jj}(s)\mathscr{U}_j(m-s)$$
$$+\sum_{\substack{k=1\\k\neq j}}^{\mu}\sum_{s=1}^{m-h_k-1}\mathscr{B}_{jk}(s)\mathscr{U}_k(m-h_k-1-s),$$

$$(2.10)\qquad \Gamma_i(h_i)\mathscr{U}_i(m+h_i)=-\sum_{k=1}^{h_i-1}\Gamma_i(k)\mathscr{U}_i(m+k)$$
$$+(m+h_{ij}+1-\Gamma_i(0)+\mathscr{B}_{ij}(0))\mathscr{U}_i(m)$$
$$+\sum_{s=1}^{m}\mathscr{B}_{ii}(s)\mathscr{U}_i(m-s)+\sum_{s=1}^{m+h_i+1}\mathscr{B}_{ij}(s)\mathscr{U}_j(m+h_i+1-s)$$
$$+\sum_{\substack{k=1\\k\neq i,j}}^{\mu}\sum_{s=1}^{m+h_i-h_k}\mathscr{B}_{ik}(s)\mathscr{U}_k(m+h_i-h_k-s)\qquad (h_i\neq 0),$$

$$(2.11)\qquad (-m-1+\Gamma_i(0)-\mathscr{B}_{ij}(0))\mathscr{U}_i(m)=\sum_{s=1}^{m}\mathscr{B}_{ii}(s)\mathscr{U}_i(m-s)$$
$$+\sum_{s=1}^{m+1}\mathscr{B}_{ij}(s)\mathscr{U}_j(m+1-s)+\sum_{\substack{k=1\\k\neq i,j}}^{\mu}\sum_{s=1}^{m-h_k}\mathscr{B}_{ik}(s)\mathscr{U}_k(m-h_k-s)$$
$$(h_i=0),$$

where $\mathscr{B}_{ij}(s)(i, j=1, 2,\dots, \mu)$ are (n_i^2) by $(n_i \times n_j)$ matrices corresponding to the matrices $B_{ij}(s)$ and $\mathscr{B}_{ij}(0)(i, j=1, 2,\dots, \mu)$ are $(n_i \times n_j)$ by $(n_i \times n_j)$ matrices constructed from the formulas

$$U_{ij}(m)J_j - J_i U_{ij}(m).$$

Since the infinite series (1.11) are convergent for $|t| > t_0$, the absolute values of all the elements in the coefficient matrices $B_{ij}(m)$ are not larger than a certain constant times η to the power m for sufficiently large $\eta > t_0$. Defining the norm of a matrix $A=(a_{ij}: i, j=1, 2,\dots, n)$ and a vector $X=(x_i: i=1, 2,\dots, n)$ by

$$\|A\| = \max_i \left(\sum_{j=1}^{n} |a_{ij}|\right),$$

$$\|X\| = \max_i (|x_i|)$$

respectively, we have

$$\|\mathscr{B}_{ij}(s)\| \leqq M\eta^s \qquad (i, j = 1, 2,\dots, \mu) \tag{2.12}$$

for a sufficiently large positive constant M.

We can now obtain a system of linear difference equations whose solutions dominate the norms of $U_i(m)(i=1, 2,\dots, \mu)$. Considering, for instance, the inequality derived from (2.9)

$$\begin{aligned}(m - \|\mathscr{B}_{jj}(0)\|)\|\mathscr{U}_j(m)\| \leqq \sum_{s=1}^{m} \|\mathscr{B}_{jj}(s)\|\, \|\mathscr{U}_j(m-s)\| \\ + \sum_{\substack{k=1 \\ k\neq j}}^{\mu} \sum_{s=1}^{m-h_k-1} \|\mathscr{B}_{jk}(s)\|\, \|\mathscr{U}_k(m-h_k-1-s)\|\end{aligned} \tag{2.13}$$

and using the estimates (2.12) and the relations

$$\|\mathscr{B}_{ij}(0)\| = 2 \qquad (i, j = 1, 2,\dots, \mu), \tag{2.14}$$

we obtain a scalar recurrence formula of the form

$$\begin{aligned}(m-2)V_j(m) = M\Big(\sum_{s=1}^{m} \eta^s V_j(m-s) \\ + \sum_{\substack{k=1 \\ k\neq j}}^{\mu} \sum_{s=1}^{m-h_k-1} \eta^s V_k(m-h_k-1-s)\Big).\end{aligned} \tag{2.15}$$

Likewise, from (2.10), (2.11) and (2.12) we obtain scalar recurrence formulas of the same type as stated above. The system of recurrence formulas derived in this manner may be called a majorant system of recurrence formulas corresponding to (2.9), (2.10) and (2.11) in the sense that giving appropriately initial values, the solutions of this system dominate the norms of $\mathscr{U}_i(m)$ $(i=1, 2,\dots, \mu)$, i.e.,

$$\|\mathscr{U}_i(m)\| \leqq V_i(m) \qquad (i = 1, 2,\dots, \mu) \tag{2.16}$$

hold for all values of m.

Moreover, replacing m by $m-1$ in (2.15) and substracting η times the formula derived just now from (2.15), we have the required linear difference equation

$$V_j(m) = \left(\frac{m-3+M}{m-2}\right)\eta V_j(m-1) + \left(\frac{M\eta}{m-2}\right)\sum_{\substack{k=1\\k\neq j}}^{\mu} V_k(m-h_k-2)\,. \tag{2.17}$$

By exactly the same procedure, we have

$$\begin{aligned} V_i(m) = {} & e_i V_i(m-1) + c_i\left(\frac{m-3+M}{m-2}\right)\eta V_j(m-1) \\ & + \sum_{k=2}^{h_i-1} d_i(h_i+1-k)V_i(m-k) \\ & + \left(\frac{m+3}{|\Gamma_i(h_i)|} + d_i(1)\right)V_i(m-h_i) \\ & + \left(c_i - \frac{m+2+|\Gamma_i(0)|}{|\Gamma_i(h_i)|}\right)V_i(m-h_i) \\ & + c_i \sum_{\substack{k=1\\k\neq i,j}}^{\mu} V_k(m-h_k-1) + c_i\left(\frac{M\eta}{m-2}\right)\sum_{\substack{k=1\\k\neq j}}^{\mu} V_k(m-h_k-2) \\ & \qquad\qquad (i\neq j) \end{aligned} \tag{2.18}$$

