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On Removable Singularities for Polyharmonic Distributions

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1. Introduction

Throughout this paper let $1<p<\infty$, $1/p+1/q=1$ and m be a positive integer. For an open set G in the n-dimensional Euclidean space R^n, we denote by $BL_m(L^q(G))$ the space of all distributions on G whose distributional derivatives of order m are all in $L^q(G)$, that is, a distribution T on G belongs to $BL_m(L^q(G))$ if and only if

$$|T|_{m,q}=|T|_{m,q,G}=(\sum_{|\alpha|=m}\|D^\alpha T\|^q_{L^q(G)})^{1/q}<\infty,$$

where α is an n-tuple $(\alpha_1, \alpha_2,\dots, \alpha_n)$ of non-negative integers with length $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$, $D^\alpha=\partial^{|\alpha|}/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}$ and $\|\cdot\|_{L_q(G)}$ denotes the L^q-norm on G. We write simply $\|\cdot\|_q$ for $\|\cdot\|_{L^q(R^n)}$. We denote by Δ^m the Laplace operator iterated m times and write simply Δ for Δ^1. The value of a distribution T on G at $\varphi\in C_0^\infty(G)$ is denoted by $<T, \varphi>$.

Let E be a compact set in R^n. L. I. Hedberg proved the following result ([5; Theorem 1]): Let $\mathscr{C}$ be the space $C_0^\infty(R^n\backslash E)$ or the space of all functions $\varphi\in C_0^\infty(R^n)$ such that $|\mathrm{grad}\,\varphi|=0$ on a neighborhood of E. Then $\mathscr{C}$ is dense in $C_0^\infty(R^n)$ with respect to the norm $|\cdot|_{1,p}$ if and only if any $T\in BL_1(L^q(R^n))$ such that $<T, \Delta\varphi>=0$ for any $\varphi\in\mathscr{C}$ is harmonic on R^n. We generalize this result as follows:

THEOREM 1. *Let $\mathscr{C}$ and $\mathscr{C}'$ be subspaces of $C_0^\infty(R^n)$ such that $\mathscr{C}\subset\mathscr{C}'$. Then $\mathscr{C}$ is dense in $\mathscr{C}'$ with respect to the norm $|\cdot|_{m,p}$ if and only if any $T\in BL_m(L^q(R^n))$ such that $<T, \Delta^m\varphi>=0$ for any $\varphi\in\mathscr{C}$ satisfies $<T, \Delta^m\psi>=0$ for any $\psi\in\mathscr{C}'$.*

As an application of this theorem, we shall give a condition, in terms of capacity, for a compact set in R^n to be removable for a class of polyharmonic distributions.

2. Proof of Theorem 1

We first suppose that $\mathscr{C}$ is dense in $\mathscr{C}'$ with respect to $|\cdot|_{m,p}$. We write

$\Delta^m = \sum_{|\alpha|=m} c_\alpha D^{2\alpha}$ with constants c_α. Let T be a distribution in $BL_m(L^q(R^n))$ such that $<T, \Delta^m\varphi> = 0$ for any $\varphi \in \mathscr{C}$. Let $\psi \in \mathscr{C}'$. Then there is a sequence $\{\varphi_j\} \subset \mathscr{C}$ such that $|\varphi_j - \psi|_{m,p} \to 0$ as $j \to \infty$. Hence we have

$$
\begin{aligned}
<T, \Delta^m\psi> &= (-1)^m \sum_{|\alpha|=m} c_\alpha <D^\alpha T, D^\alpha\psi> \\
&= \lim_{j\to\infty} (-1)^m \sum_{|\alpha|=m} c_\alpha <D^\alpha T, D^\alpha\varphi_j> = \lim_{j\to\infty} <T, \Delta^m\varphi_j> = 0.
\end{aligned}
$$

