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Unitary representations and kernel functions associated with boundaries of a bounded symmetric domain

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1. Introduction

Let $\mathscr{D}$ be the open unit disc in $\boldsymbol{C}$ and let $\mathscr{B}$ be its boundary. Then the group $G=SU(1, 1)$ of all two-by-two complex matrices of the form $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ with $|a|^2-|b|^2=1$ acts transitively both on $\mathscr{D}$ and $\mathscr{B}$ by linear fractional transformations

$$z \longrightarrow g\cdot z = \frac{az+b}{\bar{b}z+\bar{a}} \qquad \text{if} \quad g=\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}.$$

The discrete series representations of G can be realized on Hilbert spaces of holomorphic (or anti-holomorphic) functions on $\mathscr{D}$, while the principal continuous series representations can be realized on $L^2(\mathscr{B})$. Every member of the principal continuous series representations of G is irreducible except one, say V, which is given by

$$(V(g)f)(u) = j(g^{-1}, u)^{1/2}f(g^{-1}\cdot u), \quad f\in L^2(\mathscr{B}),\ g\in G,\ u\in\mathscr{B}$$

where $j(g^{-1}, u)$ denotes the complex Jacobian of the holomorphic map $z\to g^{-1}\cdot z$ at u ($j(g^{-1}, u)=(\bar{b}u+\bar{a})^{-2}$ if $g^{-1}=\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$).

The so-called holomorphic discrete series representations of G are parametrized by the integers $n\geq 2$, and the n-th representation T_n is realized on the Hilbert space

$$H_n = \left\{\text{holomorphic functions } f \text{ on } \mathscr{D};\ \int_{\mathscr{D}}|f(z)|^2(1-|z|^2)^{n-2}dxdy < \infty\right\}$$

with group action

$$(T_n(g)f)(z) = j(g^{-1}, z)^{n/2}f(g^{-1}\cdot z), \quad f\in H_n,\ g\in G,\ z\in\mathscr{D}.$$

Note that in the case $n=1$ we have $H_1=\{0\}$. However, one can associate to the integer $n=1$ a representation of G that is similar in appearance to those above. Indeed, if we let $H^2(\mathscr{D})$ be the Hardy space for $\mathscr{D}$, i.e.,

$$H^2(\mathscr{D}) = \left\{\text{holomorphic functions } f \text{ on } \mathscr{D};\ \sup_{0<r<1} \int_{\mathscr{B}} |f(ru)|^2 du < \infty\right\}$$

(du = Lebesgue measure on $\mathscr{B}$), then $H^2(\mathscr{D}) \neq \{0\}$ and the action T_1 given by

$$(T_1(g)f)(z) = j(g^{-1}, z)^{1/2} f(g^{-1} \cdot z), \quad f \in H^2(\mathscr{D}),\ g \in G,\ z \in \mathscr{D}$$

defines an irreducible unitary representation of G on $H^2(\mathscr{D})$. This representation does not belong to the holomorphic discrete series of G but is, in a sense, a limit of the holomorphic discrete series, and is of special interest due to the fact that it is unitarily equivalent to a proper subrepresentation of the "exceptional" representation $(V, L^2(\mathscr{B}))$ above. The realization of $(T_1, H^2(\mathscr{D}))$ as the subrepresentation of $(V, L^2(\mathscr{B}))$ arises from the well-known imbedding of $H^2(\mathscr{D})$ into $L^2(\mathscr{B})$ given by taking boundary values.

The possibility of such a realization of irreducible components of reducible continuous series representations was investigated by Knapp and Okamoto [15] in more generality, namely, in the context of a linear simple Lie group that acts on a hermitian symmetric space. (In this connection see also Knapp and Wallach [16] and Midorikawa [22].) On the other hand, for a simply connected real semi-simple Lie group G whose associated symmetric space G/K is hermitian, Harish-Chandra ([7], [8], [9]) constructed a certain class of (not necessarily unitary) representations that includes the holomorphic discrete series. These representations can be realized on the spaces of holomorphic sections of holomorphic vector bundles over G/K arising from finite dimensional irreducible representations of K (or on the spaces of holomorphic vector valued functions on G/K by trivializing vector bundles). Both Rossi and Vergne [25] and Wallach ([28], [29]) have studied the unitarizability of these representations and obtained complete results for the case of line bundles. In that specialized context (case of line bundles), it is also shown in [25] that those representations which are indexed by certain "integral or half-integral points" can be realized on Hardy type Hilbert spaces associated with various boundary orbits of G/K (in the unbounded realization as a Siegel domain of type II), and that they are naturally imbedded (in terms of appropriate boundary values) in certain continuous series representations. As noted in [25], the representation which corresponds to the maximal (codimension one) boundary of G/K is a member of the limits of holomorphic discrete series in the sense of Knapp and Okamoto [15]. Although Knapp and Okamoto constructed their representations on Hilbert spaces of holomorphic sections of holomorphic line bundles over G/T (T being a compact Cartan subgroup), it turns out that those representations can also be realized on vector valued Hardy type spaces associated with the maximal boundary of G/K.

Now in view of the above results of Knapp-Okamoto and Rossi-Vergne, it is natural to pose the following questions:

(1) In general (vector bundle) case, can one construct, corresponding to each boundary orbit of G/K, Hardy type Hilbert spaces on which G acts by unitary transformation?

(2) Supposing it is possible, are they imbedded in certain continuous series representations?

One of the main purposes of this paper is to give affirmative answers to these questions in the case where G is a connected simple Lie group that admits a faithful matrix representation (and whose quotient G/K by a maximal compact subgroup K carries a hermitian symmetric structure). Our construction is based on the bounded domain realization of G/K. We also consider the (operator valued) reproducing kernel functions for these Hardy type Hilbert spaces and intertwining operators associated with the kernel functions.

Here is a more detailed description of the contents of this paper. In the following it is convenient to use the notion of vector bundle, though we do not use in the text.

In section 2 we review some known facts that are needed in this paper, in a manner which is convenient for our later use. For a group G of above type, let $\mathscr{D}$ be the Harish-Chandra realization of the corresponding hermitian symmetric space G/K as a bounded domain in $\mathfrak{p}^+$ (cf. 2.1). The action of G on $\mathscr{D}$ extends smoothly to $\bar{\mathscr{D}}$, the closure of $\mathscr{D}$ in $\mathfrak{p}^+$, and the topological boundary $\bar{\mathscr{D}}-\mathscr{D}$ breaks into r ($r=\operatorname{rank} G/K$) G-orbits, say $\mathscr{B}_1,\dots,\mathscr{B}_r$, where $\bar{\mathscr{B}}_i \supset \mathscr{B}_{i+1}$ ($1\le i \le r-1$); thus $\mathscr{B}_r$ is the Silov boundary of $\mathscr{D}$. Each boundary $\mathscr{B}_i$ decomposes, in a G-equivariant manner, into complex submanifolds of $\mathfrak{p}^+$, called boundary components of $\mathscr{D}$ or holomorphic arc components of $\mathscr{B}_i$, which are themselves isomorphic to a bounded symmetric domain of rank $r-i$. For each $i=1,\dots,r$, there is a naturally associated point $o_i \in \mathscr{B}_i$ and a holomorphic arc component $\mathscr{C}_i$ of $\mathscr{B}_i$ containing o_i, and it is known that every holomorphic arc component of $\mathscr{B}_i$ is of the form $k\cdot\mathscr{C}_i$, $k\in K$. Further, there exists, for each $1\le i\le r$, a semi-simple subgroup G_i of G with $\mathscr{C}_i=G_i\cdot o_i$; thus if we let K_i be the isotropy subgroup of G_i at o_i, then $\mathscr{C}_i\cong G_i/K_i$. Let $P_i=\{g\in G;\ g\cdot\mathscr{C}_i=\mathscr{C}_i\}$ and $S_i=\{g\in G;\ g\cdot o_i=o_i\}$. Then P_i is a maximal parabolic subgroup of G, and we have a Langlands decomposition $P_i=M_iA_iN_i$ such that if we put $L_i=M_i\cap S_i$ then $S_i=L_iA_iN_i$. Each boundary $\mathscr{B}_i=G/S_i$ has a natural quasi-invariant measure $d\mu$ so that

$$\int_{\mathscr{B}_i} f(u)d\mu(u) = \int_{K\times G_i} f(kg_i\cdot o_i)dkdg_i$$

for any integrable f on $\mathscr{B}_i$, where dk, dg_i are Haar measures on K, G_i.

In Section 3 we define, for each $1\le i\le r$, a certain subset $\mathscr{F}_i(G)\subset\sqrt{-1}\mathfrak{t}^*$ ($\mathfrak{t}$=Lie algebra of a compact Cartan subgroup T of G) consisting of highest weights of irreducible representations of K. (To each member of $\mathscr{F}_i(G)$, we shall associate a Hardy type space and an irreducible unitary representation of G in Section 4.) It turns out that in the case $i=1$ the defining condition of $\mathscr{F}_1(G)$ is equivalent to the condition imposed by Knapp and Okamoto [15]. Our definition of $\mathscr{F}_i(G)$ is given in such a way that each member of $\mathscr{F}_i(G)$ is expressed explicitly in terms of fundamental highest weights; cf. (3.8).

In Section 4 we construct, corresponding to each member of $\mathscr{F}_i(G)$, unitary representations of G. Fix i, $1 \leq i \leq r$, and $\lambda \in \mathscr{F}_i(G)$. Let τ_λ be the irreducible unitary representation of K on E_λ with highest weight λ, and let $\boldsymbol{e}_\lambda$ be a nonzero highest weight vector. If we let E_λ^i be the linear span of $\{\tau_\lambda(k)\boldsymbol{e}_\lambda;\ k \in K_i\}$, then for each $\nu \in \mathfrak{a}_i^*$ ($\mathfrak{a}_i$=Lie algebra of A_i) we obtain (cf. (4.9), (4.10)) irreducible representations $\sigma_{\lambda,\nu}$ and $'\sigma_{\lambda,\nu}$ of S_i on E_λ^i, $'\sigma_{\lambda,\nu}$ being unitary, and the unitarily induced representation $U_{\lambda,\nu}=\mathrm{Ind}_{S_i \uparrow G}\,'\sigma_{\lambda,\nu}$ is realized on the Hilbert space $L^2(G, \sigma_{\lambda,\nu})$ of L^2 sections (relative to the quasi-invariant measure $d\mu$ on $\mathscr{B}_i$) of the G-homogeneous vector bundle over the boundary $\mathscr{B}_i = G/S_i$ associated with the representation $\sigma_{\lambda,\nu}$ of S_i. (In the case $\nu=0$, we write σ_λ, U_λ instead of $\sigma_{\lambda,0}$, $U_{\lambda,0}$.) Next we introduce a Hardy type Hilbert space $H^2(\mathscr{D}, \lambda)$ of E_λ-valued holomorphic functions on $\mathscr{D}$ (cf. (4.16), (4.35)), which is imbedded in $L^2(G, \sigma_\lambda)$ by taking appropriate boundary values. $H^2(\mathscr{D}, \lambda)$ is naturally identified with a space $H^2(G, \tau_\lambda)$ of holomorphic sections of the holomorphic vector bundle over G/K associated with the representation τ_λ of K (the vector bundle being holomorphically trivial). We then show (Theorem 4.49) that $H^2(\mathscr{D}, \lambda)$ is nonzero and the action T_λ of G on $H^2(\mathscr{D}, \lambda)$ given by

$$(T_\lambda(g)F)(z) = J_\lambda(g^{-1}, z)^{-1}F(g^{-1}\cdot z), \quad F \in H^2(\mathscr{D}, \lambda),\ g \in G,\ z \in \mathscr{D}$$

(J_λ=automorphic factor of type τ_λ; cf. 2.4) defines an irreducible unitary representation of G on $H^2(\mathscr{D}, \lambda)$, and that the imbedding $H^2(\mathscr{D}, \lambda) \subsetneq L^2(G, \sigma_\lambda)$ commutes with the action of G.

In Section 5 we first construct, corresponding to each $\lambda \in \mathscr{F}_i(G)$, an irreducible unitary representation μ_λ of M_i and form the continuous series representations

$$V_{\lambda,\nu} = \mathrm{Ind}_{M_iA_iN_i \uparrow G}(\mu_\lambda \otimes e^{\sqrt{-1}\nu} \otimes 1), \quad \nu \in \mathfrak{a}_i^*.$$

We then show (Proposition 5.9) that the $V_{\lambda,\nu}$ is unitarily equivalent to a subrepresentation $(U_{\lambda,\nu}, L^2(G, \sigma_{\lambda,\nu}; \mathfrak{p}_i^-))$ of the representation $(U_{\lambda,\nu}, L^2(G, \sigma_{\lambda,\nu}))$ in Section 4; here $L^2(G, \sigma_{\lambda,\nu}; \mathfrak{p}_i^-)$ is a subspace of $L^2(G, \sigma_{\lambda,\nu})$ consisting of those sections that are holomorphic on every holomorphic arc component of $\mathscr{B}_i$. Finally, we show (Theorem 5.13) that the representation T_λ of G on $H^2(\mathscr{D}, \lambda)$ is unitarily equivalent to a proper subrepresentation of V_λ $(=V_{\lambda,0})$ and hence V_λ is reducible.

In Section 6 we discuss the reproducing kernel function K_λ of $H^2(\mathscr{D}, \lambda)$ and derive (Proposition 6.4) an explicit formula for K_λ. We then specialize to the case where τ_λ ($\lambda \in \mathscr{F}_i(G)$) are one dimensional representations of K. In this situation there exists at most one, say, $\omega_i \in \mathscr{F}_i(G)$ for each $1 \leq i \leq r$; cf. (6.16). (If we take a suitable covering G° of G, then for each $1 \leq i \leq r$ there exists a unique $\lambda \in \mathscr{F}_i(G^\circ)$ with $\dim E_\lambda = 1$. But G° is not necessarily a linear group.) In the case $i = r$, $H^2(\mathscr{D}, \omega_r)$ turns out to be the usual Hardy space for the bounded symmetric domain $\mathscr{D}$ (cf. (6.17)), and hence the kernel function corresponding to ω_r is the Cauchy-Szegö kernel function of $\mathscr{D}$. For kernel functions corresponding to these ω_i, $1 \leq i \leq r$, we find more explicit formulas; cf. Proposition 6.22. The formula for the Cauchy-Szegö kernel function was first derived by Korányi [17, Proposition 5.7] by translating the results (due mostly to Gindikin [3]) on Siegel domains of type II to bounded symmetric domains by the Cayley transform (due to Korányi and Wolf [18]); for the classical domains it was first found by Hua [12].

In Section 7 we give (Theorem 7.4) an integral operator $\mathscr{P}_\lambda$: $L^2(G, \sigma_\lambda) \to H^2(G, \tau_\lambda)$ (here we identify $L^2(G, \sigma_\lambda)$ (resp. $H^2(G, \tau_\lambda)$) with a certain subspace of the space of $E_{\bar\lambda}$ (resp. E_λ) valued functions on G) which is regarded as the orthogonal projection operator if we identify $H^2(G, \tau_\lambda)$ with a subspace of $L^2(G, \sigma_\lambda)$, and also show that on the subspace $L^2(G, \sigma_\lambda; \mathfrak{p}_i^-)$, $\mathscr{P}_\lambda$ is given by

$$\mathscr{P}_\lambda \phi(g) = \beta \int_K \tau_\lambda(k) \phi(gk) dk, \quad g \in G, \ \phi \in L^2(G, \sigma_\lambda; \mathfrak{p}_i^-)$$

where β is a positive constant. Intertwining operators that take such a form as above were considered by Okamoto [24] and Knapp and Wallach [16] in other contexts, e.g. intertwining maps from non-unitary principal series representations to (limits of) discrete series representations. We note that in the special case $\lambda = \omega_r$ in the notation of Section 6, $\mathscr{P}_\lambda$ corresponds to the integral operator associated with the Cauchy-Szegö kernel function (cf. the remark at the end of this paper).

For the groups associated with classical hermitian symmetric spaces of tube type, i.e., for $Sp(n, \boldsymbol{R})$, $U(n, n)$ and $O^*(4n)$ Gross and Kunze have produced, in their study of the primary decompositions of metapletic representations ([5], [6]), some irreducible unitary representations with highest weights which are not in discrete series. (The representations that we shall construct have this property; cf. Lemma 4.52.) In a similar way Kashiwara and Vergne [14] have obtained series of such representations for $U(p, q)$ and the metapletic group $Mp(n, \boldsymbol{R})$, a two-sheeted covering of $Sp(n, \boldsymbol{R})$. In these papers, however, it is not discussed whether some of those representations have realizations in Hardy type spaces or whether they are imbedded in continuous series representations. For the conformal group $U(2, 2)$, related topics were considered by Jacobsen and Vergne [13], and by Gross, Holman and Kunze [4]. In particular, in [4] some vector-valued Hardy spaces for the corresponding Siegel domain (an unbounded realization of G/K, $G=U(2, 2)$) are introduced.

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2. Notations and preliminaries

Let G be a connected simple Lie group with a faithful matrix representation and K a maximal compact subgroup of G. We assume that G/K admits an invariant complex structure. Then K has one dimensional center, and one may choose a compact Cartan subgroup T of G with $T\subset K$. We denote the Lie algebras of G, K, T by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{t}$, and their complexifications by $\mathfrak{g}_c$, $\mathfrak{k}_c$, $\mathfrak{t}_c$; as a general notational convention the subscript c shall always mean "complexification". Since G has a faithful matrix representation, we can regard G as a subgroup of a connected group G_c with Lie algebra $\mathfrak{g}_c$. Let K_c, T_c denote the analytic subgroups of G_c corresponding to $\mathfrak{k}_c$, $\mathfrak{t}_c$.

Let Φ be the set of nonzero roots of $(\mathfrak{g}_c, \mathfrak{t}_c)$, and let Φ_c and Φ_n be the set of compact and noncompact roots, respectively; thus $\Phi=\Phi_c\cup\Phi_n$, and if $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is the Cartan decomposition corresponding to K, we have

$$\mathfrak{k}_c = \mathfrak{t}_c + \textstyle\sum_{\alpha\in\Phi_c}\mathfrak{g}_c^\alpha \quad\text{and}\quad \mathfrak{p}_c = \textstyle\sum_{\alpha\in\Phi_n}\mathfrak{g}_c^\alpha$$

where $\mathfrak{g}_c^\alpha$ denotes the complex root space for a root α. If $\alpha\in\Phi$, we denote by H_α the unique element of $\sqrt{-1}\,\mathfrak{t}$ such that

$$2(\mu, \alpha)/(\alpha, \alpha) = \langle\mu, H_\alpha\rangle \qquad \text{for all} \quad \mu\in\sqrt{-1}\,\mathfrak{t}^*$$

where $\sqrt{-1}\,\mathfrak{t}^*$ is the real vector space of all linear functions on $\mathfrak{t}_c$ which assume purely imaginary values on $\mathfrak{t}$, and $(\ ,\ )$ is the inner product on $\sqrt{-1}\,\mathfrak{t}^*$ induced by the Killing form of $\mathfrak{g}_c$. For each $\alpha\in\Phi$, we choose a root vector $X_\alpha\in\mathfrak{g}_c$ such that $[X_\alpha, X_{-\alpha}]=H_\alpha$, and such that the complex conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$ permutes X_α and $X_{-\alpha}$, whenever $\alpha\in\Phi_n$. Note that $\langle\alpha, H_\alpha\rangle=2$, $B(X_\alpha, X_{-\alpha})=2/(\alpha, \alpha)$ ($B(\ ,\ )=$Killing form of $\mathfrak{g}_c$) for all $\alpha\in\Phi$, and that $X_\alpha+X_{-\alpha}$, $\sqrt{-1}(X_\alpha-X_{-\alpha})$ are in $\mathfrak{g}$ if $\alpha\in\Phi_n$.

By our assumption on G/K, there exists an ordering of the root system Φ, such that the sum of two noncompact positive roots is never a root. We fix such an ordering once and for all, and let Φ^+ be the resulting set of positive roots. We write Φ_c^+ for $\Phi_c\cap\Phi^+$ and Φ_n^+ for $\Phi_n\cap\Phi^+$. The choice of Φ^+ determines a splitting

$$\mathfrak{p}_c = \mathfrak{p}^+ + \mathfrak{p}^- \quad\text{with}\quad \mathfrak{p}^+ = \textstyle\sum_{\alpha\in\Phi_n^+}\mathfrak{g}_c^\alpha \quad\text{and}\quad \mathfrak{p}^- = \textstyle\sum_{\alpha\in\Phi_n^+}\mathfrak{g}_c^{-\alpha}.$$

Both $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are $\mathrm{Ad}(K_c)$-invariant, abelian subalgebras of $\mathfrak{g}_c$ which are complex conjugate to each other. Let P^+ and P^- be the corresponding analytic subgroups of G_c. Then K_c normalizes $P^\pm$ and $K_c \cdot P^\pm$ is a parabolic subgroup of G_c. The G-orbit of the identity coset in $G_c/K_c \cdot P^-$ is open and can be identified with G/K, since $G \cap K_c \cdot P^- = K$. Thus the imbedding $G/K \subset G_c/K_c \cdot P^-$ induces an invariant complex structure on G/K.

2.1. *Harish-Chandra realization of* G/K (see [9], [31]). The map $\mathfrak{p}^+ \times K_c \times \mathfrak{p}^- \to G_c$, given by $(x, k, y) \to \exp x \cdot k \cdot \exp y$, is a holomorphic diffeomorphism onto a dense open subset $\Omega = P^+ \cdot K_c \cdot P^-$ of G_c, which contains G. Therefore an element $g \in \Omega$ can be written in a unique way as

$$(2.2) \qquad g = \pi_+(g) \cdot \pi_0(g) \cdot \pi_-(g), \quad \pi_0(g) \in K_c, \ \pi_\pm(g) \in P^\pm.$$

It is known that the map $\zeta\colon \Omega \to \mathfrak{p}^+$, given by $\zeta(g) = \log \pi_+(g)$, induces a holomorphic diffeomorphism of G/K onto $\zeta(G) = \mathscr{D}$, and that $\mathscr{D}$ is a bounded domain in $\mathfrak{p}^+$. This is the Harish-Chandra realization of G/K as a bounded domain. We will make the following identification:

$$G/K = \mathscr{D} \subset \mathfrak{p}^+ \subset G_c/K_c \cdot P^-.$$

Note that the action of G on $\mathscr{D}$ is given by

$$(2.3) \qquad g \cdot z = \zeta(g \exp z), \quad g \in G, \ z \in \mathscr{D}.$$

2.4. *Automorphic factor* (see [21]). For a holomorphic representation τ of K_c on a finite dimensional complex vector space E, we define the (canonical) automorphic factor of type τ, $J_\tau\colon G \times \mathscr{D} \to GL(E)$, by

$$(2.5) \qquad J_\tau(g, z) = \tau(\pi_0(g \exp z)), \quad g \in G, \ z \in \mathscr{D},$$

where π_0 is as in (2.2). It is then easily verified that J_τ has the following properties:

(2.6a) $J_\tau(g, z)$ is C^∞ in $g \in G$ and holomorphic in $z \in \mathscr{D}$;

(2.6b) $J_\tau(g_1 g_2, z) = J_\tau(g_1, g_2 \cdot z) J_\tau(g_2, z)$ for $g_1, g_2 \in G$, $z \in \mathscr{D}$;

(2.6c) $J_\tau(k, z) = \tau(k)$ for $k \in K$, $z \in \mathscr{D}$.

The formula (2.6b) will be referred to as the *cocycle formula*. We note that the definition of $g \cdot z$ and $J_\tau(g, z)$ can naturally be extended to any pair (g, z), $g \in G_c$, $z \in \mathfrak{p}^+$ such that $g \exp z \in \Omega = P^+ K_c P^-$, and that the cocycle formula (2.6b) is valid for $g_1, g_2 \in G_c$, $z \in \mathfrak{p}^+$ such that both $g_2 \exp z$ and $g_1 g_2 \exp z$ are in Ω. In particular, for a fixed $g \in G$, $J_\tau(g, \cdot)$ can be defined and is holomorphic on $\{z \in \mathfrak{p}^+;\ g \exp z \in \Omega\}$ which is an open subset of $\mathfrak{p}^+$ containing $\bar{\mathscr{D}}$, the closure of

$\mathscr{D}$ in $\mathfrak{p}^+$. Note also that

$$J_\tau(k, z) = \tau(k) \quad \text{for} \quad k \in K_{\mathbf{c}},\ z \in \mathfrak{p}^+, \tag{2.7a}$$

$$J_\tau(p, z) = I \quad \text{for} \quad p \in P^+,\ z \in \mathfrak{p}^+, \tag{2.7b}$$

where I denotes the identity transformation of E.

2.8. *Description of the root system* Φ (see [9], [23]). Two linearly independent roots α, β are called strongly orthogonal if neither $\alpha+\beta$ nor $\alpha-\beta$ is a root. We choose a maximal strongly orthogonal set

$$\{\gamma_1, \gamma_2,\dots, \gamma_r\}, \quad \gamma_1 > \gamma_2 > \cdots > \gamma_r, \quad r = \operatorname{rank} \mathscr{D} \tag{2.9}$$

of Φ_n^+ as follows. Let γ_1 be the highest root of Φ (we know that γ_1 belongs to Φ_n^+) and for each j, γ_{j+1} be the highest positive noncompact root that is strongly orthogonal to each of $\gamma_1,\dots, \gamma_j$. We write H_j, X_j, X_{-j} for H_{γ_j}, X_{γ_j}, $X_{-\gamma_j}$. Let

$$\mathfrak{a} = \textstyle\sum_{j=1}^r \boldsymbol{R}(X_j + X_{-j}). \tag{2.10}$$

This is a maximal abelian subspace of $\mathfrak{p}$. Let $\mathfrak{t}^- = \sum_{j=1}^r \boldsymbol{R}H_j$ and let π denote restriction of roots from $\mathfrak{t}_{\mathbf{c}}$ to $\mathfrak{t}^-$. Identifying each element of $\gamma_1,\dots, \gamma_r$ with its π image, we put

$$\begin{aligned}
C_0 &= \{\alpha \in \Phi_c^+;\ \pi(\alpha) = 0\}; \\
C_j &= \left\{\alpha \in \Phi_c^+;\ \pi(\alpha) = \tfrac{1}{2}\gamma_j\right\} \quad \text{for} \quad 1 \le j \le r; \\
N_j &= \left\{\alpha \in \Phi_n^+;\ \pi(\alpha) = \tfrac{1}{2}\gamma_j\right\} \quad \text{for} \quad 1 \le j \le r; \\
C_{jk} &= \left\{\alpha \in \Phi_c^+;\ \pi(\alpha) = \tfrac{1}{2}(\gamma_j - \gamma_k)\right\} \quad \text{for} \quad 1 \le j < k \le r; \\
N_{jk} &= \left\{\alpha \in \Phi_n^+;\ \pi(\alpha) = \tfrac{1}{2}(\gamma_j + \gamma_k)\right\} \quad \text{for} \quad 1 \le j < k \le r.
\end{aligned} \tag{2.11}$$

Then, by results of Harish-Chandra [9] and Moore [23], we have (since our construction of $\{\gamma_1,\dots, \gamma_r\}$ differs from that of Harish-Chandra, their results should be modified slightly; cf. Takeuchi [27] in this connection):

$$\Phi^+ = C_0 \cup \bigcup_{1\le j\le r} (C_j \cup N_j) \cup \bigcup_{1\le j<k\le r} (C_{jk} \cup N_{jk}) \cup \{\gamma_1,\dots, \gamma_r\}. \tag{2.12a}$$

(2.12b) The number of roots in C_{jk} (resp. C_j) is equal to that in N_{jk} (resp. N_j) and is independent of j, k, $1\le j<k\le r$ (resp. j, $1\le j\le r$); we call this number u (resp. v). Moreover, if $r>1$ then $u>0$ (v may be 0).

(2.12c) Nonzero π images of compact simple roots for Φ^+ are

$$\begin{cases} \left\{\frac{1}{2}(\gamma_1-\gamma_2),\dots,\frac{1}{2}(\gamma_{r-1}-\gamma_r)\right\} & \text{if } v=0 \\ \left\{\frac{1}{2}(\gamma_1-\gamma_2),\dots,\frac{1}{2}(\gamma_{r-1}-\gamma_r),\frac{1}{2}\gamma_r\right\} & \text{if } v\neq 0. \end{cases}$$

(2.12d) The γ_j all have the same length.

2.13. *Cayley transforms* (see [31], [32]). For each $1\le i\le r$, we define an element $c_i\in G_c$, which is called the (partial) Cayley transform, by

$$c_i=\prod_{1\le j\le i}\exp\frac{\pi}{4}(X_{-j}-X_j). \tag{2.14}$$

It verifies

$$\begin{aligned} &\mathrm{Ad}(c_i)H_j=X_j+X_{-j},\quad \mathrm{Ad}(c_i)(X_j+X_{-j})=-H_j && \text{for } 1\le j\le i;\\ &\mathrm{Ad}(c_i)H_j=H_j,\quad \mathrm{Ad}(c_i)(X_j+X_{-j})=X_j+X_{-j} && \text{for } i<j\le r. \end{aligned} \tag{2.15}$$

Moreover

$$G\cdot c_i\subset P^+\cdot K_c\cdot P^-, \tag{2.16}$$

and

$$\bar c_i=c_i^{-1} \tag{2.17}$$

where the bar denotes the complex conjugation of G_c with respect to G. An explicit computation in $SL(2,\boldsymbol{C})$ plus the commutativity of X_j and $X_{\pm k}$ for $j\neq k$ shows that

$$c_i=\exp\left(-\textstyle\sum_{j=1}^{i}X_j\right)\cdot\exp\left(\log\sqrt{2}\textstyle\sum_{j=1}^{i}H_j\right)\cdot\exp\left(\textstyle\sum_{j=1}^{i}X_{-j}\right). \tag{2.18}$$

$$\begin{aligned} &\exp\left(\textstyle\sum_{j=1}^{i}x_j(X_j+X_{-j})\right)\\ &\quad=\exp\left(\textstyle\sum_{j=1}^{i}(\tanh x_j)X_j\right)\cdot\exp\left(-\textstyle\sum_{j=1}^{i}\log(\cosh x_j)H_j\right)\\ &\qquad\qquad\cdot\exp\left(\textstyle\sum_{j=1}^{i}(\tanh x_j)X_{-j}\right). \end{aligned} \tag{2.19}$$

2.20 *Maximal parabolic subgroups* (see [31], [32], [1]). For each i, $1\le i\le r$, let $'C_0^{(i)}$ denote the set of positive roots which are of the form $\alpha-\beta$ with $\alpha,\beta\in C_{jk}$, $1\le j<k\le i$, and set $C_0^{(i)}=C_0-{}'C_0^{(i)}$ where C_{jk}, C_0 are as in (2.11) (it is clear that $'C_0^{(i)}\subset C_0$). Using the notation in (2.11), we define

$$\Phi_i=\pm C_0^{(i)}\cup\bigcup_{i+1\le j\le r}(\pm C_j\cup\pm N_j)\cup\bigcup_{i+1\le j<k\le r}(\pm C_{jk}\cup\pm N_{jk})\cup\{\pm\gamma_{i+1},\dots,\pm\gamma_r\}, \tag{2.21a}$$

(2.21b) $\quad {}'\Phi_i = \pm{}'C_0^{(i)} \cup \bigcup_{1\le j<k\le i} (\pm C_{jk})$.

