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Balayage, Capacity and a Duality Theorem in Dirichlet Spaces

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Introduction

H. Cartan [2] systematically applied the method of Hilbert space to the study of capacity and balayage in the classical potential theory. His idea was generalized to the axiomatic theory of Dirichlet spaces by A. Beurling and J. Deny [1]. Balayages and capacities in Dirichlet spaces are studied in [1], [4], [5], [6] and [7] to some extent. In the present paper, we proceed to study inner and outer balayages and capacities in Dirichlet spaces. We shall show that characterizations of these notions are obtained as consequences of a certain duality theorem (Theorem 3.1). As an application, we shall show that the inner balayage and the outer balayage coincide for K-analytic sets.

§1. Cones and T-cones in a Hilbert space

In this section, let H be a real Hilbert space with norm $\|\cdot\|$ and scalar product $(\cdot, \cdot)$. A cone in H is a set S in H such that $\lambda \geqq 0$ and $x \in S$ imply $\lambda x \in S$. A set S in H will be called a T-cone (T stands for "truncated") if $\lambda \geqq 1$ and $x \in S$ imply $\lambda x \in S$. Given a set S in H, we put

$$S^0 = \{y \in H;\ (x, y) \geqq 0 \quad \text{for} \quad \text{all} \quad x \in S\},$$

$$S^\Delta = \{y \in H;\ (x, y) \geqq 1 \quad \text{for} \quad \text{all} \quad x \in S\}.$$

Then the following properties are easily verified:

(1.1) S^0 is a non-empty closed convex cone containing 0; S^Δ is a closed convex T-cone.

(1.2) $S_1 \subset S_2$ implies $S_1^0 \supset S_2^0$ and $S_1^\Delta \supset S_2^\Delta$.

(1.3) $S^0 = H$ if and only if either $S = \emptyset$ or $S = \{0\}$; $S^\Delta = H$ if and only if $S = \emptyset$.

(1.4) $\bar{S}^0 = S^0$ and $\bar{S}^\Delta = S^\Delta$, where $\bar{S}$ denotes the closure of S in H.

(1.5) If S is closed convex, then $S^\Delta = \emptyset$ if and only if $S \ni 0$.

LEMMA 1.1. (a) *If* $S \neq \emptyset$, *then* S^{00} *is the smallest closed convex cone containing* S.

(b) *If* $S^{\Delta\Delta} \not\ni 0$ (*equivalently* $S^\Delta \neq \emptyset$), *then* $S^{\Delta\Delta}$ *is the smallest closed convex*

T-*cone containing* S.

PROOF. Both statements are proved by the separation theorem. Since (a) may be well-known, we give here a proof of (b). If $S=\emptyset$, then $S^{\Delta\Delta}=\emptyset$. Suppose $S\neq\emptyset$. First we observe that $S^{\Delta\Delta}$ is a closed convex T-cone containing S. Let S′ be the smallest closed convex T-cone containing S. By assumption we have $S'\not\ni 0$. Let $x_0\notin S'$. Since S′ is a T-cone, $[0, x_0]\cap S'=\emptyset$. Hence, by the separation theorem (see e.g., [8]), there is $y\in H$ such that

$$\sup_{x\in[0,x_0]}(x, y) < \inf_{x\in S'}(x, y).$$

Since the left hand side is non-negative, we may assume that $\inf_{x\in S'}(x, y)=1$. Then $y\in(S')^{\Delta}\subset S^{\Delta}$ and $(x_0, y)<1$. Hence $x_0\notin S^{\Delta\Delta}$. Thus $S^{\Delta\Delta}\subset S'$, so that $S^{\Delta\Delta}=S'$. This proves (b).

LEMMA 1.2. (a) *Let* $\{S_\alpha\}_{\alpha\in A}$ *be a family of sets in* H *and put* $S=\cup_{\alpha\in A}S_\alpha$. *Then* $S^0=\cap_{\alpha\in A}S_\alpha^0$ *and* $S^{\Delta}=\cap_{\alpha\in A}S_\alpha^{\Delta}$.

(b) *Let* $\{S_\alpha\}_{\alpha\in A}$ *be a lower directed family of closed convex cones in* H. *Then* $(\cap_{\alpha\in A}S_\alpha)^0=\cup_{\alpha\in A}S_\alpha^0$.

(c) *Let* $\{S_\alpha\}_{\alpha\in A}$ *be a lower directed family of closed convex* T-*cones in* H *and* $\cap_{\alpha\in A}S_\alpha\neq\emptyset$. *Then* $(\cap_{\alpha\in A}S_\alpha)^{\Delta}=\cup_{\alpha\in A}S_\alpha^{\Delta}$.

PROOF. (a) is easily obtained from the definition.

(b) If $S_\alpha=\emptyset$ for some α, then $(\cap_{\alpha\in A}S_\alpha)^0=\cup_{\alpha\in A}S_\alpha^0=H$. If $S_\alpha\neq\emptyset$ for all $\alpha\in A$, then the equality follows from Lemma 1.1, (a), since $S_\alpha^{00}=S_\alpha$ and $\bigcup_{\alpha\in A}S_\alpha^0$ is a convex cone.

(c) If $0\in S_\alpha$ for all $\alpha\in A$, then $(\cap_{\alpha\in A}S_\alpha)^{\Delta}=\bigcup_{\alpha\in A}S_\alpha^{\Delta}=\emptyset$. If $0\notin S_\beta$ for some $\beta\in A$, then put $A_0=\{\alpha\in A;\ \alpha\geqq\beta\}$. Then $S_\alpha^{\Delta\Delta}=S_\alpha$ for $\alpha\in A_0$ by Lemma 1.1, (b). Hence we obtain the required equality by Lemma 1.1, (b) since $\cup_{\alpha\in A}S_\alpha^{\Delta}=\bigcup_{\alpha\in A_0}S_\alpha^{\Delta}$ is a convex T-cone and $\cap_{\alpha\in A}S_\alpha\neq\emptyset$.

Now, let S be a non-empty closed convex set in H. Given $x\in H$, the projection $x_0=P_S(x)$ of x onto S is the unique element $x_0\in S$ such that $\|x-x_0\|=\min_{y\in S}\|x-y\|$. In particular, $P_S(0)$ is the unique element minimizing the norm in S. In case S is a closed convex cone, $x_0=P_S(x)$ is characterized by the following two relations:

$$(x-x_0, x_0)=0 \quad\text{and}\quad (x-x_0, y)\leqq 0 \qquad \text{for any}\quad y\in S.$$

LEMMA 1.3. (a) *Let* S *be a non-empty closed convex cone in* H. *Then*

$$P_S(x)=P_{S^0+x}(0) \qquad \textit{for any}\quad x\in H.$$

(b) *Let* S *be a non-empty closed convex set such that* $0 \notin \mathrm{S}$. *Then*

$$\mathrm{P}_{\mathrm{S}^{\Delta}}(0) = \|\mathrm{P}_{\mathrm{S}}(0)\|^{-2}\mathrm{P}_{\mathrm{S}}(0),$$

so that

$$\|\mathrm{P}_{\mathrm{S}^{\Delta}}(0)\| = \|\mathrm{P}_{\mathrm{S}}(0)\|^{-1}.$$

