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Note on Combinatorial Arrangements

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1. Introduction

A connected block design is said to be balanced if every elementary contrast is estimated with the same variance. In experimenting, the statistical usefulness of block designs with this "*balanced*" property can be seen in the definition itself. Discussions of constructions of balanced block (**BB**) designs appear in some articles throughout the literatures (cf., [3], [4], [6], [7], [8]). Especially, Kageyama [6] has extensively dealt with combinatorial properties and constructions of **BB** designs, and [7] has given miscellaneous further methods of constructing **BB** designs.

This note is concerned primarily with problems related to the articles [6] and [7] due to Kageyama. In Section 2, some further methods of constructing **BB** designs are given. They have in a sense trivial structures. In Section 3, new methods of constructing **BB** designs, balanced incomplete block (**BIB**) designs and partially balanced incomplete block (**PBIB**) designs are shown with some examples. A general bound on the latent roots for the C-matrix of a block design is presented in Section 4 which includes a correction to Kageyama [6].

Many of block designs generated by the methods of construction given here may have large values of the total number of blocks. However, they may be of both theoretical and practical importance for various aspects of experimental designs (cf., [3], [7], [9], [13]).

The definitions of block designs and notations used are coincident with those generally used, and especially with those of Kageyama [6] and [7].

2. Constructions of BB designs

This section presents some methods of constructing **BB** designs with unequal block sizes, since a **BB** design with a constant block size is a **BIB** design [6, Theorem 13.1].

THEOREM 2.1. *If there exists a* **BB** *design M with parameters* v, b, r_i, k_j, $n=\sum_{i=1}^{v} r_i=\sum_{j=1}^{b} k_j$, $\rho=(n-b)/(v-1)$ *for nonnegative integers* s, t, u *and* w *such* *that $u=t-w\rho+(sv)/2$, then there exists a **BB** design N with parameters $v'=v+1$, $b'=u+(s+t)v+wb$, $r'_p=(s+t)v$ or $s+(v-1)t+u+wr_i$, $k'_q=2$, v or k_j ($p=1, 2,\dots, v'$; $q=1, 2,\dots, b'$):*

$$N=\Bigg[\underbrace{\begin{array}{c:c:c}E_{1\times v} & \cdots & E_{1\times v}\\ \hdashline I_v & \cdots & I_v\end{array}}_{s\ times}\Bigg\vdots\underbrace{\begin{array}{c:c:c}E_{1\times v} & \cdots & E_{1\times v}\\ \hdashline G_v-I_v & \cdots & G_v-I_v\end{array}}_{t\ times}\Bigg\vdots\underbrace{\begin{array}{c:c:c}O_{1\times 1} & \cdots & O_{1\times 1}\\ \hdashline E_{v\times 1} & \cdots & E_{v\times 1}\end{array}}_{u\ times}$$

$$\Bigg\vdots\underbrace{\begin{array}{c:c:c}O_{1\times b} & \cdots & O_{1\times b}\\ \hdashline M & \cdots & M\end{array}}_{w\ times}\Bigg].$$

PROOF. For the C-matrix of N, respective constancy of its diagonal elements and off-diagonal elements implies

$$sv/2 = s/2 - t(v-1)/v + (u + w\rho)(v-1)/v$$

and

$$s/2 + t(v-1)/v = t(v-2)/v + u/v + w\rho/v,$$

which can be shown to be equivalent to $u=t-w\rho+(sv)/2$.

