HIROSHIMA MATH. J.
9 (1979), 769–785

Positive Solutions of Nonlinear Differential Inequalities

Manabu NAITO

(Received May 18, 1979)

1. Introduction

In this paper we are concerned with positive solutions of differential inequalities of the form

$$x^{(n)}(t) + p(t)f(x(g(t))) \leqq 0, \tag{I}$$

where $n \geqq 2$ and the following conditions are always assumed to hold:

(a) $p(t)$ is continuous, $p(t) \geqq 0$ for $t \geqq a$ and not eventually identically zero;

(b) $g(t)$ is continuously differentiable for $t \geqq a$ and

$$0 < \liminf_{t \to \infty} g'(t) \leqq \limsup_{t \to \infty} g'(t) < \infty;$$

(c) $f(x)$ is continuous and $f(x) > 0$ for $x > 0$.

In the oscillation theory of nonlinear differential equations one of the important problems is to find necessary and sufficient conditions for the equations under consideration to be oscillatory. With regard to the equation

$$x^{(n)}(t) + p(t)f(x(g(t))) = 0, \tag{E}$$

necessary and sufficient conditions in order that every solution of (E) be oscillatory have been established by restricting the nonlinearity $f(x)$ to various classes of functions. When $f(x) = x^{\gamma}$ (γ is a ratio of two positive odd integers and $\gamma \neq 1$), a characterization of oscillation of (E) was obtained in terms of an integral condition by Kiguradze [3] and Ličko and Švec [5]. See also Onose [8] and the references in [8] concerning further developments on this nonlinearity. When $f(x)$ is eventually nonincreasing, a characterization of oscillation of (E) was obtained by Koplatadze [4]. We refer to Mahfoud [6] for the related nonlinearity. On the other hand, when f is an arbitrary continuous function, under appropriate conditions on $p(t)$ necessary and sufficient conditions for oscillation were given by Burton and Grimmer [1, Theorem 9] and Mahfoud [7, Theorem 3]. It seems to us that little is known about the case f is arbitrary.

In this paper we investigate (I) in the latter direction to present some new results. Our main purpose is to characterize the existence of positive solutions of (I) without any restriction on $f(x)$. More precisely, when f is a general nonlinear function satisfying only (c), we give necessary and sufficient conditions for

(I) to have positive solutions with special asymptotic behavior as $t \to \infty$ (Section 2) and provide necessary and sufficient conditions for (I) to have positive solutions $x(t)$ such that $\liminf_{t\to\infty} x(t) > 0$ (Section 3).

Since parallel discussions are valid for negative solutions of the converse inequality

$$x^{(n)}(t) + p(t)f(x(g(t))) \geqq 0,$$

necessary and sufficient conditions for (E) to be oscillatory are almost automatically derived from our results. Thus the results obtained here include [1] and are independent of [7].

Note that, from the assumption (b), there exist positive constants γ_1, γ_2, g_1, g_2 and g_3 such that

$$\gamma_1 \leqq g'(t) \leqq \gamma_2 \tag{1}$$

$$g_1 t \leqq g(t) \leqq g_2 t \tag{2}$$

$$g_*(t) \equiv \min\{g(t), t\} \geqq g_3 t \tag{3}$$

for all large t. These inequalities will be frequently used throughout the paper.

2. Asymptotic Behavior of Positive Solutions

In this section we study the asymptotic behavior for $t \to \infty$ of positive solutions of (I). We begin by stating two lemmas that supply useful information on the behavior of positive solutions.

LEMMA 1. (Kiguradze [3]) *If $x(t)$ is an eventually positive solution of* (I), *then there is an integer l, $0 \leqq l \leqq n-1$, which is odd if n is even and even if n is odd, such that*

$$\begin{cases} x^{(k)}(t) \geqq 0 & (k = 0, 1, \dots, l) \\ (-1)^{n+k-1} x^{(k)}(t) \geqq 0 & (k = l+1, \dots, n) \end{cases} \tag{4}$$

for all large t.

LEMMA 2. *If $x(t)$ is an eventually positive solution of* (I) *satisfying* $\liminf_{t\to\infty} x(t) > 0$, *then there are positive numbers b_1 and b_2 such that*

$$b_1 \leqq x(t) \leqq b_2 t^{n-1} \tag{5}$$

for all large t.

The proof is easy. Notice that, when n is even, the statement "$x(t)$ is an eventually positive solution of (I) satisfying $\liminf_{t\to\infty} x(t)>0$" is equivalent to "$x(t)$ is an eventually positive solution of (I)". This is readily checked by using the nondecreasing character of $x(t)$ (see Lemma 1). From the above Lemma 2, roughly speaking, among all positive solutions of (I), those which are asymptotic to bt^{n-1} $(b>0)$ as $t\to\infty$ can be regarded as the maximal solutions and those which are asymptotic to nonzero constants as $t\to\infty$ can be regarded as the minimal solutions. The purpose of this section is to present necessary and sufficient conditions for (I) to have positive solutions of these two special types.

THEOREM 1. *Suppose that $p(t)$ is decomposable in such a way that $p(t)=c(t)q(t)$, $c(t)$ is continuous on $[a, \infty)$, $0<\liminf_{t\to\infty} c(t)\leqq\limsup_{t\to\infty} c(t)<\infty$, and $t^\sigma q(t)$ is continuous and nondecreasing on $[a, \infty)$ for some real number σ.*

Then, a necessary and sufficient condition for (I) *to have a solution $x(t)$ such that*

$$\lim_{t\to\infty}[x(t)/t^{n-1}] \qquad \textit{exists and is positive, and}$$

$$\lim_{t\to\infty}[x'(t)/t^{n-2}] \qquad \textit{exists and is positive}$$

is that

$$\int^\infty p(s)f(cs^{n-1})ds < \infty \qquad \textit{for some} \quad c>0. \tag{6}$$

PROOF. In view of the condition on $c(t)$, we see that there are positive constants c_1 and c_2 such that

$$c_1 \leqq c(t) \leqq c_2 \tag{7}$$

for all large t.