Then (2.12a) implies

$$\{\alpha \in \Phi;\ \langle \alpha, H_1 + \cdots + H_i \rangle = 0\} = \Phi_i \cup {}'\Phi_i.$$

Thus if we set $\mathfrak{h}_i = \boldsymbol{R}(H_1 + \cdots + H_i)$ and let $Z_{\mathfrak{g}_c}(\mathfrak{h}_i)$ denote the centralizer of $\mathfrak{h}_i$ in $\mathfrak{g}_c$, we have

$$Z_{\mathfrak{g}_c}(\mathfrak{h}_i) = \mathfrak{t}_c + \textstyle\sum_{\alpha\in\Phi_i} \mathfrak{g}_c^\alpha + \sum_{\alpha\in'\Phi_i} \mathfrak{g}_c^\alpha.$$

Let $\mathfrak{g}_{i,c}$ (resp. $'\mathfrak{g}_{i,c}$) be the subspace of $\mathfrak{g}_c$ generated by the H_α and the X_α for $\alpha \in \Phi_i$ (resp. $'\Phi_i$) with the convention that $\mathfrak{g}_{i,c}$ (resp. $'\mathfrak{g}_{i,c}) = \{0\}$ in case Φ_i (resp. $'\Phi_i$) is empty. As follows from (2.12a) and the definition of $'C_0^{(i)}$ and $C_0^{(i)}$, no roots of Φ_i and of $'\Phi_i$ add up to a root, whence Φ_i and $'\Phi_i$ are closed systems, i.e., if $\alpha, \beta \in \Phi_i$ (resp. $'\Phi_i$) and $\alpha+\beta \in \Phi$, then $\alpha+\beta \in \Phi_i$ (resp. $'\Phi_i$); moreover $\Phi_i = -\Phi_i$, $'\Phi_i = -{}'\Phi_i$. Hence $\mathfrak{g}_{i,c}$ and $'\mathfrak{g}_{i,c}$ are semi-simple subalgebras of $\mathfrak{g}_c$, which are contained in $Z_{\mathfrak{g}_c}(\mathfrak{h}_i)$. Furthermore, since no roots of Φ_i and of $'\Phi_i$ add up to a root, $\mathfrak{g}_{i,c}$ and $'\mathfrak{g}_{i,c}$ are ideals in $Z_{\mathfrak{g}_c}(\mathfrak{h}_i)$, and $Z_{\mathfrak{g}_c}(\mathfrak{h}_i)$ decomposes as an orthogonal (relative to the Killing form for $\mathfrak{g}_c$) direct sum

(2.22) $\quad Z_{\mathfrak{g}_c}(\mathfrak{h}_i) = \mathfrak{g}_{i,c} \oplus {}'\mathfrak{g}_{i,c} \oplus \mathfrak{i}_{i,c} \oplus \mathfrak{h}_{i,c} \quad$ with $\quad \mathfrak{i}_{i,c} \subset \mathfrak{t}_c$.

Now (2.15) implies

(2.23a) $\quad \mathrm{Ad}\,(c_i)(\sum_{j=1}^i H_j) = \sum_{j=1}^i (X_j + X_{-j})$.

Thus if we set $\mathfrak{a}_i = \boldsymbol{R}\sum_{j=1}^i (X_j + X_{-j})$, then

(3.23b) $\quad \mathrm{Ad}\,(c_i)\mathfrak{h}_i = \mathfrak{a}_i$.

Noting that $\mathrm{Ad}\,(c_i)$ acts trivially both on $\mathfrak{g}_{i,c}$ and $\mathfrak{i}_{i,c}$, we see from (2.22) and (2.23b) that

$$Z_{\mathfrak{g}_c}(\mathfrak{a}_i) = \mathfrak{g}_{i,c} \oplus \mathrm{Ad}\,(c_i)'\mathfrak{g}_{i,c} \oplus \mathfrak{i}_{i,c} \oplus \mathfrak{a}_{i,c}$$

($Z_{\mathfrak{g}_c}(\mathfrak{a}_i)$ = centralizer of $\mathfrak{a}_i$ in $\mathfrak{g}_c$). Each direct summand of this decomposition, being invariant under the complex conjugation with respect to $\mathfrak{g}$ (invariance of $\mathrm{Ad}\,(c_i)'\mathfrak{g}_{i,c}$ follows from (2.17), since, in view of (2.15), $Z_{\mathfrak{g}_c}(\mathfrak{h}_i)$ and hence $'\mathfrak{g}_{i,c}$ is preserved by $\mathrm{Ad}\,(c_i^{-2})$), arises as the complexification of a real subalgebra of $\mathfrak{g}$. Hence if we put $\mathfrak{g}_i = \mathfrak{g}_{i,c} \cap \mathfrak{g}$, $\mathfrak{g}_i' = \mathrm{Ad}\,(c_i)'\mathfrak{g}_{i,c} \cap \mathfrak{g}$ and $\mathfrak{i}_i = \mathfrak{i}_{i,c} \cap \mathfrak{g}$, then

$$Z_{\mathfrak{g}}(\mathfrak{a}_i) = \mathfrak{g}_i \oplus \mathfrak{g}_i' \oplus \mathfrak{i}_i \oplus \mathfrak{a}_i.$$

Let

(2.24a) $\quad \mathfrak{n}_i$: sum of the negative eigenspaces of $\mathrm{ad}\,(\sum_{j=1}^i (X_j + X_{-j}))$ on $\mathfrak{g}$, (2.24b) P_i: normalizer of $\mathfrak{n}_i$ in G.

Then P_i is a parabolic subgroup of G and we have the semidirect sum

$$\text{Lie algebra of } P_i = (\mathfrak{g}_i \oplus \mathfrak{g}'_i \oplus \mathfrak{i}_i \oplus \mathfrak{a}_i) + \mathfrak{n}_i.$$

Let G_i, G'_i, I_i, A_i, N_i denote the analytic subgroups of G corresponding, respectively, to $\mathfrak{g}_i$, $\mathfrak{g}'_i$, $\mathfrak{i}_i$, $\mathfrak{a}_i$, $\mathfrak{n}_i$. Letting $\mathfrak{a}$ be as in (2.10), put $F=\exp\sqrt{-1}\,\mathfrak{a}\cap K$; F is a finite subgroup normalizing G_i, G'_i, and I_i commutes with it. Now let $M_i=FI_iG_iG'_i$. Then M_i is a closed subgroup of G and we have a Langlands decomposition

$$P_i = M_iA_iN_i \tag{2.25}$$

(for Langlands decomposition of a parabolic subgroup, see Warner [30]). In our situation it is known (cf. Knapp and Okamoto [15], p. 386) that the finite group F is generated by the elements

$$\exp\pi(X_j - X_{-j}) = \exp\pi\sqrt{-1}H_j, \qquad 1 \leq j \leq r.$$

Let F_i be the subgroup of F generated by $\{\exp\pi\sqrt{-1}H_j;\ 1\leq j\leq i\}$. Then it is clear that F_i commutes with G_i. Furthermore, since $\sqrt{-1}H_j\in\mathfrak{g}_i$ for $i+1\leq j\leq r$, it follows that

$$M_i = F_iI_iG_iG'_i. \tag{2.26}$$

We note that

(2.27) the Cayley transform c_i centralizes F_i, I_i and G_i.

The group P_i is a maximal parabolic subgroup of G, and every maximal parabolic subgroup of G is conjugate to one of the groups P_i $(1\leq i\leq r)$.

2.28 *Boundary orbits, boundary components* (see [31], [32]). The formula (2.18) shows that $c_i\in P^+\cdot K_c\cdot P^-$ so that $\zeta(c_i)\in\mathfrak{p}^+$ where ζ is as in 2.1. Put $o_i=c_i\cdot o$ in the notation of (2.3), and let $\mathscr{B}_i=G\cdot o_i$ (the orbit of o_i under G). Then

$$\bar{\mathscr{D}} - \mathscr{D} = \bigcup_{1\leq i\leq r} \mathscr{B}_i \qquad \text{(disjoint union)}$$

and $\mathscr{B}_r$ is the Silov boundary.

With $\mathfrak{g}_i$, G_i being as in 2.20, put $\mathfrak{k}_i=\mathfrak{g}_i\cap\mathfrak{k}$, $\mathfrak{p}_i=\mathfrak{g}_i\cap\mathfrak{p}$, $\mathfrak{p}_i^{\pm}=\mathfrak{g}_{i,c}\cap\mathfrak{p}^{\pm}$, and let K_i denote the analytic subgroup of G_i with Lie algebra $\mathfrak{k}_i$. Then $\mathfrak{g}_i=\mathfrak{k}_i+\mathfrak{p}_i$ is a Cartan decomposition; moreover we have the direct sum decomposition

$$\mathfrak{g}_{i,c} = \mathfrak{k}_{i,c} + \mathfrak{p}_i^+ + \mathfrak{p}_i^-,$$

and so the space G_i/K_i is hermitian symmetric. It is known that the hermitian symmetric space G_i/K_i is irreducible. Now let $\mathscr{D}_i=G_i\cdot o$, $\mathscr{C}_i=G_i\cdot o_i$. Then $\mathscr{D}_i\cong G_i/K_i$ and, since $g\cdot o_i=o_i+g\cdot o$ for $g\in G_i$ by (2.27), we have

$$\mathscr{C}_i = c_i\cdot\mathscr{D}_i = o_i + \mathscr{D}_i.$$

Furthermore

$$\mathscr{B}_i = \bigcup_{k\in K} k\cdot\mathscr{C}_i.$$

The transforms of the $\mathscr{C}_i$'s by elements of G are the boundary components of $\mathscr{D}$. We note that $\mathfrak{g}_r\subset\mathfrak{k}$ by definition; hence

(2.29) $$G_r = K_r \subset K,\quad \mathscr{C}_r = \{o_r\},\quad \text{and}\quad \mathscr{B}_r = K\cdot o_r.$$

Note also that G_i is noncompact if $i\neq r$.

It is known that the parabolic subgroup P_i $(=F_iI_iG_iG_i'A_iN_i)$ in 2.20 is the normalizer of the boundary component $\mathscr{C}_i$, i.e.,

(2.30) $$P_i = \{g\in G;\ g\cdot\mathscr{C}_i = \mathscr{C}_i\}.$$

Moreover,

(2.31) $F_iI_iG_i'A_iN_i$ (this is a group) acts trivially on $\mathscr{C}_i$.

PROOF OF (2.31). We first check that N_i acts trivially on $\mathscr{C}_i$. Since N_i is normalized by G_i, and since G_i acts transitively on $\mathscr{C}_i$, it suffices to show that $n\cdot o_i=o_i$ or that $c_i^{-1}nc_i\cdot o=o$ for all $n\in N_i$; in turn, for this, it will be enough to show that $\mathrm{Ad}\,(c_i^{-1})\mathfrak{n}_i\subset\mathfrak{k}_c+\mathfrak{p}^-$, because K_cP^- is the isotropy subgroup of G_c at o. By (2.23) and the definition (2.24a) of $\mathfrak{n}_i$, $\mathrm{Ad}\,(c_i^{-1})\mathfrak{n}_i$ is contained in sum of the negative eigenspaces of $\mathrm{ad}\,(H_1+\cdots+H_i)$ on $\mathfrak{g}_c$. But, in view of (2.12a), all these eigenspaces are in $\mathfrak{k}_c+\mathfrak{p}^-$, so N_i centralizes $\mathscr{C}_i$. Next we check that $G_i'A_i$ acts trivially on $\mathscr{C}_i$. Since $G_i'A_i$ commutes with G_i it suffices, by the same reason as above, to show that $\mathrm{Ad}\,(c_i^{-1})(\mathfrak{g}_i'+\mathfrak{a}_i)\subset\mathfrak{k}_c+\mathfrak{p}^-$. But this is clear because $\mathrm{Ad}\,(c_i^{-1})\mathfrak{g}_i'\subset{}'\mathfrak{g}_{i,c}\subset\mathfrak{k}_c$ and $\mathrm{Ad}\,(c_i^{-1})\mathfrak{a}_i=\mathfrak{h}_i\subset\mathfrak{k}_c$. Finally as for F_iI_i, one needs only to note that it is contained in K and commutes with c_i and G_i. ∎

Let S_i denote the isotropy subgroup of G at o_i. Then, since $S_i\subset P_i$, (2.25), (2.26), (2.30) and (2.31) imply

(2.32) $$S_i = F_iI_iK_iG_i'A_iN_i.$$

2.33. *Normalization of measures.* We fix i, $1\leq i\leq r$. Let $\mathfrak{t}_i=\mathfrak{g}_i\cap\mathfrak{t}$; this is a Cartan subalgebra of $\mathfrak{g}_i$. If we put $\mathfrak{t}_i^-=\sum_{j=i+1}^r \boldsymbol{R}H_j$ and $\mathfrak{t}_i^+=\{H\in\mathfrak{t}_i;\ \langle\gamma_j, H\rangle=0 \text{ for all } i+1\leq j\leq r\}$, then $\mathfrak{t}_i=\mathfrak{t}_i^++\sqrt{-1}\,\mathfrak{t}_i^-$. The root subsystem Φ_i

defined by (2.21a) can be naturally identified (by restriction) with the root system of $(\mathfrak{g}_{i,c}, \mathfrak{t}_{i,c})$. In Φ_i, Φ^+ induces the system of positive roots

$$\Phi_i^+ = \Phi_i \cap \Phi^+. \tag{2.34}$$

Now assume $1 \leq i \leq r-1$; thus G_i is noncompact. Letting $\mathfrak{a}$, $\mathfrak{p}_i$ be as in (2.10) and in 2.28, we set $\mathfrak{a}_{(i)} = \mathfrak{a} \cap \mathfrak{p}_i$. Then

$$\mathfrak{a}_{(i)} = \textstyle\sum_{j=i+1}^{r} \boldsymbol{R}(X_j + X_{-j}) \tag{2.35a}$$

and $\mathfrak{a}_{(i)}$ is a maximal abelian subspace of $\mathfrak{p}_i$. If c_r is the (full) Cayley transform for $\mathfrak{g}$ given by (2.14), then (2.15) implies

$$\mathrm{Ad}\,(c_r)\mathfrak{t}_i^- = \mathfrak{a}_{(i)} \quad \text{and} \quad \mathrm{Ad}\,(c_r)\mathfrak{t}_i^+ = \mathfrak{t}_i^+.$$

Therefore $\mathrm{Ad}\,(c_r)\mathfrak{t}_{i,c} = (\mathfrak{t}_i^+ + \mathfrak{a}_{(i)})_c$ and its dual map ${}^t\mathrm{Ad}\,(c_r)$ sends the $(\mathfrak{t}_i^+ + \mathfrak{a}_{(i)})_c$ root system of $\mathfrak{g}_{i,c}$ to the $\mathfrak{t}_{i,c}$ root system Φ_i of $\mathfrak{g}_{i,c}$. Let Σ_i be the restriction to $\mathfrak{a}_{(i)}$ of the elements in ${}^t\mathrm{Ad}\,(c_r^{-1})\Phi_i$; hence Σ_i is the restricted root system of $\mathfrak{g}_i$ with respect to $\mathfrak{a}_{(i)}$. Via $\mathrm{Ad}\,(c_r)$, Φ_i^+ induces a system of positive roots Σ_i^+ in Σ_i. We denote by $\mathfrak{a}_{(i)}^+$ the corresponding positive Weyl chamber in $\mathfrak{a}_{(i)}$. Then (cf. Moore [23]) we have

$$\mathfrak{a}_{(i)}^+ = \{\textstyle\sum_{j=i+1}^{r} x_j(X_j + X_{-j}) \in \mathfrak{a}_{(i)};\ x_{i+1} > x_{i+2} > \cdots > x_r > 0\}. \tag{2.35b}$$

Let $A_{(i)}$ be the analytic subgroup of G_i corresponding to $\mathfrak{a}_{(i)}$ and let $A_{(i)}^+ = \exp \mathfrak{a}_{(i)}^+$. We define a function D_i on $A_{(i)}^+$ by

$$D_i(\exp X) = \textstyle\prod_{\alpha \in \Sigma_i^+} (\sinh \alpha(X))^{m(\alpha)}, \quad X \in \mathfrak{a}_{(i)}^+$$

where $m(\alpha)$ is the multiplicity of α, i.e., the number of roots in ${}^t\mathrm{Ad}\,(c_r^{-1})\Phi_i$ which restrict to α. If $X = \sum_{j=i+1}^{r} x_j(X_j + X_{-j}) \in \mathfrak{a}_{(i)}^+$, then one finds from (2.12ab), (2.21a) and (2.15) that

$$\begin{aligned} &D_i(\exp X) \\ &= \prod_{i+1 \leq j \leq r} (\sinh 2x_j)(\sinh x_j)^{2v} \cdot \prod_{i+1 \leq j < k \leq r} \{\sinh(x_j + x_k)\sinh(x_j - x_k)\}^u \\ &= 2^{r-i} \prod_{i+1 \leq j \leq r} (\sinh x_j)^{2v+1}(\cosh x_j) \cdot \prod_{i+1 \leq j < k \leq r} \{(\cosh x_j)^2 - (\cosh x_k)^2\}^u \end{aligned} \tag{2.36}$$

where u, v are the constants in (2.12b). For any point $X = \sum_{j=i+1}^{r} x_j(X_j + X_{-j}) \in \mathfrak{a}_{(i)}$, we regard $(x_{i+1}, x_{i+2}, \ldots, x_r)$ as the coordinates of X and denote by dX the measure $dx_{i+1} \cdots dx_r$ on $\mathfrak{a}_{(i)}$. Let da be the Haar measure on $A_{(i)}$ which corresponds to dX under the exponential mapping. We normalize the Haar measure on K_i to have total mass one. Then (cf. Helgason [11], pp. 381–382) there exists a unique determination of the Haar measure on G_i such that

(2.37) $$\int_{G_i} f(g_i)dg_i = \int_{K_i \times A^+_{(i)} \times K_i} f(k_1 a k_2) D_i(a) dk_1 da dk_2$$

for all $f \in C_c(G_i)$ (continuous with compact support). In the case $i=r$, $G_r = K_r \subset K$ (cf. (2.29)), so we normalize the Haar measure on G_r so that $\int_{G_r} dg_r = 1$.

Now return to the general case $1 \leq i \leq r$ and let $\mathfrak{a}_i$, $\mathfrak{n}_i$, P_i, S_i, be as in 2.20. Define $\rho_i \in \mathfrak{a}_i^*$ (=dual space of $\mathfrak{a}_i$) by

(2.38) $$\rho_i(H) = \frac{1}{2} \operatorname{trace}(\operatorname{ad}(H)|_{\mathfrak{n}_i}), \quad H \in \mathfrak{a}_i$$

and, using this ρ_i, define a function ρ on G as follows. Since P_i is a parabolic subgroup, each $g \in G$ can be uniquely written in the form $g = kman$ where $k \in K$, $m \in M_i \cap \exp \mathfrak{p}$, $a \in A_i$, $n \in N_i$ (cf. Warner [30], p. 78); so put $\rho(g) = e^{-2\rho_i}(a)$. Then

(2.39) ρ is a C^∞ rho-function (cf. [30], Appendix 1) on G for the subgroup P_i,

i.e., ρ is a strictly positive C^∞ function on G satisfying $\rho(e)=1$ (e=identity element of G), $\rho(gp) = \Delta_{P_i}(p)\Delta_G(p)^{-1}\rho(g)$, $g \in G$, $p \in P_i$ where Δ_G, Δ_{P_i} are the modular functions of G, P_i. In fact in the present case $\Delta_G \equiv 1$ since G is simple, and it is easy to verify that

(2.40) $\rho(p) = \Delta_{P_i}(p)$ for $p \in P_i$ and $\rho(gp) = \rho(g)\rho(p)$ for $g \in G$, $p \in P_i$.

Moreover, we see without difficulty that

(2.41) $$\Delta_{P_i}(s) = \Delta_{S_i}(s) \qquad \text{for all} \quad s \in S_i,$$

and so the ρ is a rho-function also for the subgroup S_i.

For further normalization of measures we need the following well-known measure theoretic result. Let W be a locally compact group countable at infinity, and suppose X and Y are closed subgroups such that $X \cdot Y$ is open in W, the complement of $X \cdot Y$ in W has Haar measure zero, and $X \cap Y$ is compact. Then (cf. Bourbaki [2], p. 66) we have:

(2.42) The left Haar measures of W, X, and Y may be normalized in such a way that for any integrable or non-negative Borel function f on W

$$\int_W f(w)dw = \int_{X \times Y} f(xy) \frac{\Delta_W(y)}{\Delta_Y(y)} dx dy$$

where Δ_W, Δ_Y are the modular functions on W and Y, respectively.

Now let dk denote the Haar measure on K such that $\int_K dk = 1$. Then, since

$G=K\cdot P_i$, $K\cap P_i$ is compact, and since $\Delta_G\equiv 1$, (2.40) and (2.42) imply that the left Haar measures on G and P_i can be normalized so that

$$\int_G f(g)dg = \int_{K\times P_i} f(kp)\rho(p)^{-1}dkdp \tag{2.43}$$

for any integrable f. Similarly, since $P_i=G_i\cdot S_i$, and $G_i\cap S_i$ $(=K_i)$ is compact, (2.41) and (2.42) ensure that we can normalize the left Haar measure on S_i such that

$$\int_{P_i} f(p)dp = \int_{G_i\times S_i} f(g_i s)dg_i ds \tag{2.44}$$

for any integrable f on P_i. As $\rho(g_i s)=\rho(g_i)\rho(s)=\rho(s)$ for $g_i\in G_i$ and $s\in S_i$, (2.43) and (2.44) then lead to

$$\int_G f(g)dg = \int_{K\times G_i\times S_i} f(kg_i s)\rho(s)^{-1}dkdg_i ds \tag{2.45}$$

for any integrable f on G.

As noted before, the ρ is a rho-function also for the subgroup S_i. Let $d\mu$ denote the quasi-invariant measure on $\mathscr{B}_i=G/S_i$ associated to this rho-function (cf. [30], Appendix 1), which is defined by the formula

$$\int_G f(g)\rho(g)dg = \int_{G/S_i} d\mu(\dot{g})\int_{S_i} f(gs)ds, \quad \dot{g} = gS_i \tag{2.46}$$

for all $f\in C_c(G)$. Then we observe that

$$\int_{\mathscr{B}_i} F(u)d\mu(u) = \int_{K\times G_i} F(kg_i\cdot o_i)dkdg_i \tag{2.47}$$

for any integrable F on $\mathscr{B}_i$. Indeed if we set $\dot{f}(\dot{g})=\int_{S_i} f(gs)ds$ $(\dot{g}=g\cdot o_i)$ for $f\in C_c(G)$, then

$$\begin{aligned}\int_{\mathscr{B}_i} \dot{f}(u)d\mu(u) &= \int_{\mathscr{B}_i} d\mu(\dot{g})\int_{S_i} f(gs)ds \\ &= \int_G f(g)\rho(g)dg \\ &= \int_{K\times G_i\times S_i} f(kg_i s)\rho(kg_i s)\rho(s)^{-1}dkdg_i ds \qquad \text{(by (2.45))} \\ &= \int_{K\times G_i} \dot{f}(kg_i\cdot o_i)dkdg_i\end{aligned}$$

for any $f\in C_c(G)$. Since the assignment $f\to\dot{f}$ is a surjection of $C_c(G)$ onto $C_c(\mathscr{B}_i)$, (2.47) follows.

3. Definition of $\mathscr{F}_i(G)$

In this section we define, for each $1 \leq i \leq r$, a subset $\mathscr{F}_i(G) \subset \sqrt{-1}\mathfrak{t}^*$; in the next section we shall associate to each member of $\mathscr{F}_i(G)$ a Hardy type space and an irreducible unitary representation of G.

As in 2.33 we identify the root subsystem Φ_i with the root system of $(\mathfrak{g}_{i,c}, \mathfrak{t}_{i,c})$. Similarly the root subsystem $'\Phi_i$ defined by (2.21b) can be identified with the root system of $'\mathfrak{g}_{i,c}$ with respect to the Cartan subalgebra $'\mathfrak{t}_{i,c} = {}'\mathfrak{g}_{i,c} \cap \mathfrak{t}_c$, and Φ^+ induces the system of positive roots $'\Phi_i^+ = {}'\Phi_i \cap \Phi^+$ in $'\Phi_i$. We let $\{\alpha_1, \dots, \alpha_l\}$ be an enumaration of the set of simple roots for Φ^+ such that

$$\text{(3.1a)} \quad \begin{cases} \alpha_1 \text{ is the unique noncompact simple root for } \Phi^+ \text{ and, for each} \\ 1 \leq i \leq r-1, \ \{\alpha_1, \dots, \alpha_{l_i}\} \text{ is the set of simple roots for } \Phi_i^+, \end{cases}$$

and such that

$$\text{(3.1b)} \qquad \text{for each } 2 \leq i \leq r, \ \{\alpha_{l'_i+1}, \dots, \alpha_l\} \text{ is the set of simple roots for } '\Phi_i^+;$$

this can be done because the simple roots for Φ_i^+ (resp. $'\Phi_i^+$) are the simple roots for Φ^+ that are in Φ_i^+ (resp. $'\Phi_i^+$) and α_1 is the lowest root in Φ_n^+, and because $\Phi_i^+ \subset \Phi_{i-1}^+$, $'\Phi_i^+ \supset {}'\Phi_{i-1}^+$ for $2 \leq i \leq r$. In what follows, we shall find it convenient to put $l'_1 = l$. Note that, in case $r > 1$, we have

$$\text{(3.2)} \quad 1 \leq l'_r \leq l_{r-1}, \ l_1 < l'_1 = l, \text{ and } l_i < l'_i \leq l_{i-1} \text{ for each } 2 \leq i \leq r-1;$$

this follows from (2.12c). We denote by $\{\lambda_1, \dots, \lambda_l\}$ the set of fundamental highest weights, i.e., λ_j is the weight such that $2(\lambda_j, \alpha_k)/(\alpha_k, \alpha_k) = \delta_{jk}$ for $1 \leq k \leq l$.

3.3. Lemma. *Let $H_1, \dots, H_r$ be as in* 2.8.

(1) $\langle \lambda_1, H_k \rangle = 1$ *for all* $1 \leq k \leq r$.

(2) *Fix i, $1 \leq i \leq r$, and j, $1 \leq j \leq l'_i$. Then for $1 \leq k \leq i$, $\langle \lambda_j, H_k \rangle$ is a strictly positive integer independent of k.*

(3) *Fix i, $2 \leq i \leq r$, and j, $l_{i-1}+1 \leq j \leq l$. Then $\langle \lambda_j, H_k \rangle = 0$ for all $i \leq k \leq r$.*

(4) *Fix i, $2 \leq i \leq r$. Then $\langle \lambda_j, H_1 \rangle > \langle \lambda_j, H_i \rangle$ for all $l'_i + 1 \leq j \leq l$.*

Proof. (1) Let $\{\gamma_1, \dots, \gamma_r\}$ be the strongly orthogonal noncompact positive root system as in (2.9). Since $\mathfrak{g}_c$ is simple, $\mathfrak{k}_c$ acts irreducibly on $\mathfrak{p}^+$, so every γ_k, $1 \leq k \leq r$, can be written in the form $\gamma_k = \alpha_1 + \sum_{j=2}^{l} n_j \alpha_j$ where n_j are nonnegative integers and α_1, α_j are as in (3.1). Therefore

$$\langle \lambda_1, H_k \rangle = \frac{2(\lambda_1, \gamma_k)}{(\gamma_k, \gamma_k)} = \frac{2(\lambda_1, \alpha_1)}{(\gamma_k, \gamma_k)} = \frac{(\alpha_1, \alpha_1)}{(\gamma_k, \gamma_k)}.$$

Since γ_1 and α_1 are, respectively, the highest and lowest weights of the irreducible $\mathfrak{k}_c$-module $\mathfrak{p}^+$, we have $(\alpha_1, \alpha_1)=(\gamma_1, \gamma_1)$. On the other hand $(\gamma_1, \gamma_1)=(\gamma_k, \gamma_k)$ for all $1 \le k \le r$ by (2.12d). Thus (1) follows.

(2) In view of (2.22) $\mathfrak{t}_c$ admits the orthogonal direct sum decomposition

$$\mathfrak{t}_c = \mathfrak{t}_{i,c} \oplus {'\mathfrak{t}_{i,c}} \oplus \mathfrak{i}_{i,c} \oplus \mathfrak{h}_{i,c}.$$

If $1 \le k,\ k' \le i$, then it is readily seen that $H_k - H_{k'}$ is orthogonal to the subspaces $\mathfrak{t}_{i,c}$, $\mathfrak{i}_{i,c}$ and $\mathfrak{h}_{i,c}$, so $H_k - H_{k'} \in {'\mathfrak{t}_{i,c}}$. Thus (3.1b) implies that if $1 \le k,\ k' \le i$ then $\langle \lambda_j, H_k - H_{k'} \rangle = 0$, i.e., $\langle \lambda_j, H_k \rangle = \langle \lambda_j, H_{k'} \rangle$ for every $1 \le j \le l'_i$.

Now $\langle \lambda_j, H_k \rangle$ is a nonnegative integer for each pair of indices j, k because λ_j is a highest weight. Therefore to complete the proof of (2), it suffices to show that $\langle \lambda_j, H_1 \rangle > 0$ for every $1 \le j \le l$. But this is clear since γ_1 can be written as integral linear combinations $\gamma_1 = \sum_{j=1}^{l} n_j \alpha_j$ with all $n_j > 0$ (the root system Φ is irreducible and γ_1 is the highest root).

(3) If $i \le k \le r$, $\gamma_k \in \Phi_{i-1}$. Hence we can write $\gamma_k = \sum_{j=1}^{l_{i-1}} n_j \alpha_j$ with n_j nonnegative integers, which implies (3).