PROOF. (a) Since S is a convex cone, $(\mathrm{P}_{\mathrm{S}}(x), y) \geqq (x, y)$ for all $y \in \mathrm{S}$. Hence $\mathrm{P}_{\mathrm{S}}(x) \in \mathrm{S}^0 + x$. Since $\mathrm{P}_{\mathrm{S}}(x) \in \mathrm{S}$ and $\mathrm{P}_{\mathrm{S}^0+x}(0) - x \in \mathrm{S}^0$, $(\mathrm{P}_{\mathrm{S}}(x), x) \leqq (\mathrm{P}_{\mathrm{S}}(x), \mathrm{P}_{\mathrm{S}^0+x}(0))$. Hence,

$$\|\mathrm{P}_{\mathrm{S}}(x)\|^2 = (\mathrm{P}_{\mathrm{S}}(x), x) \leqq \|\mathrm{P}_{\mathrm{S}}(x)\| \, \|\mathrm{P}_{\mathrm{S}^0+x}(0)\|,$$

so that $\|\mathrm{P}_{\mathrm{S}}(x)\| \leqq \|\mathrm{P}_{\mathrm{S}^0+x}(0)\|$. By the uniqueness of $\mathrm{P}_{\mathrm{S}^0+x}(0)$, we have $\mathrm{P}_{\mathrm{S}}(x) = \mathrm{P}_{\mathrm{S}^0+x}(0)$.

(b) Since $\mathrm{P}_{\mathrm{S}}(0) \in \mathrm{S}$ and $\mathrm{P}_{\mathrm{S}^{\Delta}}(0) \in \mathrm{S}^{\Delta}$, we have

$$1 \leqq (\mathrm{P}_{\mathrm{S}}(0), \mathrm{P}_{\mathrm{S}}{}^{\Delta}(0)) \leqq \|\mathrm{P}_{\mathrm{S}}(0)\| \, \|\mathrm{P}_{\mathrm{S}}{}^{\Delta}(0)\|.$$

On the other hand, $(\mathrm{P}_{\mathrm{S}}(0), y) \geqq \|\mathrm{P}_{\mathrm{S}}(0)\|^2$ for all $y \in \mathrm{S}$, so that $\|\mathrm{P}_{\mathrm{S}}(0)\|^{-2}\mathrm{P}_{\mathrm{S}}(0) \in \mathrm{S}^{\Delta}$. Since

$$\| \, \|\mathrm{P}_{\mathrm{S}}(0)\|^{-2}\mathrm{P}_{\mathrm{S}}(0)\| = \|\mathrm{P}_{\mathrm{S}}(0)\|^{-1} \leqq \|\mathrm{P}_{\mathrm{S}^{\Delta}}(0)\|,$$

it follows from the uniqueness of $\mathrm{P}_{\mathrm{S}^{\Delta}}(0)$ that $\|\mathrm{P}_{\mathrm{S}}(0)\|^{-2}\mathrm{P}_{\mathrm{S}}(0) = \mathrm{P}_{\mathrm{S}^{\Delta}}(0)$.

The next lemma is well-known.

LEMMA 1.4. *Let* $\{\mathrm{S}_\alpha\}_{\alpha \in A}$ *be a family of non-empty closed convex sets in* H *and let* $x \in \mathrm{H}$.

(a–1) *If* $\{\mathrm{S}_\alpha\}$ *is lower directed and* $\mathrm{S} = \bigcap_{\alpha \in A} \mathrm{S}_\alpha \neq \emptyset$, *then* $\mathrm{P}_{\mathrm{S}_\alpha}(x) \to \mathrm{P}_{\mathrm{S}}(x)$ *in* H.

(a–2) *If* $\{\mathrm{S}_\alpha\}$ *is lower directed and* $\bigcap_{\alpha \in A} \mathrm{S}_\alpha = \emptyset$, *then* $\|\mathrm{P}_{\mathrm{S}_\alpha}(x)\| \to \infty$.

(b) *If* $\{\mathrm{S}_\alpha\}$ *is upper directed and* $\mathrm{S} = \overline{\bigcup_{\alpha \in A} \mathrm{S}_\alpha}$, *then* $\mathrm{P}_{\mathrm{S}_\alpha}(x) \to \mathrm{P}_{\mathrm{S}}(x)$ *in* H.

§2. Basic facts on Dirichlet spaces

From now on, we consider a real Dirichlet space $\mathscr{D} = \mathscr{D}(X; \xi)$ in the sense of Beurling-Deny [1]. Here, X is a locally compact, σ-compact Hausdorff space and ξ is a positive Radon measure on X. The norm and scalar product in $\mathscr{D}$ are again denoted by $\|\cdot\|$ and $(\cdot, \cdot)$, respectively.

Recall that $u \in \mathscr{D}$ is called a pure potential if there exists a unique non-negative Radon measure μ on X such that

$$(u, \phi) = \int \phi d\mu$$

for any $\phi \in \mathscr{C} \cap \mathscr{D}$, where $\mathscr{C}$ is the space of all continuous functions with compact support in X. Such u is denoted by U_μ in the present paper and μ is called the associated measure of u. Let $\mathscr{P}$ be the set of all pure potentials in $\mathscr{D}$. We know the following properties ([1], [5], [6]):

(2.1) $\mathscr{P}$ is a non-empty closed convex cone in $\mathscr{D}$.

(2.2) $\mathscr{P}$ is total in $\mathscr{D}$, i.e., $\mathscr{P}-\mathscr{P}$ is dense in $\mathscr{D}$.

(2.3) If $u=U_\mu$ and $v=U_\nu$ belong to $\mathscr{P}$ and $u \leqq v$ (i.e., $u(x) \leqq v(x)$ ξ-a.e. in X), then $\|u\| \leqq \|v\|$ and $\int d\mu \leqq \int d\nu$.

(2.4) If $u, v \in \mathscr{P}$, then $\min(u, v) \in \mathscr{P}$.

(2.5) If $U_\mu \in \mathscr{P}$ and $0 \leqq \nu \leqq \mu$, then $U_\nu \in \mathscr{P}$ exists.

By [1, Lemma 2], we easily see

LEMMA 2.1. *If* S *is a non-empty closed convex set in* $\mathscr{D}$ *such that* $\mathrm{S}+\{u \in \mathscr{D}; u \geqq 0\}=\mathrm{S}$, *then* $\mathrm{P_S}(0) \in \mathscr{P}$.