(Necessity) Let $x(t)$ be a solution of (I) with $\lim_{t\to\infty}[x(t)/t^{n-1}]=b>0$ and $\lim_{t\to\infty}[x'(t)/t^{n-2}]=b'>0$. We easily find $(b/2)[g(t)]^{n-1}\leqq x(g(t))\leqq 2b[g(t)]^{n-1}$, $(b'/2)[g(t)]^{n-2}\leqq x'(g(t))\leqq 2b'[g(t)]^{n-2}$ for all large t. By the help of (1) and (2) we obtain

$$\frac{b}{2}g_1^{n-1}t^{n-1} \leqq x(g(t)) \leqq 2bg_2^{n-1}t^{n-1}, \tag{8}$$

$$\frac{b'}{2}g_1^{n-2}\gamma_1 t^{n-2} \leqq x'(g(t))g'(t) \leqq 2b'g_2^{n-2}\gamma_2 t^{n-2} \tag{9}$$

for all large t. It follows from (8) that

$$t \leqq h_1[x(g(t))]^{1/(n-1)} \quad\text{and}\quad h_2[x(g(t))]^{1/(n-1)} \leqq t$$

for $t \geqq T$, where $h_1=(bg_1^{n-1}/2)^{-1/(n-1)}$, $h_2=(2bg_2^{n-1})^{-1/(n-1)}$ are positive constants and $T>a$ is chosen so large that the inequalities (4), (7)–(9) and $h_2[(b/2)g_1^{n-1}t^{n-1}]^{1/(n-1)} \geqq a$ hold for $t \geqq T$. Using the above inequalities and the condition on $p(t)=c(t)q(t)$, we compute as follows: If $\sigma \geqq 0$, then

$$\begin{aligned} p(t) &\geqq c_1 t^{-\sigma}(h_2[x(g(t))]^{1/(n-1)})^{\sigma} q(h_2[x(g(t))]^{1/(n-1)}) \\ &\geqq c_1(h_1[x(g(t))]^{1/(n-1)})^{-\sigma}(h_2[x(g(t))]^{1/(n-1)})^{\sigma} q(h_2[x(g(t))]^{1/(n-1)}) \\ &= c_1 h_1^{-\sigma} h_2^{\sigma} q(h_2[x(g(t))]^{1/(n-1)}) \end{aligned}$$

for $t \geqq T$, and if $\sigma<0$, then

$$\begin{aligned} p(t) &\geqq c_1(h_2[x(g(t))]^{1/(n-1)})^{-\sigma}(h_2[x(g(t))]^{1/(n-1)})^{\sigma} q(h_2[x(g(t))]^{1/(n-1)}) \\ &= c_1 q(h_2[x(g(t))]^{1/(n-1)}) \end{aligned}$$

for $t \geqq T$. In either case, we get

$$p(t) \geqq dq(h_2[x(g(t))]^{1/(n-1)}) \tag{10}$$

for $t \geqq T$, where $d=c_1 h_1^{-\sigma} h_2^{\sigma}$ if $\sigma \geqq 0$ and $d=c_1$ if $\sigma<0$.

Now, integrating (I) from T to t, we obtain

$$x^{(n-1)}(t) - x^{(n-1)}(T) + \int_T^t p(s) f(x(g(s)))ds \leqq 0$$

for $t \geqq T$. Since $x^{(n-1)}(t) \geqq 0$ for $t \geqq T$ by (4), we see that

$$\int_T^\infty p(s) f(x(g(s)))ds < \infty,$$

which together with (10) implies

$$\int_T^\infty q(h_2[x(g(s))]^{1/(n-1)}) f(x(g(s)))ds < \infty.$$

By virtue of (8) and (9), we observe that there are positive constants C_1 and C_2 such that

$$C_1 \leqq \frac{d}{dt} h_2[x(g(t))]^{1/(n-1)} \leqq C_2$$

for $t \geqq T$. Hence it follows that

$$\int_T^\infty q(h_2[x(g(s))]^{1/(n-1)}) f(x(g(s))) \frac{d}{ds} h_2[x(g(s))]^{1/(n-1)} ds < \infty.$$

Letting $v=h_2[x(g(s))]^{1/(n-1)}$, we arrive at

$$\int_{T^*}^{\infty} q(v)f(h_2^{-n+1}v^{n-1})dv < \infty,$$

where $T^* = h_2[x(g(T))]^{1/(n-1)}$. The desired inequality (6) is readily derived from (7) and $p(t) = c(t)q(t)$.

(Sufficiency) Assume that (6) holds. We suppose that (1), (2) and (7) are satisfied for $t \geqq T'$. Put $b = cg_1^{-n+1}$, $h_1 = (bg_1^{n-1})^{-1/(n-1)}$, $h_2 = (2bg_2^{n-1})^{-1/(n-1)}$, $d = c_2 h_2^{-\sigma} h_1^{\sigma}$ if $\sigma \geqq 0$, $d = c_2$ if $\sigma < 0$ and $M = h_1(2bg_2^{n-1})^{(-n+2)/(n-1)} bg_1^{n-2}\gamma_1$. Choose a number $T > T'$ so large that $T_0 = \inf\{\min\{g(t), t\}: t \geqq T\} > a$,

$$(11) \qquad \int_T^{\infty} p(s)f(cs^{n-1})ds \leqq b(n-1)!\, c_1 M d^{-1}$$

and $h_2[bg_1^{n-1}t^{n-1}]^{1/(n-1)} \geqq a$ for $t \geqq T$. Let F denote the Fréchet space of all continuously differentiable functions on $[T_0, \infty)$ with the family of seminorms $\{\|\cdot\|_m: m = 1, 2, \dots\}$ defined by

$$\|x\|_m = \sup\{|x(t)| + |x'(t)|: T_0 \leqq t \leqq T_0 + m\}.$$

We have the convergence $x_k \to x$ $(k \to \infty)$ in the topology of F if and only if $x_k(t) \to x(t)$ $(k \to \infty)$ and $x_k'(t) \to x'(t)$ $(k \to \infty)$ uniformly on every compact subinterval of $[T_0, \infty)$. Let X be the subset of F such that

$$X = \{x \in F: bt^{n-1} \leqq x(t) \leqq 2bt^{n-1},$$
$$b(n-1)t^{n-2} \leqq x'(t) \leqq 2b(n-1)t^{n-2} \quad \text{for} \quad t \geqq T_0\},$$

which is a convex and closed subset of F. Define the operator Φ acting on X by the formula

$$(12) \qquad (\Phi x)(t) = \begin{cases} 2bt^{n-1} - \dfrac{1}{(n-1)!}\displaystyle\int_T^t (t-s)^{n-1}p(s)f(x(g(s)))ds & \text{for } t \geqq T \\ 2bt^{n-1} \quad \text{for} \quad T_0 \leqq t \leqq T. \end{cases}$$

We seek a fixed point of the operator Φ in X with the aid of the following fixed point theorem which is a special case of Tychonoff's theorem.