For example, Theorem 2.1 is valid for an even integer s or v and an integer w divisible by $v-1$. Note that Theorem 2.1 yields Theorem 3 of [7] when $s=t=w=1$ and $u=m$. Further, note that the juxtapositions of N in Theorem 2.1 and the following matrices clearly yield other **BB** designs:

$$\begin{bmatrix}O_{1\times v}\\ \hdashline I_v\end{bmatrix}, \qquad \begin{bmatrix}O_{(i-1)\times m}\\ \hdashline E_{1\times m}\\ \hdashline O_{(v+1-i)\times m}\end{bmatrix}, i=1, 2,\dots, v+1.$$

Since $\rho=(n-b)/(v-1)=\lambda v/k$ for a **BIB** design M with parameters v, b, r, k and λ, we get

COROLLARY 2.2. *The existence of a **BIB** design M with parameters v, b, r, k and λ for nonnegative integers s, t, u and w such that $u=t+v(sk-2w\lambda)/2k$, implies the existence of a **BB** design with parameters $v'=v+1$, $b'=u+(s+t)v+wb$, $r'_i=(s+t)v$ or $s+(v-1)t+u+wr$, $k'_j=2$, k or v ($i=1, 2,\dots, v'$; $j=1, 2,\dots, b'$).*

Examples of Corollary 2.2 can be given if there exists a **BIB** design with parameters v, b, r, k and λ such that v is divisible by $2k$. As examples of series of **BIB** designs satisfying this restriction, we can provide the following:

(1) (E_1) series of Bose [2]: $v=6t+6$, $b=2(t+1)(6t+5)$, $r=6t+5$, $k=3$, $\lambda=2$.

(2) (B_1) series of Bose [2]: $v=2\lambda+2$, $b=4\lambda+2$, $r=2\lambda+1$, $k=\lambda+1$, λ ($4\lambda+3$ is a prime or a prime power; $2\lambda+1$ is a prime or a prime power and $\lambda>1$).

(3) $v=p+1$, $b=2p$, $k=(p+1)/2$, $r=p$, $\lambda=(p-1)/2$ [1] (p is an odd prime or an odd prime power).

(4) $v=4t$, $b=2(4t-1)$, $r=4t-1$, $k=2t$, $\lambda=2t-1$ [11] ($4t-1$ is a prime power).

(5) $v=2t$, $b=4(2t-1)$, $r=2(2t-1)$, $k=t$, $\lambda=2t-2$ [11] ($2t-1$ is a prime power).

(6) $v=(s+1)(s^2+1)$, $b=(s^2+1)(s^2+s+1)$, $r=s^2+s+1$, $k=s+1$, $\lambda=1$ (s is a prime or a prime power $\geqq 3$).

(7) Existing resolvable **BIB** designs with parameters $v=nk$, $b=nr$, $r=n^2t+n+1$, $k=n[(n-1)t+1]$, $\lambda=nt+1$ (n is an even integer $\geqq 2$, $t\geqq 0$).

We can also find individual examples of **BIB** designs such that v is divisible by $2k$, from many known series of **BIB** designs and tables of **BIB** designs with the parameters of practically useful range (cf., [5], [12]).

When $w=0$ in Theorem 2.1, we have

COROLLARY 2.3. *For positive integers v, s, t and u such that $u=t+(vs)/2$, the following incidence matrix is a* ***BB*** *design with parameters $v'=v+1$, $b'=(s+t)v+u$, $r_i=(s+t)v$ or $s+(v-1)t+u$, $k_j=2$ or v $(i=1, 2,\ldots, v'; j=1, 2,\ldots, b')$:*

$$\left[\begin{array}{ccc:ccc:ccc} E_{1\times v} & \cdots & E_{1\times v} & E_{1\times v} & \cdots & E_{1\times v} & O_{1\times 1} & \cdots & O_{1\times 1} \\ \hdashline \underbrace{I_v \;\; \cdots \;\; I_v}_{s\ times} & & & \underbrace{G_v-I_v \;\; \cdots \;\; G_v-I_v}_{t\ times} & & & \underbrace{E_{v\times 1} \;\; \cdots \;\; E_{v\times 1}}_{u\ times} & & \end{array}\right].$$

Note that Corollary 2.3 is always valid for an even integer v or s, and that Theorem 2 of [7] is a special case of Corollary 2.3 when $s=t=1$ and $u=m$.