for $h_i \neq 0$ and

$$\begin{aligned} V_i(m) = {} & \left(\frac{m-2-|\Gamma_i(0)|+M}{m-1-|\Gamma_i(0)|}\right)\eta V_i(m-1) \\ & + \left(\frac{M\eta}{m-1-|\Gamma_i(0)|}\right)\left(\frac{m-3+M}{m-2}\right)\eta V_j(m-1) \\ & + \left(\frac{M\eta}{m-1-|\Gamma_i(0)|}\right)\sum_{\substack{k=1\\k\neq i,j}}^{\mu} V_k(m-h_k-1) \\ & + \left(\frac{M\eta}{m-1-|\Gamma_i(0)|}\right)\left(\frac{M\eta}{m-2}\right)\sum_{\substack{k=1\\k\neq j}}^{\mu} V_k(m-h_k-2) \end{aligned} \tag{2.19}$$

for $h_i=0$. In the expression (2.18) we put for brevity

$$e_i = \frac{|\Gamma_i(h_i-1)| + \eta|\Gamma_i(h_i)|}{|\Gamma_i(h_i)|}, \tag{2.20}$$

$$d_i(k) = \frac{|\Gamma_i(k-1)| - \eta|\Gamma_i(k)|}{|\Gamma_i(h_i)|} \qquad (1 \leqq k \leqq h_i - 1), \tag{2.21}$$

$$c_i = \frac{M\eta}{|\Gamma_i(h_i)|} \qquad (i \neq j;\, i = 1, 2,\dots, \mu). \tag{2.22}$$

We have thus obtained a majorant system of linear difference equations. In order to estimate the growth of the coefficients $\mathcal{U}_{ij}(m)(i=1, 2,\dots, \mu)$ for a fixed j as $m \to \infty$, we may investigate the behaviours of solutions of this system of linear difference equations as $m \to \infty$. To this end, the very important theorem by O. Perron [10, 11] is effectively applicable.

THEOREM (O. Perron). *Consider a single n-th order linear difference equation*

$$g(s+n) + a_1(s)g(s+n-1) + \cdots + a_n(s)g(s) = 0, \tag{2.23}$$

where the coefficients $a_i(s)$ have the properties

$$\lim_{s\to\infty} \frac{a_i(s)}{s^{k_i}} = a \text{ constant} \qquad (i = 1, 2,\dots, n). \tag{2.24}$$

We here construct an upward convex Newton-Puiseux polygon in the usual orthogonal coordinate system such that the points with the coordinates

$$(0, 0), (1, k_1), (2, k_2),\dots, (n, k_n) \tag{2.25}$$

either lie upon the polygon or below it. Assume that the polygon consists of σ sides $S_k(k=1, 2,\dots, \sigma)$ with the directional coefficients r_k respectively and the difference of abscissas of two end points of S_k is equal to n_k, i.e.,

$$n_1 + n_2 + \cdots + n_\sigma = n. \tag{2.26}$$

Then there exists a fundamental set of solutions of the linear difference equation (2. 23) *which is split into σ classes of solutions in such a way that every solution or every linear combination of solutions belonging to the k-th class has the growth property*

$$\overline{\lim_{s\to\infty}} \left| \frac{g(s)}{\Gamma(s+1)^{r_k}} \right|^{\frac{1}{s}} \leqq C_k, \tag{2.27}$$

C_k being a suitable positive constant. And that, the number of solutions included in the k-th class is n_k.

We shall now apply O. Perron's theorem to a system of linear difference equations (2.17)–(2.19). First, consider a special case when all $h_j = h$ $(i = 1, 2, \dots, \mu)$.

If we put

$$(2.28) \qquad G(m) = \begin{pmatrix} V_1(m) \\ V_2(m) \\ \vdots \\ V_\mu(m) \end{pmatrix},$$

the column vector $G(m)$ satisfies an $(h+2)$-th order linear system of difference equations

$$(2.29) \qquad G(m) = \mathscr{A}_1(m)G(m-1) + \mathscr{A}_2(m)G(m-2) + \cdots + \mathscr{A}_{h+2}(m)G(m-h-2),$$

where the μ by μ coefficient matrices $\mathscr{A}_k(m)$ $(k = 1, 2, \dots, h+2)$ are easily obtained from (2.17) and (2.18) although we do not here write their explicit forms. For the purpose of applying O. Perron's theorem, we again construct a majorant linear difference equation corresponding to (2.29)

$$(2.30) \qquad g(m) = a_1(m)g(m-1) + a_2(m)g(m-2) + \cdots + a_{h+2}(m)g(m-h-2),$$

where we set

$$(2.31) \qquad a_k(m) = \|\mathscr{A}_k(m)\| \qquad (k = 1, 2, \dots, h+2).$$

Then the just needed information is to know the growth properties of the coefficients $a_k(m)$ $(k = 1, 2, \dots, h+2)$ as $m \to \infty$. It is easy to see from (2.17) and (2.18) that

$$(2.32) \qquad \lim_{m\to\infty} a_k(m) = \textit{a constant} \qquad (1 \leqq k \leqq h-1),$$

$$\lim_{m\to\infty} m\, a_{h+2}(m) = \textit{a constant}$$

and

$$(2.33) \qquad \lim_{m\to\infty} \frac{a_k(m)}{m} = \textit{a positive constant} \qquad (k = h, h+1).$$

Hence the coordinates of the points corresponding to (2.25) are

$$(0, 0), (1, 0), \dots, (h-1, 0), (h, 1), (h+1, 1), (h+2, -1).$$

Since the Newton-Puiseux polygon then consists of three sides with the directional coefficients $\frac{1}{h}$, 0 and -2 respectively, every solution of the linear difference equation (2.30) at all events has the growth

$$\overline{\lim_{m\to\infty}} \left| \frac{g(m)}{\Gamma(m+1)^{1/h}} \right|^{\frac{1}{m}} \leqq C \tag{2.34}$$

for some constant C. We therefore obtain

$$\overline{\lim_{m\to\infty}} \left| \frac{U_{ij}(m)}{\Gamma(m+1)^{1/h}} \right|^{\frac{1}{m}} \leqq C \qquad (i = 1, 2, \dots, \mu) \tag{2.35}$$

for a fixed j. This result is the same one as derived by H. L. Turrittin in the paper [14].