FIXED POINT THEOREM. *Let F be a Fréchet space and X be a convex and closed subset of F. If Φ is a continuous mapping of X into itself and the closure $\overline{\Phi X}$ is a compact subset of X, then there exists at least one fixed point $x \in X$ of Φ.*

We show that Φ defined by (12) satisfies the conditions of the above theorem.

(i) *Φ maps X into X.* Let $x \in X$. It is obvious that $(\Phi x)(t)$ is continuously differentiable on $[T_0, \infty)$ and $(\Phi x)(t) \leqq 2bt^{n-1}$, $(\Phi x)'(t) \leqq 2b(n-1)t^{n-2}$ for $t \geqq T_0$.

To prove that $(\Phi x)(t) \geqq bt^{n-1}$, $(\Phi x)'(t) \geqq b(n-1)t^{n-2}$ for $t \geqq T_0$, consider the integral $\int_T^\infty p(s)f(x(g(s)))ds$. Since we have $bg_1^{n-1}t^{n-1} \leqq x(g(t)) \leqq 2bg_2^{n-1}t^{n-1}$, $b(n-1)g_1^{n-2}\gamma_1 t^{n-2} \leqq x'(g(t))g'(t) \leqq 2b(n-1)g_2^{n-2}\gamma_2 t^{n-2}$ for $t \geqq T$, there follows that

$$t \leqq h_1[x(g(t))]^{1/(n-1)}, \quad h_2[x(g(t))]^{1/(n-1)} \leqq t$$

for $t \geqq T$. As in the proof of the necessity part of this theorem it can be verified that

$$p(t) \leqq dq(h_1[x(g(t))]^{1/(n-1)})$$

for $t \geqq T$, where $d = c_2 h_2^{-\sigma} h_1^{\sigma}$ if $\sigma \geqq 0$ and $d = c_2$ if $\sigma < 0$. Moreover, we find

$$\frac{d}{dt} h_1[x(g(t))]^{1/(n-1)} \geqq \frac{h_1}{n-1}(2bg_2^{n-1})^{(-n+2)/(n-1)} b(n-1)g_1^{n-2}\gamma_1 = M$$

for $t \geqq T$. From these inequalities and (11) it follows that

$$\begin{aligned}
&\int_T^\infty p(s)f(x(g(s)))ds \\
&\leqq dM^{-1}\int_T^\infty q(h_1[x(g(s))]^{1/(n-1)}) f(x(g(s))) \frac{d}{ds} h_1[x(g(s))]^{1/(n-1)} ds \\
&= dM^{-1}\int_{h_1[x(g(T))]^{1/(n-1)}}^\infty q(v) f(h_1^{-n+1} v^{n-1}) dv \\
&\leqq dM^{-1}\int_T^\infty q(v) f(cv^{n-1}) dv \\
&\leqq dM^{-1} c_1^{-1}\int_T^\infty p(v) f(cv^{n-1}) dv \leqq b(n-1)!.
\end{aligned}$$

Then we easily see that $(\Phi x)(t) \geqq 2bt^{n-1} - t^{n-1}/(n-1)! \cdot b(n-1)! = bt^{n-1}$, and $(\Phi x)'(t) \geqq 2b(n-1)t^{n-2} - (n-1)t^{n-2}/(n-1)! \cdot b(n-1)! = b(n-1)t^{n-2}$ for $t \geqq T_0$.

(ii) *Φ is continuous on X.* Let x_k $(k=1, 2, \ldots)$ and x be functions in X such that $x_k(t) \to x(t)$, $x_k'(t) \to x'(t)$ as $k \to \infty$ uniformly on every compact subinterval of $[T_0, \infty)$. If $t \in [T, A]$ $(A > T$ is a fixed number), then we have

$$|(\Phi x_k)(t) - (\Phi x)(t)| \leqq \frac{(A-T)^{n-1}}{(n-1)!}\int_T^A p(s)|f(x_k(g(s))) - f(x(g(s)))|ds,$$

$$|(\Phi x_k)'(t) - (\Phi x)'(t)| \leqq \frac{(A-T)^{n-2}}{(n-2)!}\int_T^A p(s)|f(x_k(g(s))) - f(x(g(s)))|ds.$$

Using the fact that $|f(x_k(g(s))) - f(x(g(s)))| \to 0$ as $k \to \infty$ for $s \geqq T$, we conclude that $(\Phi x_k)(t) \to (\Phi x)(t)$, $(\Phi x_k)'(t) \to (\Phi x)'(t)$ as $k \to \infty$ uniformly on $[T, A]$. Thus $(\Phi x_k)(t)$ and $(\Phi x_k)'(t)$ converge to $(\Phi x)(t)$ and $(\Phi x)'(t)$ respectively as $k \to \infty$ uniformly on every compact subinterval of $[T_0, \infty)$. This proves the continuity of Φ on X.

(iii) $\overline{\Phi X}$ *is compact*. It is enough to show that, for any sequence $\{x_k: k=1, 2,\ldots\}$ in X, there exist a subsequence $\{x_{k_i}: i=1, 2,\ldots\}$ of $\{x_k: k=1, 2,\ldots\}$ and a function x in F such that $(\Phi x_{k_i})(t)\to x(t)$, $(\Phi x_{k_i})'(t)\to x'(t)$ as $i\to\infty$ uniformly on compact subintervals of $[T_0, \infty)$. Let $\{x_k: k=1, 2,\ldots\}$ be an arbitrary sequence in X. Differentiating $(\Phi x_k)(t)$ twice, we have

$$(\Phi x_k)''(t) = \begin{cases} 2b(n-1)(n-2)t^{n-3} - \dfrac{1}{(n-3)!}\displaystyle\int_T^t (t-s)^{n-3}p(s)f(x_k(g(s)))ds & \\ & \text{for} \quad t \geqq T \\ 2b(n-1)(n-2)t^{n-3} & \text{for} \quad T_0 \leqq t \leqq T \end{cases}$$

