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Minimal Cut Problems on an Infinite Network

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Introduction

Let $N=\{X, Y, K, r\}$ be an infinite network which is connected and locally finite and which has no self-loop (cf. [5]) and let A and B be mutually disjoint nonempty finite subsets of X. Denote by $\boldsymbol{Q}_{A,B}$ the set of all cuts between A and B and put $\boldsymbol{Q}^{(f)}_{A,B}=\{Q\in\boldsymbol{Q}_{A,B};\ Q$ is a finite set$\}$. Let W be a non-negative function on Y and consider the following two min-cut problems on N:

(I) Find $M^*(W;\boldsymbol{Q}_{A,B})=\inf\{\sum_Q W(y);\ Q\in\boldsymbol{Q}_{A,B}\}$.

(II) Find $M^*(W;\boldsymbol{Q}^{(f)}_{A,B})=\inf\{\sum_Q W(y);\ Q\in\boldsymbol{Q}^{(f)}_{A,B}\}$.

Then $M^*(W;\boldsymbol{Q}_{A,B})\leqq M^*(W;\boldsymbol{Q}^{(f)}_{A,B})$ and the equality does not hold in general. In order to give a sufficient condition for the equality, we shall consider the following min-cut problem on N relative to a nonempty finite subset F of X and the ideal boundary ∞ of N:

(III) Find $M^*(W;\boldsymbol{Q}_{F,\infty})=\inf\{\sum_Q W(y);\ Q\in\boldsymbol{Q}_{F,\infty}\}$,

where $\boldsymbol{Q}_{F,\infty}$ is the set of all cuts between F and ∞.

We shall prove that $M^*(W;\boldsymbol{Q}_{A,B})=M^*(W;\boldsymbol{Q}^{(f)}_{A,B})$ holds if $M^*(W;\boldsymbol{Q}_{F,\infty})=0$ for all nonempty finite subsets F of X. By the aid of this result, we shall generalize in §2 the elegant theorem in finite network theory which states that max-flow equals min-cut (cf. [2]) to an infinite network.

Throughout this paper, let p and q be positive numbers such that $1/p+1/q=1$ and $1<p<\infty$. For notation and terminology, we mainly follow [5].

§1. Min-cut problems

Let $L(Y)$ be the set of all real functions on Y. For $w\in L(Y)$, its energy $H_q(w)$ of order q is defined by

$$H_q(w)=\sum_{y\in Y} r(y)|w(y)|^q.$$

For the later use, let us put

$$L^+(Y) = \{w \in L(Y);\ w(y) \geqq 0 \text{ on } Y\},$$

$$L_1^+(Y) = \{w \in L^+(Y);\ \sum_{y \in Y} w(y) < \infty\},$$

$$L_q(Y;\ r) = \{w \in L(Y);\ H_q(w) < \infty\},$$

$$L_q^+(Y;\ r) = \{w \in L^+(Y);\ H_q(w) < \infty\}.$$

Let us recall the definition of cuts. Let A and B be mutually disjoint nonempty finite subsets of X. We say that a subset Q of Y is a cut between A and B if there exist mutually disjoint subsets $Q(A)$ and $Q(B)$ of X such that $A \subset Q(A)$, $B \subset Q(B)$, $X = Q(A) \cup Q(B)$ and the set

$$Q(A) \ominus Q(B) = \{y \in Y;\ e(y) \cap Q(A) \neq \emptyset \text{ and } e(y) \cap Q(B) \neq \emptyset\}$$

is equal to Q, where $e(y) = \{x \in X;\ K(x, y) \neq 0\}$ and $\emptyset$ denotes the empty set.

We say that a subset Q of Y is a cut between a nonempty finite subset F of X and the ideal boundary ∞ of N if there exist mutually disjoint nonempty subsets $Q(F)$ and $Q(\infty)$ of X such that $F \subset Q(F)$, $X = Q(F) \cup Q(\infty)$, $Q(F)$ is a finite set and $Q = Q(F) \ominus Q(\infty)$.

DEFINITION. We say that $W \in L^+(Y)$ satisfies condition (∞) if $M^*(W;\ \boldsymbol{Q}_{F,\infty}) = 0$ for all nonempty finite subsets F of X.