Next we consider the behaviour of $G(m)$ for sufficiently large values of m in general case. In this case an important role will be played by an integer

$$q = \min_{i \neq j} \{h_i \neq 0\}. \tag{2.36}$$

All the coefficients in the linear difference equations (2.17) and (2.19) tend to finite values as $m \to \infty$. On the other hand, the coefficients in the right hand side of the difference equation (2.18) except for the h_i-th and (h_i+1)-th coefficients are constants or tend to finite values as $m \to \infty$ and the h_i-th and (h_i+1)-th coefficients have the behaviour $mO(1)$ as $m \to \infty$. Therefore the system of linear difference equations (2.29) satisfied by $G(m)$, which is of at most $(h+2)$-th order, but exactly $\hat{h}$ $(=\max_{i \neq j}\{h_i \neq 0\}+2)$-th order, has the first $(q-1)$ coefficient matrices $\mathscr{A}_k(m)\,(k=1, 2, \dots, q-1)$ and the last coefficient $\mathscr{A}_{\hat{h}}(m)$ which tend to finite values as $m \to \infty$. Moreover, we easily have

$$\mathscr{A}_q(m) = mO(1), \tag{2.37}$$

$$\mathscr{A}_k(m) = mO(1) \qquad (q+1 \leqq k \leqq \hat{h}-1). \tag{2.38}$$

The corresponding majorant linear difference equation in the form (2.30) has the same properties as the system (2.29) has, i.e.,

$$\lim_{m\to\infty} a_k(m) = a \ \textit{constant}\ (1 \leqq k \leqq q-1), \quad \lim_{m\to\infty} m a_{\hat{h}}(m) = a \ \textit{constant}, \tag{2.39}$$

$$\lim_{m\to\infty} \frac{a_q(m)}{m} = a \ \textit{positive constant}, \tag{2.40}$$

$$\lim_{m\to\infty} \frac{a_k(m)}{m} = a \ constant \qquad (q+1 \leqq k \leqq \hat{h}-2) \tag{2.41}$$

and in particular

$$\lim_{m\to\infty} \frac{a_{\hat{h}-1}(m)}{m} = a \ positive \ constant. \tag{2.42}$$

The Newton-Puiseux polygon, constructed by the points

$$(0, 0), (1, 0), \ldots, (q-1, 0), (q, 1), \ldots, (k, \delta_k), \ldots, (\hat{h}-1, 1), (\hat{h}, -1), \tag{2.43}$$

where $\delta_k = 0$ or 1, has three sides with the directional coefficients $\frac{1}{q}$, 0 and -2 respectively. Applying O. Perron's theorem to the majorant linear difference equation (2.30), we have

$$\overline{\lim_{m\to\infty}} \left| \frac{g(m)}{\Gamma(m+1)^{1/q}} \right|^{\frac{1}{m}} \leqq C \tag{2.44}$$

for a suitable constant C.

We have thus obtained an important result with regard to the growth of the coefficients of the formal power series solutions (1.14) of the canonical system of linear differential equations (1.9). We describe it in

THEOREM 1. *For a fixed j, define a positive integer q_j by*

$$q_j = \min_{i \neq j} \{h_{ij} \neq 0\}. \tag{2.45}$$

Then we have

$$\overline{\lim_{m\to\infty}} \left| \frac{U_{ij}(m)}{\Gamma(m+1)^{1/q_j}} \right|^{\frac{1}{m}} \leqq C \qquad (i = 1, 2, \ldots, \mu) \tag{2.46}$$

for a suitable constant C.

§3. The decomposition of the original system of linear differential equations

Theorem 1 in the last section not only gives the growth of the coefficients of the formal power series solutions, but also seems to teach us the number of decomposition q.

We should like to attempt to represent solutions of the system of linear differential equations (2.1) and (2.2) by sums of Laplace integrals as follows:

$$U_{jj}(t) = \sum_{l=0}^{q_j-1} \{U_{jj}(l)t^{-l} + t^{-l} \int_0^\infty \exp(-t^{q_j}\xi) W_{jj}^l(\xi) d\xi\}, \tag{3.1}$$

(3.2) $$U_{ij}(t) = t^{-h_{ij}-1}[\sum_{l=0}^{q_j-1} \{U_{ij}(l)t^{-l} + t^{-l}\int_0^{\infty} \exp(-t^{q_j}\xi) W_{ij}^l(\xi)d\xi\}]$$

$$(i \neq j;\ i = 1, 2,\dots, \mu)$$

for each j $(j=1, 2,\dots, \mu)$.

For that purpose, we put, the second subscript j being fixed and dropped from $U_{ij}(t)$, $U_{ij}(l)$, $\gamma_{ij}(t)$, h_{ij} and q_j again,

(3.3) $$U_j(t) = \sum_{l=0}^{q-1} U_j(l)t^{-l} + Z_j(t),$$

(3.4) $$U_i(t) = t^{-h_i-1}\,[\sum_{l=0}^{q-1} U_i(l)t^{-l} + Z_i(t)]$$

$$(i \neq j;\ i = 1, 2,\dots, \mu),$$

where $Z_i(t)\,(i=1, 2,\dots, \mu)$ are moreover expressed in terms of

(3.5) $$Z_i(t) = \sum_{l=0}^{q-1} t^{-l}Z_i^l(t^q) \qquad (i = 1, 2,\dots, \mu).$$

If we first substitute the expressions (3.3) and (3.4) into the system of linear differential equations (2.1) and (2.2), we have a nonhomogeneous system of linear differential equations of the form

(3.6) $$tZ'_j = Z_jJ_j - J_jZ_j + (\sum_{m=1}^{\infty} B_{jj}(m)t^{-m})Z_j$$

$$+ \sum_{\substack{k=1\\ k\neq j}}^{\mu} (\sum_{m=1}^{\infty} B_{jk}(m)t^{-m})t^{-h_k-1}Z_k + \sum_{m=q}^{\infty} C_j(m)t^{-m},$$

(3.7) $$t^{-h_i+1}Z'_i = -\gamma_i(t)t^{-h_i}Z_i + t^{-h_i}\{Z_i(J_i + h_i + 1) - J_iZ_i\}$$

$$+ (\sum_{m=1}^{\infty} B_{ii}(m)t^{-m})t^{-h_i}Z_i + (\sum_{m=1}^{\infty} B_{ij}(m)t^{-m})tZ_j$$

$$+ \sum_{\substack{k=1\\ k\neq i,j}}^{\mu} (\sum_{m=1}^{\infty} B_{ik}(m)t^{-m})t^{-h_k}Z_k + \sum_{m=q}^{\infty} C_i(m)t^{-m}$$

$$(i \neq j;\ i = 1, 2,\dots, \mu),$$

considering the formulas determining the first q coefficients $U_i(l)\,(l=0, 1,\dots, q-1)$. Next we substitute the expressions (3.5) into the above differential equations (3.6) and (3.7) in order to obtain a system of linear differential equations satisfied by the functions $Z_i^l(t^q)\,(i=1, 2,\dots, \mu;\ l=1, 2,\dots, q-1)$.

Here we consider the functions $Z_i^l(t^q)$ as functions of a new variable s, defining $s=t^q$.