in the case $n>2$ and

$$(\Phi x_k)''(t) = \begin{cases} -p(t)f(x_k(g(t))) & \text{for} \quad t \geqq T \\ 0 & \text{for} \quad T_0 \leqq t \leqq T \end{cases}$$

in the case $n=2$. It is easy to see that, if $t\in[T, A]$, then $|(\Phi x_k)''(t)|\leqq 2b(n-1)\cdot(n-2)A^{n-3}+1/(n-3)!\cdot(A-T)^{n-2}PF$ or $|(\Phi x_k)''(t)|\leqq PF$, where $P=\max\{p(s): T\leqq s\leqq A\}$ and $F=\max\{f(u): bg_1^{n-1}T^{n-1}\leqq u\leqq 2bg_2^{n-1}A^{n-1}\}$ are constants independent of $k=1, 2,\ldots$. Hence $\{(\Phi x_k)'(t)\}$ is equicontinuous on every finite subinterval of $[T_0, \infty)$. Since the boundedness of $\{(\Phi x_k)'(t)\}$ at every point of $[T_0, \infty)$ is evident, applying Ascoli's Theorem (see, e.g. Coppel [2, p. 7]), we can choose a subsequence $\{(\Phi x_{k_i})'(t): i=1, 2,\ldots\}$ of $\{(\Phi x_k)'(t): k=1, 2,\ldots\}$ which is uniformly convergent on every compact subinterval of $[T_0, \infty)$. Let the limit function of $(\Phi x_{k_i})'(t)$ be denoted by $z(t)$, which is clearly continuous on $[T_0, \infty)$. Putting $x(t)=2bT_0^{n-1}+\int_{T_0}^t z(s)ds$, $t\geqq T_0$, we observe that $(\Phi x_{k_i})'(t)\to z(t)=x'(t)$ and $(\Phi x_{k_i})(t)=(\Phi x_{k_i})(T_0)+\int_{T_0}^t(\Phi x_{k_i})'(s)ds\to 2bT_0^{n-1}+\int_{T_0}^t z(s)ds=x(t)$ as $i\to\infty$ uniformly on compact subintervals of $[T_0, \infty)$. This proves that $\overline{\Phi X}$ is compact.

From the preceding considerations we are able to apply the Fixed Point Theorem to the operator Φ. Let $x(t)\in X$ be a fixed point of Φ. It is immediately verified that $x(t)$ is a solution of (I) for $t\geqq T$ and has the property that $\lim_{t\to\infty}[x(t)/t^{n-1}]$ and $\lim_{t\to\infty}[x'(t)/t^{n-2}]$ exist in $[b, 2b]$ and $[b(n-1), 2b(n-1)]$, respectively. Thus the proof of Theorem 1 is complete.

THEOREM 2. *A necessary and sufficient condition for* (I) *to have a solution* $x(t)$ *such that*

$$\lim_{t\to\infty} x(t) \qquad \textit{exists and is positive}$$

is that

$$\int^{\infty} p(s)s^{n-1}ds < \infty. \tag{13}$$

PROOF. (Necessity) Let $x(t)$ be a solution of (I) such that $\lim_{t\to\infty} x(t)=b$, $0<b<\infty$. We have

$$\frac{b}{2} \leqq x(t) \leqq 2b, \quad \frac{b}{2} \leqq x(g(t)) \leqq 2b \tag{14}$$

for all large t. By Lemma 1 we find an integer l, $0\leqq l\leqq n-1$, which satisfies (4). Since $x(t)$ is bounded, this integer l must be equal to 0 or 1. Therefore we obtain

$$(-1)^{n+k-1}x^{(k)}(t) \geqq 0 \qquad (k = 1, 2,\dots, n) \tag{15}$$

for all large t. Take a number $T>a$ so large that (14) and (15) hold for $t\geqq T$. Multiplying (I) by t^{n-1} and integrating from T to t, we get

$$\begin{aligned} &t^{n-1}x^{(n-1)}(t) - (n-1)t^{n-2}x^{(n-2)}(t) \\ &\quad + \cdots + (-1)^{k+1}(n-1)\cdots(n-k+1)t^{n-k}x^{(n-k)}(t) + \cdots + (-1)^n(n-1)\cdots 2tx'(t) \\ &\quad + (-1)^{n+1}(n-1)!x(t) + \int_T^t s^{n-1}p(s)f(x(g(s)))ds \leqq C, \end{aligned}$$

where C is a constant. In view of (14), (15) and the above inequality we see that

$$\int_T^{\infty} s^{n-1}p(s)f(x(g(s)))ds < \infty.$$

From this and (14) it follows that

$$m\int_T^{\infty} s^{n-1}p(s)ds < \infty,$$

where $m=\min\{f(u)\colon b/2\leqq u\leqq 2b\}$.

(Sufficiency) Assume that (13) holds. Let β be an arbitrary positive number and choose a number $T>a$ so large that $T_0=\inf\{\min\{g(t), t\}\colon t\geqq T\}>a$ and

$$\int_T^{\infty} p(s)s^{n-1}ds \leqq \beta(n-1)!M^{-1},$$

where $M=\max\{f(u)\colon \beta\leqq u\leqq 2\beta\}$. We consider the operator Φ defined by

$$(\Phi x)(t) = \begin{cases} b + \dfrac{(-1)^{n-1}}{(n-1)!}\displaystyle\int_t^\infty (s-t)^{n-1}p(s)f(x(g(s)))ds & \text{for} \quad t \geqq T \\ b + \dfrac{(-1)^{n-1}}{(n-1)!}\displaystyle\int_T^\infty (s-T)^{n-1}p(s)f(x(g(s)))ds & \text{for} \quad T_0 \leqq t \leqq T, \end{cases}$$

where $b=2\beta$ if n is even and $b=\beta$ if n is odd. With the aid of the Fixed Point Theorem stated in the proof of Theorem 1 we seek a fixed point of Φ. The underlying Fréchet space F is the set of all continuous functions on $[T_0, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T_0, \infty)$. The convex and closed subset X of F, on which Φ is well-defined, is $X=\{x \in F: \beta \leqq x(t) \leqq 2\beta \text{ for } t \geqq T_0\}$. It can be shown that (i) the operator Φ maps X into X, (ii) Φ is continuous on X, and (iii) $\overline{\Phi X}$ is a compact subset of X. Therefore, Φ has a fixed point $x \in X$ by the Fixed Point Theorem. The function $x(t)$ is a solution of (I) for $t \geqq T$ and satisfies $\lim_{t\to\infty} x(t)=b>0$. This sketches the proof of the sufficiency part. The details are left to the reader.