The capacity of an open set ω is defined as follows ([1], [5], [6]): put

$$\mathscr{V}_{\omega,1} = \{u \in \mathscr{D} : u(x) \geqq 1 \ \xi\text{-a.e. on } \omega\}$$

and

$$C(\omega) = \begin{cases} \inf\limits_{u \in \mathscr{V}_{\omega,1}} \|u\|^2 & \text{if } \mathscr{V}_{\omega,1} \neq \emptyset, \\ +\infty & \text{if } \mathscr{V}_{\omega,1} = \emptyset. \end{cases}$$

Note that $\mathscr{V}_{\omega,1}$ is a closed convex set (in fact a T-cone) and

$$C(\omega) = \|P_{\mathscr{V}_{\omega,1}}(0)\|^2$$

if $\mathscr{V}_{\omega,1} \neq \emptyset$. The outer and inner capacities are defined in the usual way: for a set $E \subset X$,

$$C^e(E) = \inf\{C(\omega); \omega \colon \text{open} \supset E\},$$

$$C^i(E) = \sup\{C^e(K); K \colon \text{compact} \subset E\}.$$

We know ([5]) that Borel sets are capacitable, i.e., $C^e(E)=C^i(E)$ if E is a Borel set. Remark that if $C^e(E)=0$, then $\mu(E)=0$ for any μ with $U_\mu \in \mathscr{P}$ (see [6, p. 169–170]).

LEMMA 2.2. (a) $C^e(\bigcup_{n=1}^{\infty} E_n) \leqq \sum_{n=1}^{\infty} C^e(E_n)$ ([5, *Théorème* 4]).

(b) *If* B_n *are Borel sets, then for any set* E, $C^i(E \cap \bigcup_{n=1}^{\infty} B_n) \leqq \sum_{n=1}^{\infty} C^i(E \cap B_n)$ (*cf.* [2, p. 253]).

(c) $C^i(E_1 \cup E_2) \leqq C^i(E_1) + C^e(E_2)$.

A property is said to hold quasi-everywhere (q.e.) (resp. nearly everywhere (n.e.)) on a set if the outer capacity (resp. inner capacity) of the exceptional set is 0. Quasi-continuity of functions is defined with respect to this capacity. We know ([1], [5], [6]) that to each $u \in \mathscr{D}$, there exists a function u^*, which is called a refinement of u, such that $u^*(x) = u(x)$ ξ-a.e. on X, u^* is quasi-continuous and $(u, U_\mu) = \int u^* d\mu$ for all $U_\mu \in \mathscr{P}$. We can choose u^* to be Borel measurable, which we shall always assume in the present paper. Note that two refinements of u coincide q.e. on X. It follows that the properties "$u^*(x) \geqq c$ q.e. on E" and "$u^*(x) \geqq c$ n.e. on E" do not depend on the choice of refinements (cf. Lemma 2.2).

§ 3. A duality theorem

For a compact set K in X, let

$$\mathscr{P}_K = \{U_\mu \in \mathscr{P};\ \operatorname{supp}\mu \subset K\} \quad \text{and} \quad \mathscr{P}_{K,1} = \{U_\mu \in \mathscr{P}_K;\ \mu(K) \geqq 1\}.$$

$\mathscr{P}_K$ is a closed convex cone and $\mathscr{P}_{K,1}$ is a closed convex T-cone. The closedness of $\mathscr{P}_{K,1}$ follows from the existence of $\phi \in \mathscr{C} \cap \mathscr{D}$ such that $\phi = 1$ on K and $\phi \geqq 0$ on X. Given a set E in X, put

$$\mathscr{P}_E^i = \overline{\bigcup_{K:\text{compact} \subset E} \mathscr{P}_K}, \qquad \mathscr{P}_{E,1}^i = \overline{\bigcup_{K:\text{compact} \subset E} \mathscr{P}_{K,1}}$$

and

$$\mathscr{P}_E^e = \bigcap_{\omega:\text{open} \supset E} \mathscr{P}_\omega^i, \qquad \mathscr{P}_{E,1}^e = \bigcap_{\omega:\text{open} \supset E} \mathscr{P}_{\omega,1}^i.$$

Then $\mathscr{P}_E^i$, $\mathscr{P}_E^e$ are closed convex cones and $\mathscr{P}_{E,1}^i$, $\mathscr{P}_{E,1}^e$ are closed convex T-cones. Obviously $\mathscr{P}_E^i \subset \mathscr{P}_E^e$ and $\mathscr{P}_{E,1}^i \subset \mathscr{P}_{E,1}^e$. The mappings $E \to \mathscr{P}_E^i$, etc., are all monotone increasing. It is easy to see that if $U_\mu \in \mathscr{P}_E^e$ (in particular, if $U_\mu \in \mathscr{P}_E^i$ or $\mathscr{P}_{E,1}^e$ or $\mathscr{P}_{E,1}^i$), then $\operatorname{supp}\mu \subset \bar{E}$. We shall consider the classes

$$\mathscr{W}_E^i = \overline{\{u \in \mathscr{D};\ \sigma(u) \subset E\}} \quad \text{and} \quad \mathscr{W}_E^e = \bigcap_{\omega:\text{open} \supset E} \mathscr{W}_\omega^i,$$

where $\sigma(u)$ denotes the spectrum of u (see [1, § 6]). These are closed linear subspaces of $\mathscr{D}$. Obviously, $\mathscr{P}_E^i \subset \mathscr{W}_E^i$ and $\mathscr{P}_E^e \subset \mathscr{W}_E^e$. The theorem of spectral synthesis (see [1, Theorem 8] or [6, p. 108]) implies

LEMMA 3.1. *$\mathscr{P}_E^i$ is total in $\mathscr{W}_E^i$.*

Next, we consider the classes

$$\mathscr{U}_E^i = \{u \in \mathscr{D};\ u^*(x) \geqq 0 \text{ n.e. on } E\},$$

$$\mathscr{U}_{E,1}^i = \{u \in \mathscr{D};\ u^*(x) \geqq 1 \text{ n.e. on } E\},$$

$$\mathscr{U}_E^e = \{u \in \mathscr{D}\,;\, u^*(x) \geqq 0 \text{ q. e. on } E\}\,,$$

$$\mathscr{U}_{E,1}^e = \{u \in \mathscr{D}\,;\, u^*(x) \geqq 1 \text{ q. e. on } E\}\,.$$

$\mathscr{U}_E^i$ and $\mathscr{U}_E^e$ are convex cones in $\mathscr{D}$ and $\mathscr{U}_{E,1}^i$ and $\mathscr{U}_{E,1}^e$ are convex T-cones in $\mathscr{D}$ (cf. Lemma 2.2). Obviously $\mathscr{U}_E^i \supset \mathscr{U}_E^e$ and $\mathscr{U}_{E,1}^i \supset \mathscr{U}_{E,1}^e$. The mappings $E \to \mathscr{U}_E^i$, etc., are all monotone decreasing. By [5, Théorème 7], we see

LEMMA 3.2. *$\mathscr{U}_E^e$ and $\mathscr{U}_{E,1}^e$ are closed.*

Now, we show

LEMMA 3.3. *Functions bounded from below in $\mathscr{U}_E^e$ (resp. $\mathscr{U}_{E,1}^e$) are dense in $\mathscr{U}_E^e$ (resp. $\mathscr{U}_{E,1}^e$).*

PROOF. Given $u \in \mathscr{U}_E^e$ (resp. $\mathscr{U}_{E,1}^e$), let $u_n = \sup(u, -n)$, $n=1, 2, \ldots$. Then $u_n \in \mathscr{U}_E^e$ (resp. $\mathscr{U}_{E,1}^e$) and $\|u_n\| \leqq \|u\|$. Since

$$(u_n, U_\mu) = \int u_n^* d\mu \longrightarrow \int u^* d\mu = (u, U_\mu)$$

for any $U_\mu \in \mathscr{P}$, $u_n \to u$ weakly in $\mathscr{D}$ by virtue of (2.2). Since $\|u_n\| \leqq \|u\|$, it follows that $u_n \to u$ strongly in $\mathscr{D}$.