Putting

$$\sum_{m=1}^{\infty} B_{jj}(m)t^{-m-\nu} = \sum_{l=0}^{q-1} t^{-l}\Big(\sum_{m=0}^{\infty} B^{\nu}_{jj}(mq+l)s^{-m}\Big), \tag{3.8}$$

$$\sum_{m=1}^{\infty} B_{jk}(m)t^{-m-h_k-\nu} = \sum_{l=0}^{q-1} t^{-l}\Big(\sum_{m=0}^{\infty} B^{\nu}_{jk}(mq+l)s^{-m}\Big) \qquad (\nu=0,1,\dots,q-1), \tag{3.9}$$

where

$$B^{\nu}_{jj}(l) = 0 \qquad (l \leqq \nu), \tag{3.10}$$

$$B^{\nu}_{jk}(l) = 0 \qquad (l \leqq \nu + h_k), \tag{3.11}$$

we immediately obtain q systems of linear differential equations

$$\begin{aligned} qs\frac{dZ^l_j}{ds} - lZ^l_j = {} & Z^l_jJ_j - J_jZ^l_j + \sum_{\nu=0}^{q-1}\Big(\sum_{m=0}^{\infty} B^{\nu}_{jj}(mq+l)s^{-m}\Big)Z^{\nu}_j \\ & + \sum_{\nu=0}^{q-1}\sum_{\substack{k=1\\k\neq i}}^{\mu}\Big(\sum_{m=0}^{\infty} B^{\nu}_{jk}(mq+l-1)s^{-m}\Big)Z^{\nu}_k \\ & + \sum_{m=1}^{\infty} C_j(mq+l)s^{-m} \qquad (l=0,1,\dots,q-1) \end{aligned} \tag{3.12}$$

from (3.6).

Consider now the case when $h_i>0$. In this case we put

$$h_i = p_iq + d_i, \tag{3.13}$$

where p_i and d_i are nonnegative integers such that

$$p_i \geqq 1, \qquad 0 \leqq d_i \leqq q-1. \tag{3.14}$$

From the definition (2.45) of the number q we have

$$h_i = q \qquad (p_i = 1,\ d_i = 0)$$

for some i. We then moreover put

$$\gamma_i(t)t^{-h_i-\nu} = \Gamma_i(h_i)t^{-\nu} + \sum_{l=0}^{q-1} t^{-l}\Big(\sum_{m=0}^{p_i+1} \Gamma^{\nu}_i(mq+l)s^{-m}\Big), \tag{3.15}$$

$$\Big(\sum_{m=1}^{\infty} B_{ij}(m)t^{-m+1}\Big)t^{-\nu} = \sum_{l=0}^{q-1} t^{-l}\Big(\sum_{m=0}^{\infty} B^{\nu}_{ij}(mq+l)s^{-m}\Big) \qquad (\nu=0,1,\dots,q-1), \tag{3.16}$$

where

$$\Gamma_i^\nu(l) = 0 \qquad (l \leqq \nu), \tag{3.17}$$

$$B_{ij}^\nu(l) = 0 \qquad (l \leqq \nu - 1) \tag{3.18}$$

and obtain q systems of linear differential equations

$$\begin{aligned} & s^{-p_i-1}\left(qs\frac{dZ_i^{q+l-d_i}}{ds} - (q + l - d_i)Z_i^{q+l-d_i}\right) \\ &= - \Gamma_i(h_i)Z_i^l - \sum_{\nu=0}^{q-1}\left(\sum_{m=0}^{p_i+1}\Gamma_i^\nu(mq + l)s^{-m}\right)Z_i^\nu \\ &\quad + s^{-p_i-1}(Z_i^{q+l-d_i}(J_j + h_i + 1) - J_iZ_i^{q+l-d_i}) \\ &\quad + \sum_{\nu=0}^{q-1}\sum_{k=1}^{\mu}\left(\sum_{m=0}^{\infty} B_{ik}^\nu(mq + l)s^{-m}\right)Z_k^\nu + \sum_{m=1}^{\infty} C_i(mq+l)s^{-m} \\ &\qquad\qquad (l = 0, 1,\dots, d_i - 1), \end{aligned} \tag{3.19}$$

$$\begin{aligned} & s^{-p_i}\left(qs\frac{dZ_i^{l-d_i}}{ds} - (l - d_i)Z_i^{l-d_i}\right) \\ &= - \Gamma_i(h_i)Z_i^l - \sum_{\nu=0}^{q-1}\left(\sum_{m=0}^{p_i+1}\Gamma_i^\nu(mq+l)s^{-m}\right)Z_i^\nu \\ &\quad + s^{-p_i}(Z_i^{l-d_i}(J_j + h_i + 1) - J_iZ_i^{l-d_i}) \\ &\quad + \sum_{\nu=0}^{q-1}\sum_{k=1}^{\mu}\left(\sum_{m=0}^{\infty} B_{ik}^\nu(mq + l)s^{-m}\right)Z_k^\nu + \sum_{m=1}^{\infty} C_i(mq + l)s^{-m} \\ &\qquad\qquad (l = d_i, d_i + 1,\dots, q - 1) \end{aligned} \tag{3.20}$$

from (3.7).

In the remaining case when $h_i=0$ we similarly obtain q systems of linear differential equations by means of putting $h_i=p_i=d_i=0$ in (3.20) and omitting the second formula in the right hand side of (3.20).

From the construction of the above decomposed system of linear differential equations it is easily seen that the infinite series

$$Z_i^l(s) = \sum_{m=1}^{\infty} U_i(mq + l)s^{-m} \qquad (i = 1, 2,\dots, \mu;\ l = 0, 1,\dots, q - 1) \tag{3.21}$$

are formal power series solutions.

We shall now turn to the representation of solutions of the decomposed systems of linear differential equations by Laplace integrals. To this end, we put

$$Z_i^l(s) = \int_0^\infty e^{-s\xi}W_i^l(\xi)d\xi \qquad (i = 1, 2,\dots, \mu;\ l = 0, 1,\dots, q - 1) \tag{3.22}$$

and investigate the behaviours of the new functions $W_i^l(\xi)(i=1, 2,\dots, \mu; l=0, 1,\dots, q-1)$. By a formal calculation we then obtain an associated system of integral equations of Volterra's type for $W_i^l(\xi)(i=1, 2,\dots, \mu; l=0, 1,\dots, q-1)$. Corresponding to (3.12), we have

$$\begin{aligned}(3.23)\qquad -q\xi W_j^l(\xi) = \int_0^\xi \{ & W_j^l(\zeta)(J_j+l) - J_j W_j^l(\zeta) \\ & + \sum_{\nu=0}^{q-1}\Big(\sum_{m=0}^{\infty} B_{jj}^{\nu}(mq+l)\frac{(\xi-\zeta)^m}{m!}\Big)W_j^{\nu}(\zeta) \\ & + \sum_{\nu=0}^{q-1}\sum_{\substack{k=1\\k\neq j}}^{\mu}\Big(\sum_{m=0}^{\infty} B_{jk}^{\nu}(mq+l-1)\frac{(\xi-\zeta)^m}{m!}\Big)W_k^{\nu}(\zeta)\}d\zeta \\ & + \sum_{m=1}^{\infty}\frac{C_j(mq+l)}{m!}\xi^m \qquad (l=0, 1,\dots, q-1).\end{aligned}$$