Next we show the converse assertion. Suppose $\mathscr{C}$ is not dense in $\mathscr{C}'$ with respect to $|\cdot|_{m,p}$. Then there is a function $u_0 \in \mathscr{C}'$ such that $M = \inf\{|u_0 - \varphi|_{m,p}^p; \varphi \in \mathscr{C}\} > 0$. Set

$$\Phi(u) = |u|_{m,p}^p, \qquad u \in BL_m(L^p(R^n)).$$

Then there is a sequence $\{u_j\} \subset \mathscr{C}$ such that $M = \lim_{j\to\infty} \Phi(u_j - u_0)$. Since $\{\Phi(u_j)\}$ is bounded, we may assume that

$$D^\alpha u_j \to u^{(\alpha)} \qquad \text{weakly in} \quad L^p(R^n) \quad \text{as} \quad j \to \infty$$

for each α with $|\alpha| = m$. Hence there is a sequence $\{a_{i,k}\}_{i=1,\dots,i_k;\, k=1,2,\dots}$ of non-negative numbers such that

$$
\begin{gathered}
a_{1,k} + a_{2,k} + \cdots + a_{i_k,k} = 1, \\
a_{1,k}D^\alpha u_1 + a_{2,k}D^\alpha u_2 + \cdots + a_{i_k,k}D^\alpha u_{i_k} \to u^{(\alpha)}
\end{gathered}
$$

strongly in $L^p(R^n)$ as $k \to \infty$ for any α with $|\alpha| = m$. Consequently

$$
\begin{aligned}
M &\leqq \lim_{k\to\infty} \Phi\Big(\sum_{i=1}^{i_k} a_{i,k}u_k - u_0\Big) = \sum_{|\alpha|=m} \|u^{(\alpha)} - D^\alpha u_0\|_p^p \\
&\leqq \lim_{j\to\infty} \Phi(u_j - u_0) = M,
\end{aligned}
$$

so that $M = \sum_{|\alpha|=m} \|u^{(\alpha)} - D^\alpha u_0\|_p^p$ and $\|D^\alpha u_j - D^\alpha u_0\|_p \to \|u^{(\alpha)} - D^\alpha u_0\|_p$ as $j \to \infty$ for any α with $|\alpha| = m$. It follows that $D^\alpha u_j \to u^{(\alpha)}$ strongly in $L^p(R^n)$ as $j \to \infty$. It is easy to check that for any $\varphi \in \mathscr{C}$

$$(1) \qquad \sum_{|\alpha|=m} \int |u^{(\alpha)} - D^\alpha u_0|^{p-2}(u^{(\alpha)} - D^\alpha u_0) D^\alpha\varphi\, dx = 0.$$

We set

$$
\begin{aligned}
h^{(\alpha)} &= |u^{(\alpha)} - D^\alpha u_0|^{p-2}(u^{(\alpha)} - D^\alpha u_0), \\
h_j^{(\alpha)} &= |D^\alpha u_j - D^\alpha u_0|^{p-2}(D^\alpha u_j - D^\alpha u_0), \\
T_j &= (-1)^m \sum_{|\alpha|=m} D^\alpha h_j^{(\alpha)}, \quad U_j = K * T_j,
\end{aligned}
$$

where K is the following fundamental solution of Δ^m, i.e.,

$$K(x) = \begin{cases} c|x|^{2m-n}, & \text{in case } n-2m>0 \text{ or} \\ & n \text{ is odd and } n-2m<0; \\ c|x|^{2m-n}\log|x|, & \text{in case } n \text{ is even and } n-2m \leqq 0 \end{cases}$$

with some constant c. Since $D^\alpha u_j \to u^{(\alpha)}$ strongly in $L^p(R^n)$ as $j\to\infty$, we can show that $h_j^{(\alpha)} \to h^{(\alpha)}$ weakly in $L^q(R^n)$ as $j\to\infty$. Furthermore,

$$\lim_{j\to\infty} \|h_j^{(\alpha)}\|_q = \lim_{j\to\infty} \|D^\alpha u_j - D^\alpha u_0\|_p^{p/q}$$
$$= \|u^{(\alpha)} - D^\alpha u_0\|_p^{p/q} = \|h^{(\alpha)}\|_q.$$