As mentioned in Kageyama [6] and [7], we can give balancing conditions for various types (e.g., juxtaposition, composition, and so on) of incidence matrices similar to the matrix presented in Corollary 2.3. As a slight modification of Theorem 6 of [7], we obtain

THEOREM 2.4. *For nonnegative integers $l(\geqq 3)$, m, e, f, g, h, i and j such that*

(i) $g/(l+1)+i/2=h/(l+1)+j/2$,
(ii) $(e-g)/(l+1)=f(1-l)/3+i/2$,
(iii) $(e-g)/(l+1)=m/l-f/3-j/2$, *and*
(iv) $(2f/3+i/2)l=(hl+2e)/(l+1)-m/l-e+f+i+j+m$,

the following incidence matrix N is a ***BB*** *design with parameters $v=l+2$, $b=(e+f+i+j)l+g+h+m$, $r_p=(e+f+i)l+g$, $(e+f+j)l+h$ or $(l-1)e+g+h$*

$+f+i+j+m$, $k_q=2, 3, l$ *or* $l+1$ $(p=1, 2,\dots, v;\ q=1, 2,\dots, b)$:

$$N=\left[\underbrace{\begin{array}{c:c:c} \multicolumn{3}{c}{} \\ E_{2\times l} & \cdots & E_{2\times l} \\ \hdashline G_l-I_l & \cdots & G_l-I_l \end{array}}_{e\ times}\;\underbrace{\begin{array}{c:c:c} \multicolumn{3}{c}{} \\ E_{2\times l} & \cdots & E_{2\times l} \\ \hdashline I_l & \cdots & I_l \end{array}}_{f\ times}\;\underbrace{\begin{array}{c:c:c} E_{1\times 1} & \cdots & E_{1\times 1} \\ \hdashline O_{1\times 1} & \cdots & O_{1\times 1} \\ \hdashline E_{l\times 1} & \cdots & E_{l\times 1} \end{array}}_{g\ times}\right.$$

$$\left.\underbrace{\begin{array}{c:c:c} O_{1\times 1} & \cdots & O_{1\times 1} \\ \hdashline E_{1\times 1} & \cdots & E_{1\times 1} \\ \hdashline E_{l\times 1} & \cdots & E_{l\times 1} \end{array}}_{h\ times}\;\underbrace{\begin{array}{c:c:c} E_{1\times l} & \cdots & E_{1\times l} \\ \hdashline O_{1\times l} & \cdots & O_{1\times l} \\ \hdashline I_l & \cdots & I_l \end{array}}_{i\ times}\;\underbrace{\begin{array}{c:c:c} O_{1\times l} & \cdots & O_{1\times l} \\ \hdashline E_{1\times l} & \cdots & E_{1\times l} \\ \hdashline I_l & \cdots & I_l \end{array}}_{j\ times}\;\begin{array}{c} O_{2\times m} \\ \hdashline E_{l\times m} \end{array}\right].$$

PROOF. For the C-matrix of N, respective constancy of its diagonal elements and off-diagonal elements yields the required conditions (i), (ii), (iii) and (iv).

REMARK. For nonnegative integers α, β and γ such that $(l-2)\alpha=3\beta+\gamma$, the combinations of values, $i=j$, $g=h$, $f=3\alpha$, $i=2\beta$, $m=l[(2-l)\alpha+2\beta]$, $h=(l+1)\gamma$ and $e=(l+1)[\gamma+\alpha(1-l)+\beta]$ satisfy conditions (i), (ii), (iii) and (iv).