COROLLARY 1. *Suppose that $p(t)$ is decomposable in such a way that $p(t)=c(t)q(t)$, $c(t)$ is continuous on $[a, \infty)$, $0<\liminf_{t\to\infty} c(t) \leqq \limsup_{t\to\infty} c(t)<\infty$ and $t^{n+\varepsilon}q(t)$ is continuous and nonincreasing on $[a, \infty)$ for some positive number ε. Then, inequality* (I) *has a solution $x(t)$ such that $\liminf_{t\to\infty} x(t)>0$.*

3. Existence of Positive Solutions

In this section we establish necessary conditions and necessary and sufficient conditions in order that (I) have a positive solution $x(t)$ such that $\liminf_{t\to\infty} x(t)>0$. First we give the following lemmas.

LEMMA 3. *Let γ be a positive number and let $z(t) \in C^1[T, \infty)$, $z(t) \geqq 0$, $z'(t) \geqq 0$ on $[T, \infty)$ and $P(t) \in C[T, \infty)$, $P(t) \geqq 0$ on $[T, \infty)$. Suppose that there is a positive continuous function $x(t)$ defined on the interval $[\inf\{g(t): t \geqq T\}, \infty)$ which satisfies*

$$x(t) \geqq z(t)\int_t^\infty P(s)\,[x(g(s))]^\gamma ds + \int_T^t z(s)P(s)\,[x(g(s))]^\gamma ds \tag{16}$$

for $t \geqq T$. Then

(i) *in case of $\gamma > 1$,* $\displaystyle\int^\infty z(g_*(s))P(s)ds < \infty$;

(ii) *in case of $0 < \gamma < 1$,* $\displaystyle\int^\infty [z(g_*(s))]^\gamma P(s)ds < \infty$,

where $g_*(t)=\min\{g(t), t\}$.

PROOF. Let $I(t)$ denote the right hand side of (16). Clearly, $I(t)$ is nondecreasing on $[T, \infty)$ and $x(t) \geqq I(t)$ for $t \geqq T$. There is a number $T_1 > T$ such that $x(g(t)) \geqq I(g(t)) \geqq I(g_*(t))$ for $t \geqq T_1$.

(i) Let $\gamma > 1$. We find

$$-\frac{d}{dt}[I(t)]^{1-\gamma} = (\gamma-1)[I(t)]^{-\gamma}z'(t)\int_t^\infty P(s)[x(g(s))]^\gamma ds$$

$$\geqq (\gamma-1)[I(t)]^{-\gamma}z'(t)\int_t^\infty P(s)[I(g(s))]^\gamma ds$$

for $t \geqq T_1$. Take $T_2 > T_1$ so large that $g_*(t) \geqq T_1$ for $t \geqq T_2$. Integrating the above inequality from T_1 to t $(t \geqq T_2)$, we see that

$$\begin{aligned}
&-[I(t)]^{1-\gamma} + [I(T_1)]^{1-\gamma} \\
&\geqq (\gamma-1)\int_{T_1}^t [I(s)]^{-\gamma}z'(s)\int_s^\infty P(u)[I(g(u))]^\gamma du ds \\
&\geqq (\gamma-1)\int_{T_1}^t \int_{T_1}^u [I(s)]^{-\gamma}z'(s)P(u)[I(g(u))]^\gamma ds du \\
&\geqq (\gamma-1)\int_{T_2}^t \int_{T_1}^{g_*(u)} [I(s)]^{-\gamma}z'(s)P(u)[I(g_*(u))]^\gamma ds du \\
&\geqq (\gamma-1)\int_{T_2}^t [z(g_*(u)) - z(T_1)]P(u)du
\end{aligned}$$

for $t \geqq T_2$. By letting $t \to \infty$ we obtain

$$\int_{T_2}^\infty [z(g_*(u)) - z(T_1)]P(u)du < \infty,$$

which implies the conclusion (i).

(ii) Let $0 < \gamma < 1$. It follows from (16) that

$$x(g(t)) \geqq z(g_*(t))\int_t^\infty P(s)[x(g(s))]^\gamma ds$$

for $t \geqq T_1$. Put $J(t) = \int_t^\infty P(s)[x(g(s))]^\gamma ds$. Then, we evaluate as follows:

$$\begin{aligned}
-\frac{d}{dt}[J(t)]^{1-\gamma} &= (1-\gamma)[J(t)]^{-\gamma}P(t)[x(g(t))]^\gamma \\
&\geqq (1-\gamma)[J(t)]^{-\gamma}P(t)[z(g_*(t))J(t)]^\gamma \\
&= (1-\gamma)[z(g_*(t))]^\gamma P(t)
\end{aligned}$$

for $t \geqq T_1$. An integration of the above readily yields the assertion (ii).

Lemma 4. *Suppose that $x(t)$ is an eventually positive solution of* (I) *and* *the integer l which is determined by Lemma 1 for $x(t)$ is not zero. Then,*

(i) *there exists a positive constant k such that*

$$x(t) \geqq k(t-T)^{n-1}\int_t^\infty p(s)f(x(g(s)))ds + k\int_T^t (s-T)^{n-1}p(s)f(x(g(s)))ds \tag{17}$$

for $t \geqq T$, provided T is sufficiently large;

(ii) *there exists a positive constant k_1 such that*

$$x(t) \geqq k_1 t x'(t) \tag{18}$$

for all large t.