From these facts it follows that $h_j^{(\alpha)} \to h^{(\alpha)}$ strongly in $L^q(R^n)$ as $j\to\infty$ for each α with $|\alpha|=m$. Note that

$$U_j(x) = \sum_{|\alpha|=m} \int D^\alpha K(x-y) h_j^{(\alpha)}(y)dy.$$

On account of [6; Lemmas 3.3 and 4.3], $\{U_j\}$ is a Cauchy sequence in $BL_m(L^q(R^n))$, and hence, by [4; Théorème 2.1 in Chap. III] there is $U_0 \in BL_m(L^q(R^n))$ such that $|U_j - U_0|_{m,q} \to 0$ as $j\to\infty$. For any $\varphi \in C_0^\infty(R^n)$, we have

$$<U_0, \Delta^m\varphi> = (-1)^m \sum_{|\alpha|=m} c_\alpha \int D^\alpha U_0(x) D^\alpha\varphi(x)dx$$
$$= (-1)^m \sum_{|\alpha|=m} c_\alpha \lim_{j\to\infty} \int D^\alpha U_j(x) D^\alpha \varphi(x)dx = \lim_{j\to\infty} <\Delta^m U_j, \varphi>$$
$$= \lim_{j\to\infty} <T_j, \varphi> = \lim_{j\to\infty} \sum_{|\alpha|=m} \int h_j^{(\alpha)}(x) D^\alpha\varphi(x)dx$$
$$= \sum_{|\alpha|=m} \int h^{(\alpha)}(x) D^\alpha \varphi(x)dx.$$

Hence,

$$M = \sum_{|\alpha|=m} \|u^{(\alpha)} - D^\alpha u_0\|_p^p = \sum_{|\alpha|=m} \int h^{(\alpha)}(x)\{u^{(\alpha)}(x) - D^\alpha u_0(x)\}dx$$
$$= \sum_{|\alpha|=m} \lim_{j\to\infty} \int h^{(\alpha)}(x)\{D^\alpha u_j(x) - D^\alpha u_0(x)\}dx$$
$$= \lim_{j\to\infty} <U_0, \Delta^m(u_j - u_0)>.$$

By (1), $<U_0, \Delta^m\varphi> = 0$ for all $\varphi \in \mathscr{C}$, while $<U_0, \Delta^m(u_j-u_0)> \neq 0$ for large j. This proves the converse part and thus our theorem is established.

COROLLARY. *Let $\mathscr{C}$ be a subspace of $C_0^\infty(R^n)$. Then $\mathscr{C}$ is dense in $C_0^\infty(R^n)$ with respect to $|\cdot|_{m,p}$ if and only if any $T\in BL_m(L^q(R^n))$ such that $<T, \Delta^m\varphi>=0$ for any $\varphi\in\mathscr{C}$ satisfies $\Delta^m T=0$ on R^n (in the distributional sense).*

3. Removable singularities

For a compact set $E\subset R^n$, we define the capacity

$$\Gamma_{m,p}(E)=\inf\{\|\varphi\|_{m,p}^p;\ \varphi\in C_0^\infty(R^n) \quad\text{and}\quad \varphi(x)\geqq 1 \quad\text{for all}\quad x\in E\},$$

where $\|\varphi\|_{m,p}=(\sum_{|\alpha|\leqq m}\|D^\alpha\varphi\|_p^p)^{1/p}$. Using [1; Theorem A] and [6; Theorem 2.4], we have

LEMMA 1. *Let E be a compact set in R^n. Then $\Gamma_{m,p}(E)=0$ if and only if $C_0^\infty(R^n\backslash E)$ is dense in $C_0^\infty(R^n)$ with respect to $\|\cdot\|_{m,p}$.*