3. Special constructions of block designs

Method A. We define the column sum mod 2, $CS(N)$, of a $v\times b$ incidence matrix N as a $v\times\binom{b}{2}$ matrix consisting of all column vectors formed by adding the corresponding elements of all pairs of b blocks of the incidence matrix N mod 2. For example, the column sum mod 2, $CS(N)$, of a **BIB** design N with parameters $v=4$, $b=6$, $r=3$, $k=2$ and $\lambda=1$ is given as follows:

$$N=\begin{bmatrix} 1&1&1&0&0&0 \\ 1&0&0&1&1&0 \\ 0&1&0&1&0&1 \\ 0&0&1&0&1&1 \end{bmatrix},\qquad CS(N)=\begin{bmatrix} 0&0&1&1&1&0&1&1&1&1&1&1&0&0&0 \\ 1&1&0&0&1&0&1&1&0&1&1&0&0&1&1 \\ 1&0&1&0&1&1&0&1&0&1&0&1&1&0&1 \\ 0&1&0&1&1&1&0&1&1&1&0&0&1&1&0 \end{bmatrix}.$$

Note that this $CS(N)$ is a **BB** design.

We obviously obtain

LEMMA 3.1. *The column sum mod 2 of an incomplete block design N with parameters v, b, r and k yields an equireplicate block design with parameters $v^*=v$, $b^*=\binom{b}{2}$ and $r^*=r(b-r)$, whose block sizes are $2(k-l_1)$, $2(k-l_2),\dots, 2(k-l_p)$ when block intersection numbers of N are l_1, $l_2,\dots, l_p$, respectively $(0\leqq l_i\leqq k,\ i=1, 2,\dots, p)$.*

LEMMA 3.2. *The column sum mod 2 of a **BIB** design N with parameters v, b, r, k and λ yields an equireplicate pairwise balanced design $CS(N)$ with* *parameters* $v^*=v$, $b^*=\binom{b}{2}$, $r^*=r(b-r)$ *and* $\lambda^*=\lambda[b-2(r-\lambda)-\lambda]+(r-\lambda)^2$.

PROOF. It is obvious that $v^*=v$, $b^*=\binom{b}{2}$ and $r^*=r(b-r)$. Noting that the frequencies of the ordered 2-plets (0, 0) and (0, 1) as columns in any 2-rowed submatrix of N are $b-2(r-\lambda)-\lambda$ and $r-\lambda$, respectively, the column sum mod2 process in N shows that the coincidence number of $CS(N)$ is $\lambda^*=\lambda[b-2(r-\lambda)-\lambda]+(r-\lambda)^2$.

From a point of view of constructing **BB** designs, it is useful to note that an equireplicate pairwise balanced design $CS(N)$ given by Lemma 3.2 may become a **BB** design with unequal block sizes. As mentioned in Lemma 3.1, values of these block sizes depend on the block structure of the original **BIB** design N. However, **BB** designs constructed in this way may have in a sense trivial structures and have large values of the total number of blocks. Three examples of Lemma 3.2 are presented.

EXAMPLE 1. As illustrated above, the design derived from a **BIB** design with parameters $v=4$, $b=6$, $r=3$, $k=2$ and $\lambda=1$ is a **BB** design with parameters $v^*=4$, $b^*=15$, $r^*=9$, $k_j^*=2, 4$, whose structure is the juxtaposition of three complete blocks and two copies of the original **BIB** design.

EXAMPLE 2. The design derived from a **BIB** design with parameters $v=6$, $b=10$, $r=5$, $k=3$ and $\lambda=2$ [12] is a **BB** design with parameters $v^*=6$, $b^*=45$, $r^*=25$, $k_j^*=2, 4$, whose structure is the juxtaposition of two kinds of unreduced **BIB** designs.

EXAMPLE 3. The design derived from a **BIB** design with parameters $v=9$, $b=12$, $r=4$, $k=3$ and $\lambda=1$ [12] is a **BB** design with parameters $v^*=9$, $b^*=66$, $r^*=32$, $k_j^*=4, 6$, whose structure is the juxtaposition of the complement of the original **BIB** design and a **BIB** design with parameters $v=9$, $b=54$, $r=24$, $k=4$ and $\lambda=9$ which should be noted to be not the three copies of a **BIB** design with parameters $v=9$, $b=18$, $r=8$, $k=4$ and $\lambda=3$.