If ω is an open set, then $\mathscr{U}_\omega^e = \mathscr{U}_\omega^i = \mathscr{V}_\omega$ and $\mathscr{U}_{\omega,1}^e = \mathscr{U}_{\omega,1}^i = \mathscr{V}_{\omega,1}$, where $\mathscr{V}_\omega = \{u \in D;\, u(x) \geqq 0 \ \xi\text{-a. e. on } \omega\}$. We have

PROPOSITION 3.1. $\mathscr{U}_E^e = \overline{\bigcup_{\omega:\text{open}\supset E} \mathscr{V}_\omega}$ *and* $\mathscr{U}_{E,1}^e = \overline{\bigcup_{\omega;\text{open}\supset E} \mathscr{V}_{\omega,1}}$.

The proof of this proposition is similar to [5, Lemme]. Note that in showing $\mathscr{U}_E^e \subset \overline{\bigcup \mathscr{V}_\omega}$ (resp. $\mathscr{U}_{E,1}^e \subset \overline{\bigcup \mathscr{V}_{\omega,1}}$), we may consider only functions bounded from below in $\mathscr{U}_E^e$ (resp. $\mathscr{U}_{E,1}^e$) by virtue of the above lemma.

PROPOSITION 3.2. $\mathscr{U}_E^i = \bigcap_{K:\text{compact}\subset E} \mathscr{U}_K^e$ *and* $\mathscr{U}_{E,1}^i = \bigcap_{K:\text{compact}\subset E} \mathscr{U}_{K,1}^e$.

PROOF. Obviously, $\mathscr{U}_E^i \subset \bigcap \mathscr{U}_K^e$ and $\mathscr{U}_{E,1}^i \subset \bigcap \mathscr{U}_{K,1}^e$. Let $u \in \bigcap \mathscr{U}_K^e$ (resp. $\bigcap \mathscr{U}_{K,1}^e$) and put $E' = \{x \in E;\, u^*(x) < 0\}$ (resp. $\{x \in E;\, u^*(x) < 1\}$). For any compact set $K \subset E'$, $C^e(K) = 0$. Hence $C^i(E') = 0$, so that $u \in \mathscr{U}_E^i$ (resp. $\mathscr{U}_{E,1}^i$).

COROLLARY 1. *$\mathscr{U}_E^i$ and $\mathscr{U}_{E,1}^i$ are closed.*

COROLLARY 2. *If $\mathscr{U}_{E,1}^e \neq \emptyset$, then $C^e(E) = \|P_{\mathscr{U}_{E,1}^e}(0)\|^2$ and if $\mathscr{U}_{E,1}^i \neq \emptyset$, then $C^i(E) = \|P_{\mathscr{U}_{E,1}^i}(0)\|^2$; if $\mathscr{U}_{E,1}^e = \emptyset$, then $C^e(E) = \infty$ and if $\mathscr{U}_{E,1}^i = \emptyset$, then $C^i(E) = \infty$.*

This corollary follows from Lemma 1.4, Propositions 3.1 and 3.2 and the definitions of C^e and C^i.

Now we are ready to prove our duality theorem.

THEOREM 3.1. *For any subset E in X,*

$$(\mathscr{P}_E^e)^0 = \mathscr{U}_E^e,\ (\mathscr{P}_E^i)^0 = \mathscr{U}_E^i;\ (\mathscr{P}_{E,1}^e)^\Delta = \mathscr{U}_{E,1}^e,\ (\mathscr{P}_{E,1}^i)^\Delta = \mathscr{U}_{E,1}^i\,.$$

PROOF. First we prove

$$(\mathscr{P}_K)^0 = \mathscr{U}_k^e \quad\text{and}\quad (\mathscr{P}_{K,1})^\Delta = \mathscr{U}_{K,1}^e$$

for compact set K. If $u\in\mathscr{U}_K^e$ (resp. $\mathscr{U}_{K,1}^e$) and $U_\mu\in\mathscr{P}_K$ (resp. $\mathscr{P}_{K,1}$), then

$$(u,\, U_\mu) = \int u^* d\mu = \int_K u^* d\mu \geqq 0 \text{ (resp. } \geqq 1).$$

Hence $\mathscr{U}_K^e\subset(\mathscr{P}_K)^0$ (resp. $\mathscr{U}_{K,1}^e\subset(\mathscr{P}_{K,1})^\Delta$).

Next, suppose $u\in(\mathscr{P}_K)^0$ (resp. $(\mathscr{P}_{K,1})^\Delta$) and put

$$F = \{x\in K;\, u^*(x) < 0\} \quad (\text{resp.} = \{x\in K;\, u^*(x) < 1\}).$$

Since F is a Borel set, $C^e(F)=C^i(F)$. Suppose $C^e(F)>0$. Then we find a compact set $K'\subset F$ such that $C^e(K')>0$. Put $v=\mathrm{P}_{\mathscr{U}_{K',1}^e}(0)$. By Lemma 2.1, we see that $v\in\mathscr{P}$. Let $v=U_\nu$. By Corollary 2 to Proposition 3.2, $\|v\|^2=C^e(K')>0$. Hence $\nu\neq 0$. Now we shall show that $\operatorname{supp}\nu\subset K'$. Let $\phi\in\mathscr{C}\cap\mathscr{D}$ and $\operatorname{supp}\phi\cap K'=\emptyset$. Then $v+t\phi\in\mathscr{U}_{K',1}^e$, so that $\|v\|\leqq\|v+t\phi\|$ for any real t. It follows that $(v,\phi)=0$, or $\int\phi d\nu=0$. Hence $\operatorname{supp}\nu\subset K'$. Therefore, $\nu\in\mathscr{P}_K$ (resp. $\nu/\nu(K')\in\mathscr{P}_{K,1}$), and so

$$0\leqq (u,\, v) = \int_{K'} u^* d\nu < 0\,,$$

$$(\text{resp. } 1\leqq (u,\, v/\nu(K')) = \int_{K'} u^* d\nu/\nu(K') < 1)\,.$$

This is a contradiction. Therefore $C^e(F)=0$, which means $u\in\mathscr{U}_K^e$ (resp. $\mathscr{U}_{K,1}^e$). Hence $(\mathscr{P}_K)^0\subset\mathscr{U}_K^e$ (resp. $(\mathscr{P}_{K,1})^\Delta\subset\mathscr{U}_{K,1}^e$).