PROOF. (i) Integrating (I) from t to τ $(t<\tau)$, we have

$$x^{(n-1)}(\tau) - x^{(n-1)}(t) + \int_t^\tau p(s)f(x(g(s)))ds \leqq 0.$$

Noting that $x^{(n-1)}(\tau) \geqq 0$ and letting $\tau \to \infty$, we obtain

$$x^{(n-1)}(t) \geqq \int_t^\infty p(s)f(x(g(s)))ds.$$

Repetition of this procedure gives

$$x^{(l)}(t) \geqq k_2\int_t^\infty (s-t)^{n-1-l}p(s)f(x(g(s)))ds,$$

where $k_2 = 1/(n-1-l)!$. Integrating the above l times from T to t, we get

$$x(t) \geqq k_3\int_T^t (t-s)^{l-1}\int_s^\infty (u-s)^{n-1-l}p(u)f(x(g(u)))duds$$

for $t \geqq T$, where $k_3 = k_2/(l-1)!$. Therefore, we have

$$\begin{aligned}
x(t) &\geqq k_3\int_T^t\int_T^u (t-s)^{l-1}(u-s)^{n-1-l}p(u)f(x(g(u)))dsdu \\
&\quad + k_3\int_t^\infty\int_T^t (t-s)^{l-1}(u-s)^{n-1-l}p(u)f(x(g(u)))dsdu \\
&\geqq k_3\int_T^t\left(\int_T^u (u-s)^{l-1}(u-s)^{n-1-l}ds\right)p(u)f(x(g(u)))du \\
&\quad + k_3\int_t^\infty\left(\int_T^t (t-s)^{l-1}(t-s)^{n-1-l}ds\right)p(u)f(x(g(u)))du \\
&= k_4\int_T^t (u-T)^{n-1}p(u)f(x(g(u)))du + k_4(t-T)^{n-1}\int_t^\infty p(u)f(x(g(u)))du
\end{aligned}$$

for $t \geqq T$, where $k_4 = k_3/(n-1)$. This proves (17).

(ii) Since the integer l in Lemma 1 is not zero, there follows that

$$x^{(l-1)}(t) = x^{(l-1)}(T) + \int_T^t x^{(l)}(s)ds$$

for $t \geqq T$. Noting that $x^{(l-1)}(T) \geqq 0$ and $x^{(l)}(s)$ is nonincreasing, we obtain $x^{(l-1)}(t) \geqq x^{(l)}(t)(t-T)$ for $t \geqq T$. Integrating this by parts from T to t, we have $x^{(l-2)}(t) - x^{(l-2)}(T) \geqq x^{(l-1)}(t)(t-T) - x^{(l-2)}(t) + x^{(l-2)}(T)$ for $t \geqq T$, which gives $x^{(l-2)}(t) \geqq (1/2)x^{(l-1)}(t)(t-T)$ for $t \geqq T$. Repeated application of the above procedure then yields $x(t) \geqq (1/l)x'(t)(t-T)$ for $t \geqq T$. From this the conclusion (18) follows immediately. The proof is complete.

LEMMA 5. *Suppose that* $p(t)$ *is decomposable in such a way that* $p(t) = c(t)q(t)$, $c(t)$ *is continuous on* $[a, \infty)$, $0 < \liminf_{t\to\infty} c(t) \leqq \limsup_{t\to\infty} c(t) < \infty$, *and* $t^{n-\varepsilon}q(t)$ *is continuous and nondecreasing on* $[a, \infty)$ *for some nonnegative number* ε. *If* $x(t)$ *is a solution of* (I) *such that* $\liminf_{t\to\infty} x(t) > 0$, *then*

(i) $\lim_{t\to\infty} x(t) = \infty$ *and the integer* l *for* $x(t)$ *in Lemma* 1 *is not zero*;

(ii) *there exist positive constants* k_2 *and* h *such that*

$$p(t) \geqq k_2 t^{-n+\varepsilon}[x(g(t))]^{(n-\varepsilon)/(n-1)} p(h[x(g(t))]^{1/(n-1)}) \tag{19}$$

for all large t.

PROOF. (i) Using (7) and the nondecreasing character of $t^{n-\varepsilon}q(t)$, we find that $x^{(n)}(t) + c_1 a^{n-\varepsilon}q(a)t^{-n+\varepsilon}f(x(g(t))) \leqq 0$ for all large t. We may assume the constant $c_1 a^{n-\varepsilon}q(a)$ is positive. Observe that $\int^\infty s^{-n+\varepsilon}s^{n-1}ds = \infty$ for $\varepsilon \geqq 0$. By virtue of Theorem 2 we see that $x(t)$ does not have a finite limit which is positive as $t \to \infty$. Since $x(t)$ is monotone by Lemma 1 and $\liminf_{t\to\infty} x(t) > 0$, we conclude that $\lim_{t\to\infty} x(t) = \infty$. Because of the unboundedness of $x(t)$, the integer l for $x(t)$ is not zero.

(ii) From Lemma 2 there is a positive constant b_2 such that $x(g(t)) \leqq b_2[g(t)]^{n-1}$ for all large t. In view of (2) we have

$$h[x(g(t))]^{1/(n-1)} \leqq t \tag{20}$$

for all large t, where $h = b_2^{-1/(n-1)}g_2^{-1}$. Let (20) and (7) be satisfied for $t \geqq T > a$. Then choose a number $T_1 > T$ so large that $h[x(g(t))]^{1/(n-1)} \geqq T$ holds for $t \geqq T_1$. This can be done by reason of $\lim_{t\to\infty} x(g(t)) = \infty$. Using the condition on $p(t) = c(t)q(t)$, (20) and (7), we obtain

$$\begin{aligned} p(t) &\geqq c_1 t^{-n+\varepsilon} t^{n-\varepsilon} q(t) \\ &\geqq c_1 h^{n-\varepsilon} t^{-n+\varepsilon}[x(g(t))]^{(n-\varepsilon)/(n-1)} q(h[x(g(t))]^{1/(n-1)}) \end{aligned}$$

$$\geqq c_1 h^{n-\varepsilon} c_2^{-1} t^{-n+\varepsilon} [x(g(t))]^{(n-\varepsilon)/(n-1)} p(h[x(g(t))]^{1/(n-1)})$$

for $t \geqq T_1$, which implies (19). This completes the proof.

THEOREM 3. *Suppose that $p(t)$ is decomposable in such a way that $p(t) = c(t)q(t)$, $c(t)$ is continuous on $[a, \infty)$, $0 < \liminf_{t\to\infty} c(t) \leqq \limsup_{t\to\infty} c(t) < \infty$ and $t^n q(t)$ is continuous and nondecreasing on $[a, \infty)$.*

Then, a necessary condition for (I) *to have a solution $x(t)$ with $\liminf_{t\to\infty} x(t) > 0$ is that*

$$(21) \qquad \int^{\infty} s^{-\lambda} p(s) f(cs^{n-1}) ds < \infty \qquad \textit{for some} \quad c > 0,$$

where λ is an arbitrary positive number.