(4) Recall (3.2) that $l'_i \le l_{i-1}$. By parts (2) and (3) of the present lemma we may assume $l'_i < l_{i-1}$ and it suffices to show that $\langle \lambda_j, H_1 \rangle > \langle \lambda_j, H_i \rangle$ for $l'_i + 1 \le j \le l_{i-1}$. Since $\mathfrak{p}^+$ is an irreducible $\mathfrak{k}_c$-module with γ_1 the highest weight, we have $\gamma_i = \gamma_1 - \sum_{j=2}^{l} n_j \alpha_j$ with $n_j \ge 0$. Hence, and because γ_1, γ_i have the same length, it is enough to show

$$\text{(3.4)} \qquad n_j > 0 \quad \text{for} \quad l'_i + 1 \le j \le l_{i-1}.$$

Take any j_0, $l'_i + 1 \le j_0 \le l_{i-1}$. Then $\alpha_{j_0} \in {'C_0^{(i)}} - {'C_0^{(i-1)}}$ by the definition (2.21) of Φ_i, $'\Phi_i$. Thus there exist an integer k $(1 \le k < i)$ and two roots β, $\beta' \in C_{ki}$ such that $\alpha_{j_0} = \beta - \beta'$. Now

$$(\gamma_i, \beta) = (\gamma_i, \frac{1}{2}(\gamma_k - \gamma_i)) = -\frac{1}{2}(\gamma_i, \gamma_i) < 0,$$

so $\gamma_i + \beta$ is a root. Since $\gamma_1 - (\gamma_i + \beta) = \sum_{j=2}^{l} n_j \alpha_j - \alpha_{j_0} - \beta'$ and γ_1 is the highest root, we must have $n_{j_0} > 0$. This establishes (3.4), and (4) follows. ∎

For each $1 \le i \le r$ and $1 \le j \le l$, let

$$\text{(3.5)} \qquad p_i = \frac{1}{2}u(i-1) + u(r-i) + v + 1 \qquad (u,\ v \text{ being as in (2.12b)})$$

$$\text{(3.6)} \qquad k_j = \langle \lambda_j, H_1 \rangle.$$

Then p_i is an integer or a half-integer and, according to Lemma 3.3, k_j is a positive integer for every $1 \le j \le l$ and in particular $k_1 = 1$. With G, T being as in Section 2, define

$$\mathscr{L}(G) = \{\lambda \in \sqrt{-1}\,\mathfrak{t}^*;\ e^{\lambda} \text{ is well defined on } T\}. \tag{3.7}$$

Now, for each $1 \le i \le r$, we define

$$\mathscr{F}_i = \left\{\sum_{j=1}^{l'_i} m_j \lambda_j \in \sqrt{-1}\,\mathfrak{t}^*;\ \begin{array}{l} m_1 \in \boldsymbol{R},\ m_j \in \boldsymbol{Z}^+ (2 \le j \le l'_i), \\ m_1 + \sum_{j=2}^{l'_i} k_j m_j = -p_i \end{array}\right\}, \tag{3.8a}$$

$$\mathscr{F}_i(G) = \mathscr{F}_i \cap \mathscr{L}(G), \tag{3.8b}$$

where $\boldsymbol{Z}^+ = \{0, 1, 2, \dots\}$. Corresponding to each element of $\mathscr{F}_i(G)$, we shall construct an irreducible unitary representation of G in the next section. The motivation for the definition (3.8) will become clear during the course of our construction of the representation. But we note here that if $\lambda \in \mathscr{F}_i$ $(2 \le i \le r)$ then λ vanishes on the Cartan subalgebra ${}'\mathfrak{t}_{i,c}$ of ${}'\mathfrak{g}_{i,c}$; this is evident from (3.1b). Let

$$\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha. \tag{3.9}$$

Then we can rephrase the definition (3.8a) as follows:

3.10. PROPOSITION. *Let $\lambda \in \sqrt{-1}\,\mathfrak{t}^*$ be such that $2(\lambda, \alpha)/(\alpha, \alpha)$ is a non-negative integer for every compact positive root α of $(\mathfrak{g}_c, \mathfrak{t}_c)$. Then λ is in $\mathscr{F}_i$ if and only if:*

(1) $\langle \lambda, H_1 \rangle = \langle \lambda, H_i \rangle$;

(2) $\langle \lambda + \delta, H_1 + \cdots + H_i \rangle = 0$.

PROOF. By assumption, λ can be written in the form $\lambda = \sum_{j=1}^{l} m_j \lambda_j$ with $m_1 \in \boldsymbol{R}$, $m_j \in \boldsymbol{Z}^+$ $(2 \le j \le l)$. Then Lemma 3.3 (2) implies

$$\langle \lambda, H_1 \rangle - \langle \lambda, H_i \rangle = \sum_{j=l'_i+1}^{l} m_j (\langle \lambda_j, H_1 \rangle - \langle \lambda_j, H_i \rangle).$$

Since $m_j \ge 0$, it then follows from Lemma 3.3 (4) that

$$\langle \lambda, H_1 \rangle = \langle \lambda, H_i \rangle \Longleftrightarrow m_j = 0 \quad \text{for} \quad l'_i + 1 \le j \le l. \tag{3.11}$$

Under the above equivalent conditions we have

$$\begin{aligned} &\langle \lambda, H_1 + \cdots + H_i \rangle \\ &= \sum_{j=1}^{l'_i} \sum_{k=1}^{i} m_j \langle \lambda_j, H_k \rangle \end{aligned} \tag{3.12}$$

$$= i(m_1 + \textstyle\sum_{j=2}^{l'_i} m_j k_j) \qquad \text{(by Lemma 3.3 and (3.6))}.$$

On the other hand if we let $\pi(\delta)$ denote the restriction of δ to $\mathfrak{t}^-$ as in 2.8, then by (2.12ab)

$$\begin{aligned} 2\pi(\delta) &= \sum_{1\le j\le r}\gamma_j + u\sum_{1\le j<k\le r}\frac{1}{2}(\gamma_j+\gamma_k) + u\sum_{1\le j<k\le r}\frac{1}{2}(\gamma_j-\gamma_k) \\ &\qquad + 2v\sum_{1\le j\le r}\frac{1}{2}\gamma_j \\ &= \sum_{1\le j\le r}(1+v+u(r-j))\gamma_j. \end{aligned} \tag{3.13}$$

Thus, and since $\langle\gamma_j, H_k\rangle = 2\delta_{jk}$, we get

$$\begin{aligned} \langle\delta, H_1+\cdots+H_i\rangle &= \sum_{1\le j\le i}(1+v+u(r-j)) \\ &= i\left(1+v+ur-\frac{1}{2}u(i+1)\right) \\ &= ip_i. \end{aligned} \tag{3.14}$$

The assertion of the proposition now follows from (3.11), (3.12) and (3.14). ∎

NOTE. In the extreme case $i=1$, the two conditions in Proposition 3.10 reduce to the single condition $\langle\lambda+\delta, H_1\rangle=0$, and so this is the case which was considered by Knapp and Okamoto [15].

Let

$$\Phi_{i,c}^+ = \Phi_i \cap \Phi_c^+, \quad \Phi_{i,n}^+ = \Phi_i \cap \Phi_n^+; \tag{3.15}$$

they can be identified with the set of compact and noncompact positive roots of $(\mathfrak{g}_{i,c}, \mathfrak{t}_{i,c})$, respectively.

The following lemma will be needed in Section 4 below.

3.16. LEMMA. *Suppose $r>1$ and fix i, $1\le i\le r-1$, and $\lambda\in\mathscr{F}_i$. Then $\langle\lambda+\delta, H_\alpha\rangle<0$ for all $\alpha\in\Phi_{i,n}^+$.*

PROOF. Recall the direct sum decomposition $\mathfrak{g}_{i,c}=\mathfrak{k}_{i,c}+\mathfrak{p}_i^+ +\mathfrak{p}_i^-$ in 2.28. Since the hermitian symmetric space G_i/K_i is irreducible, $\mathfrak{k}_c$ acts irreducibly on $\mathfrak{p}_i^+$. Note that the set of weights of this representation is naturally identified with $\Phi_{i,n}^+$ with γ_{i+1} the highest weight. Hence if $\alpha\in\Phi_{i,n}^+$, α can be written as $\alpha=\gamma_{i+1}-\sum n_j\alpha_j$ where n_j are nonnegative integers and α_j are simple roots for $\Phi_{i,c}^+$. Since $\langle\delta, H_\alpha\rangle=1$ for any simple root α of Φ^+ and since $\langle\lambda, H_\alpha\rangle\ge 0$ for all $\alpha\in\Phi_c^+$, it then follows that the condition $\langle\lambda+\delta, H_\alpha\rangle<0$ for every $\alpha\in\Phi_{i,n}^+$ is equivalent to the single condition $\langle\lambda+\delta, H_{i+1}\rangle<0$. Now (3.13) implies

$$\langle \delta, H_{i+1} \rangle = 1 + v + u(r - i - 1) = p_i - \frac{1}{2} u(i + 1).$$

On the other hand, since $\lambda = \sum_{j=1}^{l'_i} m_j \lambda_j$ with $m_1 + \sum_{j=2}^{l'_i} m_j k_j = -p_i$, using Lemma 3.3 one finds that

$$\begin{aligned} \langle \lambda, H_{i+1} \rangle &= m_1 + \sum_{j=2}^{l'_i} m_j \langle \lambda_j, H_{i+1} \rangle \\ &= m_1 + \sum_{j=2}^{l'_{i+1}} m_j k_j + \sum_{j=l'_{i+1}+1}^{l'_i} m_j \langle \lambda_j, H_{i+1} \rangle \\ &= -p_i - \sum_{j=l'_{i+1}+1}^{l'_i} m_j (\langle \lambda_j, H_1 \rangle - \langle \lambda_j, H_{i+1} \rangle) \\ &\leq -p_i. \end{aligned}$$

Hence, and because of the fact that $u > 0$ if $r > 1$ (cf. (2.12b)), it follows that

$$\langle \lambda + \delta, H_{i+1} \rangle \leq -\frac{1}{2} u(i + 1) < 0.$$

This completes the proof. ∎

We close this section with some comments on whether the set $\mathscr{F}_i(G)$ is non-empty or not. If the number p_i in (3.5) is an integer and if moreover G_c is simply connected, then $\mathscr{F}_i(G) = \mathscr{F}_i$ and hence $\mathscr{F}_i(G) \neq \emptyset$ ($\emptyset$ = the empty set) for all $1 \leq i \leq r$. On the other hand if i ($1 \leq i \leq r$) is odd then, by its definition, p_i is an integer for every simple Lie algebra $\mathfrak{g}$ (whose associated symmetric space is hermitian symmetric). Also if, for a given $\mathfrak{g}$, the integer u in (2.12b) is even, then p_i is again an integer for every $1 \leq i \leq r$. But when, for a given $\mathfrak{g}$, u is odd (this is the case when $\mathfrak{g}$ is $\mathfrak{sp}(n, \boldsymbol{R})$ or $\mathfrak{so}(2n+1, 2)$; see the Table below) and if moreover i ($1 \leq i \leq r$) is even, then we must take, corresponding to $\mathfrak{g}$, a non-linear Lie group G to ensure that $\mathscr{F}_i(G) \neq \emptyset$.

In any case, for a linear group G, if we let G° be a two-sheeted covering of the linear universal covering of G (covering group $\tilde{G}$ of G is called the linear universal covering of G if $\tilde{G}$ is again linear and if $\tilde{G}_c$ is simply connected), then $\mathscr{F}_i(G^\circ) = \mathscr{F}_i$ and hence $\mathscr{F}_i(G^\circ) \neq \emptyset$ for all $1 \leq i \leq r$. But in this paper we will mainly be concerned with linear groups.

Table

$\mathfrak{g}$	r	u	v	$p_i=\frac{1}{2}u(i-1)+u(r-i)+v+1$ $(1\leq i\leq r)$
$\mathfrak{su}(p, q)$ $(p\geq q\geq 1)$	q	2	$p-q$	$p+q-i$
$\mathfrak{so}^*(4n)$	n	4	0	$4n-2i-1$
$\mathfrak{so}^*(4n+2)$	n	4	2	$4n-2i+1$
$\mathfrak{sp}(n, \boldsymbol{R})$	n	1	0	$n-\frac{i}{2}+\frac{1}{2}$
$\mathfrak{so}(n, 2)$ $(n\geq 3)$	2	$n-2$	0	$p_1=n-1$, $p_2=\frac{n}{2}$
$\mathfrak{e}_{6(-14)}$	2	6	4	$p_1=11$, $p_2=8$
$\mathfrak{e}_{7(-25)}$	3	8	0	$p_1=17$, $p_2=13$, $p_3=9$

4. Construction of representations

We retain the notation of Sections 2, 3 and fix i, $1\leq i\leq r$, and $\lambda\in\mathscr{F}_i(G)$. Since λ is Φ_c^+-dominant, i.e., λ satisfies $(\lambda, \alpha)\geq 0$ for all $\alpha\in\Phi_c^+$, and since $\lambda\in\mathscr{L}(G)$ (notation of (3.7)), there exists an irreducible holomorphic representation τ_λ of K_c on E_λ (say) with highest weight λ. We endow E_λ with a hermitian structure such that the action of $\tau_\lambda(K)$ becomes unitary. Let $\boldsymbol{e}_\lambda\in E_\lambda$ be a highest weight vector of norm one. Letting $K_{i,c}$ be the analytic subgroup of K_c with Lie algebra $\mathfrak{k}_{i,c}$, we denote

$$(4.1)\qquad \begin{cases} \tilde{\lambda}: & \text{restriction of } \lambda \text{ to } \mathfrak{t}_{i,c}, \\ E_{\tilde{\lambda}}: & \text{linear span of } \{\tau_\lambda(k)\boldsymbol{e}_\lambda;\ k\in K_{i,c}\}, \\ \tau_{\tilde{\lambda}}: & \text{restriction of } \tau_\lambda(K_{i,c}) \text{ to } E_{\tilde{\lambda}}. \end{cases}$$

4.2. Lemma. *The representation $\tau_{\tilde{\lambda}}$ of $K_{i,c}$ on $E_{\tilde{\lambda}}$ is irreducible. The highest weight of $\tau_{\tilde{\lambda}}$ on the Cartan subalgebra $\mathfrak{t}_{i,c}$ of $\mathfrak{k}_{i,c}$, with the relative ordering, is $\tilde{\lambda}$ and $\boldsymbol{e}_\lambda$ is a highest weight vector.*

Proof. Let $\tau_{\tilde{\lambda}}$ also denote the differential of $\tau_{\tilde{\lambda}}$. To prove the lemma, it is enough to show that if β is a positive compact root of $(\mathfrak{g}_{i,c}, \mathfrak{t}_{i,c})$ with X_β as root vector then $\tau_{\tilde{\lambda}}(X_\beta)\boldsymbol{e}_\lambda=0$. But this follows from the fact that the positive compact roots of $(\mathfrak{g}_{i,c}, \mathfrak{t}_{i,c})$ are the restrictions to $\mathfrak{t}_{i,c}$ of the members of $\Phi_i\cap\Phi_c^+$ and if X_α is a root vector of $\alpha\in\Phi_i\cap\Phi_c^+$, X_α is also a root vector of the restriction to $\mathfrak{t}_{i,c}$ of α. ∎

Recall (2.25) the Langlands decomposition $P_i=M_iA_iN_i$ and the isotropy subgroup S_i of G at o_i. Let $L_i=M_i\cap S_i$. Then it follows from (2.26) and (2.32) that

$$S_i = L_iA_iN_i \quad \text{and} \quad L_i = F_iI_iK_iG_i'.$$

If c_i is the Cayley transform given by (2.14) then, since $S_i=G\cap c_iK_cP^-c_i^{-1}$,

$$c_i^{-1}S_ic_i \subset K_cP^-. \tag{4.3}$$

Thus, and because the representation τ_λ of K_c on E_λ is uniquely extended to that of K_cP^- which is trivial on P^-, we can define a representation $\tau_\lambda^{(i)}$ of S_i on E_λ by

$$\tau_\lambda^{(i)}(s) = \tau_\lambda(c_i^{-1}sc_i), \quad s\in S_i. \tag{4.4}$$

We denote by $P_{\bar\lambda}$ the orthogonal projection operator of E_λ onto $E_{\bar\lambda}$, and by $E_{\bar\lambda}^\perp$ the orthogonal complement of $E_{\bar\lambda}$ in E_λ.

4.5. LEMMA. (1) *The action of* $\tau_\lambda^{(i)}(L_i)$ *on* E_λ *is unitary and leaves the subspaces* $E_{\bar\lambda}$, $E_{\bar\lambda}^\perp$ *invariant. Moreover* $\tau_\lambda^{(i)}|_{K_i}=\tau_\lambda|_{K_i}$.

(2) $\tau_\lambda^{(i)}(a)|_{E_{\bar\lambda}}=e^{\rho_i}(a)I$ *for all* $a\in A_i$, *and* $E_{\bar\lambda}^\perp$ *is stable under* $\tau_\lambda^{(i)}(A_i)$; *here* $\rho_i\in\mathfrak{a}_i^*$ *is as in* (2.38) *and* I *denotes the identity transformation of* $E_{\bar\lambda}$.

(3) $P_{\bar\lambda}\tau_\lambda^{(i)}(n)|_{E_{\bar\lambda}}=I$ *for all* $n\in N_i$, *and* $E_{\bar\lambda}^\perp$ *is stable under* $\tau_\lambda^{(i)}(N_i)$.

PROOF. (1) The last assertion of (1) is obvious since c_i commutes with K_i. It is easily verified that c_i also commutes with subgroups F_i and I_i. As $F_iI_i\subset T$, it then follows that the action of $\tau_\lambda^{(i)}(F_iI_i)$ is unitary and leaves the subspaces $E_{\bar\lambda}$, $E_{\bar\lambda}^\perp$ invariant. Now, by definition, $\mathrm{Ad}(c_i^{-1})\mathfrak{g}_i'\subset{}'\mathfrak{g}_{i,c}\subset\mathfrak{k}_c$; cf. 2.20. But taking into account (3.1b) and the definition (3.8) of $\mathscr{F}_i(G)$, one sees that λ vanishes on the Cartan subalgebra ${}'\mathfrak{t}_{i,c}$ of ${}'\mathfrak{g}_{i,c}$; therefore $\tau_\lambda^{(i)}$ is trivial on G_i'. As $L_i=F_iI_iK_iG_i'$, the assertions of (1) are now evident.

(2) If $a\in A_i$ then $c_i^{-1}ac_i\in\exp\sqrt{-1}\mathfrak{t}$ by (2.23). Hence, and because of the fact that the subspaces $E_{\bar\lambda}$ and $E_{\bar\lambda}^\perp$ are spanned by weight vectors for τ_λ, it is clear that both $E_{\bar\lambda}$ and $E_{\bar\lambda}^\perp$ are stable under $\tau_\lambda^{(i)}(a)$. According to (1) and Lemma 4.2, $\tau_\lambda^{(i)}(K_i)|_{E_{\bar\lambda}}$ is irreducible. Since A_i commutes with K_i, it follows that $\tau_\lambda^{(i)}(a)|_{E_{\bar\lambda}}$ $(a\in A_i)$ are scalar operators. Thus to prove (2), it suffices to calculate the effect of $\tau_\lambda^{(i)}(a)$ to the highest weight vector $\boldsymbol{e}_\lambda$. Now

$$\tau_\lambda^{(i)}(a)\boldsymbol{e}_\lambda = \tau_\lambda(c_i^{-1}ac_i)\boldsymbol{e}_\lambda = e^{\langle\lambda,\mathrm{Ad}(c_i^{-1})(\log a)\rangle}\boldsymbol{e}_\lambda.$$

Hence the assertion of (2) amounts to the identity

$$\langle\rho_i, X\rangle = \langle\lambda, \mathrm{Ad}(c_i^{-1})X\rangle \qquad \text{for} \quad X\in\mathfrak{a}_i. \tag{4.6}$$

Recall (2.23a) that $\mathrm{Ad}(c_i)(\sum_{j=1}^i H_j)=\sum_{j=1}^i(X_j+X_{-j})$, and that $\mathfrak{n}_{i,c}$ is sum of the negative eigenspaces of $\mathrm{ad}\,(\sum_{j=1}^{i}(X_j+X_{-j}))$ on $\mathfrak{g}_c$. Thus $\mathrm{Ad}\,(c_i^{-1})\mathfrak{n}_{i,c}$ is sum of the negative eigenspaces of $\mathrm{ad}\,(\sum_{j=1}^{i}H_j)$ on $\mathfrak{g}_c$. Therefore, if we put $H=\mathrm{Ad}\,(c_i^{-1})X$ for $X\in\mathfrak{a}_i$, one finds

$$\begin{aligned}\langle\rho_i, X\rangle &= \frac{1}{2}\,\mathrm{trace}\,(\mathrm{ad}\,(X)|_{\mathfrak{n}_{i,c}})\\ &= \frac{1}{2}\,\mathrm{trace}\,(\mathrm{ad}\,(H)|_{\mathrm{Ad}(c_i^{-1})\mathfrak{n}_{i,c}})\\ &= -\langle\delta, H\rangle \quad (\delta \text{ is as in } (3.9))\\ &= \langle\lambda, H\rangle \quad (\text{by Proposition 3.10 and the fact that } \dim\mathfrak{a}_i=1)\,,\end{aligned}$$

which establishes (4.6), and completes the proof of (2).

(3) Let τ_λ (resp. $\tau_\lambda^{(i)}$) also denote the corresponding representation of $\mathfrak{k}_c+\mathfrak{p}^-$ (resp. the Lie algebra of S_i) on E_λ. To prove (3), it is sufficient to show that $\tau_\lambda^{(i)}(\mathfrak{n}_i)E_\lambda\subset E_\lambda^{\dagger}$; in turn, for this, it will be enough to show that $\tau_\lambda(X)E_\lambda\subset E_\lambda^{\dagger}$ for all $X\in\mathrm{Ad}\,(c_i^{-1})\mathfrak{n}_{i,c}$. As observed in the proof of (2), $\mathrm{Ad}\,(c_i^{-1})\mathfrak{n}_{i,c}$ is sum of the negative eigenspaces of $\mathrm{ad}\,(\sum_{j=1}^{i}H_j)$ on $\mathfrak{g}_c$. Thus if we let

$$\Phi_c^-(i) = \{\alpha\in\Phi_c;\ \langle\alpha, H_1+\cdots+H_i\rangle<0\}\,,$$

$$\Phi_n^-(i) = \{\alpha\in\Phi_n;\ \langle\alpha, H_1+\cdots+H_i\rangle<0\}\,,$$

then

$$\mathrm{Ad}\,(c_i^{-1})\mathfrak{n}_{i,c} = \textstyle\sum_{\alpha\in\Phi_n^-(i)\cup\Phi_c^-(i)}\mathfrak{g}_c^\alpha. \tag{4.7}$$

Further, by (2.12a),

$$\Phi_c^-(i) = \bigcup_{1\le j\le i}(-C_j)\cup\bigcup_{1\le j\le i<k\le r}(-C_{jk}),$$

$$\Phi_n^-(i) = \bigcup_{1\le j\le i}(-N_j)\cup\bigcup_{1\le j<k\le i}(-N_{jk})\cup\bigcup_{1\le j\le i<k\le r}(-N_{jk}) \cup\{-\gamma_1,\dots,-\gamma_i\}. \tag{4.8}$$

Now let $E_\lambda=\sum E^\mu$ be the orthogonal direct sum decomposition of E_λ into weight spaces for the representation τ_λ; the weights μ are all of the form $\mu=\lambda-\sum_{\alpha\in\Phi_c^+}n_\alpha\alpha$ with n_α nonnegative integers. Then $E_\lambda^{\dagger}=\sum E^\mu$, the sum taken over the weights μ of the form $\mu=\lambda-\sum_{\alpha\in\Phi_{i,c}^+}n_\alpha\alpha$. As $\Phi_{i,c}^+=C_0^{(i)}\cup\bigcup_{i+1\le j\le r}C_j\cup\bigcup_{i+1\le j<k\le r}C_{jk}$ by definition, it then follows from (4.8) and (2.12ab) that if $\alpha\in\Phi_c^-(i)$ then $E^{\mu+\alpha}\cap E_\lambda^{\dagger}=\{0\}$ for any weight μ for τ_λ; therefore we conclude that $\tau_\lambda(X)E_\lambda\subset E_\lambda^{\dagger}$ for all $X\in\sum_{\alpha\in\Phi_c^-(i)}\mathfrak{g}_c^\alpha$. On the other hand $\tau_\lambda(X)E_\lambda=\{0\}$ for all $X\in\sum_{\alpha\in\Phi_n^-(i)}\mathfrak{g}_c^\alpha$; this follows from the fact that $\Phi_n^-(i)\subset(-\Phi_n^+)$, and that $\tau_\lambda(\mathfrak{p}^-)$ acts trivially on E_λ. The assertion of (3) now follows from (4.7). ∎

Because of Lemmas 4.2 and 4.5(1), we may define an irreducible unitary representation $'\tau_{\lambda}^{(i)}$ of L_i on $E_{\tilde{\lambda}}$ by setting

$$(4.9)\qquad '\tau_{\lambda}^{(i)}(l) = \tau_{\lambda}^{(i)}(l)|_{E_{\tilde{\lambda}}}, \quad l \in L_i.$$

Now we define, for each $\nu \in \mathfrak{a}_i^*$, representations $'\sigma_{\lambda,\nu}$ and $\sigma_{\lambda,\nu}$ of $S_i = L_i A_i N_i$ on $E_{\tilde{\lambda}}$ by

$$(4.10a)\qquad '\sigma_{\lambda,\nu}(lan) = e^{\sqrt{-1}\nu}(a)\,'\tau_{\lambda}^{(i)}(l),$$

$$(4.10b)\qquad \sigma_{\lambda,\nu}(lan) = e^{\rho_i+\sqrt{-1}\nu}(a)\,'\tau_{\lambda}^{(i)}(l), \quad lan \in L_i A_i N_i$$

where $\rho_i \in \mathfrak{a}_i^*$ is as in (2.38). Note that $'\sigma_{\lambda,\nu}$ is irreducible and unitary (with respect to the inner product induced from that of E_{λ}).

We consider the unitarily induced representation

$$(4.11a)\qquad U_{\lambda,\nu} = \mathrm{Ind}_{S_i \uparrow G}\,'\sigma_{\lambda,\nu}.$$

Let us write down this representation more explicitly. Let ρ be the rho-function on G for the subgroup S_i, which was defined in 2.33, and let $d\mu$ be the corresponding quasi-invariant measure on G/S_i defined by the formula (2.46). Then (cf. Warner [30], p. 374) the representation space of $U_{\lambda,\nu}$ may be regarded as

$$(4.11b)\qquad L^2(G, \sigma_{\lambda,\nu}) = \begin{cases} \text{all Borel measurable } f\colon G \to E_{\tilde{\lambda}} \text{ such that} \\ (1)\ f(gs) = \rho(s)^{1/2}\,'\sigma_{\lambda,\nu}(s)^{-1} f(g) \text{ for } g \in G,\ s \in S_i, \\ (2)\ \|f\|^2 = \int_{G/S_i} \rho(g)^{-1} |f(g)|^2 d\mu(\dot{g}) < \infty \end{cases}$$

where we identify functions which are equal almost everywhere on G. (In connection with condition (2), note that the integral is constant on left S_i cosets and hence defines a function on G/S_i. Note too that condition (1) can also be written as $f(gs) = \sigma_{\lambda,\nu}(s)^{-1} f(g)$ because $\rho(s) = e^{-2\rho_i}(a)$ if $s = lan$.) $L^2(G, \sigma_{\lambda,\nu})$ is a Hilbert space with inner product

$$(f, f') = \int_{G/S_i} \rho(g)^{-1} (f(g), f'(g)) d\mu(\dot{g})$$

and $U_{\lambda,\nu}$ acts on $L^2(G, \sigma_{\lambda,\nu})$ by left translation: $U_{\lambda,\nu}(g) f(g') = f(g^{-1}g')$. For every $\nu \in \mathfrak{a}_i^*$, $U_{\lambda,\nu}$ is a continuous unitary representation of G. But our main concern in this paper is with the case $\nu = 0$. We write σ_{λ}, U_{λ}, $L^2(G, \sigma_{\lambda})$ for $\sigma_{\lambda,0}$, $U_{\lambda,0}$, $L^2(G, \sigma_{\lambda,0})$.

Let the normalization of measures on K and G_i be as in 2.33.

4.12. Lemma. *Fix $\nu \in \mathfrak{a}_i^*$. Then for all $f \in L^2(G, \sigma_{\lambda,\nu})$,*

$$\|f\|^2 = \int_{K\times G_i} |f(kg_i)|^2 dkdg_i.$$

PROOF. For a given $f\in L^2(G, \sigma_{\lambda,\nu})$, we may define a real valued function F on $\mathscr{B}_i = G/S_i$ by $F(g\cdot o_i)=\rho(g)^{-1}|f(g)|^2$, $g\in G$. If $k\in K$ and $g_i\in G_i$, then $\rho(kg_i)=1$ by definition of the function ρ, so $F(kg_i\cdot o_i)=|f(kg_i)|^2$. Thus, using the formula (2.47), we get

$$\|f\|^2 = \int_{\mathscr{B}_i} F(u)d\mu(u) = \int_{K\times G_i} F(kg_i\cdot o_i)dkdg_i$$
$$= \int_{K\times G_i} |f(kg_i)|^2 dkdg_i$$

for all $f\in L^2(G, \sigma_{\lambda,\nu})$. ∎

Let $\mathscr{O}(\mathscr{D}, E_\lambda)$ be the space of all E_λ-valued holomorphic functions on $\mathscr{D}$ and let $\mathscr{O}(\bar{\mathscr{D}}, E_\lambda)\subset\mathscr{O}(\mathscr{D}, E_\lambda)$ be the subspace of functions having a holomorphic extention to a neighborhood of the closure of $\mathscr{D}$ in $\mathfrak{p}^+$. We denote

J_λ: automorphic factor of type τ_λ

(cf. (2.5)). In view of (2.6a) we may define, for each $g\in G$, an action $T_\lambda(g)$ on $\mathscr{O}(\mathscr{D}, E_\lambda)$ by

$$(4.13)\qquad (T_\lambda(g)F)(z) = J_\lambda(g^{-1}, z)^{-1}F(g^{-1}\cdot z),\quad F\in\mathscr{O}(\mathscr{D}, E_\lambda),\ z\in\mathscr{D}.$$

Then it follows from (2.6b) that $T_\lambda(g_1g_2)=T_\lambda(g_1)T_\lambda(g_2)$ for $g_1, g_2\in G$. Furthermore, according to the remark after (2.6), the subspace $\mathscr{O}(\bar{\mathscr{D}}, E_\lambda)$ is stable under $T_\lambda(g)$, $g\in G$.

For $F\in\mathscr{O}(\bar{\mathscr{D}}, E_\lambda)$, we define $\tilde{F}$: $G\to E_{\tilde{\lambda}}$ by

$$(4.14)\qquad \tilde{F}(g) = P_{\tilde{\lambda}}J_\lambda(gc_i, o)^{-1}F(g\cdot o_i),\quad g\in G$$

($P_{\tilde{\lambda}}$ is the orthogonal projection of E_λ onto $E_{\tilde{\lambda}}$ and c_i is the Cayley transform). For the representation σ_λ (the case $\nu=0$ in (4.10b)) of S_i on $E_{\tilde{\lambda}}$, we put

$$C^\infty(G, \sigma_\lambda) = \{f\in C^\infty(G, E_{\tilde{\lambda}}); f(gs) = \sigma_\lambda(s)^{-1}f(g),\ g\in G,\ s\in S_i\}.$$

Then G naturally acts on $C^\infty(G, \sigma_\lambda)$ by left translation.

The key step in the construction of representations on Hardy type Hilbert spaces is the following lemma.