First we shall prove

THEOREM 1. *Let $W \in L^+(Y)$. Then W satisfies condition (∞) if and only if there exists an exhaustion $\{<X_n, Y_n>\}$ of N such that*

$$\text{(E)} \qquad \varliminf_{n \to \infty} \sum_{Z_n} W(y) = 0 \quad \text{with} \quad Z_n = Y_n - Y_{n-1} \qquad (Y_0 = \emptyset).$$

PROOF. First we assume that there exists an exhaustion $\{<X_n, Y_n>\}$ of N such that the relation (E) holds. Let F be a nonempty finite subset of X. For each n such that $F \subset X_{n-1}$, there exists $Q_n \in \boldsymbol{Q}_{F,\infty}$ such that $Q_n \subset Z_n$. It follows that

$$0 \leqq M^*(W;\ \boldsymbol{Q}_{F,\infty}) \leqq \varliminf_{n \to \infty} \sum_{Z_n} W(y) = 0.$$

Next we assume that W satisfies condition (∞). Take a finite subnetwork $<X_1, Y_1>$ of N. Since $M^*(W;\ \boldsymbol{Q}_{X_1,\infty}) = 0$ by our assumption, we can find $Q_1 \in \boldsymbol{Q}_{X_1,\infty}$ such that $\sum_{Q_1} W(y) < 2^{-1}$. We define a subset X'_1 of X as follows: $x \in X'_1$ if and only if there exists a path from X_1 to $\{x\}$ which does not intersect Q_1 (cf. [4] for the definition of a path). Set $X_2 = X_1 \cup X'_1$ and let $Y_2 = \{y \in Y;\ e(y) \subset X_2\}$. Then $Y_1 \subset Y_2$ and $<X_2, Y_2>$ is a finite subnetwork of N. Let $Q_1 = Q_1(X_1) \ominus Q_1(\infty)$ and put $Q'_1(X_1) = X_2$ and $Q'_1(\infty) = X - X_2$. It is clear that $Q'_1 = Q'_1(X_1)$

$\ominus Q'_1(\infty) \in \boldsymbol{Q}_{X_1,\infty}$. We show that $Q'_1 \subset Q_1$. Let $y \in Q'_1$ and $e(y)=\{a, b\}$ with $a \in X_2$ and $b \in X-X_2$. Since $X_2 \subset Q_1(X_1)$, it suffices to show that $b \in Q_1(\infty)$. Suppose that $b \in Q_1(X_1)$. In case $a \in X_1$, we see easily that $b \in X'_1$, which is a contradiction. In case $a \in X'_1$, there exists a path P from X_1 to $\{a\}$ which does not intersect Q_1. Let $\bar{P}$ be the path from X_1 to $\{b\}$ which is generated by P and $\{y\}$. Since $y \notin Q_1$, we see that $\bar{P}$ does not intersect Q_1, and hence $b \in X'_1 \subset X_2$. This is again a contradiction. Therefore $Q'_1 \subset Q_1$. Let us define finite subnetworks $<X_3, Y_3>$ and $<X_4, Y_4>$ of N by

$$(*)\qquad \begin{aligned} Y_i &= \{y \in Y;\, K(x, y) \neq 0 \text{ for some } x \in X_{i-1}\}, \\ X_i &= \{x \in X;\, K(x, y) \neq 0 \text{ for some } y \in Y_i\} \end{aligned}$$

for $i=3, 4$. We have $Q'_1 = Y_3 - Y_2$ and $\sum_{Q'_1} W(y) < 2^{-1}$. By repeating this process, we obtain a sequence $\{<X_n, Y_n>\}$ of finite subnetworks of N such that $Y_{3n-2} \subset Y_{3n-1}$, $Q'_n = Y_{3n} - Y_{3n-1} \in \boldsymbol{Q}_{X_{3n-2},\infty}$, $\sum_{Q'_n} W(y) < 2^{-n}$ and the relation $(*)$ holds for $i=3n, 3n+1$ $(n=1, 2,\dots)$. Consider a subsequence $\{<\bar{X}_n, \bar{Y}_n>\}$ of $\{<X_n, Y_n>\}$ defined by $\bar{X}_{2n-1} = X_{3n-1}$, $\bar{Y}_{2n-1} = Y_{3n-1}$, $\bar{X}_{2n} = X_{3n}$, $\bar{Y}_{2n} = Y_{3n}$ for $n=1, 2,\dots$. It is easily seen that $\{<\bar{X}_n, \bar{Y}_n>\}$ is an exhaustion of N such that $Z_{2n} = \bar{Y}_{2n} - \bar{Y}_{2n-1} = Q'_n$ and $\sum_{Z_{2n}} W(y) < 2^{-n}$. Thus the relation (E) holds.