PROOF. Let $x(t)$ be a solution of (I) such that $\liminf_{t\to\infty} x(t) > 0$ and let λ be an arbitrary positive number. Applying Lemma 5 to the case $\varepsilon = 0$, we see that $\lim_{t\to\infty} x(t) = \infty$, the integer l in Lemma 1 is not zero, and

$$(22) \qquad p(t) \geqq k_2 t^{-n} [x(g(t))]^{n/(n-1)} p(h[x(g(t))]^{1/(n-1)})$$

for all large t. Notice that (1)–(3) hold. In view of Lemma 4 (ii) we find that

$$(23) \qquad x(g(t)) \geqq k_3 t \frac{d}{dt} x(g(t))$$

for all large t, where k_3 is a positive constant such that $k_1 g(t) \geqq k_1 g_1 t \geqq k_3 \gamma_2 t \geqq k_3 g'(t) t$ for all large t. Moreover, by Lemma 4 (i), we get

$$x(t) \geqq k(t-T)^{n-1} \int_t^{\infty} p(s) f(x(g(s))) [x(g(s))]^{-1-\lambda/(n-1)} [x(g(s))]^{1+\lambda/(n-1)} ds$$
$$+ k \int_T^t (s-T)^{n-1} p(s) f(x(g(s))) [x(g(s))]^{-1-\lambda/(n-1)} [x(g(s))]^{1+\lambda/(n-1)} ds$$

for $t \geqq T$. According to Lemma 3 (i) applied to the case $\gamma = 1 + \lambda/(n-1)$, $z(t) = k(t-T)^{n-1}$ and $P(t) = p(t) f(x(g(t))) [x(g(t))]^{-1-\lambda/(n-1)}$, we conclude that

$$(24) \qquad \int^{\infty} [g_*(s)]^{n-1} p(s) f(x(g(s))) [x(g(s))]^{-1-\lambda/(n-1)} ds < \infty.$$

By the aid of (22), (23) and (3) we see that

$$[g_*(s)]^{n-1} p(s) [x(g(s))]^{-1-\lambda/(n-1)}$$
$$\geqq g_3^{n-1} k_2 s^{n-1} s^{-n} [x(g(s))]^{n/(n-1)} p(h[x(g(s))]^{1/(n-1)}) [x(g(s))]^{-1-\lambda/(n-1)}$$
$$\geqq g_3^{n-1} k_2 k_3 \frac{d}{ds} x(g(s)) \cdot [x(g(s))]^{1/(n-1)-1-\lambda/(n-1)} p(h[x(g(s))]^{1/(n-1)})$$

for all large t. Therefore, the inequality (24) shows that the integral

$$\int^t \frac{d}{ds} x(g(s)) \cdot [x(g(s))]^{1/(n-1)-1-\lambda/(n-1)} p(h[x(g(s))]^{1/(n-1)}) f(x(g(s))) ds$$

$$= (n-1)h^{-1}h^{\lambda} \int^{h[x(g(t))]^{1/(n-1)}} v^{-\lambda} p(v) f(h^{-n+1} v^{n-1}) dv$$

is bounded above as $t \to \infty$. Since $\lim_{t\to\infty} x(g(t)) = \infty$, we arrive at the desired integral condition (21). The proof of Theorem 3 is complete.

REMARK 1. In Theorem 3, it is impossible to take $\lambda = 0$, as can be seen by the Euler equation

$$x''(t) + kt^{-2}x(t) = 0. \tag{25}$$

In fact, every nontrivial solution of (25) is oscillatory if $k > 1/4$ and nonoscillatory if $k \leqq 1/4$.

However, we can establish the following theorem.

THEOREM 4. *Suppose that* $p(t)$ *is decomposable in such a way that* $p(t) = c(t)q(t)$, $c(t)$ *is continuous on* $[a, \infty)$, $0 < \liminf_{t\to\infty} c(t) \leqq \limsup_{t\to\infty} c(t) < \infty$ *and* $t^{n-\varepsilon}q(t)$ *is continuous and nondecreasing on* $[a, \infty)$ *for some* $\varepsilon > 0$.

Then, a necessary and sufficient condition for (I) *to have a solution* $x(t)$ *with* $\liminf_{t\to\infty} x(t) > 0$ *is that*

$$\int^{\infty} p(s) f(cs^{n-1}) ds < \infty \qquad \textit{for some} \quad c > 0. \tag{6}$$

PROOF. The sufficiency part is contained in Theorem 1. It remains to prove the necessity part. Let $x(t)$ be a solution of (I) which satisfies $\liminf_{t\to\infty} x(t) > 0$. Proceeding as in the proof of Theorem 3, we see that (23) holds for all large t. By the aid of Lemma 4 (i) we have

$$x(t) \geqq k(t-T)^{n-1} \int_t^{\infty} p(s) f(x(g(s))) [x(g(s))]^{-1+\varepsilon/(n-1)} [x(g(s))]^{1-\varepsilon/(n-1)} ds$$

$$+ k\int_T^t (s-T)^{n-1} p(s) f(x(g(s))) [x(g(s))]^{-1+\varepsilon/(n-1)} [x(g(s))]^{1-\varepsilon/(n-1)} ds$$

for $t \geqq T$. Without loss of generality, we may suppose that $0 < \varepsilon < n-1$. Applying Lemma 3 (ii) to the case $\gamma = 1 - \varepsilon/(n-1)$, $z(t) = k(t-T)^{n-1}$ and $P(t) = p(t) f(x(g(t))) [x(g(t))]^{-1+\varepsilon/(n-1)}$, we obtain

$$\int^{\infty} [g_*(s)]^{(n-1)(1-\varepsilon/(n-1))} p(s) f(x(g(s))) [x(g(s))]^{-1+\varepsilon/(n-1)} ds < \infty. \tag{26}$$

From (3), (23) and Lemma 5 (ii) it follows that

$$(27)\quad [g_*(s)]^{(n-1)(1-\varepsilon/(n-1))}p(s)\,[x(g(s))]^{-1+\varepsilon/(n-1)}$$

$$\geqq g_3^{n-1-\varepsilon}k_2 s^{n-1-\varepsilon}s^{-n+\varepsilon}[x(g(s))]^{(n-\varepsilon)/(n-1)}p(h[x(g(s))]^{1/(n-1)})\,[x(g(s))]^{-1+\varepsilon/(n-1)}$$

$$\geqq g_3^{n-1-\varepsilon}k_2k_3\frac{d}{ds}x(g(s))\cdot[x(g(s))]^{1/(n-1)-1}p(h[x(g(s))]^{1/(n-1)})$$

for all large t. Using (26), (27) and the fact that $\lim_{t\to\infty} x(g(t))=\infty$, we arrive at

$$\int^{\infty} p(v)f(h^{-n+1}v^{n-1})dv < \infty,$$

which proves (6). The proof of Theorem 4 is complete.