4.15. LEMMA. *If $F\in\mathscr{O}(\bar{\mathscr{D}}, E_\lambda)$, then the function $\tilde{F}$ defined by* (4.14) *lies in $C^\infty(G, \sigma_\lambda)$. Moreover the mapping $F\to\tilde{F}$ is equivariant with respect to the action of G.*

PROOF. Let $F \in \mathcal{O}(\bar{\mathscr{D}}, E_\lambda)$; then clearly $\tilde{F} \in C^\infty(G, E_{\bar\lambda})$. If $g \in G$ and $lan \in S_i = L_i A_i N_i$ then, since $c_i^{-1} lanc_i \in K_e P^-$ (cf. (4.3)), we get

$$\begin{aligned} J_\lambda(glanc_i, o)^{-1} &= J_\lambda(gc_i c_i^{-1} lanc_i, o)^{-1} \\ &= J_\lambda(c_i^{-1} lanc_i, o)^{-1} J_\lambda(gc_i, o)^{-1} \\ &\qquad \text{(by the cocycle formula (2.6b))} \\ &= \tau_\lambda^{(i)}(lan)^{-1} J_\lambda(gc_i, o)^{-1} \\ &\qquad \text{(by (2.7a) and the definition (4.4))}. \end{aligned}$$

Thus, using Lemma 4.5 and recalling that S_i is the isotropy subgroup at o_i, we have

$$\begin{aligned} \tilde{F}(glan) &= P_{\bar\lambda} \tau_\lambda^{(i)}(lan)^{-1} J_\lambda(gc_i, o)^{-1} F(glan \cdot o_i) \\ &= e^{-\rho_i}(a)' \tau_\lambda^{(i)}(l)^{-1} P_{\bar\lambda} J_\lambda(gc_i, o)^{-1} F(g \cdot o_i) \\ &= \sigma_\lambda(lan)^{-1} \tilde{F}(g), \end{aligned}$$

and so $\tilde{F} \in C^\infty(G, \sigma_\lambda)$. The G-equivariance of the mapping $F \to \tilde{F}$ follows from the cocycle formula (2.6b). ∎

Keeping in mind Lemmas 4.12 and 4.15, we define a seminorm $\| \ \|_\lambda$ on $\mathcal{O}(\bar{\mathscr{D}}, E_\lambda)$ by setting

$$\|F\|_\lambda^2 = \int_{K \times G_i} |P_{\bar\lambda} J_\lambda(kg_i c_i, o)^{-1} F(kg_i \cdot o_i)|^2 dk dg_i, \quad F \in \mathcal{O}(\bar{\mathscr{D}}, E_\lambda).$$

Now let

$$\mathcal{O}^2(\bar{\mathscr{D}}, E_\lambda) = \{F \in \mathcal{O}(\bar{\mathscr{D}}, E_\lambda);\ \|F\|_\lambda < \infty\}. \tag{4.16}$$

We will later show that the above seminorm is actually a norm (Corollary 4.34), and that $\mathcal{O}^2(\bar{\mathscr{D}}, E_\lambda) \neq \{0\}$ for all $\lambda \in \mathscr{F}_i(G)$ (Corollary 4.39). Since U_λ is unitary on $L^2(G, \sigma_\lambda)$, Lemmas 4.12, 4.15 and the definition of the seminorm imply

(4.17) $\left\{\begin{array}{l}\text{the action } T_\lambda \text{ of } G \text{ on } \mathcal{O}(\bar{\mathscr{D}}, E_\lambda) \text{ leaves the subspace } \mathcal{O}^2(\bar{\mathscr{D}}, E_\lambda) \text{ invariant} \\ \text{and preserves the seminorm.}\end{array}\right.$

Moreover, if $F \in \mathcal{O}^2(\bar{\mathscr{D}}, E_\lambda)$ then the function $\tilde{F}$ defined by (4.14) belongs to $L^2(G, \sigma_\lambda)$. Let $\mathscr{I}_\lambda$ denote the mapping $F \to \tilde{F}$ of $\mathcal{O}^2(\bar{\mathscr{D}}, E_\lambda)$ into $L^2(G, \sigma_\lambda)$. Then by Lemma 4.15,

(4.18) $\left\{\begin{array}{l}\mathscr{I}_\lambda\colon\ \mathcal{O}^2(\bar{\mathscr{D}}, E_\lambda) \to L^2(G, \sigma_\lambda) \text{ is equivariant with respect to the action} \\ \text{of } G.\end{array}\right.$

If $X \in \mathfrak{g}_c$, $X = X_1 + \sqrt{-1}\,X_2$ $(X_1, X_2 \in \mathfrak{g})$, and if f is a differentiable function on G, define functions $r(X)f$ and $l(X)f$ on G by

$$(4.19)\quad \begin{aligned} (r(X)f)(g) &= \frac{d}{dt}(f(g \exp tX_1) + \sqrt{-1}f(g \exp tX_2))|_{t=0}, \\ (l(X)f)(g) &= \frac{d}{dt}(f(\exp(-tX_1)g) + \sqrt{-1}f(\exp(-tX_2)g))|_{t=0}. \end{aligned}$$

As in 2.28 we put $\mathfrak{p}_i^- = \mathfrak{g}_{i,c} \cap \mathfrak{p}^-$.

The following lemma will be used in Section 5.

4.20. Lemma. *Let* $F \in \mathcal{O}(\bar{\mathscr{D}}, E_\lambda)$, *and define* $\tilde{F}: G \to E_{\bar{\lambda}}$ *by* (4.14). *Then* $r(X)\tilde{F} = 0$ *for all* $X \in \mathfrak{p}_i^-$.

Proof. For $F \in \mathcal{O}(\bar{\mathscr{D}}, E_\lambda)$ and $g \in G$, let us put $'F(g) = F(g \cdot o_i)$ and $J(g) = J_\lambda(gc_i, o)$. Then, for $X \in \mathfrak{g}_c$,

$$(4.21)\quad (r(X)\tilde{F})(g) = -P_{\bar{\lambda}}J(g)^{-1}(r(X)J)(g)J(g)^{-1}\,'F(g) + P_{\bar{\lambda}}J(g)^{-1}(r(X)\,'F)(g).$$

Let $\pi: G \to \mathscr{D} = G/K$ be the canonical projection. Then π restricts to the canonical projection of G_i onto $\mathscr{D}_i = G_i/K_i$. Hence if $X \in \mathfrak{g}_{i,c}$, $d\pi_e X$ (e=identity of G) may be identified with a complex tangent vector of $\mathscr{D}_i$ at o.

Now fix $g \in G$ and define $\gamma: \mathscr{D}_i \to g \cdot \mathscr{C}_i$ by $\gamma(z) = gc_i \cdot z$; γ is a holomorphic diffeomorphism of $\mathscr{D}_i$ onto the boundary component $g \cdot \mathscr{C}_i$ containing $g \cdot o_i$. Then, since c_i commutes with G_i, it follows that

$$(r(X)\,'F)(g) = [(d\gamma_o \circ d\pi_e X)F](g \cdot o_i) \qquad \text{for all} \quad X \in \mathfrak{g}_{i,c}.$$

But if $X \in \mathfrak{p}_i^-$ it is not difficult to show that $d\pi_e X$ is an antiholomorphic tangent vector of $\mathscr{D}_i$ at o, hence $d\gamma_o \circ d\pi_e X$ is that of $g \cdot \mathscr{C}_i$ at $g \cdot o_i$. Since F restricted to the boundary component $g \cdot \mathscr{C}_i$ is holomorphic, we conclude

$$(4.22)\qquad (r(X)\,'F)(g) = 0 \qquad \text{for all} \quad X \in \mathfrak{p}_i^-.$$

On the other hand, since c_i commutes with G_i, by the same argument as the one in the proof of Lemma 5.7 of [1] one finds that

$$(4.23)\qquad r(X)J = 0 \qquad \text{for all} \quad X \in \mathfrak{p}_i^-,$$

and our assertion follows from (4.21)–(4.23). ∎

Now let $A_{(i)}^+$, D_i, da be as in 2.33, and π_0 be as in (2.2).

4.24. Proposition. *Let* $\lambda \in \mathscr{F}_i(G)$ *and, for each* $\boldsymbol{e} \in E_\lambda$, *let* $\mathbf{1}_{\boldsymbol{e}}$ *denote the constant function* $\mathbf{1}_{\boldsymbol{e}}(z) = \boldsymbol{e}$, $z \in \bar{\mathscr{D}}$. *Then we have*

$$\|\mathbf{1}_e\|_\lambda^2 = \begin{cases} \beta_i^2 d(\lambda)^{-1}|\boldsymbol{e}|^2\int_{A_{(i)}^+}\chi_{\tilde\lambda}(\pi_0(a)^{-2})D_i(a)da & \text{if } i \neq r \\ \beta_r^2 d(\lambda)^{-1}d(\tilde\lambda)\,|\boldsymbol{e}|^2 & \text{if } i = r, \end{cases}$$

where $\chi_{\tilde\lambda}$ *denotes the character of* $\tau_{\tilde\lambda}$, $d(\lambda)$ (*resp.* $d(\tilde\lambda)$) *the degree of* τ_λ (*resp.* $\tau_{\tilde\lambda}$), *and* $\beta_i=(\sqrt{2})^{ip_i}$ *with* p_i *as in* (3.5).

The proof requires the following lemma.

4.25. Lemma. (1) $J_\lambda(g, o_i)=J_\lambda(g, o)$ *for all* $g\in G_i$.
(2) $J_\lambda(c_i, o)|_{E_{\tilde\lambda}}=\beta_i^{-1}I$ *and* $E_{\tilde\lambda}^\perp$ *is stable under* $J_\lambda(c_i, o)$.

Proof. (1) As it follows from (2.18), $\pi_0(gc_i)=\pi_0(g)\pi_0(c_i)$ for all $g\in G_i$. Thus $J_\lambda(gc_i, o)=J_\lambda(g, o)J_\lambda(c_i, o)$, while $J_\lambda(gc_i, o)=J_\lambda(g, o_i)J_\lambda(c_i, o)$ by the cocycle formula, so (1) follows.

(2) Since $J_\lambda(c_i, o)=\tau_\lambda(\pi_0(c_i))$ by definition, and since $\pi_0(c_i)=\exp(\log\sqrt{2}\cdot\sum_{j=1}^i H_j)\in\exp\sqrt{-1}\mathfrak{t}$ by (2.18), it is clear that both $E_{\tilde\lambda}$ and $E_{\tilde\lambda}^\perp$ are stable under $J_\lambda(c_i, o)$. As c_i commutes with G_i, it follows that $\pi_0(g)\pi_0(c_i)=\pi_0(c_i)\pi_0(g)$ for all $g\in G_i$; in particular $\pi_0(c_i)$ commutes with K_i. Since $\tau_\lambda(K_i)|_{E_{\tilde\lambda}}$ is irreducible by Lemma 4.2, it then follows that $J_\lambda(c_i, o)|_{E_{\tilde\lambda}}$ is a scalar operator. But, using Proposition 3.10 and (3.14), we obtain

$$\begin{aligned} J_\lambda(c_i, o)\boldsymbol{e}_\lambda &= \tau_\lambda(\exp(\log\sqrt{2}\textstyle\sum_{j=1}^i H_j))\boldsymbol{e}_\lambda = (\sqrt{2})^{\langle\lambda, H_1+\cdots+H_i\rangle}\boldsymbol{e}_\lambda \\ &= (\sqrt{2})^{-ip_i}\boldsymbol{e}_\lambda \end{aligned}$$

where $\boldsymbol{e}_\lambda$ is the highest weight vector for τ_λ. Thus (2) follows. ∎

Proof of Proposition 4.24. For each $\boldsymbol{e}\in E_\lambda$, define $\phi_{\boldsymbol{e}}\colon G\to E_{\tilde\lambda}$ by $\phi_{\boldsymbol{e}}(g)=P_{\tilde\lambda}J_\lambda(gc_i, o)^{-1}\boldsymbol{e}$. Then by definition of the seminorm,

$$\|\mathbf{1}_e\|_\lambda^2 = \int_{K\times G_i}|\phi_{\boldsymbol{e}}(kg_i)|^2dkdg_i.$$

We first suppose $i\neq r$. Then, applying the integration formula (2.37), we get

$$\|\mathbf{1}_e\|_\lambda^2 = \int_{K\times K_i\times A_{(i)}^+\times K_i}|\phi_{\boldsymbol{e}}(kk_1ak_2)|^2D_i(a)dkdk_1dadk_2.$$

Since $\phi_{\boldsymbol{e}}\in C^\infty(G, \sigma_\lambda)$ by Lemma 4.15, it follows that

$$|\phi_{\boldsymbol{e}}(kk_1ak_2)| = |\sigma_\lambda(k_2)^{-1}\phi_{\boldsymbol{e}}(kk_1a)| = |\phi_{\boldsymbol{e}}(kk_1a)|$$

because $K_i\subset L_i$ and $\sigma_\lambda(k_2)$ is unitary. Thus, noting that $K_i\subset K$ and using the invariance of Haar measure on K, we obtain

$$
\begin{aligned}
\|\mathbf{1}_e\|_\lambda^2 &= \int_{K\times A_{(i)}^+} |\phi_e(ka)|^2 D_i(a)dkda \\
&= \int_{K\times A_{(i)}^+} |P_{\tilde{\lambda}} J_\lambda(kac_i, o)^{-1}\boldsymbol{e}|^2 D_i(a)dkda \qquad (4.26)\\
&= \beta_i^2 \int_{K\times A_{(i)}^+} |P_{\tilde{\lambda}} J_\lambda(a, o)^{-1}\tau_\lambda(k)^{-1}\boldsymbol{e}|^2 D_i(a)dkda
\end{aligned}
$$

where $\beta_i^2 = 2^{ip_i}$, and in the last step we have used Lemma 4.25 and the cocycle formula. Because of Lemma 4.2, we can choose an orthonormal basis $\boldsymbol{e}_1 = \boldsymbol{e}_\lambda$, $\boldsymbol{e}_2, \ldots, \boldsymbol{e}_{d(\lambda)}$ of weight vectors for τ_λ in such a way that $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{d(\tilde{\lambda})}$ are in $E_{\tilde{\lambda}}$. Let $\boldsymbol{e} = \sum_{i=1}^{d(\lambda)} x_i \boldsymbol{e}_i$, $x_i \in \boldsymbol{C}$, and write for $k \in K$, $\tau_\lambda(k)\boldsymbol{e}_i = \sum_{j=1}^{d(\lambda)} \tau_{ji}(k)\boldsymbol{e}_j$. Then, since $\tau_\lambda(k)^{-1}\boldsymbol{e}_i = \sum_{j=1}^{d(\lambda)} \overline{\tau_{ij}(k)}\boldsymbol{e}_j$, we obtain from (4.26)

$$
\begin{aligned}
\|\mathbf{1}_e\|_\lambda^2 &= \beta_i^2 \int_{K\times A_{(i)}^+} \Big|\sum_{1\le i,j\le d(\lambda)} x_i\overline{\tau_{ij}(k)} P_{\tilde{\lambda}} J_\lambda(a, o)^{-1}\boldsymbol{e}_j\Big|^2 D_i(a)dkda \\
&= \beta_i^2 d(\lambda)^{-1}\Big(\sum_{i=1}^{d(\lambda)} |x_i|^2\Big)\int_{A_{(i)}^+} \Big|\sum_{j=1}^{d(\lambda)} P_{\tilde{\lambda}} J_\lambda(a, o)^{-1}\boldsymbol{e}_j\Big|^2 D_i(a)da,
\end{aligned}
$$

the second equality following from Schur orthogonality relations for τ_λ. Now $J_\lambda(a, o) = \tau_\lambda(\pi_0(a))$ by definition, and, in view of (2.19), $\pi_0(a) \in \exp\sqrt{-1}\,\mathfrak{t}_i$. Thus $J_\lambda(a, o)$ is diagonal relative to the basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{d(\lambda)}$, so we conclude that

$$
\|\mathbf{1}_e\|_\lambda^2 = \beta_i^2 d(\lambda)^{-1} |\boldsymbol{e}|^2 \int_{A_{(i)}^+} \chi_{\tilde{\lambda}}(\pi_0(a)^{-2}) D_i(a)da.
$$

Next let $i = r$. Then $G_r = K_r \subset K$ by (2.29), so using the invariance of Haar measure and arguing as above, one finds that

$$
\begin{aligned}
\|\mathbf{1}_e\|_\lambda^2 &= \int_K |\phi_e(k)|^2 dk = \int_K |P_{\tilde{\lambda}} J_\lambda(kc_r, o)^{-1}\boldsymbol{e}|^2 dk \\
&= \beta_r^2 \int_K |P_{\tilde{\lambda}} \tau_\lambda(k)^{-1}\boldsymbol{e}|^2 dk = \beta_r^2 d(\tilde{\lambda}) d(\lambda)^{-1} |\boldsymbol{e}|^2.
\end{aligned}
$$

This completes the proof of the proposition. ∎

4.27. Corollary. *Let $\boldsymbol{e}_\lambda$ be a highest weight vector of τ_λ with $|\boldsymbol{e}_\lambda| = 1$, and let $\mathbf{1}_\lambda$ denote the constant function $\mathbf{1}_\lambda(z) = \boldsymbol{e}_\lambda$, $z \in \bar{\mathscr{D}}$. Then*

$$
\|\mathbf{1}_\lambda\|_\lambda^2 = \begin{cases} \beta_i^2 d(\lambda)^{-1} \int_{A_{(i)}^+} \chi_{\tilde{\lambda}}(\pi_0(a)^{-2}) D_i(a)da & \text{if } i \neq r \\ \beta_r^2 d(\tilde{\lambda}) d(\lambda)^{-1} & \text{if } i = r. \end{cases}
$$

Furthermore $\|\mathbf{1}_e\|_\lambda = \|\mathbf{1}_\lambda\|_\lambda |\boldsymbol{e}|$ for all $\boldsymbol{e} \in E_\lambda$.

We note that if $\mathbf{1}_\lambda$ is as in the preceding corollary, then

(4.28) $$\|\mathbf{1}_\lambda\|_\lambda \neq 0.$$

Indeed, if we define a function ϕ on $K\times G_i$ by $\phi(k, g_i)=|P_{\bar\lambda}J_\lambda(kg_ic_i, o)^{-1}\boldsymbol{e}_\lambda|^2$, then ϕ is continuous and $\|\mathbf{1}_\lambda\|_\lambda^2=\int_{K\times G_i}\phi(k, g_i)dkdg_i$. As $\phi(e, e)=\beta_i^2\neq 0$ by Lemma 4.25 (2), it follows that $\|\mathbf{1}_\lambda\|_\lambda\neq 0$.

4.29. Lemma. $\|F\|_\lambda\geq\|\mathbf{1}_\lambda\|_\lambda|F(o)|$ *for all* $F\in\mathcal{O}^2(\bar{\mathcal{D}}, E_\lambda)$.

Proof. It is known that K contains the group T^1 of rotation $z\to e^{i\theta}z$ ($z\in\mathfrak{p}^+$, $0\leq\theta<2\pi$) as a central subgroup; cf. Korányi and Wolf [18], p. 269. Let dt denote the Haar measure on T^1 such that $\int_{T^1}dt=1$. We first note that if $F\in\mathcal{O}(\bar{\mathcal{D}}, E_\lambda)$, then

(4.30) $$F(o)=\int_{T^1}F(t\cdot z)dt \qquad \text{for all} \quad z\in\bar{\mathcal{D}},$$

because the restriction of F to the complex line spanned by z is an E_λ-valued holomorphic function of one complex variable.

Now, for $F\in\mathcal{O}^2(\bar{\mathcal{D}}, E_\lambda)$, we have

$$\begin{aligned}
\|F\|_\lambda^2 &= \int_{K\times G_i}|P_{\bar\lambda}J_\lambda(kg_ic_i, o)^{-1}F(kg_i\cdot o_i)|^2dkdg_i \\
&= \int_{K\times G_i}\left[\int_{T^1}|P_{\bar\lambda}J_\lambda(tkg_ic_i, o)^{-1}F(tkg_i\cdot o_i)|^2dt\right]dkdg_i \\
&= \int_{K\times G_i}\left[\int_{T^1}|P_{\bar\lambda}J_\lambda(kg_ic_i, o)^{-1}F(tkg_i\cdot o_i)|^2dt\right]dkdg_i \\
&\qquad\qquad \text{(since } T^1 \text{ is a central subgroup of } K\text{)} \\
&\geq \int_{K\times G_i}\left|\int_{T^1}P_{\bar\lambda}J_\lambda(kg_ic_i, o)^{-1}F(tkg_i\cdot o_i)dt\right|^2dkdg_i \\
&= \int_{K\times G_i}|P_{\bar\lambda}J_\lambda(kg_ic_i, o)^{-1}F(o)|^2dkdg_i \qquad \text{(by (4.30))} \\
&= \|\mathbf{1}_\lambda\|_\lambda^2|F(o)|^2 \qquad \text{(by Corollary 4.27)}. \qquad \blacksquare
\end{aligned}$$

4.31. Corollary. $\mathcal{O}^2(\bar{\mathcal{D}}, E_\lambda)\neq\{0\}$ *if and only if* $\|\mathbf{1}_\lambda\|_\lambda<\infty$.

Proof. If $\|\mathbf{1}_\lambda\|_\lambda<\infty$, then $\mathbf{1}_\lambda\in\mathcal{O}^2(\bar{\mathcal{D}}, E_\lambda)$. Conversely, if $F\in\mathcal{O}^2(\bar{\mathcal{D}}, E_\lambda)$ and $F\neq 0$, then there exists a $g\in G$ such that $F(g\cdot o)\neq 0$. Thus, using (4.17) and Lemma 4.29, we get

$$\|F\|_\lambda=\|T_\lambda(g^{-1})F\|_\lambda\geq\|\mathbf{1}_\lambda\|_\lambda|(T_\lambda(g^{-1})F)(o)|=\|\mathbf{1}_\lambda\|_\lambda|J_\lambda(g, o)^{-1}F(g\cdot o)|.$$

Since $\|F\|_\lambda<\infty$ and $|J_\lambda(g, o)^{-1}F(g\cdot o)|\neq 0$, it then follows that $\|\mathbf{1}_\lambda\|_\lambda<\infty$. $\blacksquare$

4.32. Corollary. *For any compact subset X of* $\mathcal{D}$, *there exists a constant* $C_X < \infty$ *such that*

$$|F(z)| \leq C_X \|F\|_\lambda \tag{4.33a}$$

for all $F \in \mathcal{O}(\bar{\mathcal{D}}, E_\lambda)$ *and all* $z \in X$.

Proof. Let $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{p}$ as in (2.10) and let $A = \exp \mathfrak{a}$. Then we know that $G = KAK$. Since $K \cdot o = o$, there then exists, for a given compact subset X of $\mathcal{D}$, a compact subset Y of A such that $X \subset KY \cdot o$. Put $C_X = \sup_{a \in Y} |J_\lambda(a, o)| \cdot \|\mathbf{1}_\lambda\|_\lambda^{-1}$ where $|J_\lambda(a, o)|$ denotes the Hilbert-Schmidt norm of $J_\lambda(a, o)$. If $z \in X$ then $z = ka \cdot o$ for some $k \in K$ and $a \in Y$, and we have

$$\begin{aligned}
|F(z)| &= |J_\lambda(ka, o)(T_\lambda(ka)^{-1}F)(o)| \\
&= |\tau_\lambda(k) J_\lambda(a, o)(T_\lambda(ka)^{-1}F)(o)| \\
&= |J_\lambda(a, o)(T_\lambda(ka)^{-1}F)(o)| \\
&\leq |J_\lambda(a, o)| \cdot |(T_\lambda(ka)^{-1}F)(o)| \\
&\leq \sup_{a \in Y} |J_\lambda(a, o)| \cdot \|\mathbf{1}_\lambda\|_\lambda^{-1} \cdot \|T_\lambda(ka)^{-1}F\|_\lambda \qquad \text{(by Lemma 4.29)} \\
&= C_X \|F\|_\lambda \qquad \text{(by (4.17))}
\end{aligned}$$

for all $F \in \mathcal{O}(\bar{\mathcal{D}}, E_\lambda)$. ∎

Corollary 4.32 implies

4.34. Corollary. *Let* $F \in \mathcal{O}^2(\bar{\mathcal{D}}, E_\lambda)$; *if* $\|F\|_\lambda = 0$, *then* $F \equiv 0$. *Consequently the seminorm* $\| \ \|_\lambda$ *on* $\mathcal{O}^2(\bar{\mathcal{D}}, E_\lambda)$ *is a norm.*

We denote

(4.35) $H^2(\mathcal{D}, \lambda)$: the completion of $\mathcal{O}^2(\bar{\mathcal{D}}, E_\lambda)$ in the norm $\| \ \|_\lambda$.

$H^2(\mathcal{D}, \lambda)$ is a Hilbert space, whose inner product is given on the dense subspace $\mathcal{O}^2(\bar{\mathcal{D}}, E_\lambda)$ by

$$\begin{aligned}
&(F, F') \\
&= \int_{K \times G_i} (P_{\bar{\lambda}} J_\lambda(kg_i c_i, o)^{-1} F(kg_i \cdot o_i), P_{\bar{\lambda}} J_\lambda(kg_i c_i, o)^{-1} F'(kg_i \cdot o_i)) dk dg_i.
\end{aligned} \tag{4.36}$$

From Corollary 4.32 one easily proves in the standard way that $H^2(\mathcal{D}, \lambda)$ can be identified with a subspace of $\mathcal{O}(\mathcal{D}, E_\lambda)$. Note that

(4.33b) the inequality (4.33a) is valid for every $F \in H^2(\mathcal{D}, \lambda)$.

We also note that the mapping $\mathscr{I}_\lambda$ in (4.18) extends uniquely to a G-equivariant isometric isomorphism, again denoted by $\mathscr{I}_\lambda$, of $H^2(\mathscr{D}, \lambda)$ onto a closed subspace of $L^2(G, \sigma_\lambda)$. Hence, and because $(U_\lambda, L^2(G, \sigma_\lambda))$ is a unitary representation of G, we conclude that

$$(4.37) \quad \begin{cases} T_\lambda \text{ defines a unitary representation of } G \text{ on } H^2(\mathscr{D}, \lambda) \text{ and the mapping} \\ \mathscr{I}_\lambda \text{ is an isometric intertwining operator from } H^2(\mathscr{D}, \lambda) \text{ into } L^2(G, \sigma_\lambda). \end{cases}$$

Of course, (4.37) is insignificant unless $H^2(\mathscr{D}, \lambda)$ is nonzero. For that, we will prove

4.38. PROPOSITION. $\|\mathbf{1}_\lambda\|_\lambda < \infty$ *for all* $\lambda \in \mathscr{F}_i(G)$.

This proposition, together with Corollaries 4.31 and 4.27, immediately implies

4.39. COROLLARY. *For any* $\lambda \in \mathscr{F}_i(G)$, $H^2(\mathscr{D}, \lambda) \neq \{0\}$ *and all constant functions* $\mathbf{1}_e$ $(e \in E_\lambda)$ *belong to* $H^2(\mathscr{D}, \lambda)$.

NOTE. Because of Corollary 4.27, in order to prove Proposition 4.38 we may assume $i \neq r$ and it suffices to show that

$$(4.40) \qquad \text{the integral } \int_{A^+_{(i)}} \chi_{\bar{\lambda}}(\pi_0(a)^{-2}) D_i(a) da \text{ is finite.}$$

Such an integral occurs in Harish-Chandra's work [9] on holomorphic discrete series. There he gives a criterion for the finiteness of the integral and also calculates its value explicitly. Using his criterion one may verify (4.40). Our integral in question, however, looks slightly different from Harish-Chandra's; so for the sake of completeness we directly establish (4.40) following the lines of Harish-Chandra's computation.

We start the proof of Proposition 4.38 with a lemma. Recall the systems of positive roots Φ^+ and Φ_i^+ in Section 2 and define $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$, $\delta_i = \frac{1}{2}\sum_{\alpha\in\Phi_i^+}\alpha$. Similarly define $\delta_{i,c}$, $\delta_{i,n}$ using $\Phi_{i,c}^+$, $\Phi_{i,n}^+$ in (3.15); so that $\delta_i = \delta_{i,c} + \delta_{i,n}$.

4.41. LEMMA. $\delta|_{\mathfrak{t}_{i,e}} = \delta_i|_{\mathfrak{t}_{i,e}}$ *for all* $1 \leq i \leq r$.

PROOF. Let $\{\gamma_1, \ldots, \gamma_r\}$, C_j, N_j, C_{jk}, N_{jk} be as in (2.9), (2.11). For each γ_j, let s_j denote the Weyl reflection corresponding to γ_j. If $\alpha \in C_j$, then $\langle \alpha, H_j \rangle = 1$; hence $-s_j(\alpha) = \langle \alpha, H_j \rangle \gamma_j - \alpha = \gamma_j - \alpha$, so $\gamma_j - \alpha$ is a root which is clearly in N_j. Conversely if $\beta \in N_j$, then $\langle \beta, H_j \rangle = 1$; therefore $-s_j(\beta) = \gamma_j - \beta$ is a root, which belongs to C_j. Consequently, for each $1 \leq j \leq r$, the mapping $C_j \to N_j$ given by $\alpha \to \gamma_j - \alpha$ is bijective. Thus if $\alpha \in C_j$ with $1 \leq j \leq i$, then $-s_j(\alpha) \in N_j$ and $\alpha + (-s_j(\alpha))|_{\mathfrak{t}_{i,e}} = \gamma_j|_{\mathfrak{t}_{i,e}} = 0$ (the second equality following from its definition of $\mathfrak{t}_{i,e}$).