COROLLARY. *If $W \in L_1^+(Y)$, then W satisfies condition (∞).*

THEOREM 2. *Assume that $W(y)>0$ on Y. Then W satisfies condition (∞) if and only if there exists a nonempty finite subset F of X such that $M^*(W; \boldsymbol{Q}_{F,\infty}) = 0$.*

PROOF. It suffices to show the "if" part. Let F' be a nonempty finite subset of X. Take a finite subnetwork $<X', Y'>$ of N such that $F \cup F' \subset X'$ and let $\varepsilon_0 = \min\{W(y);\, y \in Y'\}$. For any ε such that $0<\varepsilon<\varepsilon_0$, there exists $Q \in \boldsymbol{Q}_{F,\infty}$ such that $\sum_Q W(y) < \varepsilon$. Let $Q = Q(F) \ominus Q(\infty)$. Then $F' \subset X' \subset Q(F)$, so that $Q \in \boldsymbol{Q}_{F',\infty}$ and $M^*(W; \boldsymbol{Q}_{F',\infty}) < \varepsilon$. Thus $M^*(W; \boldsymbol{Q}_{F',\infty}) = 0$ and W satisfies condition (∞).

We have

LEMMA 1. *Assume that N is of parabolic type of order p. If $W \in L_q^+(Y; r)$, then W satisfies condition (∞).*

PROOF. Define $V \in L(Y)$ by $V(y) = W(y)^{1/(p-1)}$. Then $H_p(V) = H_q(W) < \infty$. On account of Corollary 2 of Theorem 4.1 in [5], we have $M^*(W; \boldsymbol{Q}_{F,\infty}) = M^*(V^{p-1}; \boldsymbol{Q}_{F,\infty}) = 0$ for all nonempty finite subsets F of X.

Now we shall prove

THEOREM 3. *If* $W \in L^+(Y)$ *satisfies condition* (∞), *then* $M^*(W; \boldsymbol{Q}_{A,B}) = M^*(W; \boldsymbol{Q}_{A,B}^{(f)})$.

PROOF. We have only to show that $M^*(W; \boldsymbol{Q}_{A,B}^{(f)}) \leqq M^*(W; \boldsymbol{Q}_{A,B})$. There exists $\hat{Q} \in \boldsymbol{Q}_{A,B}$ such that $M^*(W; \boldsymbol{Q}_{A,B}) = \sum_{\hat{Q}} W(y)$ by Lemma 10 in [4]. For any $\varepsilon > 0$, there exists $Q \in \boldsymbol{Q}_{A,\infty}$ such that $\sum_Q W(y) < \varepsilon$, since $M^*(W; \boldsymbol{Q}_{A,\infty}) = 0$. Let $\hat{Q} = \hat{Q}(A) \ominus \hat{Q}(B)$ and $Q = Q(A) \ominus Q(\infty)$ and define $\bar{Q}(A)$ and $\bar{Q}(B)$ by

$$\bar{Q}(A) = \hat{Q}(A) \cap Q(A) \quad \text{and} \quad \bar{Q}(B) = \hat{Q}(B) \cup Q(\infty).$$

Then $\bar{Q} = \bar{Q}(A) \ominus \bar{Q}(B) \in \boldsymbol{Q}_{A,B}^{(f)}$ and $\bar{Q} \subset \hat{Q} \cup Q$. It follows that

$$0 \leqq M^*(W; \boldsymbol{Q}_{A,B}^{(f)}) - M^*(W; \boldsymbol{Q}_{A,B}) \leqq \sum_{\bar{Q}} W(y) - \sum_{\hat{Q}} W(y)$$

$$\leqq \sum_Q W(y) < \varepsilon.$$

By the arbitrariness of ε, we conclude that $M^*(W; \boldsymbol{Q}_{A,B}) = M^*(W; \boldsymbol{Q}_{A,B}^{(f)})$.

By this theorem and the corollary of Theorem 1, we obtain

COROLLARY 1. *If* $W \in L_1^+(Y)$, *then* $M^*(W; \boldsymbol{Q}_{A,B}) = M^*(W; \boldsymbol{Q}_{A,B}^{(f)})$.

By this theorem and Lemma 1, we obtain

COROLLARY 2. *Assume that* N *is of parabolic type of order* p. *If* $W \in L_q^+(Y; r)$, *then* $M^*(W; \boldsymbol{Q}_{A,B}) = M^*(W; \boldsymbol{Q}_{A,B}^{(f)})$.