A block design is said to be a linked block design if every pair of blocks has exactly μ treatments in common. Note that if N in Lemma 3.1 is a linked block design (i.e., $l_1=l_2=\cdots=l_p=\mu$), then $CS(N)$ has a constant block size $k^*=2(k-\mu)$. For a symmetrical **BIB** design with parameters v, k and λ every pair of whose blocks has exactly λ treatments in common, we have

THEOREM 3.3. *The existence of a symmetrical* ***BIB*** *design with parameters v, k and λ implies the existence of a* ***BIB*** *design with parameters*:

$$v^* = v,\ b^* = \binom{v}{2},\ r^* = (v-1)(k-\lambda),\ k^* = 2(k-\lambda),$$
$$\lambda^* = (k-\lambda)[2(k-\lambda)-1].$$

PROOF. Consider the column sum mod 2 of a symmetrical **BIB** design with parameters v, k and λ. The parameters v^*, b^* and k^* need no proof. From Lemma 3.2 we have $r^*=k(v-k)$ and $\lambda^*=\lambda[v-2(k-\lambda)-\lambda]+(k-\lambda)^2$ which can be shown to be equal to $(v-1)(k-\lambda)$ and $(k-\lambda)[2(k-\lambda)-1]$, respectively, since $\lambda(v-1)=k(k-1)$.

REMARK. Theorem 3.3 shows that the existence of a trivial symmetrical **BIB** design with parameters v, $k=v-1$ and $\lambda=v-2$ implies the existence of an unreduced **BIB** design with parameters $v^*=v$, $b^*=\binom{v}{2}$, $r^*=v-1$, $k^*=2$ and $\lambda^*=1$.

It is noteworthy that applications of Theorem 3.3 to known series or individual examples of symmetrical **BIB** designs yield many other series or individual examples of **BIB** designs which may be new. For example, since it is known (cf., [2]) that when $4t+3$ is a prime or a prime power there exists a symmetrical **BIB** design with parameters $v=4t+3$, $k=2t+1$ and $\lambda=t$, we get a **BIB** design with parameters $v^*=4t+3$, $b^*=(2t+1)(4t+3)$, $r^*=2(t+1)(2t+1)$, $k^*=2(t+1)$, $\lambda^*=(t+1)(2t+1)$, the complement of which implies

COROLLARY 3.4. *There exists a **BIB** design with parameters*

$$v^*=4t+3,\ b^*=(2t+1)(4t+3),\ r^*=(2t+1)^2,\ k^*=2t+1,\ \lambda^*=t(2t+1)$$

for a prime or a prime power $4t+3$.

Similarly, the existence of a symmetrical **BIB** design with parameters $v=(q^{n+1}-1)/(q-1)$, $k=(q^n-1)/(q-1)$ and $\lambda=(q^{n-1}-1)/(q-1)$ obtained from a finite projective n-dimensional geometry over a Galois field GF(q) (cf., [2], [9]) yields

COROLLARY 3.5. *When q is a prime or a prime power, there exists a **BIB** design with parameters*:

$$v^*=\frac{q^{n+1}-1}{q-1},\quad b^*=\frac{q(q^n-1)(q^{n+1}-1)}{2(q-1)^2},\quad r^*=\frac{q^n(q^n-1)}{q-1},\quad k^*=2q^{n-1},$$
$$\lambda^*=q^{n-1}(2q^{n-1}-1),$$

for $n\geqq 2$.

We now give another theorem which can be easily proved following the arguments in Lemma 3.2 and Theorem 3.3.