Corollary 2. *Suppose that there exists a positive number* ε *such that*

$$\liminf_{t\to\infty} t^{n-\varepsilon}p(t) > 0.$$

If inequality (I) *has a solution* $x(t)$ *such that* $\liminf_{t\to\infty} x(t)>0$, *then*

$$\int^{\infty} s^{-n+\varepsilon}f(cs^{n-1})ds < \infty \qquad \text{for some} \quad c > 0.$$

Proof. There is a positive constant k such that $p(t)\geqq kt^{-n+\varepsilon}$ for all large t; hence we have $x^{(n)}(t)+kt^{-n+\varepsilon}f(x(g(t)))\leqq 0$. Apply the necessity part of Theorem 4 to this inequality.

Remark 2. When $n=2$ and $\varepsilon=2$, Corollary 2 was given by Wong [9, Corollary 3].

Corollary 3. *Suppose that there exists a positive number* ε *such that* $t^{n-\varepsilon}p(t)\in C^1[a, \infty)$, $p(t)>0$ *on* $[a, \infty)$ *and*

$$\int^{\infty} \frac{(s^{n-\varepsilon}p(s))'_-}{s^{n-\varepsilon}p(s)}ds < \infty,$$

where $(s^{n-\varepsilon}p(s))'_- = \max\{-(s^{n-\varepsilon}p(s))', 0\}$.

Then, inequality (I) *has a solution* $x(t)$ *such that* $\liminf_{t\to\infty} x(t)>0$ *if and only if*

$$\int^{\infty} p(s)f(cs^{n-1})ds < \infty \qquad \text{for some} \quad c > 0.$$

Proof. We have only to apply Theorem 4 to the case

$$c(t)=\exp\left(-\int_a^t \frac{(s^{n-\varepsilon}p(s))'_-}{s^{n-\varepsilon}p(s)}ds\right),\ q(t)=a^{n-\varepsilon}p(a)t^{-n+\varepsilon}\exp\left(\int_a^t \frac{(s^{n-\varepsilon}p(s))'_+}{s^{n-\varepsilon}p(s)}ds\right),$$

where $(s^{n-\varepsilon}p(s))'_+=\max\{(s^{n-\varepsilon}p(s))', 0\}$.

REMARK 3. When $n=2$ and $\varepsilon=2$, Corollary 3 was proved by Burton and Grimmer [1, Theorem 9].

REMARK 4. The Euler equation (25) shows that in Theorem 4 and Corollaries 2 and 3 the assumption that $\varepsilon>0$ cannot be weakened to $\varepsilon\geqq 0$.

EXAMPLE. Consider the equation

$$x^{(n)}(t) + kt^{\alpha}f(x(g(t))) = 0, \tag{28}$$

where k is a positive constant and α is a real number. If $\alpha<-n$, then (28) has a solution $x(t)$ such that $\liminf_{t\to\infty} x(t)>0$ (Theorem 2 or Corollary 1). If $\alpha=-n$, then a sufficient condition and a necessary condition for (28) to have a solution $x(t)$ such that $\liminf_{t\to\infty} x(t)>0$ are that $\int^\infty s^{-n}f(cs^{n-1})ds<\infty$ and $\int^\infty s^{-n-\lambda}f(cs^{n-1})ds<\infty$ $(\lambda>0)$ for some $c>0$ respectively (Theorem 1 and Theorem 3). If $\alpha>-n$, then a necessary and sufficient condition for (28) to have a solution $x(t)$ such that $\liminf_{t\to\infty} x(t)>0$ is that $\int^\infty s^{\alpha}f(cs^{n-1})ds<\infty$ for some $c>0$ (Theorem 4 or Corollary 3).

ACKNOWLEDGMENT. The author would like to thank Professor Takaŝi Kusano for his helpful suggestions concerning this work.

References

[1] T. Burton and R. Grimmer, Oscillation, continuation, and uniqueness of solutions of retarded differential equations, Trans. Amer. Math. Soc. **179** (1973), 193–209.
[2] W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, Heath, Boston, 1965.
[3] I. T. Kiguradze, Oscillation properties of solutions of certain ordinary differential equations, Dokl. Akad. Nauk SSSR **144** (1962), 33–36. (Russian)=Soviet Math. Dokl. **3** (1962), 649–652.
[4] R. G. Koplatadze, On the oscillation of solutions of second-order differential equations with retarded argument, Soobšč. Akad. Nauk Gruzin. SSR **60** (1970), 269–272. (Russian)
[5] I. Ličko and M. Švec, Le caractère oscillatoire des solutions de l'équation $y^{(n)}+f(x)y^{\alpha}=0$, $n>1$, Czechoslovak Math. J. **13** (1963), 481–491.
[6] W. E. Mahfoud, Oscillation and asymptotic behavior of solutions of nth order nonlinear delay differential equations, J. Differential Equations **24** (1977), 75–98.
[7] W. E. Mahfoud, Characterization of oscillation of solutions of the delay equation $x^{(n)}(t)+a(t)f(x(q(t)))=0$, J. Differential Equations **28** (1978), 437–451.
[8] H. Onose, Oscillation and asymptotic behavior of solutions of retarded differential

equations of arbitrary order, Hiroshima Math. J. **3** (1973), 333–360.
[9] J. S. W. Wong, On second order nonlinear oscillation, Funkcialaj Ekvacioj **11** (1968), 207–234.

Department of Mathematics,
Faculty of Science,
Hiroshima University