Corresponding to (3.19) and (3.20) for $h_i>0$, we have

$$\begin{aligned}(3.24)\qquad \Gamma_i(h_i)W_i^l(\xi) = & -\sum_{\nu=0}^{l-1}\Gamma_i^{\nu}(l)W_i^{\nu}(\xi) + \sum_{\nu=0}^{l} B_{ij}^{\nu}(l)W_j^{\nu}(\xi) \\ & + \sum_{\substack{k=1\\k\neq j}}^{\mu}\sum_{\nu=0}^{l-h_k-1} B_{ik}^{\nu}(l)W_k^{\nu}(\xi) \\ & - \sum_{\nu=0}^{q-1}\sum_{m=1}^{p_i+1}\Gamma_i^{\nu}(mq+l)\int_0^\xi \frac{(\xi-\zeta)^{m-1}}{(m-1)!}W_i^{\nu}(\zeta)d\zeta \\ & + q\int_0^\xi \frac{(\xi-\zeta)^{p_i-1}}{(p_i-1)!}\zeta W_i^{q+l-d_i}(\zeta)d\zeta \\ & + \int_0^\xi \frac{(\xi-\zeta)^{p_i}}{p_i!}(W_i^{q+l-d_i}(\zeta)(J_j+(p_i+1)q+l) - J_i W_i^{q+l-d_i}(\zeta))d\zeta \\ & + \sum_{\nu=0}^{q-1}\sum_{k=1}^{\mu}\sum_{m=1}^{\infty} B_{ik}^{\nu}(mq+l)\int_0^\xi \frac{(\xi-\zeta)^{m-1}}{(m-1)!}W_k^{\nu}(\zeta)d\zeta \\ & + \sum_{m=1}^{\infty}\frac{C_i(mq+l)}{(m-1)!}\xi^{m-1} \qquad (l=0, 1,\dots, d_i-1),\end{aligned}$$

$$\begin{aligned}(3.25)\qquad \Gamma_i(h_i)W_i^l(\xi) = & -\sum_{\nu=0}^{l-1}\Gamma_i^{\nu}(\xi)W_i^{\nu}(l) + \sum_{\nu=0}^{l} B_{ij}^{\nu}(l)W_j^{\nu}(\xi) \\ & + \sum_{\substack{k=1\\k\neq j}}^{\mu}\sum_{\nu=0}^{l-h_k-1} B_{ik}^{\nu}(l)W_k^{\nu}(\xi)\end{aligned}$$

$$- \sum_{\nu=0}^{q-1} \sum_{m=1}^{p_i+1} \Gamma_i^\nu(mq+l) \int_0^\xi \frac{(\xi-\zeta)^{m-1}}{(m-1)!} W_i^\nu(\zeta)d\zeta$$

$$+ q\int_0^\xi \frac{(\xi-\zeta)^{p_i-2}}{(p_i-2)!} W_i^{l-d_i}(\zeta)d\zeta$$

$$+ \int_0^\xi \frac{(\xi-\zeta)^{p_i-1}}{(p_i-1)!} (W_i^{l-d_i}(\zeta)(J_j+p_iq+l) - J_iW_i^{l-d_i}(\zeta))d\zeta$$

$$+ \sum_{\nu=0}^{q-1} \sum_{k=1}^{\mu} \sum_{m=1}^{\infty} B_{ik}^\nu(mq+l) \int_0^\xi \frac{(\xi-\zeta)^{m-1}}{(m-1)!} W_k^\nu(\zeta)d\zeta$$

$$+ \sum_{m=1}^{\infty} \frac{C_i(mq+l)}{(m-1)!} \xi^{m-1} \qquad (l = d_i, d_i+1, \dots, q-1).$$

Lastly, corresponding to (3.20) for $h_i=0$, we have

$$(3.26) \qquad -q\xi W_i^l(\xi) = \int_0^\xi \Big\{ W_i^l(\zeta)(J_j+l+1-\Gamma_i(0)) - J_iW_i^l(\zeta)$$

$$+ \sum_{\nu=0}^{q-1} \sum_{k=1}^{\mu} \Big(\sum_{m=0}^{\infty} B_{ik}^\nu(mq+l) \frac{(\xi-\zeta)^m}{m!} \Big) W_k^\nu(\zeta) \Big\} d\zeta$$

$$+ \sum_{m=1}^{\infty} \frac{C_i(mq+l)}{m!} \xi^m \qquad (l = 0, 1, \dots, q-1).$$

We here remark that (i) the third formulas in the right hand sides of (3.24) and (3.25) have meaning only for $h_k=0$ since if $h_k>0$ i.e., $h_k \geqq q$, $B_{ik}^\nu(l)=0$ $(0 \leqq l \leqq q-1)$ hold for any ν $(0 \leqq \nu \leqq q-1)$ and (ii) the fifth formula in the right hand side of (3.25) must be replaced by

$$(3.27) \qquad q\xi W_i^{l-d_i}(\xi)$$

for $p_i=1$.

We have thus obtained the associated system of linear integral equations (3.23)–(3.26) with entire functions as their coefficients. The problem to be solved is to seek solutions of this system of integral equations with such properties as to guarantee the well-definedness of the integrals (3.22) and the validity of formal calculations done above.

We can immediately prove a local existence of solutions of the associated system of linear integral equations in a neighborhood of the origin $\xi=0$. Let us define series by

$$(3.28) \quad W_i^l(\xi) = \sum_{m=1}^{\infty} \frac{U_i(mq+l)}{(m-1)!} \xi^{m-1} \qquad (i = 1, 2, \dots, \mu; l = 0, 1, \dots, q-1).$$

Considering a formal calculation leading to

$$\int_0^\infty e^{-s\xi}\left(\sum_{m=1}^\infty \frac{U_i(mq+l)}{(m-1)!}\xi^{m-1}\right)d\xi = \sum_{m=1}^\infty U_i(mq+l)s^{-m}$$

and the derivation of the associated system of linear integral equations, it will be clear that the series (3.28) are formal solutions of the system of integral equations. However by means of Theorem 1 and the asymptotic behaviour of the gamma function we can prove

$$\begin{aligned}\overline{\lim_{m\to\infty}}\left|\frac{U_i(mq+l)}{\Gamma(m)}\right|^{\frac{1}{m}} &\leqq \overline{\lim_{m\to\infty}}\left|\frac{U_i(mq+l)}{\Gamma(mq+l+1)^{1/q}}\right|^{\left(\frac{1}{mq+l}\right)\left(\frac{mq+l}{m}\right)} \\ &\quad\times \overline{\lim_{m\to\infty}}\left|\frac{\Gamma(mq+l+1)^{\frac{1}{q}}}{\Gamma(m)}\right|^{\frac{1}{m}} \\ &\leqq C^q q\end{aligned}$$

and therefore the series (3.28) actually eonverge for $|\xi|<C^{-q}q^{-1}$.