REMARK 1. Condition (∞) is not necessary for our equality. If $W \in L^+(Y)$ and if $\sum_Q W(y) = \infty$ for every $Q \in \boldsymbol{Q}_{A,B}$ such that $Q \notin \boldsymbol{Q}_{A,B}^{(f)}$, then $M^*(W; \boldsymbol{Q}_{A,B}) = M^*(W; \boldsymbol{Q}_{A,B}^{(f)})$. In particular, if $\inf\{W(y); y \in Y\} > 0$, then $M^*(W; \boldsymbol{Q}_{A,B}) = M^*(W; \boldsymbol{Q}_{A,B}^{(f)})$.

§2. Max-flow problems

We say that $w \in L(Y)$ is a flow from A to B of strength $I(w)$ if

$$\sum_{y \in Y} K(x, y) w(y) = 0 \qquad \text{for all} \quad x \in X - A - B,$$

$$I(w) = -\sum_{x \in A} \sum_{y \in Y} K(x, y) w(y) = \sum_{x \in B} \sum_{y \in Y} K(x, y) w(y).$$

Denote by $F(A, B)$ the set of all flows from A to B and by $G(A, B)$ the set of all $w \in F(A, B)$ such that $\{y \in Y; w(y) \neq 0\}$ is a finite subset of Y. Let $F_q(A, B)$ be the closure of $G(A, B)$ in $L_q(Y; r)$. For any $w \in F_q(A, B)$, there exists a sequence $\{w_n\}$ in $G(A, B)$ such that $H_q(w - w_n) \to 0$ as $n \to \infty$. It follows that $w \in F(A, B)$ and $I(w_n) \to I(w)$ as $n \to \infty$.

REMARK 2. The spaces of flows on an infinite network have been analyzed by H. Flanders [1] and A. H. Zemanian [6].

Let $W \in L^+(Y)$ and consider the following max-flow problem:

(IV) Find $M(W; F(A, B)) = \sup\{I(w);\ w \in F(A, B) \text{ and } |w| \leqq W \text{ on } Y\}$.

We define $M(W; G(A, B))$ and $M(W; F_q(A, B))$ similarly. Then $M(W; G(A, B)) \leqq M(W; F_q(A, B)) \leqq M(W; F(A, B))$.

We proved in [4]

THEOREM 4. $M(W; G(A, B)) = M^*(W; \boldsymbol{Q}_{A,B})$.

It was also shown in [4] that $M(W; F_2(A, B)) = M^*(W; \boldsymbol{Q}_{A,B}^{(f)})$ does not hold in general.

We shall prove the following duality theorem.

THEOREM 5. *If $W \in L^+(Y)$ satisfies condition (∞), then $M(W; F(A, B)) = M^*(W; \boldsymbol{Q}_{A,B})$.*

PROOF. By Theorems 3 and 4, it suffices to prove $M(W; F(A, B)) \leqq M^*(W; \boldsymbol{Q}_{A,B}^{(f)})$. On account of Theorem 1, we can find an exhaustion $\{<X_n, Y_n>\}$ of N such that $A \cup B \subset X_1$ and the relation (E) holds. Let $w \in F(A, B)$ such that $|w(y)| \leqq W(y)$ on Y and let $Q = Q(A) \ominus Q(B) \in \boldsymbol{Q}_{A,B}^{(f)}$. Since Q is a finite set, there is n_0 such that $Q \subset Y_n$ for all $n \geqq n_0$. Notice that

$$(Q(A) \cup X_n) \ominus (Q(B) \cap (X - X_n)) \subset Z_n \cup (Q \cap (Y - Y_n)) = Z_n$$

for all $n \geqq n_0$. Define functions u, u_n and v_n on X by

$$u = 0 \quad \text{on} \quad Q(A), \qquad u = 1 \quad \text{on} \quad Q(B),$$

$$u_n = u \quad \text{on} \quad X_n,\ u_n = 0 \quad \text{on} \quad X - X_n,$$

$$v_n = u - u_n \quad \text{on} \quad X.$$

Then $v_n = 0$ on $Q(A) \cup X_n$ and $v_n = 1$ on $Q(B) \cap (X - X_n)$. We have by Lemma 3.1 in [3]

$$I(w) = \sum_{x \in X} u_n(x) \sum_{y \in Y} K(x, y) w(y) = \sum_{y \in Y} w(y) \sum_{y \in X} K(x, y) u_n(x)$$

and by the above observation

$$|I(w) - \sum_{y \in Y} w(y) \sum_{x \in X} K(x, y) u(x)| = |\sum_{y \in Y} w(y) \sum_{x \in X} K(x, y) v_n(x)|$$

$$\leqq \sum_{y \in Y} |w(y)|\,|\sum_{x \in X} K(x, y) v_n(x)| \leqq \sum_{Z_n} W(y).$$

for all $n \geqq n_0$. It follows from the relation (E) that

$$I(w) = \sum_{y \in Y} w(y) \sum_{x \in X} K(x, y)u(x) \leqq \sum_{y \in Y} |w(y)| \, |\sum_{x \in X} K(x, y)u(x)|$$
$$\leqq \sum_{Q} W(y).$$

Thus we have $M(W; F(A, B)) \leqq M^*(W; \boldsymbol{Q}_{A,B}^{(f)})$.