For any subset E, by virtue of Lemma 1.2 (a) and (b), Propositions 3.1 and 3.2, we have

$$(\mathscr{P}_E^e)^0 = \mathscr{U}_E^e\,,\ (\mathscr{P}_E^i)^0 = \mathscr{U}_E^i\,,\ (\mathscr{P}_{E,1}^i)^\Delta = \mathscr{U}_{E,1}^i\,.$$

If $\mathscr{P}_{E,1}^e\neq\emptyset$, then by Lemma 1.2 (c), Proposition 3.1 and the above fact, we have

$$(\mathscr{P}_{E,1}^e)^\Delta = \mathscr{U}_{E,1}^e\,.$$

If $\mathscr{P}_{E,1}^e=\emptyset$, then by Lemma 1.3 (b) and the above fact and the definition of outer capacity, $C^e(E)=0$. Hence $(\mathscr{P}_{E,1}^e)^\Delta=\mathscr{U}_{E,1}^e=\mathscr{D}$.

COROLLARY (cf. [6, Lemme 4]). *$(\mathscr{W}_E^i)^\perp=\{u\in\mathscr{D};\, u^*(x)=0$ n.e. on $E\}$, where $\perp$ denotes the orthogonal complement.*

PROOF. By Lemma 3.1 and the above theorem, we have

$$(\mathscr{W}^i_E)^\perp = (\mathscr{P}^i_E)^0 \cap -(\mathscr{P}^i_E)^0 = \mathscr{U}^i_E \cap (-\mathscr{U}^i_E)$$
$$= \{u \in \mathscr{D}; u^*(x) = 0 \quad \text{n.e. on } E\}.$$

§4. Inner and outer balayages and applications

For $u \in \mathscr{P}$ and $E \subset X$, the inner and outer balayages of u to E are defined by

$$u^i_E = \mathrm{P}_{\mathscr{P}^i_E}(u) \quad \text{and} \quad u^e_E = \mathrm{P}_{\mathscr{P}^e_E}(u),$$

respectively. If $u = U_\mu$, then the associated measures of u^i_E and u^e_E are denoted by μ^i_E and μ^e_E respectively. By Lemma 1.3 (a) and Theorem 3.1, we immediately have the following characterizations of balayages.

THEOREM 4.1. *For $u \in \mathscr{P}$, u^i_E (resp. u^e_E) is the unique element which attains the minimum norm in*

$$u + \mathscr{U}^i_E = \{v \in \mathscr{D}; v^* \geqq u^* \text{ n.e. on } E\}$$
$$(\textit{resp. } u + \mathscr{U}^e_E = \{v \in \mathscr{D}; v^* \geqq u^* \text{ q.e. on } E\}).$$

Now, we obtain the following properties of balayages, which are well-known in the classical case (see [2]).

THEOREM 4.2. *Let $u \in \mathscr{P}$ and $E \subset X$. Then*
(1) $u^i_E \leqq u$ *and* $u^e_E \leqq u$,
(2) $(u^i_E)^*(x) = u^*(x)$ *n.e. on* E, $(u^e_E)^*(x) = u^*(x)$ *q.e. on* E,
(3) $\operatorname{supp} \mu^i_E \subset \bar{E}$, $\operatorname{supp} \mu^e_E \subset \bar{E}$,
(4) $\int d\mu^i_E \leqq \int d\mu$, $\int d\mu^e_E \int \leqq d\mu$.

PROOF. (1) is proved in the same way as the proof of [6, p. 164, théorème du balayage]. Then (2) follows from the above theorem. (3) is trivial and (4) is a consequence of (2.3).

PROPOSITION 4.1. *For $u \in \mathscr{P}$, $u^i_E = \mathrm{P}_{\mathscr{W}^i_E}(u)$ and $u^e = \mathrm{P}_{\mathscr{W}^e_E}(u)$.*

PROOF. By (2) of the above theorem and the corollary to Theorem 3.1, we see that $u - u^i_E \in (\mathscr{W}^i_E)^\perp$. Since $\mathscr{W}^i_E$ is a linear space, it follows that $u^i_E = \mathrm{P}_{\mathscr{W}^i_E}(u)$. By Lemma 1.4 (a-1), we see that $u^e_E = \lim u^i_\omega$ and $\mathrm{P}_{\mathscr{W}^e_E}(u) = \lim \mathrm{P}_{\mathscr{W}^i_\omega}(u)$, where limits are taken with respect to the directed set of open sets ω containing E. Hence $u^e_E = \mathrm{P}_{\mathscr{W}^e_E}(u)$.

COROLLARY. *The mappings $u \to u^i_E$ and $u \to u^e_E$ are additive on $\mathscr{P}$.*

PROOF. Since $\mathscr{W}^i_E$, $\mathscr{W}^e_E$ are linear spaces, $\mathrm{P}_{\mathscr{W}^i_E}$, $\mathrm{P}_{\mathscr{W}^e_E}$ are linear maps.

PROPOSITION 4.2. *If* $U_\mu \in \mathscr{P}_E^i$ (*resp.* $\mathscr{P}_E^e$) *and* $0 \leqq \nu \leqq \mu$, *then* $U_\nu \in \mathscr{P}_E^i$ (*resp.* $\mathscr{P}_E^e$).

PROOF. Let $u = U_\mu$, $v = U_\nu$ and $w = u - v$. Then $v, w \in \mathscr{P}$ by (2.5). Hence by the above corollary, $u = u_E^i = v_E^i + w_E^i$. Since $v_E^i \leqq v$, $w_E^i \leqq w$ and $v + w = u$, it follows that $v = v_E^i \in \mathscr{P}_E^i$. The proof for $\mathscr{P}_E^e$ is similar.

THEOREM 4.3. *For* $u, v \in \mathscr{D}$ *and* $E \subset X$, *the following assertions* (a_i), (b_i), (c_i), (d_i) (*resp.* (a_e), (b_e), (c_e), (d_e)) *are mutually equivalent*:

(a_i) $u^* \leqq v^*$ *n.e. on* E,	(a_e) $u^* \leqq v^*$ *q.e. on* E,
(b_i) $(u, U_\mu) \leqq (v, U_\mu)$ *for all* $U_\mu \in \mathscr{P}_E^i$,	(b_e) $(u, U_\mu) \leqq (v, U_\mu)$ *for all* $U_\mu \in \mathscr{P}_E^e$,
(c_i) $\int u^* d\mu \leqq \int v^* d\mu$ *for all* $U_\mu \in \mathscr{P}_E^i$,	(c_e) $\int u^* d\mu \leqq \int v^* d\mu$ *for all* $U_\mu \in \mathscr{P}_E^e$,
(d_i) $u^* \leqq v^*$ μ-*a.e. on* X *for all* $U_\mu \in \mathscr{P}_E^i$,	(d_e) $u^* \leqq v^*$ μ-*a.e. on* X *for all* $U_\mu \in \mathscr{P}_E^e$.