We state the above result in the following

THEOREM 2. *The series defined in* (3.28) *are actual solutions of the associated system of linear integral equations* (3.25)–(3.26) *in the neighborhood of the origin* $\xi=0$, *i.e.*, $|\xi|<C^{-q}q^{-1}$.

Now, for our objective, it must moreover be shown that solutions of the system of integral equations exist in the large and have at most the exponential growth of ξ as $\xi\to\infty$ in order to guarantee the validity of Laplace transformations (3.22) and apply N. E. Nörlund's theorem. We can easily show the existence of solutions in the large by the successive approximation method. Such direct proof will be referred to J. Horn's papers and also W. Wasow [16]. But it seems to be difficult to prove that the solutions have such properties as described in N. E. Nörlund's theorem. In the next section we shall study a special case in which $h_i=q$ or $h_i=0$ $(i=1, 2,\dots, \mu)$ and make progress in H. L. Turrittin's work to a small extent.

§4. Generalized factorial series expansions

We consider a case when $h_i=q$ or $h_i=0$ $(i=1, 2,\dots, \mu)$ for a fixed j. Let us denote $n_i\times n_j$ elements of the matrices $W_i^l(\xi)$ $(i=1, 2,\dots, \mu; l=0, 1,\dots, q-1)$ by $n_i\times n_j$-dimensional column vectors $\omega_i^l(\xi)$ $(i=1, 2,\dots, \mu; l=0, 1,\dots, q-1)$. Then the integral equations (3.23) can be rewritten in the form

$$(4.1)\qquad -q\xi\omega_j^l(\xi) = \int_0^\xi \mathscr{E}_j^l\omega_j^l(\zeta)d\zeta + \sum_{\nu=0}^{q-1}\sum_{k=1}^{\mu}\int_0^\xi \mathfrak{B}_{jk}^{l\nu}(\xi-\zeta)\omega_k^\nu(\zeta)d\zeta + \mathscr{C}_j^l(\xi)$$
$$(l=0, 1,\dots, q-1),$$

where $\mathscr{E}^l_j$ $(l=0, 1,\ldots, q-1)$ are n_j^2 by n_j^2 constant matrices relating to the formulas $W^l_j(\xi)(J_j+l)-J_jW^l_j(\xi)$, and the n_j^2 by $n_k\times n_j$ matrices $\mathfrak{B}^{l\nu}_{jk}(\xi)$ $(k=1, 2,\ldots, \mu;\ l,\ \nu=0, 1,\ldots, q-1)$ and the n_j^2-dimensional column vectors $\mathscr{C}^l_j(\xi)$ $(l=0, 1,\ldots, q-1)$ consist of entire functions of ξ. We hereafter use the same notation as above. It is easy to see that the integral equations (3.26) corresponding to $h_i=0$ can be rewritten in the form similar to (4.1).

From (3.25) and (3.27), the integral equations corresponding to $h_i=q$ $(p_i=1, d_i=0)$ can be rewritten in the form

$$
\begin{aligned}
(4.2)\qquad (\Gamma_i(h_i)-q\xi)\omega^l_i(\xi) = &-\sum_{\nu=0}^{l-1}\Gamma^\nu_i(l)\omega^\nu_i(\xi)\\
&+\sum_{\nu=0}^{l}\mathscr{D}^{l\nu}_{ij}\omega^\nu_j(\zeta)+\sum_{\substack{k=1\\k\neq j}}^{\mu}\sum_{\nu=0}^{l-h_k-1}\mathscr{D}^\nu_{ik}(\xi)\omega^\nu_k(\xi)\\
&+\sum_{\nu=0}^{q-1}\int_0^\xi\mathfrak{G}^{l\nu}_i(\xi-\zeta)\omega^\nu_i(\zeta)d\zeta\\
&+\sum_{\nu=0}^{q-1}\sum_{k=1}^{\mu}\int_0^\xi\mathfrak{B}^{l\nu}_{ik}(\xi-\zeta)\omega^\nu_k(\zeta)d\zeta+\mathscr{C}^l_i(\xi)\\
&(l=0, 1,\ldots, q-1),
\end{aligned}
$$

where $\mathscr{D}^{l\nu}_{ik}$ $(l,\ \nu=0, 1,\ldots, q-1)$ are $n_i\times n_j$ by $n_k\times n_j$ constant matrices and $\mathfrak{G}^{l\nu}_i(\xi)$ $(l,\ \nu=0, 1,\ldots, q-1)$ are $n_i\times n_j$ by $n_i\times n_j$ matrices consisting of polynomials in ξ of degree at most 1. For our further analysis, in terms of successive substitutions after multiplying both sides of (4.2) by ξ and then replacing the second and the third formulas in the right hand side of (4.2) by the right members of (4.1) and the like, we obtain the following more suitable expressions of the integral equations:

$$
\begin{aligned}
(4.3)\qquad (\Gamma_i(h_i)-q\xi)^{l+1}\xi\omega^l_i(\xi) = &\int_0^\xi P^l_i(\xi,\ \mathfrak{G}^{\lambda l}_i(\xi-\zeta),\ \mathfrak{B}^{\lambda l}_{ii}(\xi-\zeta))\omega^l_i(\zeta)d\zeta\\
&+\sum_{\nu=0}^{q-1}\sum_{k=1}^{\mu}\int_0^\xi Q^{l\nu}_k(\xi,\ \mathfrak{G}^{\lambda\nu}_i(\xi-\zeta),\ \mathfrak{B}^{\lambda\nu}_{hk}(\xi-\zeta))\omega^\nu_k(\zeta)d\zeta\\
&+R^l_i(\xi,\ \mathscr{C}^\lambda_k(\xi))\qquad (l=0, 1,\ldots, q-1),
\end{aligned}
$$

where $P^l_i(\xi,\ y^\lambda,\ z^\lambda)$ and $R^l_i(\xi,\ z^\lambda)$ $(l=0, 1,\ldots, q-1)$ are linear in y^λ and z^λ $(\lambda=0, 1,\ldots, l)$ with coefficients of polynomials in ξ of degree at most l and $(l+1)$ respectively, and $Q^{l\nu}_k(\xi,\ y^\lambda,\ z^\lambda_h)$ $(k=1, 2,\ldots, \mu;\ l,\ \nu=0, 1,\ldots, q-1)$ are polynomials in y^λ and z^λ_h $(h=1, 2,\ldots, \mu;\ \lambda=0, 1,\ldots, l)$ of the first degree with coefficients of polynomials in ξ of degree at most $(l+1)$.