Hence by grouping the summands in pairs, we see that

$$\sum_{\alpha\in C_j\cup N_j}\alpha|_{\mathfrak{t}_{i,\mathfrak{c}}}=0 \qquad \text{for every} \quad 1\le j\le i.$$

Similarly, for $1\le j<k\le r$, the mapping $C_{jk}\to N_{jk}$ given by $\alpha\to -s_j(\alpha)=\gamma_j-\alpha$ is bijective, and so one finds as above that

$$\sum_{\alpha\in C_{jk}\cup N_{jk}}\alpha|_{\mathfrak{t}_{i,\mathfrak{c}}}=0 \qquad \text{provided} \quad 1\le j<k\le i \quad \text{or} \quad 1\le j\le i<k\le r.$$

Now, in view of (2.12a), (2.21a) and (2.34),

$$\Phi^+-\Phi_i^+={}'C_0^{(i)}\cup\bigcup_{1\le j\le i}(C_j\cup N_j)\cup\bigcup_{1\le j<k\le i}(C_{jk}\cup N_{jk})$$
$$\cup\bigcup_{1\le j\le i<k\le r}(C_{jk}\cup N_{jk})\cup\{\gamma_1,\dots,\gamma_i\}.$$

Thus we conclude that

$$\sum_{\alpha\in\Phi^+-\Phi_i^+}\alpha|_{\mathfrak{t}_{i,\mathfrak{c}}}=0$$

and hence that

$$\sum_{\alpha\in\Phi^+}\alpha|_{\mathfrak{t}_{i,\mathfrak{c}}}=\sum_{\alpha\in\Phi_i^+}\alpha|_{\mathfrak{t}_{i,\mathfrak{c}}},$$

which implies the lemma. ∎

PROOF OF PROPOSITION 4.38. As noted after Corollary 4.39, it is sufficient to show that

$$\int_{A_{(i)}^+}\chi_\lambda(\pi_0(a)^{-2})D_i(a)da<\infty \qquad (1\le i\le r-1).$$

Let $X=\sum_{j=i+1}^r x_j(X_j+X_{-j})\in\mathfrak{a}_{(i)}^+$ and put $H(X)=2\sum_{j=i+1}^r\log(\cosh x_j)H_j$. Then, since $\pi_0(\exp X)=\exp(-\sum_{j=i+1}^r\log(\cosh x_j)H_j)$ by (2.19), it follows from our normalization of the measure da that

$$\int_{A_{(i)}^+}\chi_\lambda(\pi_0(a)^{-2})D_i(a)da=\int_{\mathfrak{a}_{(i)}^+}\chi_\lambda(\exp H(X))D_i(\exp X)dX. \tag{4.42}$$

Here, according to (2.36),

$$D_i(\exp X)=2^{r-i}\prod_{i+1\le j\le r}(\sinh x_j)^{2v+1}(\cosh x_j)$$
$$\prod_{i+1\le j<k\le r}\{(\cosh x_j)^2-(\cosh x_k)^2\}^u. \tag{4.43}$$

If we define, for $H\in\mathfrak{t}_{i,\mathfrak{c}}$,

$$\Delta(\exp H)=\prod_{\alpha\in\Phi_{i,\mathfrak{c}}^+-C_0^{(i)}}\{e^{\alpha(H)/2}-e^{-\alpha(H)/2}\}$$

($C_0^{(i)}$ is as in (2.21a)), then using (2.12ab) we get

$$\Delta(\exp H(X)) = \prod_{i+1\leq j\leq r} \left\{\frac{(\sinh x_j)^2}{\cosh x_j}\right\}^v \prod_{i+1\leq j<k\leq r} \left\{\frac{(\cosh x_j)^2-(\cosh x_k)^2}{\cosh x_j \cosh x_k}\right\}^u$$

This combined with (4.43) gives

$$D_i(\exp X) = 2^{r-i}\Delta(\exp H(X)) \prod_{i+1\leq j\leq r} (\sinh x_j)(\cosh x_j)^{1+v+u(r-i-1)}.$$

But

$$1+v+u(r-i-1) = \langle 2\delta_{i,n}, H_j\rangle - 1 \qquad \text{for all} \quad i+1\leq j\leq r.$$

In fact, by (2.12b), (2.21a) and (3.15),

$$\begin{aligned}\langle 2\delta_{i,n}, H_j\rangle &= \langle \sum_{i+1\leq j\leq r}\gamma_j + u\sum_{i+1\leq j<k\leq r}\frac{1}{2}(\gamma_j+\gamma_k) + v\sum_{i+1\leq j\leq r}\frac{1}{2}\gamma_j, H_j\rangle \\ &= \left(1+\frac{1}{2}u(r-i-1)+\frac{1}{2}v\right)\langle \sum_{i+1\leq j\leq r}\gamma_j, H_j\rangle \\ &= 2+u(r-i-1)+v.\end{aligned}$$

Hence

$$D_i(\exp X) = 2^{r-i}\Delta(\exp H(X)) \prod_{i+1\leq j\leq r} \sinh x_j (\cosh x_j)^{\langle 2\delta_{i,n}, H_j\rangle - 1}. \tag{4.44}$$

On the other hand, using the same argument as in the proof of Lemma 25 of Harish-Chandra [9], one derives from Weyl's character formula the identity

$$\begin{aligned}&\chi_\lambda(\exp H)\Delta(\exp H) \\ &= \{w_0 \prod_{\alpha\in C_0^{(i)}}\langle\delta_0, H_\alpha\rangle\}^{-1}\sum_{s\in W_i}\varepsilon(s)\{\prod_{\alpha\in C_0^{(i)}}\langle\lambda+\delta_{i,c}, sH_\alpha\rangle\}e^{\langle\lambda+\delta_{i,c}, sH\rangle}\end{aligned} \tag{4.45}$$

for all $H\in\sum_{j=i+1}^r \boldsymbol{C}H_j$; here W_i denotes the Weyl group of $(\mathfrak{k}_{i,c}, \mathfrak{t}_{i,c})$, w_0 the order of the subgroup of W_i generated by Weyl reflections corresponding to the roots in $C_0^{(i)}$, $\varepsilon(s)$ the sign of s, and $\delta_0=\frac{1}{2}\sum_{\alpha\in C_0^{(i)}}\alpha$.

Now, since $\mathfrak{k}_{i,c}$ normalizes $\mathfrak{p}_i^+$, each element of W_i permutes the members of $\Phi_{i,n}^+$ and leaves $2\delta_{i,n}$ invariant. Thus if we put, for each $s\in W_i$,

$$C(\lambda, s) = 2^{r-i}\varepsilon(s)\{w_0\prod_{\alpha\in C_0^{(i)}}\langle\delta_0, H_\alpha\rangle\}^{-1}\prod_{\alpha\in C_0^{(i)}}\langle\lambda+\delta_{i,c}, sH_\alpha\rangle,$$

then, since $\delta_i=\delta_{i,c}+\delta_{i,n}$, it follows from (4.44) and (4.45) that

$$\begin{aligned}&\chi_\lambda(\exp H(X))D_i(\exp X) \\ &= \sum_{s\in W_i} C(\lambda, s)\prod_{i+1\leq j\leq r}\sinh x_j(\cosh x_j)^{2\langle\lambda+\delta_i, sH_j\rangle-1}.\end{aligned} \tag{4.46}$$

If we make the change of variables $y_j=(\cosh x_j)^{-1}$ then, in view of (2.35b), $\mathfrak{a}_{(i)}^+$ corresponds to the region $\{\sum_{j=i+1}^r y_j(X_j+X_{-j});\ 0<y_{i+1}<y_{i+2}<\cdots<y_r<1\}$. We denote this region by $\mathfrak{b}$ and put $dy=dy_{i+1}\cdots dy_r$. Then we conclude from (4.42) and (4.46) that

$$(4.47)\qquad \int_{A_{(i)}^+}\chi_\lambda(\pi_0(a)^{-2})D_i(a)da=\sum_{s\in W_i}C(\lambda,s)\int_{\mathfrak{b}}\prod_{i+1\leq j\leq r}y_j^{-2\langle\lambda+\delta_i,sH_j\rangle-1}dy.$$

Now Lemma 4.41 implies $\langle\lambda+\delta_i, sH_j\rangle=\langle\lambda+\delta, sH_j\rangle$ for all $s\in W_i$ and $i+1\leq j\leq r$, while, according to Lemma 3.16, $\langle\lambda+\delta, H_\alpha\rangle<0$ for every $\alpha\in\Phi_{i,n}^+$. Since $\Phi_{i,n}^+$ is invariant under each $s\in W_i$, it then follows that $\langle\lambda+\delta_i, sH_j\rangle<0$ for all $s\in W_i$ and $i+1\leq j\leq r$. From this it is now clear that the right-hand side of (4.47) is finite, thus finishing the proof of the proposition. ∎

Finally we observe that

(4.48) the representation $(T_\lambda, H^2(\mathscr{D},\lambda))$ is irreducible.

In fact the proof in Kunze [19] can be modified slightly to cover this case. Alternatively, one may proceed by a standard method as follows. Let $\mathfrak{H}$ be a nonzero closed invariant subspace of $H^2(\mathscr{D},\lambda)$. Take a function $F\in\mathfrak{H}$ such that $F(o)\neq 0$ and let T^1 be the central subgroup of K as in the proof of Lemma 4.29. Then, for $t\in T^1$, the function $z\to\tau_\lambda(t)(T_\lambda(t)^{-1}F)(z)$ again lies in $\mathfrak{H}$ because $\tau_\lambda(t)$ is a scalar operator. Hence the function $\int_{T^1}\tau_\lambda(t)(T_\lambda(t)^{-1}F)dt$, being the limit of a sum in $\mathfrak{H}$, also belongs to $\mathfrak{H}$. Since $\tau_\lambda(t)(T_\lambda(t)^{-1}F)(z)=F(t\cdot z)$ and since $\int_{T^1}F(t\cdot z)dt=F(o)$ (cf. (4.30)), we then see that the constant function $z\to F(o)$ belongs to $\mathfrak{H}$; hence $\mathbf{1}_\lambda\in\mathfrak{H}$. If the orthogonal complement $\mathfrak{H}^\perp$ of $\mathfrak{H}$ is nonzero, the above argument shows that $\mathbf{1}_\lambda$ is also in $\mathfrak{H}^\perp$, which is a contradiction; this gives (4.48).

The results obtained above can be summarized as follows:

4.49. Theorem. *For any $\lambda\in\mathscr{F}_i(G)$ $(1\leq i\leq r)$, $H^2(\mathscr{D},\lambda)$ is nonzero and $(T_\lambda, H^2(\mathscr{D},\lambda))$ is an irreducible unitary representation of G. Furthermore $(T_\lambda, H^2(\mathscr{D},\lambda))$ is unitarily equivalent to a subrepresentation of $(U_\lambda, L^2(G,\sigma_\lambda))$.*

For later use, we mention here other realization of $(T_\lambda, H^2(\mathscr{D},\lambda))$. Let

$$\mathcal{O}(G,\tau_\lambda)=\left\{f\in C^\infty(G,E_\lambda);\ \begin{matrix} f(gk)=\tau_\lambda(k)^{-1}f(g), & g\in G, & k\in K, \\ r(X)f=0 & \text{for all} & X\in\mathfrak{p}^- \end{matrix}\right\}$$

where $r(X)$ is as in (4.19). For $F\in\mathcal{O}(\mathscr{D},E_\lambda)$, define a function $\mathscr{J}_\lambda F\colon G\to E_\lambda$ by

$$(4.50)\qquad \mathscr{J}_\lambda F(g)=J_\lambda(g,o)^{-1}F(g\cdot o),\quad g\in G.$$

Then (cf. Baily and Borel [1], p. 493) $\mathcal{J}_\lambda F$ lies in $\mathcal{O}(G, \tau_\lambda)$ and the resulting mapping $\mathcal{J}_\lambda\colon \mathcal{O}(\mathcal{D}, E_\lambda)\to\mathcal{O}(G, \tau_\lambda)$ is bijective. We put

$$(4.51a) \qquad H^2(G, \tau_\lambda) = \mathcal{J}_\lambda(H^2(\mathcal{D}, \lambda)).$$

One can define an inner product on $H^2(G, \tau_\lambda)$ so that $\mathcal{J}_\lambda$ (restricted to $H^2(\mathcal{D}, \lambda)$) is a unitary isomorphism. Via $\mathcal{J}_\lambda$, $(T_\lambda, H^2(\mathcal{D}, \lambda))$ may be transported to a unitary representation of G on $H^2(G, \tau_\lambda)$. We observe that the transport of T_λ to $H^2(G, \tau_\lambda)$ is given by left translation:

$$(4.51b) \qquad (T_\lambda(g)f)(g') = f(g^{-1}g'), \quad g, g' \in G, \ f\in H^2(G, \tau_\lambda).$$

The following lemma will be needed in Section 5. For $X \in \mathfrak{g}_c$, we define $l(X)$ as in (4.19).

4.52. Lemma. *Letting* $\mathbf{1}_\lambda$ *be the constant function as in Corollary* 4.27, *put* $\psi_\lambda=\mathcal{J}_\lambda(\mathbf{1}_\lambda)$. *Then* ψ_λ *is K-finite of type* τ_λ. *Moreover* ψ_λ *satisfies* $l(H)\psi_\lambda =\langle\lambda, H\rangle\psi_\lambda$ *for* $H\in \mathfrak{t}_c$ *and* $l(X_\alpha)\psi_\lambda=0$ *for all positive roots* α *of* $(\mathfrak{g}_c, \mathfrak{t}_c)$.

Proof. The first assertion is obvious. As for the second, define a function $\tilde{\psi}_\lambda\colon \Omega\to E_\lambda$ by $\tilde{\psi}_\lambda(\omega)=J_\lambda(\omega, o)^{-1}\boldsymbol{e}_\lambda$ where $\Omega=P^+K_cP^-$ and $\boldsymbol{e}_\lambda$ is the highest weight vector for τ_λ. Then ψ_λ is the restriction of $\tilde{\psi}_\lambda$ to G. As Ω is open in G_c, we can define, for $X\in\mathfrak{g}_c$, $l(X)\tilde{\psi}_\lambda$ as in (4.19). Since $\tilde{\psi}_\lambda$ is clearly holomorphic, it then follows that $l(X)\tilde{\psi}_\lambda(\omega)=\frac{d}{dt}\tilde{\psi}_\lambda(\exp(-tX)\omega)|_{t=0}$ for $\omega\in\Omega$, $X\in\mathfrak{g}_c$. Since $\boldsymbol{e}_\lambda$ is the highest weight vector for τ_λ, these observations, together with (2.7), imply the second assertion of the lemma. ∎

5. Imbedding in continuous series

We again fix i, $1\le i\le r$, and $\lambda\in\mathcal{F}_i(G)$. In this section we construct an irreducible unitary representation μ_λ of M_i and show that the representation $(T_\lambda, H^2(\mathcal{D}, \lambda))$, which was constructed in Section 4, is unitarily equivalent to a proper subrepresentation of the induced representation $V_\lambda=\mathrm{Ind}_{M_iA_iN_i\uparrow G}(\mu_\lambda\otimes 1\otimes 1)$ and hence V_λ is reducible.

With $\tilde{\lambda}$, $E_{\tilde{\lambda}}$ being as in (4.1), let

$$(5.1a) \qquad L^2(\mathcal{C}_i, \tilde{\lambda}) = \left\{ F\colon \mathcal{C}_i\to E_{\tilde{\lambda}};\ \begin{array}{c} F \text{ Borel measurable,} \\ \|F\|_{\tilde{\lambda}}^2 = \int_{G_i} |J_\lambda(g_ic_i, o)^{-1}F(g_i\cdot o_i)|^2 dg_i < \infty \end{array} \right\},$$

and let

$$(5.1b) \qquad H^2(\mathcal{C}_i, \tilde{\lambda}) = L^2(\mathcal{C}_i, \tilde{\lambda}) \cap \mathcal{O}(\mathcal{C}_i, E_{\tilde{\lambda}})$$

where $\mathcal{C}_i$ is the boundary component containing o_i and $\mathcal{O}(\mathcal{C}_i, E_{\tilde{\lambda}})$ denotes the space of all $E_{\tilde{\lambda}}$-valued holomorphic functions on $\mathscr{C}_i$. Then $L^2(\mathscr{C}_i, \tilde{\lambda})$ (functions which are equal almost everywhere being identified) is a Hilbert space with inner product

$$(F, F') = \int_{G_i} (J_\lambda(g_i c_i, o)^{-1} F(g_i \cdot o_i), J_\lambda(g_i c_i, o)^{-1} F'(g_i \cdot o_i)) dg_i.$$

Given $F \in L^2(\mathscr{C}_i, \tilde{\lambda})$ and $g \in G_i$, define $T_{\tilde{\lambda}}(g)F\colon \mathscr{C}_i \to E_\lambda$ by

$$(T_{\tilde{\lambda}}(g)F)(z) = J_\lambda(g^{-1}, z)^{-1} F(g^{-1} \cdot z), \quad z \in \mathscr{C}_i. \tag{5.2}$$

Then, as follows from 4.25(1) and the definition of $E_{\tilde{\lambda}}$, $T_{\tilde{\lambda}}(g)F$ takes values in $E_{\tilde{\lambda}}$, and it is easily seen that $T_{\tilde{\lambda}}$ defines a unitary representation of G_i on $L^2(\mathscr{C}_i, \tilde{\lambda})$ and $H^2(\mathscr{C}_i, \tilde{\lambda})$ is an invariant subspace. Therefore, arguing as in Section 4, one finds that the norm convergence in $H^2(\mathscr{C}_i, \tilde{\lambda})$ implies uniform convergence on every compact subset of $\mathscr{C}_i$ and hence $H^2(\mathscr{C}_i, \tilde{\lambda})$ is a closed subspace of $L^2(\mathscr{C}_i, \tilde{\lambda})$.

5.3. LEMMA. $H^2(\mathscr{C}_i, \tilde{\lambda}) \neq \{0\}$.

PROOF. Let $\mathbf{1}_\lambda$ be the constant function on $\bar{\mathscr{D}}$ as in Corollary 4.27. The restriction of $\mathbf{1}_\lambda$ to $\mathscr{C}_i$, which we denote again by $\mathbf{1}_\lambda$, clearly lies in $\mathscr{O}(\mathscr{C}_i, E_{\tilde{\lambda}})$. Therefore, to prove the lemma, it suffices to show that $\|\mathbf{1}_\lambda\|_{\tilde{\lambda}} < \infty$. If $i \neq r$ then, using the integration formula (2.37) and carrying over the similar computation as in the proof of Proposition 4.24, we get

$$\begin{aligned}
\|\mathbf{1}_\lambda\|_{\tilde{\lambda}}^2 &= \int_{K_i \times A_{(i)}^+ \times K_i} |J_\lambda(k_1 a k_2 c_i, o)^{-1} \mathbf{e}_\lambda|^2 D_i(a) dk_1 da dk_2 \\
&= \beta_i^2 \int_{K_i \times A_{(i)}^+} |J_\lambda(a, o)^{-1} \tau_\lambda(k_1)^{-1} \mathbf{e}_\lambda|^2 D_i(a) dk_1 da \\
&= \beta_i^2 d(\tilde{\lambda})^{-1} \int_{A_{(i)}^+} \chi_{\tilde{\lambda}}(\pi_0(a)^{-2}) D_i(a) da \\
&= d(\lambda) d(\tilde{\lambda})^{-1} \|\mathbf{1}_\lambda\|_{\tilde{\lambda}}^2 \qquad \text{(by Corollary 4.27)}.
\end{aligned}$$

Likewise, also in the case $i = r$, we obtain $\|\mathbf{1}_\lambda\|_{\tilde{\lambda}}^2 = d(\lambda) d(\tilde{\lambda})^{-1} \|\mathbf{1}_\lambda\|_{\tilde{\lambda}}^2$. Since $\|\mathbf{1}_\lambda\|_\lambda < \infty$ by Proposition 4.38, the lemma follows. ∎

Let $'\tau_{\tilde{\lambda}}^{(i)}$ be the irreducible unitary representation of L_i on $E_{\tilde{\lambda}}$ defined by (4.9) and form the unitarily induced representation $\tilde{\mu}_\lambda = \mathrm{Ind}_{L_i \uparrow M_i} {}'\tau_{\tilde{\lambda}}^{(i)}$. We denote by $L^2(M_i, {}'\tau_{\tilde{\lambda}}^{(i)})$ the representation space of $\tilde{\mu}_\lambda$. Since $M_i = F_i I_i G_i' G_i$, and since $F_i I_i G_i'$ acts trivially on $\mathscr{C}_i = M_i / L_i = G_i / K_i$ (cf. 2.28), M_i preserves every G_i-invariant measure on $\mathscr{C}_i$. Thus we may take

$$(5.4a)\qquad L^2(M_i, {}'\tau_\lambda^{(i)}) = \left\{ \begin{array}{c} f \text{ Borel measurable,} \\ f\colon M_i \to E_{\tilde{\lambda}};\ f(ml) = {}'\tau_\lambda^{(i)}(l)^{-1} f(m),\ \ m \in M_i,\ l \in L_i, \\ \int_{G_i} |f(g_i)|^2 dg_i < \infty \end{array} \right\}$$

and $\tilde{\mu}_\lambda$ acts by left translation. We also define

$$(5.4b)\qquad \begin{aligned} &H^2(M_i, {}'\tau_\lambda^{(i)}) \\ &= L^2(M_i, {}'\tau_\lambda^{(i)}) \cap \{f \in C^\infty(M_i, E_{\tilde{\lambda}});\ r(X)f = 0 \text{ for all } X \in \mathfrak{p}_i^-\} \end{aligned}$$

where $\mathfrak{p}_i^- = \mathfrak{g}_{i,c} \cap \mathfrak{p}^-$, and $r(X)f$ is defined similarly as in (4.19). Then it is clear that the subspace $H^2(M_i, {}'\tau_\lambda^{(i)})$ is invariant under the representation $\tilde{\mu}_\lambda$.

Now, for $F \in L^2(\mathscr{C}_i, \tilde{\lambda})$, define $\mathscr{J}_{\tilde{\lambda}}F\colon M_i \to E_{\tilde{\lambda}}$ by

$$(5.5)\qquad \mathscr{J}_{\tilde{\lambda}}F(m) = J_\lambda(mc_i, o)^{-1} F(m \cdot o_i), \quad m \in M_i.$$

(Since $M_i = F_i I_i G_i' G_i$, Lemmas 4.5(1) and 4.25 ensure that $\mathscr{J}_{\tilde{\lambda}}F$ actually takes values in $E_{\tilde{\lambda}}$.) Then, for $m \in M_i$ and $l \in L_i$, we have

$$\begin{aligned} \mathscr{J}_{\tilde{\lambda}}F(ml) &= J_\lambda(mlc_i, o)^{-1} F(ml \cdot o_i) \\ &= J_\lambda(mc_i c_i^{-1} l c_i, o)^{-1} F(m \cdot o_i) \\ &= J_\lambda(c_i^{-1} l c_i, o)^{-1} J_\lambda(mc_i, o)^{-1} F(m \cdot o_i) \\ &= {}'\tau_\lambda^{(i)}(l)^{-1} \mathscr{J}_{\tilde{\lambda}}F(m), \end{aligned}$$

and so it follows that $\mathscr{J}_{\tilde{\lambda}}F \in L^2(M_i, {}'\tau_\lambda^{(i)})$. Therefore one finds without difficulty that the mapping $\mathscr{J}_{\tilde{\lambda}}\colon L^2(\mathscr{C}_i, \tilde{\lambda}) \to L^2(M_i, {}'\tau_\lambda^{(i)})$ is a surjective isometry.

5.6. LEMMA. *The subspace $H^2(M_i, {}'\tau_\lambda^{(i)})$ of $L^2(M_i, {}'\tau_\lambda^{(i)})$ is closed, non-zero and corresponds to the subspace $H^2(\mathscr{C}_i, \tilde{\lambda})$ under the unitary isomorphism $\mathscr{J}_{\tilde{\lambda}}\colon L^2(\mathscr{C}_i, \tilde{\lambda}) \to L^2(M_i, {}'\tau_\lambda^{(i)})$.*

PROOF. In view of Lemma 5.3, it suffices to show that $F \in C^\infty(\mathscr{C}_i, E_{\tilde{\lambda}})$ is holomorphic if and only if $r(X)\mathscr{J}_{\tilde{\lambda}}F = 0$ for all $X \in \mathfrak{p}_i^-$ ($\mathscr{J}_{\tilde{\lambda}}F$ stands for the function defined by (5.5)). But this follows from an argument similar to that used in proving Lemma 4.20. ∎

5.7. LEMMA. *The restriction of the representation $\tilde{\mu}_\lambda$ to the subspace $H^2(M_i, {}'\tau_\lambda^{(i)})$ is irreducible.*

PROOF. By means of the unitary isomorphism $\mathscr{J}_{\tilde{\lambda}}\colon L^2(\mathscr{C}_i, \tilde{\lambda}) \to L^2(M_i, {}'\tau_\lambda^{(i)})$, $\tilde{\mu}_\lambda$ may be transported to a unitary representation of M_i on $L^2(\mathscr{C}_i, \tilde{\lambda})$. It is easy to see that the transport of $\tilde{\mu}_\lambda$ to $L^2(\mathscr{C}_i, \tilde{\lambda})$ is given by

$$(\tilde{\mu}_\lambda(m)F)(z) = J_\lambda(m^{-1}, z)^{-1}F(m^{-1}\cdot z), \quad F\in L^2(\mathscr{C}_i, \tilde{\lambda}),\ m\in M_i,\ z\in\mathscr{C}_i.$$

Now to prove the lemma it is enough to show that the representation $(\tilde{\mu}_\lambda, H^2(\mathscr{C}_i, \tilde{\lambda}))$ of M_i is irreducible on the subgroup G_i. But this follows as in the proof of (4.48). ∎

Let μ_λ denote the restriction of $\tilde{\mu}_\lambda$ to $H^2(M_i, '\tau_\lambda^{(i)})$. Then by the above lemma μ_λ is an irreducible unitary representation of M_i. For each $\nu\in\mathfrak{a}_i^*$, we define an irreducible unitary representation $\mu_\lambda\otimes e^{\sqrt{-1}\nu}\otimes 1$ of $P_i=M_iA_iN_i$ on $H^2(M_i, '\tau_\lambda^{(i)})$ by

$$(\mu_\lambda\otimes e^{\sqrt{-1}\nu}\otimes 1)(man) = \mu_\lambda(m)e^{\sqrt{-1}\nu}(a), \quad man\in M_iA_iN_i$$

and form the continuous series representation

$$V_{\lambda,\nu} = \operatorname{Ind}_{M_iA_iN_i\uparrow G}(\mu_\lambda\otimes e^{\sqrt{-1}\nu}\otimes 1).$$

Let $\mathfrak{H}_{\lambda,\nu}$ denote the representation space of $V_{\lambda,\nu}$. Then, in view of (2.39) and (2.43), we may take

$$\mathfrak{H}_{\lambda,\nu} = \left\{ \Psi: G\to H^2(M_i, '\tau_\lambda^{(i)});\ \begin{array}{l} \Psi \text{ Borel measurable,} \\ \Psi(gman) = \{e^{\rho_i+\sqrt{-1}\nu}(a)\mu_\lambda(m)\}^{-1}\Psi(g), \\ \qquad\qquad g\in G,\ man\in P_i, \\ \|\Psi\|^2 = \int_K \|\Psi(k)\|^2 dk < \infty \end{array} \right\}.$$

Next we show that the representation $(V_{\lambda,\nu}, \mathfrak{H}_{\lambda,\nu})$ is unitarily equivalent to a subrepresentation of $(U_{\lambda,\nu}, L^2(G, \sigma_{\lambda,\nu}))$ defined by (4.11). For this purpose we consider the space

$$\begin{aligned} (5.8)\qquad & C_2^\infty(G, \sigma_{\lambda,\nu}; \mathfrak{p}_i^-) \\ & = L^2(G, \sigma_{\lambda,\nu}) \cap \{f\in C^\infty(G, E_{\tilde{\lambda}});\ r(X)f = 0 \text{ for all } X\in\mathfrak{p}_i^-\}. \end{aligned}$$

The closure of $C_2^\infty(G, \sigma_{\lambda,\nu}; \mathfrak{p}_i^-)$ in $L^2(G, \sigma_{\lambda,\nu})$, which we denote by $L^2(G, \sigma_{\lambda,\nu}; \mathfrak{p}_i^-)$, is clearly a closed, $U_{\lambda,\nu}$-stable subspace.

5.9. PROPOSITION. *For every $\nu\in\mathfrak{a}_i^*$, the representation $(V_{\lambda,\nu}, \mathfrak{H}_{\lambda,\nu})$ of G is unitarily equivalent to $(U_{\lambda,\nu}, L^2(G, \sigma_{\lambda,\nu}; \mathfrak{p}_i^-))$.*

PROOF. For $f\in C_2^\infty(G, \sigma_{\lambda,\nu}; \mathfrak{p}_i^-)$ and $g\in G$, define $f_g: M_i\to E_{\tilde{\lambda}}$ by $f_g(m) = f(gm)$. It is easily checked that $f_g\in H^2(M_i, '\tau_\lambda^{(i)})$. Next define $\Psi_f: G\to H^2(M_i, '\tau_\lambda^{(i)})$ by $\Psi_f(g)=f_g$. Then we claim:

$$(5.10)\qquad \Psi_f\in\mathfrak{H}_{\lambda,\nu} \quad\text{and}\quad \|\Psi_f\| = \|f\|.$$

Indeed if $man \in M_iA_iN_i$ then, for $m_1 \in M_i$,

$$\begin{aligned}\Psi_f(gman)(m_1) &= f(gmanm_1) = f(gmm_1am_1^{-1}nm_1)\\ &= e^{-\rho_i-\sqrt{-1}\nu}(a)f_g(mm_1) \qquad (m_1^{-1}nm_1 \in N_i)\\ &= e^{-\rho_i-\sqrt{-1}\nu}(a)(\mu_\lambda(m)^{-1}f_g)(m_1)\\ &= [e^{-\rho_i-\sqrt{-1}\nu}(a)\mu_\lambda(m)^{-1}\Psi_f(g)](m_1).\end{aligned}$$

Moreover

$$\begin{aligned}\|\Psi_f\|^2 = \int_K \|\Psi_f(k)\|^2 dk &= \int_K \left\{\int_{G_i} |f_k(g_i)|^2 dg_i\right\} dk\\ &= \int_{K\times G_i} |f(kg_i)|^2 dk dg_i = \|f\|^2.\end{aligned}$$

This implies the claim (5.10).

In view of (5.10), one can define an isometry $\mathscr{R}$ of $L^2(G, \sigma_{\lambda,\nu};\, \mathfrak{p}_i^-)$ into $\mathfrak{H}_{\lambda,\nu}$, satisfying $\mathscr{R}(f) = \Psi_f$ for $f \in C_2^\infty(G, \sigma_{\lambda,\nu};\, \mathfrak{p}_i^-)$. Since it is obvious that $\mathscr{R}$ intertwines $U_{\lambda,\nu}$ and $V_{\lambda,\nu}$, the proof of the proposition will be complete when it is established that $\mathscr{R}$ is surjective. For $\phi \in C_c^\infty(G)$ and $h \in H^2(M_i, {}'\tau_\lambda^{(i)})$, define $F_{\phi,h}\colon G \to H^2(M_i, {}'\tau_\lambda^{(i)})$ by

$$F_{\phi,h}(g) = \int_{P_i} \rho(p)^{-1/2}\phi(gp)[(\mu_\lambda \otimes e^{\sqrt{-1}\nu} \otimes 1)(p)]h dp, \quad g \in G$$

(ρ is the rho-function in (2.39)), and further define $f_{\phi,h}\colon G \to E_\lambda$ by $f_{\phi,h}(g) = (F_{\phi,h}(g))(e)$ (e = identity of M_i). We are going to show that

$$(5.11) \qquad f_{\phi,h} \in C_2^\infty(G, \sigma_{\lambda,\nu};\, \mathfrak{p}_i^-) \quad \text{and} \quad \mathscr{R}(f_{\phi,h}) = F_{\phi,h}.$$

Since $F_{\phi,h}$ ($\phi \in C_c^\infty(G)$, $h \in H^2(M_i, {}'\tau_\lambda^{(i)})$) span a dense subspace of $\mathfrak{H}_{\lambda,\nu}$ (cf. Warner [30], p. 371), this will imply that $\mathscr{R}\colon L^2(G, \sigma_{\lambda,\nu};\, \mathfrak{p}_i^-) \to \mathfrak{H}_{\lambda,\nu}$ is surjective and, in view of our earlier remark, thus serve to complete the proof of the proposition.