PROOF. The equivalence of (a_i) and (b_i) (resp. (a_e) and (b_e)) is nothing but Theorem 3.1. The equivalence of (b_i) and (c_i) (resp. (b_e) and (c_e)) is obvious. The implication (d_i)$\Rightarrow$(c_i) (resp. (d_e)$\Rightarrow$(c_e)) is also trivial. We shall show (c_i) $\Rightarrow$(d_i) (resp. (c_e)$\Rightarrow$(d_e)). Put $E' = \{x \in X; u^*(x) > v^*(x)\}$ and suppose there is $U_\mu \in \mathscr{P}_E^i$ (resp. $\mathscr{P}_E^e$) such that $\mu(E') > 0$. Put $\mu' = \mu|_{E'}$. By the above proposition, $U_{\mu'} \in \mathscr{P}_E^i$ (resp. $\mathscr{P}_E^e$). On the other hand, $\int u^* d\mu' > \int v^* d\mu'$, which contradicts ($c_i$) (resp. ($c_e$)).

REMARK. We can see that (a_i)$\sim$(d_i) are also equivalent to
(d_i)$'$ $u^* \leqq v^*$ μ-a.e. on X for any $U_\mu \in \mathscr{P}$ with $\operatorname{supp} \mu \subset E$.

As applications of Proposition 4.1, we have the following propositions.

PROPOSITION 4.3. $\mathscr{P}_E^e$ *is total in* $\mathscr{W}_E^e$.

PROOF. Let $u \in \mathscr{W}_E^e$. Since $\mathscr{P}$ is total in $\mathscr{D}$, there exists a sequence $\{u_n\}$ in $\mathscr{P} - \mathscr{P}$ converging to u. Let $u_n = v_n - w_n$ with $v_n, w_n \in \mathscr{P}$. By the linearity of $P_{\mathscr{W}_E^e}$ and Proposition 4.1, we have

$$P_{\mathscr{W}_E^e}(u_n) = P_{\mathscr{W}_E^e}(v_n) - P_{\mathscr{W}_E^e}(w_n) = (v_n)_E^e - (w_n)_E^e \in \mathscr{P}_E^e - \mathscr{P}_E^e .$$

Obviously $P_{\mathscr{W}_E^e}(u_n) \to P_{\mathscr{W}_E^e}(u) = u$. Hence $\mathscr{P}_E^e$ is total in $\mathscr{W}_E^e$.

COROLLARY. $(\mathscr{W}_E^e)^\perp = \{u \in \mathscr{D}; u^* = 0 \ q.e. \ on \ E\}$.

PROOF. We can show this in the same way as the corollary to Theorem 3.1 using the above proposition and Theorem 3.1.

PROPOSITION 4.4. $\mathscr{P} \cap \mathscr{W}_E^i = \mathscr{P}_E^i$; $\mathscr{P} \cap \mathscr{W}_E^e = \mathscr{P}_E^e$.

PROOF. Obviously, $\mathscr{P} \cap \mathscr{W}_E^i \supset \mathscr{P}_E^i$. If $u \in \mathscr{P} \cap \mathscr{W}_E^i$, then $u = \mathrm{P}_{\mathscr{W}_E^i}(u) = \mathrm{P}_{\mathscr{P}_E^i}(u) \in \mathscr{P}_E^i$ by Proposition 4.1. Hence $\mathscr{P} \cap \mathscr{W}_E^i = \mathscr{P}_E^i$. Similarly we have $\mathscr{P} \cap \mathscr{W}_E^e = \mathscr{P}_E^e$.

PROPOSITION 4.5. *Let* T *be a normalized contraction (see* [1]). *If* $u \in \mathscr{W}_E^i$ *(resp.* $\mathscr{W}_E^e$*) and* $u^*(x) = \mathrm{T}u^*(x)$ *n.e. on* E *(resp. q.e. on* E*), then* $u = \mathrm{T}u$.

PROOF. By the corollary to Theorem 3.1 (resp. the corollary to Proposition 4.3), $u - \mathrm{T}u \in (\mathscr{W}_E^i)^\perp$ (resp. $(\mathscr{W}_E^e)^\perp$). Hence $(u, u - \mathrm{T}u) = 0$. It then follows that $u = \mathrm{T}u$ (cf. [6, p. 173, (c)]).

PROPOSITION 4.6. *Let* $u \in \mathscr{D}$. *In order that* $\mathrm{P}_{\mathscr{W}_E^i}(u) = \mathrm{P}_{\mathscr{P}_E^i}(u)$ *(resp.* $\mathrm{P}_{\mathscr{W}_E^e}(u) = \mathrm{P}_{\mathscr{P}_E^e}(u)$*), it is necessary and sufficient that* $u \in \mathscr{P} + (\mathscr{W}_E^i)^\perp$ *(resp.* $\mathscr{P} + (\mathscr{W}_E^e)^\perp$*).*

PROOF. Since $\mathrm{P}_{\mathscr{W}_E^i}$ is linear and $\mathrm{P}_{\mathscr{W}_E^i}(w) = 0$ if $w \in (\mathscr{W}_E^i)^\perp$, the sufficiency follows from Proposition 4.1. Now, suppose $\mathrm{P}_{\mathscr{W}_E^i}(u) = \mathrm{P}_{\mathscr{P}_E^i}(u)$. Then $u = \mathrm{P}_{\mathscr{W}_E^i}(u) + \mathrm{P}_{(\mathscr{W}_E^i)^\perp}(u) = \mathrm{P}_{\mathscr{P}_E^i}(u) + \mathrm{P}_{(\mathscr{W}_E^i)^\perp}(u) \in \mathscr{P} + (\mathscr{W}_E^i)^\perp$. Similarly, we see that $\mathrm{P}_{\mathscr{W}_E^e}(u) = \mathrm{P}_{\mathscr{P}_E^e}(u)$ if and only if $u \in \mathscr{P} + (\mathscr{W}_E^e)^\perp$.

§5. Characterizations of inner and outer capacities

First, we give characterizations of sets of inner or outer capacity 0.

THEOREM 5.1. *For a set* $E \subset X$, *the following assertions* $(1_i) \sim (6_i)$ *(resp.* $(1_e) \sim (6_e)$*) are equivalent:*

(1_i) $C^i(E) = 0$,	(1_e) $C^e(E) = 0$,
(2_i) $\mathscr{P}_E^i = \{0\}$,	(2_e) $\mathscr{P}_E^e = \{0\}$,
(3_i) $\mathscr{P}_{E,1}^i = \emptyset$,	(3_e) $\mathscr{P}_{E,1}^e = \emptyset$,
(4_i) $\mathscr{W}_E^i = \{0\}$,	(4_e) $\mathscr{W}_E^e = \{0\}$,
(5_i) $\mathscr{U}_E^i = \mathscr{D}$,	(5_e) $\mathscr{U}_E^e = \mathscr{D}$,
(6_i) $\mathscr{U}_{E,1}^i = \mathscr{D}$,	(6_e) $\mathscr{U}_{E,1}^e = \mathscr{D}$.