THEOREM 3.6. *The column sum mod 2 of an m-associate **PBIB** design N with parameters v, b, r, k, λ_1, $\lambda_2,\dots,\lambda_m$, yields an equireplicate block design $CS(N)$ with parameters* $v^*=v$, $b^*=\binom{b}{2}$, $r^*=r(b-r)$, $\lambda_i^*=\lambda_i[b-2(r-\lambda_i)-\lambda_i]$

$+(r-\lambda_i)^2$. *Furthermore, if N is a linked block design with the block intersection number μ, then $CS(N)$ is a* **PBIB** *design, having the same association scheme as N, with parameters*:

$$v^* = v,\ b^* = \binom{b}{2},\ r^* = r(b-r),\ k^* = 2(k-\mu),$$
$$\lambda_i^* = \lambda_i[b - 2(r-\lambda_i) - \lambda_i] + (r-\lambda_i)^2, \qquad i = 1, 2, \ldots, m,$$

in which $\mu = k(r-1)/(b-1)$.

For **PBIB** designs of the linked block type, we refer to Roy and Laha [10] which also includes the list of 2-associate **PBIB** designs of the linked block type.

Method B. We can define the row sum mod 2, $RS(N)$, of a $v \times b$ incidence matrix N as a $\binom{v}{2} \times b$ matrix consisting of all row vectors formed by adding the corresponding elements of all pairs of v rows of the incidence matrix N mod 2. For example, the row sum mod 2, $RS(N)$, of a **BIB** design N with parameters $v=6$, $b=10$, $r=5$, $k=3$ and $\lambda=2$ is given as follows:

$$N = \begin{bmatrix} 1&0&0&0&1&1&0&1&0&1 \\ 1&1&0&0&0&1&1&0&1&0 \\ 0&1&1&0&0&0&1&1&0&1 \\ 0&0&1&1&0&1&0&1&1&0 \\ 0&0&0&1&1&0&1&0&1&1 \\ 1&1&1&1&1&0&0&0&0&0 \end{bmatrix},$$

Association Scheme

x	1	2	3	4	5
1	x	6	7	8	9
2	6	x	10	11	12
3	7	10	x	13	14
4	8	11	13	x	15
5	9	12	14	15	x

$$RS(N) = \begin{array}{r} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \end{array} \begin{bmatrix} 0&1&0&0&1&0&1&1&1&1 \\ 1&1&1&0&1&1&1&0&0&0 \\ 1&0&1&1&1&0&0&0&1&1 \\ 1&0&0&1&0&1&1&1&1&0 \\ 0&1&1&1&0&1&0&1&0&1 \\ 1&0&1&0&0&1&0&1&1&1 \\ 1&1&1&1&0&0&1&1&0&0 \\ 1&1&0&1&1&1&0&0&0&1 \\ 0&0&1&1&1&1&1&0&1&0 \\ 0&1&0&1&0&1&1&0&1&1 \\ 0&1&1&1&1&0&0&1&1&0 \\ 1&0&0&1&1&0&1&1&0&1 \\ 0&0&1&0&1&1&1&1&0&1 \\ 1&1&0&0&1&1&0&1&1&0 \\ 1&1&1&0&0&0&1&0&1&1 \end{bmatrix}.$$

Note that this $RS(N)$ is a 2-associate **PBIB** design with parameters $v'=15$, $b'=10$, $r'=6$, $k'=9$, $\lambda_1=3$, $\lambda_2=4$, having the above triangular association scheme.

We obviously obtain the following.

LEMMA 3.7. *The row sum mod 2 of an incomplete block design N with* *parameters* v, b, r *and* k *yields a block design with parameters* $v'=\binom{v}{2}$, $b'=b$ *and* $k'=k(v-k)$, *whose replication numbers are* $2(r-m_1)$, $2(r-m_2),\dots, 2(r-m_p)$ *when coincidence numbers of* N *are* $m_1, m_2,\dots, m_p$, *respectively* $(0\leqq m_i\leqq r$, $i=1, 2,\dots, p)$.