First observe that if $lan \in S_i = L_iA_iN_i$, then

$$\begin{aligned}f_{\phi,h}(glan) &= [e^{-\rho_i-\sqrt{-1}\nu}(a)\mu_\lambda(l)^{-1}F_{\phi,h}(g)](e)\\ &= e^{-\rho_i-\sqrt{-1}\nu}(a)\,{}'\tau_\lambda^{(i)}(l)^{-1}[(F_{\phi,h}(g))(e)]\\ &= \sigma_{\lambda,\nu}(lan)^{-1}f_{\phi,h}(g)\end{aligned}$$

for $g \in G$. Since convergence in $H^2(M_i, {}'\tau_\lambda^{(i)})$ implies uniform convergence on every compact subset of M_i, we have, for $m \in M_i$,

$$(F_{\phi,h}(g))(m) = \int_{P_i} \rho(p)^{-1/2}\phi(gp)[(\mu_\lambda \otimes e^{\sqrt{-1}\nu} \otimes 1)(p)h](m)dp,$$

which in particular implies $f_{\phi,h} \in C^\infty(G, E_\lambda)$. Now

$$(5.12) \qquad f_{\phi,h}(gm) = (\mu_\lambda(m)^{-1}F_{\phi,h}(g))(e) = (F_{\phi,h}(g))(m)$$

for $g \in G$, $m \in M_i$. Thus, and since $r(X)(F_{\phi,h}(g))=0$ for $X \in \mathfrak{p}_i^-$, it follows that $r(X)f_{\phi,h}=0$ for all $X \in \mathfrak{p}_i^-$. Furthermore

$$\int_{K\times G_i} |f_{\phi,h}(kg_i)|^2 dkdg_i = \int_{K\times G_i} |(F_{\phi,h}(k))(g_i)|^2 dkdg_i$$
$$= \int_K \|F_{\phi,h}(k)\|^2 dk < \infty.$$

These observations show that $f_{\phi,h} \in C_2^\infty(G, \sigma_{\lambda,\nu};\, \mathfrak{p}_i^-)$. Also (5.12) implies $\mathscr{R}(f_{\phi,h}) = F_{\phi,h}$. This establishes (5.11) and completes the proof. ∎

We write V_λ, $\mathfrak{H}_\lambda$ for $V_{\lambda,0}$, $\mathfrak{H}_{\lambda,0}$ (the case $\nu=0$). According to Harish-Chandra's irreducibility criterion [10, Lemma 3, p. 145] of the representations induced from a parabolic subgroup of G, the representations $V_{\lambda,\nu}=\mathrm{Ind}_{M_iA_iN_i\uparrow G}(\mu_\lambda \otimes e^{\sqrt{-1}\nu} \otimes 1)$ are irreducible for all $\nu \neq 0$, at least for the case that the representation μ_λ of M_i is square-integrable (this is the case when $i=1$; cf. Knapp and Okamoto [15]). However, for the exceptional case V_λ $(=V_{\lambda,0})$, we have

5.13. Theorem. *The representation $(T_\lambda, H^2(\mathscr{D}, \lambda))$ which was constructed in Section 4 is unitarily equivalent to a proper subrepresentation of $(V_\lambda, \mathfrak{H}_\lambda)$, and consequently $(V_\lambda, \mathfrak{H}_\lambda)$ is reducible.*

In order to prove this theorem we need some notation and a lemma. Letting $\mathfrak{g}_i'$ be as in 2.20, put $\mathfrak{k}_i'=\mathfrak{g}_i' \cap \mathfrak{k}$, $\mathfrak{p}_i'=\mathfrak{g}_i' \cap \mathfrak{p}$. Then $\mathfrak{g}_i'=\mathfrak{k}_i'+\mathfrak{p}_i'$ is a Cartan decomposition and we have a corresponding decomposition of the complexification $\mathfrak{g}_{i,c}'=\mathfrak{k}_{i,c}'+\mathfrak{p}_{i,c}'$. Let K_i' be the analytic subgroup of G_i' with Lie algebra $\mathfrak{k}_i'$ and let $K_{M_i}=K \cap M_i$. Then, since $K_i'=G_i' \cap K$, (2.26) implies $K_{M_i}=F_iI_iK_iK_i'$.

5.14. Lemma. *Let B_i be the analytic subgroup of T with Lie algebra $\sqrt{-1}\boldsymbol{R}(H_1+\cdots+H_i)$. Then B_i is isomorphic to a circle group. Furthermore B_i commutes with K_{M_i}.*

Proof. As is easily seen, $\mathrm{Ad}(\exp(2\pi\sqrt{-1}(H_1+\cdots+H_i)))$ operates on $\mathfrak{g}_c$ as the identity, whence the first assertion follows. As for the second assertion, it suffices to show that B_i commutes with K_i' since B_i commutes with $F_iI_iK_i$. For this it will be enough to show that $H_1+\cdots+H_i$ commutes with $\mathfrak{k}_{i,c}'$. Since $\mathfrak{g}_{i,c}'=\mathrm{Ad}(c_i)'\mathfrak{g}_{i,c}$ by definition, $\mathfrak{g}_{i,c}'$ is spanned by vectors $\mathrm{Ad}(c_i)X_\alpha$, $\mathrm{Ad}(c_i)H_\alpha$ with $\alpha \in {}'\Phi_i = \pm{}'C_0^{(i)} \cup \bigcup_{1\le j<k\le i}(\pm C_{jk})$; cf. 2.20. If $\alpha \in \pm{}'C_0^{(i)}$ then $\mathrm{Ad}(c_i)X_\alpha$

$=X_\alpha$, $\mathrm{Ad}(c_i)H_\alpha=H_\alpha$ because every root in C_0 is strongly orthogonal to all γ_j $(1\le j\le r)$; cf. Moore [23]. If α is a root in $'\Phi_i$ with $\pi(\alpha)=\frac{1}{2}(\gamma_j-\gamma_k)$ $(1\le j,\ k\le i,\ j\ne k)$, then using the formula $[X_{-\alpha},[X_\alpha,X_\beta]]=q(1-p)X_\beta$ on p. 143 of [11] (p, q are integers so that $\beta+n\alpha$, $p\le n\le q$, is the α-series containing β) we obtain from (2.18) that

$$\mathrm{Ad}(c_i)X_\alpha=\frac{1}{2}(X_\alpha-[X_k,[X_{-j},X_\alpha]]+[X_{-j},X_\alpha]-[X_k,X_\alpha]),$$

$$\mathrm{Ad}(c_i)H_\alpha=H_\alpha-\frac{1}{2}(\langle\gamma_j,H_\alpha\rangle H_j+\langle\gamma_k,H_\alpha\rangle H_k)$$
$$+\frac{1}{2}\langle\gamma_j,H_\alpha\rangle(X_j+X_{-j})+\frac{1}{2}\langle\gamma_k,H_\alpha\rangle(X_k+X_{-k}).$$

Here $X_\alpha-[X_k,[X_{-j},X_\alpha]]$, $H_\alpha-\frac{1}{2}(\langle\gamma_j,H_\alpha\rangle H_j+\langle\gamma_k,H_\alpha\rangle H_k)\in\mathfrak{k}'_{i,c}$, and $[X_{-j},X_\alpha]-[X_k,X_\alpha]$, $\langle\gamma_j,H_\alpha\rangle(X_j+X_{-j})+\langle\gamma_k,H_\alpha\rangle(X_k+X_{-k})\in\mathfrak{p}'_{i,c}$. Thus, and since $\mathfrak{g}'_{i,c}=\mathfrak{k}'_{i,c}+\mathfrak{p}'_{i,c}$ (vector space direct sum), $\mathfrak{k}'_{i,c}$ is spanned by vectors

$$X_\alpha,\ H_\alpha\qquad\text{with}\quad \alpha\in\pm'C_0^{(i)},$$

$$\left.\begin{array}{l}X_\alpha-[X_k,[X_{-j},X_\alpha]]\\ H_\alpha-\frac{1}{2}(\langle\gamma_j,H_\alpha\rangle H_j+\langle\gamma_k,H_\alpha\rangle H_k)\end{array}\right\}\ \text{with } \pi(\alpha)=\frac{1}{2}(\gamma_j-\gamma_k),\ 1\le j,k\le i,\ j\ne k.$$

As $H_1+\cdots+H_i$ clearly commutes with these vectors, we conclude that $H_1+\cdots+H_i$ commutes with $\mathfrak{k}'_{i,c}$. This completes the proof of the lemma. ∎

PROOF OF THEOREM 5.13. In order to prove that $(T_\lambda, H^2(\mathscr{D},\lambda))$ is unitarily equivalent to a subrepresentation of $(V_\lambda,\mathfrak{H}_\lambda)$, it is enough, in view of Proposition 5.9, to show that the range of the mapping $\mathscr{I}_\lambda$ in (4.37) is contained in $L^2(G,\sigma_\lambda;\mathfrak{p}_i^-)$. But this follows from Lemma 4.20 and the definition of $L^2(G,\sigma_\lambda;\mathfrak{p}_i^-)$.

We have yet to show that $(T_\lambda, H^2(\mathscr{D},\lambda))$ is not unitarily equivalent to $(V_\lambda,\mathfrak{H}_\lambda)$. Let B_i be as in Lemma 5.14. The argument that follows is adapted from Knapp and Okamoto [15] who study the case $i=1$, and consists of examining the restrictions to B_i of T_λ and V_λ to see that they are different. Since B_i is isomorphic to a circle group, we can think of its character group as the integers. In an obvious sense, the integers extend in two directions from 0. Hence the proof of the theorem will be complete if we verify the following two statements.

(5.15) The restriction $V_\lambda|_{B_i}$ contains infinitely many characters of B_i in both directions with positive multiplicity.

(5.16) In positive direction, the restriction $T_\lambda|_{B_i}$ contains no character of B_i.

The proofs of above statements are practically identical with the proofs of Lemmas 7.2 and 7.3 in [15], so we will just sketch their main outlines.

Let $K_{M_i}=K\cap M_i$ as before. Then, since $G=KM_iA_iN_i$ and $K\cap M_iA_iN_i=K_{M_i}$, one can prove (5.15) by the same argument as the one in the proof of Lemma 7.2 of [15], if we can show that

(5.17) $B_iK_{M_i}$ is a compact group and $B_i\cap K_{M_i}$ is a finite cyclic group.

But the first assertion of (5.17) is an immediate consequence of Lemma 5.14. As for the second assertion of (5.17), one only needs to note that $H_1+\cdots+H_i$ does not commute with $\mathfrak{a}_i$ and hence that the Lie algebra of $B_i\cap K_{M_i}$ is 0. This verifies (5.17) and hence (5.15) follows.

Now we turn to the verification of (5.16). First observe that

$$\langle\alpha, H_1+\cdots+H_i\rangle\geq 0 \quad \text{for every positive root} \quad \alpha; \tag{5.18}$$

this follows from (2.12a). Thus using Lemma 4.52 and (5.18) one can show, just as in the proof of Lemma 7.3 of [15], that the eigenvalues of $T_\lambda(H_1+\cdots+H_i)$ are all$\leq\langle\lambda, H_1+\cdots+H_i\rangle$ on a dense subspace of $H^2(\mathscr{D}, \lambda)$. Since $\langle\lambda, H_1+\cdots+H_i\rangle=-\langle\delta, H_1+\cdots+H_i\rangle=-ip_i<0$ by Proposition 3.10 and (3.14), and since $B_i=\exp\sqrt{-1}\boldsymbol{R}(H_1+\cdots+H_i)$, this implies (5.16) and completes the proof of the theorem. ∎

6. Kernel functions

Retain the setup of Sections 3 and 4. According to Corollary 4.39, every constant function $\mathbf{1}_{\boldsymbol{e}}$, $\boldsymbol{e}\in E_\lambda$, lies in $H^2(\mathscr{D}, \lambda)$. Hence, and because of (4.33b), for each point $z\in\mathscr{D}$, the point evaluation

$$E_z\colon F\longrightarrow F(z), \quad F\in H^2(\mathscr{D}, \lambda)$$

is a continuous linear mapping from $H^2(\mathscr{D}, \lambda)$ *onto* E_λ. Therefore E_z has the continuous non-singular adjoint $E_z^*\colon E_\lambda\to H^2(\mathscr{D}, \lambda)$ such that

$$(F(z), \boldsymbol{e})_{E_\lambda}=(F, E_z^*\boldsymbol{e})_{H^2(\mathscr{D},\lambda)} \tag{6.1a}$$

for all $F\in H^2(\mathscr{D}, \lambda)$ and $\boldsymbol{e}\in E_\lambda$. We define the function $K_\lambda\colon\mathscr{D}\times\mathscr{D}\to GL(E_\lambda)$ by

$$K_\lambda(z, w)=E_zE_w^*, \quad z, w\in\mathscr{D}. \tag{6.2}$$

Then the formula (6.1a) is rewritten as

$$(F(z), \boldsymbol{e})_{E_\lambda}=(F(\cdot), K_\lambda(\cdot, z)\boldsymbol{e})_{H^2(\mathscr{D},\lambda)}. \tag{6.1b}$$

K_λ will be called the reproducing kernel function of $H^2(\mathscr{D}, \lambda)$. (For the general theory of operator valued kernel functions and the connection with unitary representations, see Kunze [20].) It is obvious that

(6.3a) $\quad K_\lambda(z, w)$ is holomorphic in z and $K_\lambda(w, z) = K_\lambda(z, w)^*$

where $K_\lambda(z, w)^*$ denotes the Hilbert space adjoint of $K_\lambda(z, w)$. Since T_λ given by (4.13) defines a unitary representation of G on $H^2(\mathscr{D}, \lambda)$, one also obtains

$$K_\lambda(g \cdot z, g \cdot w) = J_\lambda(g, z) K_\lambda(z, w) J_\lambda^*(g, w) \tag{6.3b}$$

for all $z, w \in \mathscr{D}$ and all $g \in G$ ($J_\lambda^*(g, w)$=adjoint of $J_\lambda(g, w)$). In fact, since $E_{g \cdot z} = J_\lambda(g, z) E_z T_\lambda(g^{-1})$ by definition of T_λ, we have

$$\begin{aligned} K_\lambda(g \cdot z, g \cdot w) &= E_{g \cdot z} E_{g \cdot w}^* \\ &= J_\lambda(g, z) E_z T_\lambda(g^{-1}) T_\lambda(g^{-1})^* E_w^* J_\lambda^*(g, w) \\ &= J_\lambda(g, z) E_z E_w^* J_\lambda^*(g, w) \qquad (T_\lambda(g^{-1}) \text{ is unitary}) \\ &= J_\lambda(g, z) K_\lambda(z, w) J_\lambda^*(g, w). \end{aligned}$$

The following proposition gives a formula for the kernel function K_λ in terms of the automorphic factor J_λ. Recall that $\mathscr{D}$ is realized as a bounded domain in $\mathfrak{p}^+$; thus if we let $w \to \bar{w}$ denote the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$, then for $w \in \mathscr{D}$, $\exp \bar{w}$ makes sense and lies in P^-.

6.4. PROPOSITION. *The reproducing kernel function K_λ of $H^2(\mathscr{D}, \lambda)$ is given by*

$$K_\lambda(z, w) = \|\mathbf{1}_\lambda\|_\lambda^{-2} J_\lambda(\exp(-\bar{w}), z)^{-1} \tag{6.5}$$

for $z, w \in \mathscr{D}$, where $\|\mathbf{1}_\lambda\|_\lambda$ is as in Corollary 4.27.

It is not difficult to show that the function K_λ given by (6.5) satisfies (6.3a) and (6.3b). Then we may prove Proposition 6.4 by showing, as in Satake [26, Proposition 2, p. 86], that the function K_λ: $\mathscr{D} \times \mathscr{D} \to GL(E_\lambda)$ satisfying (6.3a) and (6.3b) is unique up to constant factors. Here, however, we derive the formula (6.5) directly from (6.3a) and (6.3b) (some intermediate steps in the proof will be needed later).

Our proof of Proposition 6.4 rests on the following lemmas.

6.6. LEMMA. *For all $z \in \mathscr{D}$,*

$$K_\lambda(z, o) = K_\lambda(o, z) = \|\mathbf{1}_\lambda\|_\lambda^{-2} I$$

where I is the identity transformation of E_λ.

PROOF. If we take $g = k \in K$, $z = w = o$ in the formula (6.3b), then

$$K_\lambda(o, o) = \tau_\lambda(k)K_\lambda(o, o)\tau_\lambda(k)^* \quad \text{and hence} \quad K_\lambda(o, o)\tau_\lambda(k) = \tau_\lambda(k)K_\lambda(o, o)$$

for all $k \in K$. As τ_λ is irreducible, bearing (6.3a) in mind one finds that

$$K_\lambda(o, o) = cI \quad \text{with} \quad c \in \boldsymbol{R}. \tag{6.7}$$

Similarly we get

$$K_\lambda(z, o) = \tau_\lambda(k)^{-1}K_\lambda(k \cdot z, o)\tau_\lambda(k)$$

for all $z \in \mathscr{D}$, $k \in K$. Hence, using the same technique as in the proof of Lemma 4.29, we have, for all $z \in \mathscr{D}$,

$$\begin{aligned}
K_\lambda(z, o) &= \int_K \tau_\lambda(k)^{-1}K_\lambda(k \cdot z, o)\tau_\lambda(k)dk \\
&= \int_K \tau_\lambda(tk)^{-1}K_\lambda(tk \cdot z, o)\tau_\lambda(tk)dk \qquad \text{for all} \quad t \in T^1 \\
&\qquad\qquad\qquad (T^1 \text{ is as in the proof of Lemma 4.29}) \\
&= \int_K \tau_\lambda(k)^{-1}K_\lambda(tk \cdot z, o)\tau_\lambda(k)dk \qquad \text{for all} \quad t \in T^1 \\
&\qquad\qquad\qquad (\text{since } T^1 \text{ is a central subgroup of } K) \\
&= \int_{T^1 \times K} \tau_\lambda(k)^{-1}K_\lambda(tk \cdot z, o)\tau_\lambda(k)dtdk \\
&= \int_K \tau_\lambda(k)^{-1}K_\lambda(o, o)\tau_\lambda(k)dk \qquad (K_\lambda(z, w) \text{ is holomorphic in } z) \\
&= cI \qquad\qquad (\text{by } (6.7)).
\end{aligned}$$

As $c \in \boldsymbol{R}$, it then follows from (6.3a) that

$$K_\lambda(z, o) = K_\lambda(o, z) = cI. \tag{6.8}$$

It remains to show that $c = \|\mathbf{1}_\lambda\|_\lambda^{-2}$. For this, take $\boldsymbol{e} = \boldsymbol{e}_\lambda$, $F = \mathbf{1}_\lambda$, $z = o$ in (6.1b) to get

$$\begin{aligned}
1 &= (\mathbf{1}_\lambda(\cdot), K_\lambda(\cdot, o)\boldsymbol{e}_\lambda) \\
&= (\mathbf{1}_\lambda(\cdot), c\mathbf{1}_\lambda(\cdot)) \qquad (\text{by } (6.8)) \\
&= c\|\mathbf{1}_\lambda\|_\lambda^2.
\end{aligned}$$

Thus $c = \|\mathbf{1}_\lambda\|_\lambda^{-2}$, as desired. ∎

The complex conjugation in $\mathfrak{g}_c$ with respect to $\mathfrak{g}$ lifts to an involutive automorphism of the underlying real Lie group of G_c, which we shall denote by $g \to \sigma(g)$. In the next lemma it is convenient to denote the anti-automorphism

$g \to g^{-1}$ of G_c by $g \to \iota(g)$. Recall (2.2) that any element $g \in \Omega = P^+K_cP^-$ can be written in a unique way as

$$g = \pi_+(g) \cdot \pi_0(g) \cdot \pi_-(g), \quad \pi_0(g) \in K_c, \ \pi_\pm(g) \in P^\pm.$$

6.9. LEMMA. *If $g \in \Omega = P^+K_cP^-$, then $\iota\sigma(g) \in \Omega$ and*

$$\iota\sigma\pi_+(g) = \pi_-\iota\sigma(g), \quad \iota\sigma\pi_0(g) = \pi_0\iota\sigma(g), \quad \iota\sigma\pi_-(g) = \pi_+\iota\sigma(g).$$

PROOF. If $g \in \Omega$, then

$$\begin{aligned} \iota\sigma(g) &= \iota\sigma(\pi_+(g) \cdot \pi_0(g) \cdot \pi_-(g)) \\ &= \iota\sigma\pi_-(g) \cdot \iota\sigma\pi_0(g) \cdot \iota\sigma\pi_+(g). \end{aligned}$$

Since $\sigma(P^\pm) = P^\mp$, $\sigma(K_c) = K_c$, and since $P^\pm$, K_c are groups, it follows that $\iota\sigma\pi_\pm(g) \in P^\mp$, $\iota\sigma\pi_0(g) \in K_c$ and hence that $\iota\sigma(g) \in \Omega$. Then the lemma follows from the uniqueness of the factorization

$$\iota\sigma(g) = \pi_+\iota\sigma(g) \cdot \pi_0\iota\sigma(g) \cdot \pi_-\iota\sigma(g). \qquad \blacksquare$$

6.10. LEMMA. *Let τ be a holomorphic representation of K_c on a finite dimensional Hilbert space E and suppose that τ is unitary on K. Then for $k \in K_c$*

$$\tau(k)^* = \tau(\sigma(k))^{-1}$$

where $\tau(k)^*$ is the Hilbert space adjoint of $\tau(k)$.

PROOF. Let $\dot{\tau}$ denote the corresponding representation of $\mathfrak{k}_c$ on E. Given $Z \in \mathfrak{k}_c$, write $Z = X + \sqrt{-1}Y$ with $X, Y \in \mathfrak{k}$. Then

$$\begin{aligned} \dot{\tau}(Z)^* &= (\dot{\tau}(X) + \sqrt{-1}\dot{\tau}(Y))^* \\ &= \dot{\tau}(X)^* - \sqrt{-1}\dot{\tau}(Y)^* \\ &= -\dot{\tau}(X) + \sqrt{-1}\dot{\tau}(Y) \qquad (\dot{\tau} \text{ is skew-adjoint on } \mathfrak{k}) \\ &= \dot{\tau}(-\bar{Z}), \end{aligned}$$

from which the lemma follows. $\blacksquare$

PROOF OF PROPOSITION 6.4. Fix $z, w \in \mathscr{D}$ and choose $g \in G$ such that $g \cdot w = o$. Then, using the formula (6.3b) and Lemma 6.6, one obtains

$$(6.11) \qquad \begin{aligned} K_\lambda(z, w) &= \|\mathbf{1}_\lambda\|_\lambda^{-2} J_\lambda(g, z)^{-1} J_\lambda^*(g, g^{-1} \cdot o)^{-1} \\ &= \|\mathbf{1}_\lambda\|_\lambda^{-2} J_\lambda(g, z)^{-1} J_\lambda^*(g^{-1}, o). \end{aligned}$$

Let $g=\pi_+(g)\cdot\pi_0(g)\cdot\pi_-(g)$ be the factorization as in (2.2). Then, since $g\exp z \in\Omega=P^+K_cP^-$, it is clear that both $\pi_-(g)\exp z$ and $\pi_0(g)\pi_-(g)\exp z$ are in Ω. Therefore, by the remark after (2.6) together with (2.7)

$$\begin{aligned}&J_\lambda(g, z)^{-1}\\ &\quad= J_\lambda(\pi_-(g), z)^{-1}J_\lambda(\pi_0(g), \pi_-(g)\cdot z)^{-1}J_\lambda(\pi_+(g), \pi_0(g)\pi_-(g)\cdot z)^{-1}\\ &\quad= J_\lambda(\pi_-(g), z)^{-1}\tau_\lambda(\pi_0(g))^{-1}.\end{aligned}$$

On the other hand, using Lemmas 6.9, 6.10, and noting that $\sigma(g)=g$, one finds

$$\begin{aligned}J_\lambda^*(g^{-1}, o) &= \tau_\lambda(\pi_0(g^{-1}))^* = \tau_\lambda(\pi_0\iota\sigma(g))^*\\ &= \tau_\lambda(\iota\sigma\pi_0(g))^* = \tau_\lambda(\pi_0(g)).\end{aligned}$$

Thus we get

$$K_\lambda(z, w) = \|\mathbf{1}_\lambda\|_\lambda^{-2}J_\lambda(\pi_-(g), z)^{-1}.$$

But, again by Lemma 6.9,

$$\pi_-(g) = \pi_-\iota\sigma(g^{-1}) = \iota\sigma\pi_+(g^{-1}),$$

while $\pi_+(g^{-1})=\exp w$ since $w=g^{-1}\cdot o$ (cf. (2.3)); therefore $\pi_-(g)=\exp(-\bar{w})$. Hence the proposition is proved. ∎

In view of (6.2), (6.11) and the remark after (2.6), for each fixed $w\in\mathscr{D}$ and $\boldsymbol{e}\in E_\lambda$,

(6.12) $\begin{cases}\text{the function } z\to K_\lambda(z, w) \text{ is holomorphic on } \bar{\mathscr{D}} \text{ and the function } z\to\\ K_\lambda(z, w)\boldsymbol{e} \text{ belongs to } \mathcal{O}^2(\bar{\mathscr{D}}, E_\lambda)\end{cases}$

where $\mathcal{O}^2(\bar{\mathscr{D}}, E_\lambda)$ is as in (4.16). Thus if $F\in\mathcal{O}^2(\bar{\mathscr{D}}, E_\lambda)$, we see from (6.1b) and (4.36) that

$$\begin{aligned}&(F(z), \boldsymbol{e})\\ (6.13)\quad &=\int_{K\times G_i}(P_\lambda^* J_\lambda(kg_ic_i, o)^{-1}F(kg_i\cdot o_i), P_\lambda^* J_\lambda(kg_ic_i, o)^{-1}K_\lambda(kg_i\cdot o_i, z)\boldsymbol{e})dkdg_i\\ &=\left(\int_{K\times G_i}K_\lambda(z, kg_i\cdot o_i)J_\lambda^*(kg_ic_i, o)^{-1}P_\lambda^* J_\lambda(kg_ic_i, o)^{-1}F(kg_i\cdot o_i)dkdg_i, \boldsymbol{e}\right)\end{aligned}$$

for all $\boldsymbol{e}\in E_\lambda$. Define a function $M_\lambda\colon\mathscr{B}_i\to\mathrm{End}(E_\lambda)$ as follows. Given $u\in\mathscr{B}_i$, there exist $k\in K$ and $g_i\in G_i$ such that $u=kg_i\cdot o_i$; then put

$$(6.14)\qquad M_\lambda(u) = J_\lambda^*(kg_ic_i, o)^{-1}P_\lambda^* J_\lambda(kg_ic_i, o)^{-1}.$$

(It is easily checked that M_λ is a well defined function on $\mathscr{B}_i$.) Then, from

(6.13), (6.14) and (2.47) we obtain

$$(6.15)\qquad F(z) = \int_{\mathscr{B}_i} K_\lambda(z, u) M_\lambda(u) F(u) d\mu(u) \qquad \text{for all} \quad F \in \mathcal{O}^2(\bar{\mathscr{D}}, E_\lambda)$$

where $d\mu$ is the quasi-invariant measure on $\mathscr{B}_i$ defined by the formula (2.46).

In the remainder of this section we specialize to the case where τ_λ ($\lambda \in \mathscr{F}_i(G)$) is a one dimensional representation of K, and want to find more explicit formula for the kernel function of $H^2(\mathscr{D}, \lambda)$ than the one given by (6.5). It turns out that this kernel function is proportional to a positive power of the Bergman kernel function of $\mathscr{D}$.

Before proceeding further we note that if we write, for a given $\lambda \in \mathscr{F}_i(G)$, $\lambda = \sum_{j=1}^{l'_i} m_j \lambda_j$ with $m_1 + \sum_{j=2}^{l'_i} k_j m_j = -p_i$ as in (3.8a), then the degree $d(\lambda)$ of τ_λ is 1 just when $m_j = 0$ for $2 \leq j \leq l'_i$. Thus if $\mathscr{F}_i \subset \mathscr{L}(G)$ and hence $\mathscr{F}_i(G) = \mathscr{F}_i$ (see (3.7) and (3.8) for notation), then there exists a unique $\omega_i \in \mathscr{F}_i(G)$ with $d(\omega_i) = 1$, i.e.,

$$(6.16)\qquad \omega_i = -p_i \lambda_1.$$

We also note that

(6.17) $\begin{cases} H^2(\mathscr{D}, \omega_r) \text{ is the usual Hardy space for the bounded symmetric domain} \\ \mathscr{D} = G/K \text{ (provided } \omega_r \in \mathscr{L}(G)). \end{cases}$

Indeed, if we identify the representation space E_{ω_r} of τ_{ω_r} with complex numbers and if we let $\mathcal{O}(\bar{\mathscr{D}})$ denote the space of all holomorphic functions on $\bar{\mathscr{D}}$, then recalling that $G_r = K_r \subset K$ (cf. (2.29)), we have for all $f \in \mathcal{O}(\bar{\mathscr{D}})$,

$$\begin{aligned} \|f\|^2_{\omega_r} &= \int_{K \times G_r} |J_{\omega_r}(kg_r c_r, o)^{-1} f(kg_r \cdot o_r)|^2 dk dg_r \\ &= \int_K |J_{\omega_r}(kc_r, o)^{-1} f(k \cdot o_r)|^2 dk \\ &= \beta_r^2 \int_K |\tau_{\omega_r}(k)^{-1} f(k \cdot o_r)|^2 dk \qquad \text{(by Lemma 4.25)} \\ &= \beta_r^2 \int_K |f(k \cdot o_r)|^2 dk. \end{aligned}$$

Since this is just a Hardy type norm for $\mathcal{O}(\bar{\mathscr{D}})$, (6.17) follows. Therefore the kernel function of $H^2(\mathscr{D}, \omega_r)$ is the Cauchy-Szegö kernel function of the bounded symmetric domain $\mathscr{D}$. But under the assumption that G is a matrix group, it may happen that $\mathscr{F}_r(G) = \emptyset$ as remarked at the end of Section 3; thus in order to treat the case of arbitrary irreducible $\mathscr{D} = G/K$, one must drop the assumption that G is linear. But we assume for the moment that G is a matrix group and that $\omega_i \in \mathscr{S}(G)$ for all $1 \leq i \leq r$; we shall indicate later how to drop this assumption.

We shall denote by $\boldsymbol{k}_i$ the kernel function of $H^2(\mathscr{D}, \omega_i)$. To make the formula (6.5) more explicit in the present situation, letting δ_n be half the sum of positive noncompact roots of $(\mathfrak{g}_c, \mathfrak{t}_c)$, we define a particular one dimensional representation $\tau_{2\delta_n}$ of K_c by

$$\tau_{2\delta_n}(k) = \det(\mathrm{Ad}(k)|_{\mathfrak{p}^+}), \quad k \in K_c$$

(thus $2\delta_n$ is the weight of $\tau_{2\delta_n}$). We also define a function $\boldsymbol{k}$: $\mathscr{D} \times \mathscr{D} \to \boldsymbol{C}$ by

$$\boldsymbol{k}(z, w) = J_{2\delta_n}(\exp(-\bar{w}), z). \tag{6.18}$$

We first show that this function $\boldsymbol{k}$ is, up to a constant factor, the Bergman kernel function of $\mathscr{D}$. We begin by recalling the definition of Bergman kernel function; see Helgason [11]. Let

$$\mathscr{O}^2(\mathscr{D}) = \{\text{holomorphic functions } f\colon \mathscr{D} \to \boldsymbol{C};\ \|f\|^2 = \int_{\mathscr{D}} |f(z)|^2 dz < \infty\}$$

where dz denotes the Euclidean measure on $\mathfrak{p}^+$. Then $\mathscr{O}^2(\mathscr{D})$ is a complete Hilbert space and for each $z \in \mathscr{D}$ the point evaluation $f \to f(z)$ is a bounded linear functional on $\mathscr{O}^2(\mathscr{D})$; so there exists a unique reproducing kernel function of $\mathscr{O}^2(\mathscr{D})$. This function is by definition the Bergman kernel of $\mathscr{D}$. We shall denote this kernel function by $\boldsymbol{b}$.

The following formula for the Bergman kernel function is perhaps known, but we include a proof, as it is not readily available in the literature.