Let S be a closed domain in the complex ξ-plane which consists of a circular disk about the origin $\xi=0$ with the radius less than $C^{-q}q^{-1}$ and sectors with vertex at the origin and appropriate central angles so that S has a positive distance from the points $\Gamma_i(h_i)/q$ corresponding to $h_i=q$. It is easily verified from (2.12) that all the entire functions $\mathfrak{B}_{hk}^{\lambda\nu}(\xi)$, $\mathscr{C}_h^{\lambda}(\xi)$ $(h, k=1, 2,\dots, \mu;\ \lambda, \nu=0, 1,\dots, q-1)$have the exponential growth as $\xi\to\infty$. We can choose positive numbers K and θ such that

$$(4.5)\qquad \|\mathfrak{B}_{hk}^{\lambda\nu}(\xi)\| \leqq Ke^{\theta|\xi|}$$

and

$$(4.6)\qquad \|\mathscr{C}_h^{\lambda}(\xi)\| \leqq Ke^{\theta|\xi|} \qquad (h, k = 1, 2,\dots, \mu;\ \lambda, \nu = 0, 1,\dots, q-1)$$

hold. Moreover, taking, if necessary, a larger positive number K than that of (4.5) and (4.6), we have for ξ in S

$$(4.7)\qquad \|\mathfrak{G}_i^{\lambda\nu}(\xi-\zeta)\| \leqq Ke^{\theta|\xi-\zeta|},$$

$$(4.8)\qquad \left\|\frac{P_i^l(\xi, \mathfrak{G}_i^{\lambda l}(\xi-\zeta), \mathfrak{B}_{ii}^{\lambda l}(\xi-\zeta))}{(\Gamma_i(h_i)-q\xi)^l}\right\| \leqq Ke^{\theta|\xi-\zeta|},$$

$$(4.9)\qquad \left\|\frac{Q_k^{l\nu}(\xi, \mathfrak{G}_i^{\lambda\nu}(\xi-\zeta), \mathfrak{G}_{hk}^{\lambda\nu}(\xi-\zeta))}{(\Gamma_i(h_i)-q\xi)^{l+1}}\right\| \leqq Ke^{\theta|\xi-\zeta|}$$

and

$$(4.10)\qquad \left\|\frac{R_i^l(\xi, \mathscr{C}_h^{\lambda}(\xi))}{(\Gamma_i(h_i)-q\xi)^{l+1}}\right\| \leqq Ke^{\theta|\xi|} \qquad (l, \lambda, \nu = 0, 1,\dots, q-1).$$

We here remark that since the entire functions $C_j^l(\xi)$ and $C_i^l(\xi)$ corresponding to $h_i=0$ have the factor ξ (see (3.23) and (3.26)), $R_i^l(\xi, \mathscr{C}_h^{\lambda}(\xi))$ can be written as

$$(4.11)\qquad R_i^l(\xi, \mathscr{C}_h^{\lambda}(\xi)) = \xi\hat{R}_i^l(\xi, \hat{\mathscr{C}}_h^{\lambda}(\xi)),$$

where $\hat{R}_i^l(\xi, \hat{z}^{\lambda})$ $(l=0, 1,\dots, q-1)$ are linear in $\hat{z}^{\lambda}$ $(\lambda=0, 1,\dots, l)$ with coefficients of polynomials in ξ of degree at most l and $\hat{\mathscr{C}}_h^{\lambda}(\xi)$ $(h=1, 2,\dots, \mu;\ \lambda=0, 1,\dots, l)$ are entire functions, and

$$(4.12)\qquad \left\|\frac{\hat{R}_i^l(\xi, \mathscr{C}_h^{\lambda}(\xi))}{(\Gamma_i(h_i)-q\xi)^l}\right\| \leqq Ke^{\theta|\xi|} \qquad (l=0, 1,\dots, q-1)$$

hold.

Under these preparations, we can apply the usual successive approximation method in order to prove the existence of solutions of the associated system of integral equations (3.23)–(3.26) and at the same time to estimate the growth order of these solutions as $\xi\to\infty$. Since the proof of the existence is not difficult, we here follow the analysis used first by J. Horn and later by W. J. Trjitzinsky and H. L. Turrittin (see Lemma 1 in [14]) to obtain the behaviours of the solutions $W_i^l(\xi)$ $(i=1, 2,\dots, \mu;\ l=0, 1,\dots, q-1)$.

THEOREM 3. *The matrices* $W_i^l(\xi)$ *satisfying the associated system of integral equations* (3.23)–(3.26) *and defined in the neighborhood of the origin by the series* (3.28) *can be analytically continued along every ray in S and satisfy the inequalities*

$$\|W_i^l(\xi)\| \leqq Me^{\phi|\xi|} \qquad (i=1, 2,\dots, \mu;\ l=0, 1, \dots, q-1) \tag{4.13}$$

for sufficiently large positive numbers M and ϕ.

PROOF. We consider the integral equations (4.1) and the like corresponding to $h_i=0$ and (4.3). From the definition of the solutions $\omega_i^l(\xi)$ in the neighborhood of the origin $\xi=0$, we can choose a constant M such that

$$\|\omega_i^l(\xi)\| < M \quad \text{and} \quad \|\omega_i^l(\xi)\| < Me^{r|\xi|} \tag{4.14}$$

$$(i=1, 2,\dots, \mu;\ l=0, 1,\dots, q-1)$$

hold for all $r\geqq 0$ and $|\xi|\leqq\xi_0$, ξ_0 being a positive number less than $C^{-q}q^{-1}$. Suppose that all $\omega_i^l(\xi)$ $(i=1, 2,\dots, \mu;\ l=0, 1,\dots, q-1)$ do not satisfy the inequalities similar to (4.13). Then, for some pair of i and l, say $i=i'$ and $l=l'$, we have

$$\|\omega_{i'}^{l'}(\xi')\| = Me^{\phi|\xi'|} \tag{4.15}$$

for some point ξ' $(|\xi'|>\xi_0)$ in S, while

$$\|\omega_i^l(\xi)\| < Me^{\phi|\xi|} \qquad (i=1, 2,\dots, \mu;\ l=0, 1, \dots, q-1) \tag{4.16}$$

for all $|\xi|<|\xi'|$, when we analytically continue $\omega_i^l(\xi)$ along the ray connecting the origin with the point ξ'.