LEMMA 3.8. *The row sum mod 2 of a* **BIB** *design* N *with parameters* v, b, r, k *and* λ *yields an incomplete block design* $RS(N)$ *with parameters* $v'=\binom{v}{2}$, $b'=b$, $r'=2(r-\lambda)$ *and* $k'=k(v-k)$. *Furthermore, if* N *is symmetrical, then* $RS(N)$ *has the constant block intersection number* $(k-\lambda)[2(k-\lambda)-1]$, *that is,* $RS(N)$ *is the design of the linked block type.*

Note from Lemma 3.8 that when N is a symmetrical **BIB** design, the parameters of the dual of $RS(N)$ coincides with those of the **BIB** design constructed by Theorem 3.3.

As illustrated above, the row sum mod 2 of a **BIB** design with parameters v, b, r, k and λ *often* yields a 2-associate **PBIB** design with parameters $v'=\binom{v}{2}$, $b'=b$, $r'=2(r-\lambda)$, $k'=k(v-k)$, $\lambda_1=r-\lambda$, $\lambda_2=[4(r-\lambda)(k-1)(v-k-1)]/[(v-2)(v-3)]$, having a triangular association scheme. This observation may be useful to construct triangular **PBIB** designs, especially, designs of the linked block type.

CONCLUDING REMARK. In the operator of column (or row) sum mod 2 described above, we consider the addition of *all* pairs of column (or row) vectors of the incidence matrix of a block design. Without adding *all* pairs, we can also consider the addition of $b/2$ (or $v/2$) disjoint pairs of b column (or v row) vectors of the $v\times b$ incidence matrix of a block design. This operator is valid when b (or v) is even. Empirically, the application of this operator to **BIB** designs often yields **BB** designs. We can apply this operator to various block designs in order to produce other block designs. Operators similar to the above can be variously defined. Further discussions about these operators will appear in a later paper.

4. Inequality

In Kageyama [6], there needs some corrections for Theorem 18.5 on page 614. The last line in Theorem 18.5 should read:

$$\text{“}0 < \rho_l \leqq \max_{1\leqq i\leqq v} r_i, \qquad l = 1, 2,\dots, m.\text{”}$$

Furthermore, concerning its proof, the lines 9–12 on page 615 should read:

$$\text{“}\rho_l \leqq \max_{1\leqq i\leqq v} r_i, \qquad l = 0, 1,\dots, m \quad (\leqq v-1),$$

since it is known (cf., [60]) that $0 \leqq \mathrm{ch}_l(CD_r^{-1}) \leqq 1$ for $l=0, 1, \ldots, v-1$."

Note that this correction of Theorem 18.5 of [6] immediately shows that for a connected block design N with parameters v, b, r_i, k_j ($i=1, 2, \ldots, v; j=1, 2, \ldots, b$) latent roots ρ_l's of the matrix $C(=D_r - ND_k^{-1}N')$ satisfy $0 \leqq \rho_l \leqq \max_{1 \leqq i \leqq v} r_i$. Furthermore, when a block design N is disconnected, we can put N without loss of generality as

$$N = \begin{bmatrix} N_1 & O \\ O & N_2 \end{bmatrix},$$

where N_i's ($i=1, 2$) are connected components. Then the C-matrix of N can be shown to be

$$C = \begin{bmatrix} C_1 & O \\ O & C_2 \end{bmatrix},$$

where C_i's ($i=1, 2$) are C-matrices of N_i's. This matrix form of C implies that a condition of connectedness of a block design can be deleted from the assumption described above. We therefore get

THEOREM 4.1. *For a block design N with parameters v, b, r_i, k_j ($i=1, 2, \ldots, v; j=1, 2, \ldots, b$) latent roots ρ_l's of the matrix $C(=D_r - ND_k^{-1}N')$ satisfy the following inequality*:

$$0 \leqq \rho_l \leqq \max_{1 \leqq i \leqq v} r_i.$$

This fact may be powerful in various discussions of incomplete block designs. Note that Theorem 4.1 remains valid for a **BB** design or a **PBB** design.

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