6.19. PROPOSITION. *Let $\boldsymbol{k}$ by the function defined by* (6.18). *Then the Bergman kernel function $\boldsymbol{b}$ of $\mathscr{D}$ is given by*

$$\boldsymbol{b}(z, w) = \mathrm{vol}(\mathscr{D})^{-1}\boldsymbol{k}(z, w).$$

PROOF. From the general theory of Bergman kernel, one knows that $\boldsymbol{b}(z, w)$ is holomorphic in z, $\boldsymbol{b}(w, z) = \overline{\boldsymbol{b}(z, w)}$ for all $z, w \in \mathscr{D}$, and that $\boldsymbol{b}$ satisfies

$$\boldsymbol{b}(g \cdot z, g \cdot w) = j(g, z)^{-1}\boldsymbol{b}(z, w)\overline{j(g, w)^{-1}}$$

for all $z, w \in \mathscr{D}$, $g \in G$ where $j(g, z)$ denotes the complex Jacobian of the holomorphic map $z \to g \cdot z$ at $z \in \mathscr{D}$. In the present situation it is also known (cf. Baily and Borel [1], Lemma 1.9) that

$$j(g, z) = J_{2\delta_n}(g, z).$$

Therefore by the same argument as in the proof of Proposition 6.4, we obtain

$$\boldsymbol{b}(z, w) = \|\mathbf{1}\|^{-2} J_{2\delta_n}(\exp(-\bar{w}), z)$$

where $\mathbf{1}$ denotes the constant function $\mathbf{1}(z) = 1$, $z \in \mathscr{D}$. But clearly $\|\mathbf{1}\|^2 =$

vol($\mathscr{D}$), finishing the proof. ∎

Now we turn to the consideration of the kernel function $\boldsymbol{k}_i$. Letting $n=\dim_{\boldsymbol{C}}\mathscr{D}$, $n_i=\dim_{\boldsymbol{C}}\mathscr{C}_i$ and $d_i=\dim_{\boldsymbol{R}}\mathscr{B}_i$, we set

$$q_i = \frac{n - n_i}{3n - n_i - d_i} \tag{6.20}$$

where $\mathscr{C}_i$ and $\mathscr{B}_i$ are, respectively, the boundary component and the boundary orbit containing o_i. (It can be shown that $\frac{1}{2}\leq q_i<1$; cf. Remark after Lemma 6.24 below.) As $\mathscr{D}\times\mathscr{D}$ is simply connected, we can uniquely define powers $\boldsymbol{k}(z, w)^{q_i}$ of $\boldsymbol{k}(z, w)$ with $\boldsymbol{k}(o, o)^{q_i}=1$.

Using the notation of Corollary 4.27, put

$$\kappa_i = \begin{cases} \beta_i^2\int_{A^+_{(i)}} \tau_{\omega_i}(\pi_0(a)^{-2})D_i(a)da & \text{if } \ i \neq r \\ \beta_r^2 & \text{if } \ i = r. \end{cases} \tag{6.21}$$

Then, by the same corollary, $\kappa_i=\|\mathbf{1}_{\omega_i}\|^2_{\omega_i}$ since $d(\omega_i)=1$. Thus $0<\kappa_i<\infty$ by (4.28) and Proposition 4.38. Before stating the next proposition, let us notice that $\boldsymbol{k}_i(z, w)$, being holomorphic in z and anti-holomorphic in w, is completely determined by its restriction to the diagonal of $\mathscr{D}\times\mathscr{D}$. Note also that every point in $\mathscr{D}$ can be written in the form $k\cdot(\sum_{j=1}^r t_jX_j)=\mathrm{Ad}\,(k)(\sum_{j=1}^r t_jX_j)$ with $k\in K$, $-1<t_j<1$; cf. Korányi and Wolf [18], p. 269.

6.22. PROPOSITION. *Let $\boldsymbol{k}$ be the function defined by* (6.18) *and let q_i, κ_i, p_i be the constants as in* (6.20), (6.21), (3.5). *Then the reproducing kernel function $\boldsymbol{k}_i$ of $H^2(\mathscr{D}, \omega_i)$ is given by*

$$\boldsymbol{k}_i(z, w) = \kappa_i^{-1}\boldsymbol{k}(z, w)^{q_i} \tag{6.23a}$$

for all $z, w\in\mathscr{D}$. Moreover, if $z=k\cdot(\sum_{j=1}^r t_jX_j)$ with $k\in K$, $-1<t_j<1$, then

$$\boldsymbol{k}_i(z, z) = \kappa_i^{-1}\prod_{1\leq j\leq r}(1 - t_j^2)^{-p_i}. \tag{6.23b}$$

NOTES. (1) The constant p_i in the formula (6.23b) (which is an integer or a half-integer by definition (3.5)) can also be written as $p_i=\frac{3n-d_r}{r}q_i=\frac{(3n-d_r)\,(n-n_i)}{r(3n-n_i-d_i)}$ (see Lemma 6.24 below), in particular $p_r=\frac{n}{r}$ since $n_r=0$.

(2) In the extreme case $i=r$, $\boldsymbol{k}_r$ is the Cauchy-Szegö kernel function of $\mathscr{D}$ as remarked before. The formula (6.23b) for $\boldsymbol{k}_r$ was obtained by Korányi [17, Proposition 5.7] using different methods (note that the constant κ_i depends on the normalization of measures).

The proof of Proposition 6.22 rests on the following lemmas. Recall the set of fundamental highest weights, $\{\lambda_1,\ldots,\lambda_l\}$, in Section 3.

6.24. LEMMA. *Letting u, v be the constants as in* (2.12b), *put* $s=u(r-1)+v+2$. *Then*:

(1) $2\delta_n=s\lambda_1$.

(2) *For each i, $1\leq i\leq r$, we have $p_i=sq_i$. Moreover $s=\frac{3n-d_r}{r}$, where $n=\dim_{\boldsymbol{C}}\mathscr{D}$ and $d_r=\dim_{\boldsymbol{R}}\mathscr{B}_r$ ($\mathscr{B}_r$ = Silov boundary of $\mathscr{D}$).*

PROOF. Since $\mathfrak{k}_c$ normalizes $\mathfrak{p}^+$, its Weyl group permutes the elements of Φ_n^+ and leaves $2\delta_n$ invariant. Therefore $(2\delta_n,\alpha)=0$ for all $\alpha\in\Phi_c$, and it follows from the definition of λ_1 that $2\delta_n=t\lambda_1$ for some $t\in\boldsymbol{R}$. We must show that $t=u(r-1)+v+2$. For this we will calculate the effects of $2\delta_n$ and λ_1 to the vector H_1. According to (2.12ab),

$$\begin{aligned}2\pi(\delta_n) &= \sum_{1\leq j\leq r}\gamma_j + u\sum_{1\leq j<k\leq r}\frac{1}{2}(\gamma_j+\gamma_k) + v\sum_{1\leq j\leq r}\frac{1}{2}\gamma_j \\ &= \left(1+\frac{1}{2}u(r-1)+\frac{1}{2}v\right)\sum_{1\leq j\leq r}\gamma_j,\end{aligned} \tag{6.25}$$

so we get

$$\langle 2\delta_n, H_1\rangle = u(r-1)+v+2.$$

On the other hand $\langle\lambda_1, H_1\rangle=1$ by Lemma 3.3(1), whence (1) follows.

(2) By definition, $p_i=\frac{1}{2}u(i-1)+u(r-i)+v+1$ and $q_i=\frac{n-n_i}{3n-n_i-d_i}$. On the other hand, using (2.12ab), one finds that

$$n=\dim_{\boldsymbol{C}}\mathfrak{p}^+=|\Phi_n^+|=\frac{1}{2}r\{u(r-1)+2v+2\},$$

$$n_i=\dim_{\boldsymbol{C}}\mathfrak{p}_i^+=|\Phi_{i,n}^+|=\frac{1}{2}(r-i)\{u(r-i-1)+2v+2\},$$

and

$$\begin{aligned}d_i &= \dim_{\boldsymbol{R}} G/S_i = \dim_{\boldsymbol{R}} G/P_i + \dim_{\boldsymbol{R}} P_i/S_i \\ &= \dim_{\boldsymbol{R}}\mathfrak{n}_i + \dim_{\boldsymbol{R}}\mathscr{C}_i \qquad (\mathfrak{n}_i \text{ is as in (2.24a)}) \\ &= \frac{1}{2}i\{u(i-1)+4u(r-i)+4v+2\}+2n_i \\ &= 2n-\frac{1}{2}i\{u(i-1)+2\}.\end{aligned}$$

The assertion of (2) then follows from straightforward computation. ∎

Remark. From Lemma 6.24(2), it is easily seen that $\frac{1}{2} \leq q_i < 1$ for each $1 \leq i \leq r$.

6.26. Lemma. $\omega_i = -2q_i\delta_n$.

Proof. Since $\omega_i = -p_i\lambda_1$ by (6.16), this is an immediate consequence of Lemma 6.24. ∎

Proof of Proposition 6.22. To prove the first assertion, it is enough to show that both functions coincide on the diagonal of $\mathscr{D} \times \mathscr{D}$. Fix $z \in \mathscr{D}$ and choose $g \in G$ so that $z = g \cdot o$. Let us write $J_i(g, z)$ in place of $J_{\omega_i}(g, z)$. Then, taking into account (6.3b), Lemma 6.6, and the definition (6.21) of κ_i, we get

$$\begin{aligned} \boldsymbol{k}_i(z, z) &= J_i(g, o)\boldsymbol{k}_i(o, o)\overline{J_i(g, o)} \\ &= \kappa_i^{-1}|J_i(g, o)|^2. \end{aligned}$$

Let A be the abelian subgroup of G as in the proof of Corollary 4.32. Then, since $G = KAK$, we can write $g = kak'$ with $k, k' \in K$, $a \in A$, and it follows from the cocycle formula that

$$|J_i(g, o)| = |J_i(k, a \cdot o)|\,|J_i(a, o)|\,|J_i(k', o)|.$$

But, in view of (2.19), $J_i(a, o)$ is real, while $|J_i(k, a \cdot o)| = |J_i(k', o)| = 1$, and so one finds

$$\boldsymbol{k}_i(z, z) = \kappa_i^{-1}J_i(a, o)^2.$$

Now by Lemma 6.26,

$$J_i(a, o) = J_{2\delta_n}(a, o)^{-q_i}. \tag{6.27}$$

Therefore we conclude that

$$\boldsymbol{k}_i(z, z) = \kappa_i^{-1}J_{2\delta_n}(a, o)^{-2q_i}.$$

On the other hand, since $\boldsymbol{k}(g \cdot o, g \cdot o) = J_{2\delta_n}(g, o)^{-1}\overline{J_{2\delta_n}(g, o)^{-1}}$, the same argument as above yields

$$\boldsymbol{k}(z, z)^{q_i} = J_{2\delta_n}(a, o)^{-2q_i},$$

and the first assertion follows.

Given $a \in A$, write $a = \exp(\sum_{j=1}^r x_j(X_j + X_{-j}))$; then, in view of (2.19), $a \cdot o = \sum_{j=1}^r (\tanh x_j)X_j$. Thus if $-1 < t_j < 1$, there exists a unique $a \in A$ such that $a \cdot o = \sum_{j=1}^r t_jX_j$. Moreover, as we have already observed,

$$\boldsymbol{k}_i(ka \cdot o, ka \cdot o) = \kappa_i^{-1}J_i(a, o)^2 \qquad \text{for} \quad k \in K,\ a \in A.$$

Therefore to prove the second assertion of the proposition it suffices to show that if $a=\exp(\sum_{j=1}^r x_j(X_j+X_{-j}))$, then

$$J_i(a, o) = \prod_{1\leq j\leq r} \{1-(\tanh x_j)^2\}^{-p_i/2}$$

or equivalently that

$$J_i(a, o) = \prod_{1\leq j\leq r} (\cosh x_j)^{p_i}.$$

By (6.27) and the definition of the automorphic factor, we have

$$J_i(a, o) = J_{2\delta_n}(a, o)^{-q_i} = \tau_{2\delta_n}(\pi_0(a))^{-q_i};$$

here, according to (2.19), $\pi_0(a)=\exp(-\sum_{j=1}^r \log(\cosh x_j)H_j)$. But (6.25) implies

$$\langle 2\delta_n, -\textstyle\sum_{j=1}^r \log(\cosh x_j)H_j\rangle = -s\textstyle\sum_{j=1}^r \log(\cosh x_j).$$

Thus, and since $p_i=sq_i$ by Lemma 6.24(2), we obtain

$$J_i(a, o) = \prod_{1\leq j\leq r} (\cosh x_j)^{p_i},$$

as we wished to show. ∎

EXAMPLE. To illustrate how one gets a precise formula for the kernel function $\boldsymbol{k}_i$ in concrete cases, we consider the case where $G=SU(p, q)$ $(p\geq q\geq 1)$ and $K=\left\{\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix};\ a\in U(p),\ d\in U(q),\ (\det a)(\det d)=1\right\}$. In this case rank G/K $=q$ and we have $G_c=SL(p+q, \boldsymbol{C})$, $K_c=\left\{\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix};\ a\in GL(p, \boldsymbol{C}),\ d\in GL(q, \boldsymbol{C}),\ (\det a)(\det d)=1\right\}$. If g is a $(p+q)\times(p+q)$ complex matrix, we write $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ where the matrix blocks are of the size given by

$$a \text{ is } p\times p,\quad b \text{ is } p\times q,\quad c \text{ is } q\times p,\quad d \text{ is } q\times q.$$

Then $\mathfrak{g}_c=\mathfrak{sl}(p+q, \boldsymbol{C})$, $\mathfrak{k}_c=\left\{\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix};\ \text{trace } a+\text{trace } d=0\right\}$, $\mathfrak{p}_c=\left\{\begin{pmatrix} 0 & b \\ c & 0\end{pmatrix}\right\}$, and we may put $\mathfrak{p}^+=\left\{\begin{pmatrix} 0 & b \\ 0 & 0\end{pmatrix}\right\}$, $\mathfrak{p}^-=\left\{\begin{pmatrix} 0 & 0 \\ c & 0\end{pmatrix}\right\}$; hence $P^+=\left\{\begin{pmatrix} 1_p & b \\ 0 & 1_q\end{pmatrix}\right\}$, $P^-=\left\{\begin{pmatrix} 1_p & 0 \\ c & 1_q\end{pmatrix}\right\}$. Each $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in G$ is written uniquely as

$$g = \begin{pmatrix} 1_p & bd^{-1} \\ 0 & 1_q\end{pmatrix}\begin{pmatrix} a-bd^{-1}c & 0 \\ 0 & d\end{pmatrix}\begin{pmatrix} 1_p & 0 \\ d^{-1}c & 1_q\end{pmatrix},$$

so $\zeta(g)=\begin{pmatrix} 0 & bd^{-1} \\ 0 & 0\end{pmatrix}$ where ζ is as in 2.1. It then follows (cf. Wolf [31]) that the

Harish-Chandra realization of G/K is given by

$$\mathscr{D} = \zeta(G) = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \in \mathfrak{p}^+;\ 1_q - z^*z > 0 \right\},$$

where z^* is the conjugate transpose of z and ">0" means "is positive definite". Moreover, if we identify $\mathfrak{p}^+$ with the space $M_{p,q}(\boldsymbol{C})$ of $p\times q$ complex matrices, then (cf. [31], p. 327) for each $1\leq i\leq q$

$$\mathscr{B}_i = \{z\in M_{p,q}(\boldsymbol{C});\ 1_q - z^*z \text{ is positive semidefinite and } \operatorname{rank}(1_q - z^*z) = q - i\} \tag{6.28a}$$

and we can put

$$\mathscr{C}_i = \left\{ \begin{pmatrix} 1_i & 0 \\ 0 & z \end{pmatrix} \in M_{p,q}(\boldsymbol{C});\ 1_{q-i} - z^*z > 0 \right\}. \tag{6.28b}$$

Now let $\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \in \mathscr{D}$, and put $z' = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}$, $w' = \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}$. Then, since $\bar{w}' = \begin{pmatrix} 0 & 0 \\ w^* & 0 \end{pmatrix}$ (bar denoting conjugation of $\mathfrak{sl}(p+q, \boldsymbol{C})$ with respect to $\mathfrak{su}(p, q)$),

$$\begin{aligned}
&\pi_0(\exp(-\bar{w}')\exp z') \\
&= K_c\text{-component of } \begin{pmatrix} 1_p & 0 \\ -w^* & 1_q \end{pmatrix}\begin{pmatrix} 1_p & z \\ 0 & 1_q \end{pmatrix} \text{ in the factorization (2.2)} \\
&= \begin{pmatrix} 1_p + z(1_q - w^*z)^{-1}w^* & 0 \\ 0 & 1_q - w^*z \end{pmatrix}.
\end{aligned}$$

But if $k = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in K_c$, then

$$\tau_{2\delta_n}(k) = \det(\operatorname{Ad}(k)|_{\mathfrak{p}^+}) = (\det a)^q(\det d)^{-p} = (\det d)^{-(p+q)}.$$

Thus

$$\begin{aligned}
J_{2\delta_n}(\exp(-\bar{w}'), z') &= \tau_{2\delta_n}(\pi_0(\exp(-\bar{w}')\exp z')) \\
&= \det(1_q - w^*z)^{-(p+q)}.
\end{aligned}$$

According to Proposition 6.19, this is the Bergman kernel function of $\mathscr{D}$ up to a constant factor.

Now $\dim_{\boldsymbol{C}} \mathscr{D} = pq$, while by (6.28) $\dim_{\boldsymbol{R}} \mathscr{B}_i = 2pq - i^2$ and $\dim_{\boldsymbol{C}} \mathscr{C}_i = (p-i)\cdot(q-i)$. Hence, in the present case, the constant q_i in (6.20) turns out to be equal to $\frac{p+q-i}{p+q}$. Therefore we conclude from Proposition 6.22 that, up to a constant factor, the kernel function $\boldsymbol{k}_i$ is given (under the identification $\mathscr{D}=\{z\in M_{p,q}(\boldsymbol{C});\ 1_q-z^*z>0\}$) by

$$\boldsymbol{k}_i(z, w) = \det(1_q - w^*z)^{-(p+q-i)} \qquad (1 \leq i \leq q)$$

for $z, w\in\mathscr{D}$. In the extreme case $i=q$, $\boldsymbol{k}_q$ is the Cauchy-Szegö kernel function of $\mathscr{D}$ and is given by

$$\boldsymbol{k}_q(z, w) = \det(1_q - w^*z)^{-p}. \qquad \blacksquare$$

As remarked before, in order to account for all $\boldsymbol{k}_i$ $(1\leq i\leq r)$ in the case of arbitrary irreducible $\mathscr{D}=G/K$, one must drop the requirement that G is linear; this is the case when $\omega_i=-p_i\lambda_1$ with p_i a half-integer. We shall briefly indicate that, even in this case, one can define the kernel function $\boldsymbol{k}_i$ and its explicit formula is again given by (6.23). For this purpose we take, in place of G, a two-sheeted covering G° of the linear universal covering of G; then $\omega_i\in\mathscr{F}_i(G^\circ)$ for all $1\leq i\leq r$; cf. the remark at the end of Section 3. G° naturally acts on $\bar{\mathscr{D}}$; $g\cdot z=p(g)\cdot z$ (p: $G^\circ\to G$ being the covering homomorphism). Let K°, G_i°, A_i°, N_i° denote the analytic subgroups of G° corresponding to $\mathfrak{k}$, $\mathfrak{g}_i$, $\mathfrak{a}_i$, $\mathfrak{n}_i$ respectively, and let $M_i^\circ=p^{-1}(M_i)$, $S_i^\circ=p^{-1}(S_i)$, $L_i^\circ=M_i^\circ\cap S_i^\circ$. If we define $J_{2\delta_n}^\circ$: $G^\circ\times\bar{\mathscr{D}}\to\boldsymbol{C}$ by $J_{2\delta_n}^\circ(g, z)=J_{2\delta_n}(p(g), z)$ then, since p_i is a half-integer and since $\omega_i=-p_i\lambda_1=-2q_i\delta_n$ by Lemma 6.26, we can uniquely define a continuous function J_i°: $G^\circ\times\bar{\mathscr{D}}\to\boldsymbol{C}$ by $J_i^\circ(g, z)=J_{2\delta_n}^\circ(g, z)^{-q_i}$ with $J_i^\circ(e, z)=1$ (e=identity of G°). One can check that J_i° satisfies the following conditions:

$$(6.29)\quad \begin{cases} J_i^\circ(g, z) \text{ is } C^\infty \text{ in } g\in G^\circ \text{ and holomorphic in } z\in\bar{\mathscr{D}}; \\ J_i^\circ(g_1g_2, z) = J_i^\circ(g_1, g_2\cdot z)J_i^\circ(g_2, z) \quad \text{for} \quad g_1, g_2\in G^\circ,\ z\in\bar{\mathscr{D}}; \\ J_i^\circ(k, z) = \tau_{\omega_i}(k) \quad \text{for} \quad k\in K^\circ,\ z\in\bar{\mathscr{D}}. \end{cases}$$

Now let $\mathcal{O}(\mathscr{D})$ denote the space of all holomorphic functions on $\mathscr{D}$ and $\mathcal{O}(\bar{\mathscr{D}})\subset\mathcal{O}(\mathscr{D})$ the subspace of functions holomorphic on $\bar{\mathscr{D}}$. By (6.29) we can define (in algebraic sense) a representation T_i of G° on $\mathcal{O}(\mathscr{D})$ by

$$(T_i(g)f)(z) = J_i^\circ(g^{-1}, z)^{-1}f(g^{-1}\cdot z), \quad f\in\mathcal{O}(\mathscr{D}),\ g\in G^\circ\ z\in\mathscr{D}.$$

Note that the subspace $\mathcal{O}(\bar{\mathscr{D}})$ is stable under the representation T_i. Let

$$\mathcal{O}^2(\bar{\mathscr{D}}, \omega_i) = \left\{f\in\mathcal{O}(\bar{\mathscr{D}});\ \|f\|_i^2 = \beta_i^2\int_{K^\circ\times G_i^\circ}|J_i^\circ(kg_i, o_i)^{-1}f(kg_i\cdot o_i)|^2dkdg_i < \infty\right\},$$

where β_i is as in Proposition 4.24 and dk, dg_i are Haar measures on K°, G_i°.

6.30. LEMMA. *$\mathcal{O}^2(\bar{\mathscr{D}}, \omega_i)$ is stable under $T_i(g)$ $(g \in G^\circ)$ and $T_i(g)$ preserve the seminorm $\| \;\|_i$.*

PROOF. We may define a C^∞ rho-function ρ° on G° for the subgroup S_i° in the same way as in 2.33. Thus, for $g \in G^\circ$, write $g = kman$ ($k \in K^\circ$, $m \in M_i^\circ \cap \exp \mathfrak{p}$, $a \in A_i^\circ$, $n \in N_i^\circ$); put $\rho^\circ(g) = e^{-2\rho_i}(a)$ with ρ_i as in (2.38). Then, just as in (2.45), we can normalize various Haar measures in such a way that

$$\int_{G^\circ} f(g)dg = \int_{K^\circ \times G_i^\circ \times S_i^\circ} f(kg_i s)\rho^\circ(s)^{-1} dk dg_i ds \tag{6.31}$$

for any integrable f.

Now fix $F \in \mathcal{O}^2(\bar{\mathscr{D}}, \omega_i)$ and $x \in G^\circ$. If we put $\tilde{F}(g) = J_i^\circ(g, o_i)^{-1} F(g \cdot o_i)$ for $g \in G^\circ$, then

$$\begin{aligned} &\| T_i(x)F \|_i^2 \\ &= \beta_i^2 \int_{K^\circ \times G_i^\circ} |J_i^\circ(kg_i, o_i)^{-1} J_i^\circ(x^{-1}, kg_i \cdot o_i)^{-1} F(x^{-1}kg_i \cdot o_i)|^2 dk dg_i \\ &= \beta_i^2 \int_{K^\circ \times G_i^\circ} |J_i^\circ(x^{-1}kg_i, o_i)^{-1} F(x^{-1}kg_i \cdot o_i)|^2 dk dg_i \\ &= \beta_i^2 \int_{K^\circ \times G_i^\circ} |\tilde{F}(x^{-1}kg_i)|^2 dk dg_i. \end{aligned}$$

Therefore to prove the lemma it suffices to show that

$$\int_{K^\circ \times G_i^\circ} |\tilde{F}(kg_i)|^2 dk dg_i = \int_{K^\circ \times G_i^\circ} |\tilde{F}(x^{-1}kg_i)|^2 dk dg_i \tag{6.32}$$

for all $x \in G^\circ$. First we claim:

$$|\tilde{F}(gs)|^2 = \rho^\circ(s)\, |\tilde{F}(g)|^2 \qquad \text{for all} \quad g \in G^\circ \quad \text{and} \quad s \in S_i^\circ. \tag{6.33}$$

In fact, for $g \in G^\circ$, $s \in S_i^\circ$,

$$\begin{aligned} \tilde{F}(gs) &= J_i^\circ(gs, o_i)^{-1} F(gs \cdot o_i) \\ &= J_i^\circ(s, o_i)^{-1} J_i^\circ(g, o_i)^{-1} F(g \cdot o_i) \\ &= J_i^\circ(s, o_i)^{-1} \tilde{F}(g) \end{aligned}$$

and if we write $s = lan$ with $l \in L_i^\circ$, $a \in A_i^\circ$, $n \in N_i^\circ$, then

$$J_i^\circ(s, o_i) = J_i^\circ(l, o_i) J_i^\circ(a, o_i) J_i^\circ(n, o_i).$$

But it is easy to check that $|J_i^\circ(l, o_i)| = |J_i^\circ(n, o_i)| = 1$, while, by Lemma 4.5 (2) together with the fact that $\omega_i = -2q_i\delta_n$, we have

$$J_i^\circ(a, o_i) = J_{2\delta_n}(p(a), o_i)^{-q_i} = J_{2\delta_n}(c_i^{-1}p(a)c_i, o)^{-q_i} = e^{\rho_i}(a).$$

Therefore

$$|\tilde{F}(gs)|^2 = |J_i^\circ(s, o_i)|^{-2}|\tilde{F}(g)|^2 = e^{-2\rho_i}(a)\,|\tilde{F}(g)|^2 = \rho^\circ(s)\,|\tilde{F}(g)|^2$$

for all $g \in G^\circ$ and $s \in S_i^\circ$, establishing (6.33).

Now put $\dot{\phi}(g \cdot o_i) = \int_{S_i^\circ} \phi(gs)ds$ for $\phi \in C_c(G^\circ)$ and $g \in G^\circ$. Then using (6.31) and (6.33) we get

$$\begin{aligned}
&\int_{K^\circ \times G_i^\circ} |\tilde{F}(kg_i)|^2 \dot{\phi}(kg_i \cdot o_i) dk dg_i \\
&= \int_{K^\circ \times G_i^\circ \times S_i^\circ} |\tilde{F}(kg_i s)|^2 \phi(kg_i s) \rho^\circ(s)^{-1} dk dg_i ds \\
&= \int_{G^\circ} |\tilde{F}(g)|^2 \phi(g) dg \\
&= \int_{G^\circ} |\tilde{F}(x^{-1}g)|^2 \phi(x^{-1}g) dg \\
&= \int_{K^\circ \times G_i^\circ} |\tilde{F}(x^{-1}kg_i)|^2 \dot{\phi}(x^{-1}kg_i \cdot o_i) dk dg_i.
\end{aligned}$$

Taking the supremum over all $\phi \in C_c(G^\circ)$ such that $0 \le \dot{\phi} \le 1$, we have the assertion of (6.32) and complete the proof. ∎

From Lemma 6.30, following the same arguments used in Section 4, we deduce:

(6.34) The completion $H^2(\mathscr{D}, \omega_i)$ of $\mathcal{O}(\bar{\mathscr{D}}, \omega_i)$ can be identified with a subspace of $\mathcal{O}(\mathscr{D})$ and is a nonzero Hilbert space with the property that for each $z \in \mathscr{D}$, the linear map $f \to f(z)$ is continuous from $H^2(\mathscr{D}, \omega_i)$ onto $\boldsymbol{C}$. Furthermore T_i defines an irreducible unitary representation of G° on $H^2(\mathscr{D}, \omega_i)$.

It follows from (6.34) that the reproducing kernel function $\boldsymbol{k}_i$ of $H^2(\mathscr{D}, \omega_i)$ exists and satisfies

$$\boldsymbol{k}_i(g \cdot z, g \cdot w) = J_i^\circ(g, z)\boldsymbol{k}_i(z, w)\overline{J_i^\circ(g, w)} \tag{6.35}$$

for all $g \in G^\circ$ and $z, w \in \mathscr{D}$. Once we have the formula (6.35), the proof of Proposition 6.22 goes through without change also in the present situation. Hence we conclude that the kernel function $\boldsymbol{k}_i$ of $H^2(\mathscr{D}, \omega_i)$ is given by (6.23) for any irreducible $\mathscr{D}$ and $1 \le i \le r$.

7. Intertwining operators

Let $(U_\lambda, L^2(G, \sigma_\lambda))$ and $(T_\lambda, H^2(\mathscr{D}, \lambda))$ be the unitary representations of G as in Theorem 4.49. As noted after that theorem, $(T_\lambda, H^2(\mathscr{D}, \lambda))$ is unitarily equivalent to the representation $(T_\lambda, H^2(G, \tau_\lambda))$ defined by (4.51), and so, in view of Theorem 4.49, $H^2(G, \tau_\lambda)$ can be identified with a closed subspace of $L^2(G, \sigma_\lambda)$.

In this section we construct an integral operator $\mathscr{P}_\lambda$: $L^2(G, \sigma_\lambda) \to H^2(G, \tau_\lambda)$ which is regarded as the orthogonal projection operator if we identify $H^2(G, \tau_\lambda)$ with a subspace of $L^2(G, \sigma_\lambda)$. In view of Proposition 5.9, the restriction of $\mathscr{P}_\lambda$ to the subspace $L^2(G, \sigma_\lambda; \mathfrak{p}_i^-)$ may be more important.

Now, letting β_i, $\|\mathbf{1}_\lambda\|_\lambda$ be as in Corollary 4.27, put $\gamma = \beta_i \|\mathbf{1}_\lambda\|_\lambda^{-2}$. Given $\phi \in L^2(G, \sigma_\lambda)$, we define

$$\mathscr{P}_\lambda \phi(g) = \gamma \int_{K \times G_i} \tau_\lambda(k) J_\lambda(g_i^{-1}, o)^{-1} \phi(gkg_i) dk dg_i \tag{7.1a}$$

for all $g \in G$ for which this integral exists.