For instance, let $\omega_{i'}^{l'}(\xi)$ be represented by the integral equations (4.3). Then from (4.8)–(4.10)

$$\begin{aligned} |\xi'|\,\|\omega_{i'}^{l'}(\xi')\| &= |\xi'|Me^{\phi|\xi'|} \\ &\leqq \int_0^{|\xi'|}\left|\frac{\xi'}{\Gamma_i(h_i)-q\xi'}\right| Ke^{\theta|\xi'-\zeta|}\|\omega_{i'}^{l'}(\zeta)\|d|\zeta| \\ &\quad + \sum_{\nu=0}^{q-1}\sum_{k=1}^{\mu}\int_0^{|\xi'|} Ke^{\theta|\xi'-\zeta|}\|\omega_k^{\nu}(\zeta)\|d|\zeta| + Ke^{\theta|\xi'|} \\ &< MK\int_0^{|\xi'|}\left|\frac{\xi'}{\Gamma_i(h_i)-q\xi'}\right| e^{\theta(|\xi'|-|\zeta|)}e^{\phi|\zeta|}d|\zeta| \\ &\quad + q\mu MK\int_0^{|\xi'|} e^{\theta(|\xi'|-|\zeta|)}e^{\phi|\zeta|}d|\zeta| + Ke^{\theta|\xi'|}, \end{aligned} \tag{4.17}$$

where the path of integration is the ray connecting the origin with the point ξ' in S. If we take $\phi>\theta$ and moreover choose ϕ sufficiently large from the outset, the above inequality leads to

$$1<\frac{K}{|\Gamma_i(h_i)-q\xi'|}\int_0^{|\xi'|}e^{(\theta-\phi)(|\xi'|-|\zeta|)}d|\zeta|$$

$$+\frac{q\mu K}{\xi_0}\int_0^{|\xi'|}e^{(\theta-\phi)(|\xi'|-|\zeta|)}d|\zeta|+\frac{K}{\xi_0}e^{(\theta-\phi)|\xi'|}$$

$$<\left(\frac{1}{|\Gamma_i(h_i)-q\xi'|}+\frac{q\mu}{\xi_0}\right)\frac{K}{\phi-\theta}+\frac{K}{\xi_0}e^{-(\phi-\theta)|\xi'|},$$

which is a contradiction since the right member will be less than 1 for a sufficiently large value of ϕ. Analogously, if $\omega_i^{l'}(\xi)$ is represented by the integral equation (4.1) and the like corresponding to $h_i=0$, the inequality similar to (4.17) leads to a contradiction from (4.5) and (4.6). Thus the proof is completed.

Now we shall return to the study of Laplace integrals

$$Z_i^l(s)=\int_0^\infty e^{-s\xi}W_i^l(\xi)d\xi \qquad (i=1, 2, .., \mu;\ l=0, 1,..., q-1),$$

where the path of integration is some ray $\arg\xi=\Psi$ in S. From Theorem 3, it is easily seen that the above Laplace integrals are absolutely convergent and hence define analytic functions in the half-plane

$$\text{(4.18)} \qquad \operatorname{Re}(se^{i\Psi})>\Phi=\phi+\varepsilon,$$

ε being an arbitrarily small positive number. Formal calculations done until now are legitimate and, as a consequence of it, the analytic functions $Z_i^l(s)$ are solutions of the decomposed system of linear differential equations (3.12), (3.19) and (3.20) in the half-plane (4.18). Moreover we can easily check the conditions in N. E. Nörlund's theorem to obtain the convergent factorial series expansions

$$\text{(4.19)} \qquad Z_i^l(s)=\sum_{r=0}^\infty\frac{A_i^l(r)}{\left(\frac{se^{i\Psi}}{\omega}\right)\left(\frac{se^{i\Psi}}{\omega}+1\right)\cdots\left(\frac{se^{i\Psi}}{\omega}+r\right)}$$

$$(i=1, 2,..., \mu;\ l=0, 1,..., q-1)$$

in the half-plane (4.18), where the positive constant ω is sufficiently large and the constant coefficient matrices $A_i^l(r)$ also depend on Ψ and ω.

We have thus obtained the following main theorem in this note.

THEOREM 4. *If for some j ($j=1, 2, .., \mu$), $h_{ij}=q_j$ or $h_{ij}=0$ ($i=1, 2,..., \mu$),* *where* h_{ij} *and* q_j *are defined in* (1.18) *and* (2.45) *respectively, then there exists an* n *by* n_j *matrix of solutions of the canonical system of linear differential equations* (1.9)

$$(4.20)\qquad Y_j(t)=\begin{pmatrix} U_{1j}(t)\\ U_{2j}(t)\\ \vdots\\ U_{\mu j}(t)\end{pmatrix}\exp\left\{\left(\frac{\alpha_i^h}{h}t^h+\frac{\alpha_j^{h-1}}{h-1}t^{h-1}+\cdots\right.\right.$$
$$\left.\left.+\alpha_j^1 t+\alpha_j^0\log t\right)I_j+J_j\log t\right\},$$

where $U_{ij}(t)$ $(i=1, 2,\dots, \mu)$ *can be represented in terms of convergent generalized factorial series as follows*:

$$(4.21)\qquad U_{jj}(t)=\sum_{l=0}^{q_j-1}\{U_{jj}(l)t^{-l}+t^{-l}Z^l_{jj}(t)\},$$

$$(4.22)\qquad U_{ij}(t)=t^{-h_{ij}-1}\left[\sum_{l=0}^{q_j-1}\{U_{ij}(l)t^{-l}+t^{-l}Z_{ij}(t)\}\right]\qquad(i\neq j),$$

$$(4.23)\qquad Z^l_{ij}(t)=\sum_{r=0}^{\infty}\frac{A^l_{ij}(r)}{\left(\dfrac{t^{q_j}e^{i\Psi}}{\omega}\right)\left(\dfrac{t^{q_j}e^{i\Psi}}{\omega}+1\right)\cdots\left(\dfrac{t^{q_j}e^{i\Psi}}{\omega}+r\right)}$$

$$(i=1, 2,\dots, \mu;\ l=0, 1,\dots, q_j-1)$$

provided that

(i) $\Psi\neq\arg\left(\dfrac{\Gamma_{ij}(q_j)}{q_j}\right)$ $(i\neq j\colon i=1, 2,\dots, \mu)$, $\Gamma_{ij}(q_j)$ *being defined in* (1.18);

(ii) $\mathrm{Re}(e^{i\Psi}t^{q_j})>\Phi$, Φ *being defined in* (4.18);

(iii) *the positive constant* ω *is sufficiently large.*

In particular, if $h_{ij}=q$ *or* $h_{ij}=0$ $(i, j=1, 2,\dots, \mu)$, *then a fundamental set of solutions of the canonical system of linear differential equations* (1.9) *near an irregular singular point can be represented in terms of convergent generalized factorial series of the above form.*

H. L. Turrittin [14] proved the convergent generalized factorial series expansions of solutions only when $h=1$ or $\mu=2$ without restriction on the nature of the characteristic constants, or when $h>1$ and the characteristic constants α_i^h $(i=1, 2, \dots, \mu)$ are mutually distinct. In these cases, however, $h_{ij}=h$ or $h_{ij}=0$ necessarily and hence his results are included in our main theorem.

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