COROLLARY 1. *If* $W \in L_1^+(Y)$, *then* $M(W; F(A, B)) = M^*(W; \boldsymbol{Q}_{A,B})$.

COROLLARY 2. *Assume that* N *is of parabolic type of order* p. *If* $W \in L_q^+(Y; r)$, *then* $M(W; F(A, B)) = M(W; F_q(A, B)) = M(W; G(A, B)) = M^*(W; \boldsymbol{Q}_{A,B}^{(f)})$.

REMARK 3. In view of Corollary 1 of Theorem 4.1 in [5], we see that Corollary 2 of Theorem 5 is an improvement of Theorem 7 in [4].

We can not omit in Theorem 5 the condition that W satisfies condition (∞). This is shown by

EXAMPLE 1. Denote by Z the set of all integers and let

$$X = \{x_n; n \in Z\}, \quad Y = \{y_n; n \in Z\},$$
$$K(x_n, y_n) = 1 \quad \text{and} \quad K(x_{n-1}, y_n) = -1 \quad \text{for} \quad n \in Z,$$
$$K(x, y) = 0 \quad \text{for any other pair } (x, y),$$
$$r = 1 \quad \text{on} \quad Y.$$

Then $N = \{X, Y, K, r\}$ is an infinite network. Let us take $A = \{x_0\}$ and $B = \{x_1\}$ and define $W \in L(Y)$ by $W = 1$ on Y. Then W does not satisfy condition (∞). We have $M^*(W; \boldsymbol{Q}_{A,B}) = M^*(W; \boldsymbol{Q}_{A,B}^{(f)}) = M(W; F_q(A, B)) = 1 < 2 = M(W; F(A, B))$.

EXAMPLE 2. Let $Z^+ = \{n \in Z; n > 0\}$, $X = \{x_n; n \in Z\}$ and $Y = \{y_n; n \in Z\} \cup \{y'_n; n \in Z^+\}$. Define K by

$$K(x_n, y_n) = 1 \quad \text{and} \quad K(x_{n-1}, y_n) = -1 \quad \text{for} \quad n \in Z,$$
$$K(x_n, y'_n) = 1 \quad \text{and} \quad K(x_{-n}, y'_n) = -1 \quad \text{for} \quad n \in Z^+,$$
$$K(x, y) = 0 \quad \text{for any other pair } (x, y).$$

Assume that $r \in L_1^+(Y)$ and $r(y) > 0$ on Y. Then $N = \{X, Y, K, r\}$ is an infinite network which is totally hyperbolic (cf. [5]). Let us take $A = \{x_0\}$ and $B = \{x_1\}$ and define $W \in L(Y)$ by $W(y_1) = W(y'_n) = 0$ for all $n \in Z^+$ and $W(y_n) = 1$ for all $n \in Z$ such that $n \neq 1$. Then W does not satisfy condition (∞) and $W \in L_q^+(Y; r)$. We have $M^*(W; \boldsymbol{Q}_{A,B}) = 0 < 1 = M^*(W; \boldsymbol{Q}_{A,B}^{(f)}) = M(W; F_q(A, B)) = M(W; F(A, B))$.

References

[1] H. Flanders: Infinite networks: I — Resistive networks, IEEE Trans. Circuit Theory CT-**18** (1971), 326–331.

[2] L. R. Ford and D. R. Fulkerson: Flows in networks, Princeton Univ. Press, Princeton, N. J., 1962.

[3] T. Nakamura and M. Yamasaki: Generalized extremal length of an infinite network, Hiroshima Math. J. **6** (1976), 95–111.

[4] M. Yamasaki: Extremum problems on an infinite network, ibid. **5** (1975), 223–250.

[5] M. Yamasaki: Parabolic and hyperbolic infinite networks, ibid. **7** (1977), 135–146.

[6] A. H. Zemanian: Infinite networks of positive operators, Internat. J. Circuit Theory and Applications **2** (1974), 69–78.

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