By using Poincaré's inequality (cf. [4; p. 318]), we obtain

LEMMA 2. *Let E be a compact set in R^n. If $C_0^\infty(R^n\backslash E)$ is dense in $C_0^\infty(R^n)$ with respect to $\|\cdot\|_{m,p}$, then $C_0^\infty(G\backslash E)$ is dense in $C_0^\infty(G)$ with respect to $|\cdot|_{m,p}$ for any open set $G\supset E$. Conversely, if $C_0^\infty(G\backslash E)$ is dense in $C_0^\infty(G)$ with respect to $|\cdot|_{m,p}$ for some bounded open set $G\supset E$, then $C_0^\infty(R^n\backslash E)$ is dense in $C_0^\infty(R^n)$ with respect to $\|\cdot\|_{m,p}$.*

We shall show

THEOREM 2. *Let E be a compact set in R^n. If $\Gamma_{m,p}(E)=0$, then for any open set $G\supset E$, any distribution $T\in BL_m(L^q(G))$ such that $\Delta^m T=0$ on $G\backslash E$ satisfies $\Delta^m T=0$ on G. Conversely, if for some bounded open set $G\supset E$, any $T\in BL_m(L^q(G))$ such that $\Delta^m T=0$ on $G\backslash E$ satisfies $\Delta^m T=0$ on G, then $\Gamma_{m,p}(E)=0$.*

PROOF. We first suppose $\Gamma_{m,p}(E)=0$. Let G be an open set in R^n which contains E. By Lemma 1, $C_0^\infty(R^n\backslash E)$ is dense in $C_0^\infty(R^n)$ with respect to $\|\cdot\|_{m,p}$. Hence Lemma 2 implies that $C_0^\infty(G\backslash E)$ is dense in $C_0^\infty(G)$ with respect to $|\cdot|_{m,p}$. Since $\Delta^m T=0$ on $G\backslash E$ (G resp.) if and only if $<T, \Delta^m\varphi>=0$ for any $\varphi\in C_0^\infty(G\backslash E)$ ($C_0^\infty(G)$ resp.), the first assertion in our theorem follows from Theorem 1. The second assertion follows also from Lemmas 1, 2 and Theorem 1.

A function f on an open set $G\subset R^n$ is said to be (m, q)-quasi continuous if given $\varepsilon>0$, there is an open set ω such that $\Gamma_{m,q}(\omega)<\varepsilon$ and f is continuous as a function on $G\backslash\omega$. If $T\in BL_m(L^q(G))$, then there is an (m, q)-quasi continuous function f in $L_{loc}^q(G)$ such that $<T, \varphi>=\int f(x)\varphi(x)dx$ for any $\varphi\in C_0^\infty(G)$ (cf. [6; Lemma 2.3]). We shall say that a function f on G is *ACL* (absolutely continuous on lines) when f is absolutely continuous on each component of the part in G of almost every line parallel to each coordinate axis.

LEMMA 3. *Let k be a positive integer and G be an open set in R^n. If f is a (k, q)-quasi continuous function in $BL_k(L^q(G))$, then f is ACL on G.*

PROOF. Take $\varphi \in C_0^\infty(G)$ and set $u=\varphi \cdot f$. Then $u \in BL_k(L^q(R^n))$ and is (k, q)-quasi continuous on R^n. It suffices to show that u is ACL on R^n. By [6; Theorem 3.1], there is a set $E \subset R^n$ with $\Gamma_{k,q}(E)=0$ such that if $x \in R^n \backslash E$, then

$$\int |x-y|^{k-n}\Big(\sum_{|\alpha|=k}|D^\alpha u(y)|\Big)dy < \infty$$

and

$$u(x) = \sum_{|\alpha|=k} a_\alpha \int \frac{(x-y)^\alpha}{|x-y|^n} D^\alpha u(y)dy, \tag{2}$$

where a_α are constants. With the aid of [3; Lemma at p. 297] and [6; Theorem 2.4], u is seen to be ACL on R^n from the proof of [2; Theorem 1 in § 7] with $G_\alpha g$ replaced by the right-hand side of (2).