7.2. Lemma. *For any $\phi \in L^2(G, \sigma_\lambda)$, the integral defining $\mathscr{P}_\lambda \phi(g)$ exists for all $g \in G$. Moreover $\mathscr{P}_\lambda$ is given by an integral operator with a kernel $\gamma J_{\bar\lambda}^*(g^{-1}kg_i, o_i)^{-1}$, i.e.,*

$$\mathscr{P}_\lambda \phi(g) = \gamma \int_{K \times G_i} J_{\bar\lambda}^*(g^{-1}kg_i, o_i)^{-1} \phi(kg_i) dk dg_i. \tag{7.1b}$$

Proof. According to Corollary 4.39, the constant function $\mathbf{1}_e$ belongs to $H^2(\mathscr{D}, \lambda)$ for every $e \in E_\lambda$. Hence if we define, for each $e \in E_\lambda$, ϕ_e: $G \to E_{\bar\lambda}$ by $\phi_e(g) = P_{\bar\lambda} J_\lambda(gc_i, o)^{-1} e$, then (4.37) implies $\phi_e \in L^2(G, \sigma_\lambda)$ since $\phi_e = \mathscr{I}_\lambda(\mathbf{1}_e)$. On the other hand, if $\phi \in L^2(G, \sigma_\lambda)$ then the left translates $U_\lambda(g)\phi$ $(g \in G)$ also belong to $L^2(G, \sigma_\lambda)$, so $(U_\lambda(g^{-1})\phi, \phi_e)$ (inner product in $L^2(G, \sigma_\lambda)$) exist for all $g \in G$ and $e \in E_\lambda$. Now

$$\begin{aligned}(U_\lambda(g^{-1})\phi, \phi_e) &= \int_{K \times G_i} (\phi(gkg_i), P_{\bar\lambda} J_\lambda(kg_ic_i, o)^{-1} e) dk dg_i \\ &= \left(\int_{K \times G_i} J_{\bar\lambda}^*(kg_ic_i, o)^{-1} \phi(gkg_i) dk dg_i, e \right).\end{aligned}$$

But, using Lemmas 4.25, 6.9 and 6.10, one finds

$$\begin{aligned} J_{\bar\lambda}^*(kg_ic_i, o)^{-1}|_{E_{\bar\lambda}} &= \beta_i J_{\bar\lambda}^*(k, g_i \cdot o_i)^{-1} J_{\bar\lambda}^*(g_i, o_i)^{-1}|_{E_{\bar\lambda}} \\ &= \beta_i \tau_{\bar\lambda}^*(k)^{-1} J_{\bar\lambda}^*(g_i, o)^{-1}|_{E_{\bar\lambda}} \\ &= \beta_i \tau_\lambda(k) J_\lambda(g_i^{-1}, o)^{-1}|_{E_{\bar\lambda}}. \end{aligned}$$

Therefore we obtain

$$\|\mathbf{1}_\lambda\|_\lambda^{-2}(U_\lambda(g^{-1})\phi,\ \phi_{\boldsymbol{e}}) = (\mathscr{P}_\lambda\phi(g),\ \boldsymbol{e}) \tag{7.3}$$

for all $\phi \in L^2(G, \sigma_\lambda)$, $g \in G$, $\boldsymbol{e} \in E_\lambda$, from which the first assertion of the lemma follows. As for the second assertion, recall that $U_\lambda(g)$ is a unitary operator of $L^2(G, \sigma_\lambda)$ for every $g \in G$. Thus

$$\begin{aligned}\|\mathbf{1}_\lambda\|_\lambda^{-2}(U_\lambda(g^{-1})\phi,\ \phi_{\boldsymbol{e}}) &= \|\mathbf{1}_\lambda\|_\lambda^{-2}(\phi,\ U_\lambda(g)\phi_{\boldsymbol{e}})\\ &= \|\mathbf{1}_\lambda\|_\lambda^{-2}\int_{K\times G_i}(\phi(kg_i),\ P_{\tilde{\lambda}}J_\lambda(g^{-1}kg_ic_i,\ o)^{-1}\boldsymbol{e})dkdg_i\\ &= \|\mathbf{1}_\lambda\|_\lambda^{-2}\int_{K\times G_i}(J_{\tilde{\lambda}}^*(g^{-1}kg_ic_i,\ o)^{-1}\phi(kg_i),\ \boldsymbol{e})dkdg_i\\ &= \left(\gamma\int_{K\times G_i}J_{\tilde{\lambda}}^*(g^{-1}kg_i,\ o_i)^{-1}\phi(kg_i)dkdg_i,\ \boldsymbol{e}\right)\end{aligned}$$

for all $\boldsymbol{e} \in E_\lambda$. This combined with (7.3) gives the second assertion of the lemma. ∎

Our main result in this section is the following theorem.

7.4. THEOREM. *Fix any i, $1 \leq i \leq r$, and $\lambda \in \mathscr{F}_i(G)$. Then:*

(1) *For any $\phi \in L^2(G, \sigma_\lambda)$, $\mathscr{P}_\lambda\phi$ belongs to $H^2(G, \tau_\lambda)$ and the resulting mapping*

$$\mathscr{P}_\lambda:\ L^2(G, \sigma_\lambda) \longrightarrow H^2(G, \tau_\lambda)$$

is a surjective G-intertwining operator. Furthermore, if we regard $H^2(G, \tau_\lambda)$ as a closed subspace of $L^2(G, \sigma_\lambda)$ then $\mathscr{P}_\lambda$ is the orthogonal projection operator onto $H^2(G, \tau_\lambda)$.

(2) *On the subspace $L^2(G, \sigma_\lambda;\ \mathfrak{p}_i^-)$ of $L^2(G, \sigma_\lambda)$, $\mathscr{P}_\lambda$ is given by*

$$\mathscr{P}_\lambda\phi(g) = \beta\int_K \tau_\lambda(k)\phi(gk)dk \tag{7.5}$$

with $\beta = \dfrac{d(\lambda)}{\beta_i d(\tilde{\lambda})}$.

The proof of this theorem requires some preparation. For an $E_{\tilde{\lambda}}$-valued Borel function F on the boundary orbit $\mathscr{B}_i$, define (as in (4.14)) $\tilde{F}$: $G \to E_{\tilde{\lambda}}$ by

$$\tilde{F}(g) = P_{\tilde{\lambda}}J_\lambda(gc_i,\ o)^{-1}F(g\cdot o_i),\quad g \in G \tag{7.6}$$

and let

$$L^2(\mathscr{B}_i, \lambda) = \left\{\text{Borel functions } F\colon \mathscr{B}_i \to E_\lambda;\ \|F\|^2 = \int_{K\times G_i} |\tilde{F}(kg_i)|^2 dk dg_i < \infty\right\}.$$

Then if $F \in L^2(\mathscr{B}_i, \lambda)$, the argument used in proving Lemma 4.15 implies $\tilde{F} \in L^2(G, \sigma_\lambda)$. Given $F, F' \in L^2(\mathscr{B}_i, \lambda)$, let us say that F and F' are equivalent ($F \sim F'$) if $\|F - F'\| = 0$. Now define $L^2(\mathscr{B}_i, \lambda)^\circ$ as the set of equivalence classes (relative to $\sim$) of elements in $L^2(\mathscr{B}_i, \lambda)$. Then $L^2(\mathscr{B}_i, \lambda)^\circ$ is an inner product space. For $\phi \in L^2(G, \sigma_\lambda)$ we define a function $\mathscr{L}_\lambda\phi\colon \mathscr{B}_i \to E_\lambda$ as follows. Every $u \in \mathscr{B}_i$ is represented as $u = kg_i \cdot o_i$, $k \in K$, $g_i \in G_i$; so put $\mathscr{L}_\lambda\phi(u) = J_\lambda(kg_ic_i, o) \cdot \phi(kg_i)$. It is easily verified that $\mathscr{L}_\lambda\phi$ is a well defined function on $\mathscr{B}_i$ and belongs to $L^2(\mathscr{B}_i, \lambda)$. Let $\mathscr{L}_\lambda^\circ\colon L^2(G, \sigma_\lambda) \to L^2(\mathscr{B}_i, \lambda)^\circ$ be the composition of the mapping $\mathscr{L}_\lambda\colon L^2(G, \sigma_\lambda) \to L^2(\mathscr{B}_i, \lambda)$ with the canonical projection of $L^2(\mathscr{B}_i, \lambda)$ onto $L^2(\mathscr{B}_i, \lambda)^\circ$.

7.7. Lemma. *$\mathscr{L}_\lambda^\circ$ is a linear isometry of $L^2(G, \sigma_\lambda)$ onto $L^2(\mathscr{B}_i, \lambda)^\circ$. In particular $L^2(\mathscr{B}_i, \lambda)^\circ$ is a Hilbert space.*

Proof. If $F \in L^2(\mathscr{B}_i, \lambda)$ then $\tilde{F} \in L^2(G, \sigma_\lambda)$ as remarked before, and it is easily seen that $F \sim \mathscr{L}_\lambda\tilde{F}$. Thus $\mathscr{L}_\lambda^\circ$ is surjective. The rest of the lemma is obvious. ∎

Given $F \in L^2(\mathscr{B}_i, \lambda)$, we define (using the notation in (6.15))

$$\mathscr{Q}_\lambda F(z) = \int_{\mathscr{B}_i} K_\lambda(z, u) M_\lambda(u) F(u) d\mu(u)$$

for all $z \in \mathscr{D}$ for which this integral exists.

7.8. Lemma. *For any $F \in L^2(\mathscr{B}_i, \lambda)$, the integral defining $\mathscr{Q}_\lambda F(z)$ exists for all $z \in \mathscr{D}$, and if $F, F' \in L^2(\mathscr{B}_i, \lambda)$ with $F \sim F'$, then $\mathscr{Q}_\lambda F = \mathscr{Q}_\lambda F'$.*

Proof. According to (6.2) and (6.12), $E_w E_z^* \boldsymbol{e} = K_\lambda(w, z)\boldsymbol{e}$ and $E_z^* \boldsymbol{e} \in \mathcal{O}^2(\bar{\mathscr{D}}, E_\lambda)$ for all $z, w \in \mathscr{D}$ and all $\boldsymbol{e} \in E_\lambda$. If we set $F_{z,\boldsymbol{e}} = E_z^* \boldsymbol{e}|_{\mathscr{B}_i}$, then clearly $F_{z,\boldsymbol{e}} \in L^2(\mathscr{B}_i, \lambda)$. Hence if $F \in L^2(\mathscr{B}_i, \lambda)$, then $(\tilde{F}, \tilde{F}_{z,\boldsymbol{e}})$ (inner product in $L^2(G, \sigma_\lambda)$) exists for all $z \in \mathscr{D}$, $\boldsymbol{e} \in E_\lambda$ where $\tilde{F}$, $\tilde{F}_{z,\boldsymbol{e}}$ are defined as in (7.6). Now recalling the definition (6.14) of M_λ, we have, for all $z \in \mathscr{D}$, $\boldsymbol{e} \in E_\lambda$,

$$\begin{aligned}
&(\tilde{F}, \tilde{F}_{z,\boldsymbol{e}}) \\
&\quad = \int_{K\times G_i} (P_\lambda^* J_\lambda(kg_ic_i, o)^{-1} F(kg_i \cdot o_i), P_\lambda^* J_\lambda(kg_ic_i, o)^{-1} K_\lambda(kg_i \cdot o_i, z)\boldsymbol{e}) dk dg_i \\
&\quad = \left(\int_{\mathscr{B}_i} K_\lambda(z, u) M_\lambda(u) F(u) d\mu(u), \boldsymbol{e}\right) \\
&\quad = (\mathscr{Q}_\lambda F(z), \boldsymbol{e}),
\end{aligned}$$

and from this the first assertion follows at once. Since $F \sim F' \Leftrightarrow \tilde{F} = \tilde{F}'$ (in $L^2(G, \sigma_\lambda)$), the second assertion also follows. ∎

By Lemma 7.8 we may define, for $F^\circ \in L^2(\mathscr{B}_i, \lambda)^\circ$, a function $\mathscr{Q}_\lambda^\circ F^\circ : \mathscr{D} \to E_\lambda$ by setting $\mathscr{Q}_\lambda^\circ F^\circ = \mathscr{Q}_\lambda F$ where F is any representative of the class F°. In view of (4.37) and Lemma 7.7, $H^2(\mathscr{D}, \lambda)$ can be identified with a closed subspace of $L^2(\mathscr{B}_i, \lambda)^\circ$. Note that this identification is induced by taking boundary values of functions belonging to the dense subspace $\mathscr{O}^2(\bar{\mathscr{D}}, E_\lambda)$ of $H^2(\mathscr{D}, \lambda)$.

7.9. LEMMA. *For any $F^\circ \in L^2(\mathscr{B}_i, \lambda)^\circ$, the function $\mathscr{Q}_\lambda^\circ F^\circ$ belongs to $H^2(\mathscr{D}, \lambda)$. Moreover, the mapping $\mathscr{Q}_\lambda^\circ : L^2(\mathscr{B}_i, \lambda)^\circ \to H^2(\mathscr{D}, \lambda)$ is the orthogonal projection operator onto $H^2(\mathscr{D}, \lambda)$ if we regard $H^2(\mathscr{D}, \lambda)$ as a closed subspace of $L^2(\mathscr{B}_i, \lambda)^\circ$.*

PROOF. According to (6.15), $\mathscr{Q}_\lambda^\circ F = F$ for all $F \in \mathscr{O}^2(\bar{\mathscr{D}}, E_\lambda)$. Since $\mathscr{O}^2(\bar{\mathscr{D}}, E_\lambda)$ is dense in $H^2(\mathscr{D}, \lambda)$, it follows that $\mathscr{Q}_\lambda^\circ F = F$ for all $F \in H^2(\mathscr{D}, \lambda)$. On the other hand if $F^\circ \in L^2(\mathscr{B}_i, \lambda)^\circ$ is orthogonal to $H^2(\mathscr{D}, \lambda)$ then, since $E_z^* \boldsymbol{e} \in H^2(\mathscr{D}, \lambda)$ for every $\boldsymbol{e} \in E_\lambda$, we have

$$\begin{aligned} 0 &= (F^\circ, E_z^* \boldsymbol{e}) \\ &= \int_{K \times G_i} (P_\lambda^+ J_\lambda(kg_i c_i, o)^{-1} F(kg_i \cdot o_i), P_\lambda^+ J_\lambda(kg_i c_i, o)^{-1} K_\lambda(kg_i \cdot o_i, z)\boldsymbol{e}) dk dg_i \\ &\qquad\qquad (F \text{ is a representative of } F^\circ) \\ &= \left(\int_{\mathscr{B}_i} K_\lambda(z, u) M_\lambda(u) F(u) d\mu(u), \boldsymbol{e} \right) \\ &= (\mathscr{Q}_\lambda^\circ F^\circ(z), \boldsymbol{e}) \end{aligned}$$

for all $z \in \mathscr{D}$ and $\boldsymbol{e} \in E_\lambda$. Thus $\mathscr{Q}_\lambda^\circ F^\circ = 0$ and the lemma follows. ∎

With this preparation, we can now prove part (1) of Theorem 7.4.

PROOF OF THEOREM 7.4(1). Consider the following mappings:

$$L^2(G, \sigma_\lambda) \xrightarrow{\mathscr{L}_\lambda^\circ} L^2(\mathscr{B}_i, \lambda)^\circ \xrightarrow{\mathscr{Q}_\lambda^\circ} H^2(\mathscr{D}, \lambda) \xrightarrow{\mathscr{J}_\lambda} H^2(G, \tau_\lambda)$$

where $\mathscr{J}_\lambda$ is the unitary isomorphism defined by (4.50). We are going to prove that $\mathscr{J}_\lambda \circ \mathscr{Q}_\lambda^\circ \circ \mathscr{L}_\lambda^\circ = \mathscr{P}_\lambda$; since the G-equivariance of $\mathscr{P}_\lambda$ is obvious, in view of Lemmas 7.7 and 7.9 this will imply the result. So letting $\phi \in L^2(G, \sigma_\lambda)$ and $g \in G$, we calculate

$$\begin{aligned} &(\mathscr{J}_\lambda \circ \mathscr{Q}_\lambda^\circ \circ \mathscr{L}_\lambda^\circ(\phi))(g) \\ &\quad = J_\lambda(g, o)^{-1} (\mathscr{Q}_\lambda \circ \mathscr{L}_\lambda(\phi))(g \cdot o) \end{aligned}$$

$$= J_\lambda(g, o)^{-1}\int_{\mathscr{B}_i} K_\lambda(g\cdot o, u)M_\lambda(u)\mathscr{L}_\lambda\phi(u)d\mu(u)$$

$$= J_\lambda(g, o)^{-1}\int_{K\times G_i} K_\lambda(g\cdot o, kg_i\cdot o_i)M_\lambda(kg_i\cdot o_i)\mathscr{L}_\lambda\phi(kg_i\cdot o_i)dkdg_i.$$

Now (6.3b), Lemma 6.6 and (6.12) imply

$$K_\lambda(g\cdot o, kg_i\cdot o_i) = J_\lambda(g, o)K_\lambda(o, g^{-1}kg_i\cdot o_i)J_{\bar\lambda}^*(g, g^{-1}kg_i\cdot o_i)$$
$$= \|\mathbf{1}_\lambda\|_\lambda^{-2}J_\lambda(g, o)J_{\bar\lambda}^*(g^{-1}, kg_i\cdot o_i)^{-1},$$

while by definition

$$M_\lambda(kg_i\cdot o_i) = J_{\bar\lambda}^*(kg_ic_i, o)^{-1}P_{\bar\lambda}J_\lambda(kg_ic_i, o)^{-1},$$
$$\mathscr{L}_\lambda\phi(kg_i\cdot o_i) = J_\lambda(kg_ic_i, o)\phi(kg_i).$$

Therefore, using the cocycle formula and Lemma 7.2, we get

$$(\mathscr{J}_\lambda\circ\mathscr{Q}_\lambda^\circ\circ\mathscr{L}_\lambda^\circ(\phi))(g) = \|\mathbf{1}_\lambda\|_\lambda^{-2}\int_{K\times G_i} J_{\bar\lambda}^*(g^{-1}kg_ic_i, o)^{-1}\phi(kg_i)dkdg_i$$
$$= \gamma\int_{K\times G_i} J_{\bar\lambda}^*(g^{-1}kg_i, o_i)^{-1}\phi(kg_i)dkdg_i$$
$$= \mathscr{P}_\lambda\phi(g)$$

for all $\phi\in L^2(G, \sigma_\lambda)$ and $g\in G$. Thus $\mathscr{J}_\lambda\circ\mathscr{Q}_\lambda^\circ\circ\mathscr{L}_\lambda^\circ = \mathscr{P}_\lambda$ as desired. ∎

For part (2) we need some notation and one more lemma. With $\tau_{\bar\lambda}$ being as in Lemma 4.2, we define

$$L^2(G_i, \tau_{\bar\lambda}) = \left\{ \begin{array}{c} f \text{ Borel measurable,} \\ f\colon G_i\to E_{\bar\lambda};\ f(gk) = \tau_{\bar\lambda}(k)^{-1}f(g),\ g\in G_i,\ k\in K_i, \\ \int_{G_i}|f(g_i)|^2dg_i < \infty \end{array} \right\},$$

and put

$$H^2(G_i, \tau_{\bar\lambda}) = L^2(G_i, \tau_{\bar\lambda})\cap\{f\in C^\infty(G_i, E_{\bar\lambda});\ r(X)f = 0 \text{ for all } X\in\mathfrak{p}_i^-\}$$

($r(X)$ is defined similarly as in (4.19)). Recall the spaces $L^2(M_i, '\tau_\lambda^{(i)})$ and $H^2(M_i, '\tau_\lambda^{(i)})$ defined in (5.4). Note that each function in $L^2(M_i, '\tau_\lambda^{(i)})$ is determined completely by its restriction to the subgroup G_i. Hence, and because $'\tau_\lambda^{(i)}|_{K_i} = \tau_{\bar\lambda}$ (cf. Lemma 4.5(1) and (4.9)), it is obvious that

$$(7.10)\quad \begin{cases} \text{restricting elements in } L^2(M_i, {'\tau_\lambda^{(i)}}) \text{ to } G_i \text{ sets up a unitary isomor-} \\ \text{phism of } L^2(M_i, {'\tau_\lambda^{(i)}}) \text{ onto } L^2(G_i, \tau_{\tilde\lambda}) \text{ and } H^2(M_i, {'\tau_\lambda^{(i)}}) \text{ corresponds} \\ \text{to } H^2(G_i, \tau_{\tilde\lambda}) \text{ under this mapping.} \end{cases}$$

Therefore Lemma 5.6 implies that $H^2(G_i, \tau_{\tilde\lambda})$ is a nonzero closed subspace of $L^2(G_i, \tau_{\tilde\lambda})$.

7.11. LEMMA. *For $\phi \in L^2(G_i, \tau_{\tilde\lambda})$ and $g \in G_i$, set*

$$\mathcal{Q}_{\tilde\lambda}\phi(g) = \beta_i^2 \|\mathbf{1}_\lambda\|_{\tilde\lambda}^{-2} \int_{G_i} J_\lambda(g_i^{-1}, o)^{-1}\phi(gg_i)dg_i$$

where $\|\mathbf{1}_\lambda\|_{\tilde\lambda}$ is as in the proof of Lemma 5.3. Then $\mathcal{Q}_{\tilde\lambda}$ defines the orthogonal projection operator of $L^2(G_i, \tau_{\tilde\lambda})$ onto $H^2(G_i, \tau_{\tilde\lambda})$.

PROOF. Let $L^2(\mathscr{C}_i, \tilde\lambda)$ and $H^2(\mathscr{C}_i, \tilde\lambda)$ be as in (5.1). For $F \in L^2(\mathscr{C}_i, \tilde\lambda)$, define (using the same notation as in (5.5)) $\mathscr{J}_{\tilde\lambda}F\colon G_i \to E_{\tilde\lambda}$ by $\mathscr{J}_{\tilde\lambda}F(g) = J_\lambda(gc_i, o)^{-1}F(g \cdot o_i)$. Then, in view of (7.10) and Lemma 5.6, $\mathscr{J}_{\tilde\lambda}$ is a unitary isomorphism of $L^2(\mathscr{C}_i, \tilde\lambda)$ onto $L^2(G_i, \tau_{\tilde\lambda})$, and $H^2(\mathscr{C}_i, \tilde\lambda)$ corresponds to $H^2(G_i, \tau_{\tilde\lambda})$ under $\mathscr{J}_{\tilde\lambda}$. Recall the representation $(T_{\tilde\lambda}, H^2(\mathscr{C}_i, \tilde\lambda))$ of G_i defined by (5.2). Since $T_{\tilde\lambda}$ is unitary, just as in the case of the kernel function K_λ of $H^2(\mathscr{D}, \lambda)$ one can prove: $H^2(\mathscr{C}_i, \tilde\lambda)$ has a reproducing kernel $K_{\tilde\lambda}\colon \mathscr{C}_i \times \mathscr{C}_i \to GL(E_{\tilde\lambda})$ such that

$$(F(z), \boldsymbol{e})_{E_{\tilde\lambda}} = (F(\cdot), K_{\tilde\lambda}(\cdot, z)\boldsymbol{e})_{H^2(\mathscr{C}_i, \tilde\lambda)}$$

for $F \in H^2(\mathscr{C}_i, \tilde\lambda)$, $\boldsymbol{e} \in E_{\tilde\lambda}$, $z \in \mathscr{C}_i$, and $K_{\tilde\lambda}$ satisfies

$$K_{\tilde\lambda}(g \cdot z, g \cdot w) = J_{\tilde\lambda}(g, z)K_{\tilde\lambda}(z, w)J_{\tilde\lambda}^*(g, w)$$

for $g \in G_i$, $z, w \in \mathscr{C}_i$ where $J_{\tilde\lambda}(g, z)$ denotes the restriction of $J_\lambda(g, z)$ to $E_{\tilde\lambda}$. Moreover

$$K_{\tilde\lambda}(z, o_i) = K_{\tilde\lambda}(o_i, z) = \|\mathbf{1}_\lambda\|_{\tilde\lambda}^{-2} I$$

for all $z \in \mathscr{C}_i$.

We define a measure on $\mathscr{C}_i$ by

$$\int_{\mathscr{C}_i} f(w)dw = \int_{G_i} f(g_i \cdot o_i)dg_i$$

for $f \in C_c(\mathscr{C}_i)$, and define a function $M_{\tilde\lambda}\colon \mathscr{C}_i \to GL(E_{\tilde\lambda})$ by $M_{\tilde\lambda}(w) = J_{\tilde\lambda}^*(gc_i, o)^{-1} \cdot J_{\tilde\lambda}(gc_i, o)^{-1}$ with $w = g \cdot o_i$, $g \in G_i$ (this is well defined). Then by the same argument as in the proof of Lemma 7.9, one finds that the orthogonal projection $Q_{\tilde\lambda}$ of $L^2(\mathscr{C}_i, \tilde\lambda)$ onto $H^2(\mathscr{C}_i, \tilde\lambda)$ is given by

$$Q_{\tilde\lambda}F(z) = \int_{\mathscr{C}_i} K_{\tilde\lambda}(z, w)M_{\tilde\lambda}(w)F(w)dw$$

for all $F \in L^2(\mathscr{C}_i, \tilde{\lambda})$.

Now consider the following mappings:

$$L^2(G_i, \tau_{\tilde{\lambda}}) \xrightarrow{\mathscr{J}_{\tilde{\lambda}}^{-1}} L^2(\mathscr{C}_i, \tilde{\lambda}) \xrightarrow{Q_{\tilde{\lambda}}} H^2(\mathscr{C}_i, \tilde{\lambda}) \xrightarrow{\mathscr{J}_{\tilde{\lambda}}} H^2(G_i, \tau_{\tilde{\lambda}}).$$

To prove the lemma it is enough to show that $\mathscr{J}_{\tilde{\lambda}} \circ Q_{\tilde{\lambda}} \circ \mathscr{J}_{\tilde{\lambda}}^{-1} = \mathscr{Q}_{\tilde{\lambda}}$. If $\phi \in L^2(G_i, \tau_{\tilde{\lambda}})$ and $g \in G_i$ then, calculating as in the proof of Theorem 7.4(1), we get

$$(\mathscr{J}_{\tilde{\lambda}} \circ Q_{\tilde{\lambda}} \circ \mathscr{J}_{\tilde{\lambda}}^{-1}(\phi))(g)$$

$$= J_{\tilde{\lambda}}(gc_i, o)^{-1} \int_{G_i} K_{\tilde{\lambda}}(g \cdot o_i, g_i \cdot o_i) M_{\tilde{\lambda}}(g_i \cdot o_i) J_{\tilde{\lambda}}(g_i c_i, o) \phi(g_i) dg_i$$

$$= \|\mathbf{1}_\lambda\|_{\tilde{\lambda}}^{-2} J_{\tilde{\lambda}}(gc_i, o)^{-1} J_{\tilde{\lambda}}(g, o_i) \int_{G_i} J_{\tilde{\lambda}}^*(g^{-1} g_i c_i, o)^{-1} \phi(g_i) dg_i$$

$$= \beta_i^2 \|\mathbf{1}_\lambda\|_{\tilde{\lambda}}^{-2} \int_{G_i} J_{\tilde{\lambda}}^*(g_i, o)^{-1} \phi(g g_i) dg_i$$

(by Lemma 4.25 and the invariance of measure)

$$= \beta_i^2 \|\mathbf{1}_\lambda\|_{\tilde{\lambda}}^{-2} \int_{G_i} J_\lambda(g_i^{-1}, o)^{-1} \phi(g g_i) dg_i$$

(by Lemmas 6.9 and 6.10)

$$= \mathscr{Q}_{\tilde{\lambda}} \phi(g).$$

Thus $\mathscr{J}_{\tilde{\lambda}} \circ Q_{\tilde{\lambda}} \circ \mathscr{J}_{\tilde{\lambda}}^{-1} = \mathscr{Q}_{\tilde{\lambda}}$ and the lemma follows. ∎

Proof of Theorem 7.4(2). Since $C_2^\infty(G, \sigma_\lambda; \mathfrak{p}_i^-)$ (notation of (5.8)) is dense in $L^2(G, \sigma_\lambda; \mathfrak{p}_i^-)$, it suffices to show that the formula (7.5) is valid for all $\phi \in C_2^\infty(G, \sigma_\lambda; \mathfrak{p}_i^-)$. For $\phi \in C_2^\infty(G, \sigma_\lambda; \mathfrak{p}_i^-)$ and $g \in G$, define $\phi_g: G_i \to E_{\tilde{\lambda}}$ by $\phi_g(g_i) = \phi(g g_i)$. Then it is not difficult to show that $\phi_g \in H^2(G_i, \tau_{\tilde{\lambda}})$. Therefore if $\phi \in C_2^\infty(G, \sigma_\lambda; \mathfrak{p}_i^-)$ then, using Lemma 7.11, we get

$$\phi(g) = \phi_g(e) = \mathscr{Q}_{\tilde{\lambda}} \phi_g(e) = \beta_i^2 \|\mathbf{1}_\lambda\|_{\tilde{\lambda}}^{-2} \int_{G_i} J_\lambda(g_i^{-1}, o)^{-1} \phi(g g_i) dg_i$$

for all $g \in G$. Thus, and since $\|\mathbf{1}_\lambda\|_{\tilde{\lambda}}^2 = d(\lambda) d(\tilde{\lambda})^{-1} \|\mathbf{1}_\lambda\|_{\lambda}^2$ (cf. the proof of Lemma 5.3), we obtain

$$\mathscr{P}_\lambda \phi(g) = \beta_i \|\mathbf{1}_\lambda\|_{\lambda}^{-2} \int_{K \times G_i} \tau_\lambda(k) J_\lambda(g_i^{-1}, o)^{-1} \phi(g k g_i) dk dg_i$$

$$= \beta_i^{-1} d(\tilde{\lambda})^{-1} d(\lambda) \int_K \tau_\lambda(k) \phi(g k) dk$$

for all $\phi \in C_2^\infty(G, \sigma_\lambda; \mathfrak{p}_i^-)$ and $g \in G$. This completes the proof. ∎

REMARK. In the extreme case $i=r$, we have $G_r=K_r\subset K$ (cf. (2.29)) and hence $\mathfrak{p}_r^-=\{0\}$; therefore $L^2(G, \sigma_\lambda)=L^2(G, \sigma_\lambda; \mathfrak{p}_r^-)$. Thus if $\lambda\in\mathscr{F}_r(G)$ then, since $\|\mathbf{1}_\lambda\|_\lambda^2=\beta_r^2 d(\tilde{\lambda})d(\lambda)^{-1}$ (cf. Corollary 4.27) and since $\int_{G_r} dg_r=1$, taking (7.1b) into account we see that the operator $\mathscr{P}_\lambda$: $L^2(G, \sigma_\lambda)\to H^2(G, \tau_\lambda)$ defined by (7.1a) is given by

$$\mathscr{P}_\lambda\phi(g) = \beta\int_K \tau_\lambda(k)\phi(gk)dk = \beta\int_K J_\lambda^*(g^{-1}k, o_r)^{-1}\phi(k)dk$$

with $\beta=\beta_r^{-1}d(\tilde{\lambda})^{-1}d(\lambda)$ (this is consistent with the formula (7.5)). In particular, if $\lambda=\omega_r$ (notation of (6.16)) then

$$H^2(\mathscr{D}, \omega_r) = \text{Hardy space of } \mathscr{D} \text{ (cf. (6.17))},$$

$$L^2(\mathscr{B}_r, \omega_r) = L^2(\mathscr{B}_r, \omega_r)^\circ$$

$$= \left\{\text{Borel functions } f\colon \mathscr{B}_r\to \boldsymbol{C};\ \int_K |f(k\cdot o_r)|^2 dk < \infty\right\}$$

and, as is clear from our proof of Theorem 7.4(1), $\mathscr{P}_{\omega_r}$ corresponds to the integral operator $\mathscr{Q}_{\omega_r}$: $L^2(\mathscr{B}_r, \omega_r)\to H^2(\mathscr{D}, \omega_r)$ associated with the Cauchy-Szegö kernel function.

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