PROOF. The equivalence of (1_i) and (5_i) (resp. (1_e) and (5_e)) as well as the equivalence of (1_i) and (6_i) (resp. (1_e) and (6_e)) is trivial. Equivalences $(2_i) \Leftrightarrow (5_i)$

(resp. $(2_e)\Leftrightarrow(5_e)$) and $(3_i)\Leftrightarrow(6_i)$ (resp. $(3_e)\Leftrightarrow(6_e)$) are consequences of Theorem 3.1 and (1.3). $(2_i)\Leftrightarrow(4_i)$ follows from Lemma 3.1 and $(2_e)\Leftrightarrow(4_e)$ follows from Proposition 4.3.

Next, we give characterizations of sets whose inner or outer capacities are infinite.

THEOREM 5.2. *For a set $E\subset X$, the following assertions* $(\mathrm{i}_i)\sim(\mathrm{iv}_i)$ (*resp.* $(\mathrm{i}_e)\sim(\mathrm{iv}_e)$) *are equivalent*:

(i_i) $C^i(E)=\infty$,	(i_e) $C^e(E)=\infty$,
(ii_i) $\mathscr{P}^i_{E,1}\ni 0$,	(ii_e) $\mathscr{P}^e_{E,1}\ni 0$,
(iii_i) $\mathscr{U}^i_{E,1}=\emptyset$,	(iii_e) $\mathscr{U}^e_{E,1}=\emptyset$,
(iv_i) $\mathscr{P}^i_{E,1}=\mathscr{P}^i_E$,	(iv_e) $\mathscr{P}^e_{E,1}=\mathscr{P}^e_E$.

PROOF. $(\mathrm{i}_i)\Leftrightarrow(\mathrm{iii}_i)$ (resp. $(\mathrm{i}_e)\Leftrightarrow(\mathrm{iii}_e)$) is given by Corollary 2 to Proposition 3.2. $(\mathrm{ii}_i)\Leftrightarrow(\mathrm{iii}_i)$ (resp. $(\mathrm{ii}_e)\Leftrightarrow(\mathrm{iii}_e)$) follows from Theorem 3.1 and (1.5). $(\mathrm{iv}_i)\Rightarrow(\mathrm{ii}_i)$ (resp. $(\mathrm{iv}_e)\Rightarrow(\mathrm{ii}_e)$) is clear. If we assume (ii_i), then there exists a sequence $\{U_{\mu_n}\}$ such that supp μ_n is compact and contained in E, $\mu_n(X)\geqq 1$ and $U_{\mu_n}\to 0$ strongly as $n\to\infty$. Let $U_\mu\in\mathscr{P}$ and supp μ be compact and contained in E. Considering $U_{\nu_n}=U_\mu+U_{\mu_n}$, we see that $U_\mu\in\mathscr{P}^i_{E,1}$. Therefore $\mathscr{P}^i_E=\overline{\cup_{K\subset E}\mathscr{P}_K}\subset\mathscr{P}^i_{E,1}$. Hence $(\mathrm{ii}_i)\Rightarrow(\mathrm{iv}_i)$. $(\mathrm{ii}_e)\Rightarrow(\mathrm{iv}_e)$ is easily seen from $(\mathrm{ii}_i)\Rightarrow(\mathrm{iv}_i)$.

By the above two theorems we see that if $C^i(E)>0$ (resp. $C^e(E)>0$), then $\mathscr{P}^i_{E,1}\neq\emptyset$ (resp. $\mathscr{P}^e_{E,1}\neq\emptyset$) and if $C^i(E)<\infty$ (resp. $C^e(E)<\infty$), then $\mathscr{U}^i_{e,1}\neq\emptyset$ (resp. $\mathscr{U}^e_{E,1}\neq\emptyset$). When $C^i(E)<\infty$ (resp. $C^e(E)<\infty$), we call $\mathrm{P}_{\mathscr{U}^i_{E,1}}(0)$ (resp. $\mathrm{P}_{\mathscr{U}^e_{E,1}}(0)$) the inner (resp. outer) capacitary potential of E and denote it by $\ell^i_{E,1}$ (resp. $\ell^e_{E,1}$). By Lemma 1.3 (b) and Theorem 3.1, these are pure potentials. The associated measures of $\ell^i_{E,1}$ and $\ell^e_{E,1}$ are denoted by $\lambda^i_{E,1}$ and $\lambda^e_{E,1}$, respectively. Lemma 1.3 (b), Theorem 3.1 and Corollary 2 to Proposition 3.2 yield

THEOREM 5.3. *If $0<C^i(E)<\infty$, then $C^i(E)=\|\ell^i_{E,1}\|^2=\|\mathrm{P}_{\mathscr{P}^i_{E,1}}(0)\|^{-2}$. If $0<C^e(E)<\infty$, then $C^e(E)=\|\ell^e_{E,1}\|^2=\|\mathrm{P}_{\mathscr{P}^e_{E,1}}(0)\|^{-2}$.*

We obtain the following properties of inner and outer capacitary potentials of E.

THEOREM 5.4. (1) *$\ell^i_{E,1}\leqq 1$ and $\ell^e_{E,1}\leqq 1$.*
(2) *$(\ell^i_{E,1})^*(x)=1$ n.e. on E and $(\ell^e_{E,1})^*(x)=1$ q.e. on E.*
(3) *The supports of $\lambda^i_{E,1}$ and $\lambda^e_{E,1}$ are contained in $\bar{E}$.*

PROOF. (1) is proved in the same way as the proof of [6, p. 162, théorème d'équilibre]. Then (2) is trivial. (3) follows from Lemma 1.3 (b) and Theorem 3.1.

Next we give a relation between $\mathscr{P}^i_{E,1}$ and $\mathscr{P}^i_E$ (resp. $\mathscr{P}^e_{E,1}$ and $\mathscr{P}^e_E$). We put $\mathscr{P}_{X,1}=\{U_\mu\in\mathscr{P};\ \mu(X)\geqq 1\}$. In Theorem 5.2 we have already seen that $\mathscr{P}^i_{E,1}=\mathscr{P}^i_E$ (resp. $\mathscr{P}^e_{E,1}=\mathscr{P}^e_E$) in case $C^i(E)=\infty$ (resp. $C^e(E)=\infty$). Furthermore we have

LEMMA 5.1. *If* $C^i(E)<\infty$ *(resp.* $C^e(E)<\infty$*), then*

$$\mathscr{P}^i_{E,1}=\mathscr{P}^i_E\cap\mathscr{P}_{X,1}\ (resp.\ \mathscr{P}^e_{E,1}=\mathscr{P}^e_E\cap\mathscr{P}_{X,1}).$$