By Lemma 3 and [6; Theorem 3.3], we have

COROLLARY. *Let G be an open set in R^n and let f be an (m, q)-quasi continuous function in $BL_m(L^q(G))$. Then f, together with its derivatives of order less than m, is ACL on G.*

LEMMA 4. *If E is a compact set in R^n with $\Gamma_{m,p}(E)=0$ and if $T \in BL_m(L^q(R^n \backslash E))$, then T can be extended to an element in $BL_m(L^q(R^n))$.*

PROOF. Let f be an (m, q)-quasi continuous function in $BL_m(L^q(R^n \backslash E))$ such that $f=T$ in the distributional sense. Consider the function

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in R^n \backslash E \\ 0 & \text{for } x \in E. \end{cases}$$

Then $\tilde{f}$ and its derivatives of order less than m are ACL on R^n in view of Corollary to Lemma 3. It is easy to see that $\tilde{f} \in BL_m(L^q(R^n))$. Thus $\tilde{f}$ gives an extension of T to the whole space.

LEMMA 5. *If any $T \in BL_m(L^q(R^n \backslash E))$ such that $\Delta^m T=0$ on $R^n \backslash E$ can be extended to a distribution $\tilde{T} \in BL_m(L^q(R^n))$ such that $\Delta^m \tilde{T}=0$ on R^n, then the n-dimensional (Lebesgue) measure of E is zero.*

PPOOF. Suppose the n-dimensional measure of E is positive. We consider the function $U(x)=K*\chi_E(x)=\int_E K(x-y)dy$. Then $D^\alpha U \in BL_m(L^q(R^n))$ for

$|\alpha|=m$ according to [6; Lemmas 3.3 and 4.3]. Furthermore

$$\Delta^m(D^\alpha U) = D^\alpha \chi_E = 0 \qquad \text{on} \quad R^n \backslash E.$$

By the assumption, $D^\alpha U$ can be extended to a distribution $T_\alpha \in BL_m(L^q(R^n))$ such that $\Delta^m T_\alpha = 0$ on R^n, where $|\alpha|=m$. In view of [6; Lemma 4.1], T_α is a polynomial. Hence U is equal to a polynomial outside E, which is a contradiction.

We now show

THEOREM 3. *Let E be a compact set in R^n. If $\Gamma_{m,p}(E)=0$, then for any open set $G \supset E$, any $T \in BL_m(L^q(G \backslash E))$ such that $\Delta^m T=0$ on $G \backslash E$ can be extended to a distribution $\tilde{T} \in BL_m(L^q(G))$ such that $\Delta^m \tilde{T}=0$ on G. Conversely, if for some bounded open set $G \supset E$, any $T \in BL_m(L^q(G \backslash E))$ such that $\Delta^m T=0$ on $G \backslash E$ can be extended to a distribution $\tilde{T}$ such that $\Delta^m \tilde{T}=0$ on G, then $\Gamma_{m,p}(E)=0$.*

PROOF. The first assertion follows from Theorem 2 and Lemma 4. To prove the converse part, suppose that any $T \in BL_m(L^q(G \backslash E))$ such that $\Delta^m T=0$ on $G \backslash E$ can be extended to $\tilde{T}$ such that $\Delta^m \tilde{T}=0$ on G. Take any $T^* \in BL_m(L^q(G))$ satisfying $\Delta^m T^*=0$ on $G \backslash E$. Let T^{**} be the restriction of T^* to $G \backslash E$, and $\tilde{T}^{**}$ be an extension of T^{**} such that $\Delta^m \tilde{T}^{**}=0$ on G. Since E is of measure zero by Lemma 5, $\Delta^m T^* = \Delta^m \tilde{T}^{**}$ on G. Thus $\Delta^m T^*=0$ on G. This implies that $\Gamma_{m,p}(E)=0$ by Theorem 2.

REMARK. In case $m=1$, Theorem 3 is a consequence of [5; Theorem 1].

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