PROOF. If $C^i(E)=0$ (resp. $C^e(E)=0$), then the conclusion is clear by Theorem 5.1. Let $0<C^i(E)<\infty$. Given $U_\mu\in\mathscr{P}^i_{E,1}$, Theorems 3.1 and 5.4 imply

$$1\leqq(U_\mu,\ \ell^i_{E,1})=\int(\ell^i_{E,1})^*d\mu\int\leqq d\mu\,.$$

Hence $\mathscr{P}^i_{E,1}\subset\mathscr{P}^i_E\cap\mathscr{P}_{X,1}$. Conversely let $U_\mu\in\mathscr{P}^i_E$ and $\mu(X)\geqq 1$. There exists a sequence $\{U_{\mu_n}\}\subset\mathscr{P}$ such that $\operatorname{supp}\mu_n$ is compact and contained in E and $U_{\mu_n}\to U_\mu$ strongly as $n\to\infty$. We have $\mu(X)\leqq\varliminf_{n\to\infty}\mu_n(X)=\eta$. Taking a subsequence if necessary, we may assume $\mu_n(X)\to\eta$ as $n\to\infty$. We put $\nu_n=\mu_n/\mu_n(X)$. Then $U_{\nu_n}\in\mathscr{P}^i_{E,1}$. Since $0\notin\mathscr{P}^i_{E,1}$ by Theorem 5.2, we see that η is finite. Therefore $U_\mu/\eta=\lim_{n\to\infty}U_{\nu_n}\in\mathscr{P}^i_{E,1}$. Since $\eta\geqq 1$, $U_\mu=\eta U_\mu/\eta\in\mathscr{P}^i_{E,1}$. Thus $\mathscr{P}^i_E\cap\mathscr{P}_{X,1}\subset\mathscr{P}^i_{E,1}$ Next let $0<C^e(E)<\infty$. Then we have

$$\begin{aligned}\mathscr{P}^e_E\cap\mathscr{P}_{X,1}&=\bigcap_{\substack{\omega\supset E\\ C(\omega)<\infty}}\mathscr{P}^i_\omega\cap\mathscr{P}_{X,1}=\bigcap_{\substack{\omega\supset E\\ C(\omega)<\infty}}(\mathscr{P}^i_\omega\cap\mathscr{P}_{X,1})\\&=\bigcap_{\substack{\omega\supset E\\ C(\omega)<\infty}}\mathscr{P}^i_{\omega,1}=\mathscr{P}^e_{E,1}\,.\end{aligned}$$

Now we obtain a result corresponding to Theorem 4.3.

THEOREM 5.5. *For* $u\in\mathscr{D}$ *and* $E\subset X$ *with* $C^i(E)<\infty$ *(resp.* $C^e(E)<\infty$*), the following assertions* (a_i), (b_i) *and* (c_i) *(resp.* (a_e), (b_e) *and* (c_e)*) are mutually equivalent:*

(a_i) $u^*(x)\geqq 1$ *n.e. on* E,	(a_e) $u^*(x)\geqq 1$ *q.e. on* E,
(b_i) $u^*(x)\geqq 1$ μ*-a.e. on* X *for all* $U_\mu\in\mathscr{P}^i_{E,1}$,	(b_e) $u^*(x)\geqq 1$ μ*-a.e. on* X *for all* $U_\mu\in\mathscr{P}^e_{E,1}$,
(c_i) $(u,\ U_\mu)\geqq 1$ *for all* $U_\mu\in\mathscr{P}^i_{E,1}$,	(c_e) $(u,\ U_\mu)\geqq 1$ *for all* $U_\mu\in\mathscr{P}^e_{E,1}$.

PROOF. $(a_i)\Leftrightarrow(c_i)$ and $(a_e)\Leftrightarrow(c_e)$ are nothing but Theorem 3.1. Since $(u, U_\mu)=\int u^* d\mu$, $(b_i)\Rightarrow(c_i)$ and $(b_e)\Rightarrow(c_e)$ are trivial. Assume (c_i) (resp. (c_e)). Put $E'=\{x\in X;\ u^*(x)<1\}$ and suppose that there is $U_\mu\in\mathscr{P}^i_{E,1}$ (resp. $\mathscr{P}^e_{E,1}$) such that $\mu(E')>0$. Since $0\notin\mathscr{P}^i_{E,1}$ (resp. $0\notin\mathscr{P}^e_{E,1}$) by Theorem 5.2, we see that $\mu(X)$ is finite. Put $\mu'=\mu(E')^{-1}\mu|_{E'}$. Then by Proposition 4.2 and Lemma 5.1, we see that $U_{\mu'}\in\mathscr{P}^i_{E,1}$ (resp. $\mathscr{P}^e_{E,1}$). On the other hand, $(u, U_{\mu'})=\int u^* d\mu'<1$, which contradicts (c_i) (resp. (c_e)). Hence $(c_i)\Rightarrow(b_i)$ (resp. $(c_e)\Rightarrow(b_e)$).

§6. Balayable sets

We shall say that a set E is O-capacitable if $E\cap\omega$ is capacitable for any open set ω, and balayable if $\mathscr{P}^i_E=\mathscr{P}^e_E$. We shall investigate relations between these notions. First, from Lemma 5.1 and Theorem 5.2, we obtain

PROPOSITION 6.1. *A set E is balayable if and only if $\mathscr{P}^i_{E,1}=\mathscr{P}^e_{E,1}$.*

COROLLARY. *A balayable set is capacitable.*

REMARK. A capacitable set is not necessarily balayable (cf. [3, 35. 1. Example]).

We obtain the strong sub-additivity of outer capacity by virtue of Theorems 5.3 and 5.4 (cf. [6, p. 163] for proofs). Hence the union of two capacitable sets is again capacitable (cf. [3, p. 219]).

THEOREM 6.1. *An O-capacitable set is balayable.*

PROOF. Let E be an O-capacitable set. By Lemma 1.1 and Theorem 3.1, $(\mathscr{U}^i_E)^0=\mathscr{P}^i_E$ and $(\mathscr{U}^e_E)^0=\mathscr{P}^e_E$. Hence it is sufficient to prove $\mathscr{U}^i_E=\mathscr{U}^e_E$. Obviously $\mathscr{U}^e_E\subset\mathscr{U}^i_E$. Let $u\in\mathscr{U}^i_E$. We put $F=\{x\in E;\ u^*(x)<0\}$. Then $C^i(F)=0$. From the quasi-continuity of u^*, for any $\varepsilon>0$, there exists an open set ω such that $C(\omega)<\varepsilon$ and $u^*|_{\omega^c}$ is continuous. Then $\omega'=\omega\cup\{x\in\omega^c;\ u^*(x)<0\}$ is an open set. Since $F\cup\omega=(E\cap\omega')\cup\omega$, $F\cup\omega$ is capacitable by the assumption of the theorem. Hence, by Lemma 2.2,

$$C^e(F)\leqq C^e(F\cup\omega)=C^i(F\cup\omega)\leqq C^i(F)+C(\omega)=C(\omega)<\varepsilon.$$

Thus $C^e(F)=0$, and hence $u\in\mathscr{U}^e_E$.

COROLLARY. *Any* K-*analytic set is balayable.*

PROOF. A K-analytic set is obviously O-capacitable (cf. [3, p. 139]).

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