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On glueings of prime ideals

Dedicated to Professor Yoshikazu Nakai on his sixtieth birthday

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In the article [11] Traverso defined the notion of seminormal subrings of a commutative ring and developed an interesting theory on such subrings. In particular, to study the structure of seminormal subrings of a noetherian ring, he used a nice method by which a subring A' is constructed from a ring B by glueing prime ideals $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ of B lying over a prime ideal $\mathfrak{p}$ of a subring A of B (for the precise definition see § 1). Such a subring A' is called the ring obtained from B by glueing over $\mathfrak{p}$, or simply a glueing of prime ideals $\mathfrak{p}_1,\dots,\mathfrak{p}_n$. Traverso showed that any seminormal subring A in a noetherian ring B is obtained from B by a finite number of glueings, if B is a finite A-module (cf. Theorem 2.1 in [11]).

The aim of this paper is to show some results on glueings of prime ideals in the above sense. In § 1 we give a necessary and sufficient condition for a finite number of prime ideals $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ of a noetherian ring B to be glued. In other words, we give a condition under which there is a subring A of B such that B is a finite A-module and that $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ are the prime ideals of B lying over a prime ideal $\mathfrak{p}$ of A. Moreover we investigate when the ring A' obtained from B by glueing over $\mathfrak{p}$ coincides with A. Although Pedrini [8] and Tamone [10] have already attacked the same problem, the rings treated by them are very special ones. So, in § 2, we apply our results in § 1 to these special cases and show how our results work there in a unified way. Next, in § 3, we show that Serre's property (S_2) goes down from a noetherian ring B to a glueing A of prime ideals of height 1. This result also has been shown by Pedrini [8] in very special cases of integral domains, but we give a complete proof of this without any assumption. Moreover we show that if A is a glueing of prime ideals $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ of a noetherian ring B one of which has height>1 and if any $\mathfrak{p}_i$ contains a regular element of B, then A does not have the property (S_2) or A coincides with B. In the last section we study local rings which are glueings of maximal ideals of semilocal rings. When A is a local subring of a semilocal ring B which is finite over A, we give several conditions for A to be a glueing of maximal ideals of B. In particular a condition for A to be such a glueing of a regular semilocal ring B will be given in terms of multiplicity of A and the conductor of A in B in the case where A is the locality of a closed point of an algebraic variety.

All the rings in this paper are commutative with unit.

§1. A condition for prime ideals to be glued

First we shall recall some basic results on glueing of a noetherian ring B over a prime ideal of a subring of B. Let A be a noetherian ring and B an overring of A such that B is a finite A-module. Let $\mathfrak{p}$ be a prime ideal of A, and denote by $\kappa(\mathfrak{p})$ the quotient field of $A/\mathfrak{p}$. Then Traverso [11] proved the existence of the largest subring A' of B containing A satisfying the following conditions:

i) *There is exactly one prime ideal* $\mathfrak{p}'$ *of* A' *lying over* $\mathfrak{p}$.

ii) *The quotient field* $\kappa(\mathfrak{p}')$ *of* $A'/\mathfrak{p}'$ *is isomorphic to* $\kappa(\mathfrak{p})$ *by the canonical homomorphism* $\kappa(\mathfrak{p})\to\kappa(\mathfrak{p}')$.

This largest subring A' of B is called *the ring obtained from B by glueing over* $\mathfrak{p}$.

LEMMA 1. *Let A, B and $\mathfrak{p}$ be as above, and let $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ be the prime ideals of B lying over $\mathfrak{p}$. Let $\kappa(\mathfrak{p}_i)$ be the quotient field of $B/\mathfrak{p}_i$, and let π_i and w_i be the canonical homomorphisms $B\to B/\mathfrak{p}_i$ and $\kappa(\mathfrak{p})\to\kappa(\mathfrak{p}_i)$, respectively, for each i. If A' is a subring of B, then the followings are equivalent:*

(i) *A' is the ring obtained from B by glueing over $\mathfrak{p}$.*

(ii) *A' is the subring of B consisting of all $b\in B$ satisfying the followings:*

a) *$\pi_i(b)$ belongs to $w_i(\kappa(\mathfrak{p}))$ for each i.*

b) *$w_i^{-1}(\pi_i(b))=w_j^{-1}(\pi_j(b))$ for any i and j.*

The proof is easy and given in [11], p. 588.

Next we give another characterization of A'. For this purpose put $C=\kappa(\mathfrak{p}_1)\times\cdots\times\kappa(\mathfrak{p}_n)$ and let ρ_i be the canonical projection of C to $\kappa(\mathfrak{p}_i)$. Let ϕ be the homomorphism of B to C such that $\rho_i\phi$ is the canonical homomorphism $B\to B/\mathfrak{p}_i\to\kappa(\mathfrak{p}_i)$, and let ψ be the homomorphism of $\kappa(\mathfrak{p})$ to C such that $\rho_i\psi$ is w_i. Let the following diagram be the pullback one of commutative rings with respect to ϕ and ψ:

$$\begin{array}{ccc} B & \xleftarrow{\quad i \quad} & D \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle j} \\ C & \xleftarrow[\quad \psi \quad]{} & \kappa(\mathfrak{p}) \end{array}$$

Then we have the following

LEMMA 2. *The homomorphism i is injective, and the image $i(D)$ is the ring A' obtained from B by glueing over $\mathfrak{p}$.*

PROOF. Since ψ is injective, so is i clearly. Now, by definition, D consists of the elements (b, α) of $B\times\kappa(\mathfrak{p})$ such that $\phi(b)=\psi(\alpha)$. If b is an element of A', then $\pi_i(b)$ is an element of $w_i(\kappa(\mathfrak{p}))$ and $\alpha=w_i^{-1}(\pi_i(b))$ is independent of i by

Lemma 1. Then it is easy to see that $\phi(b)=\psi(\alpha)$, and hence A' is contained in $i(D)$. Conversely let b be an element of $i(D)$. If $i((b, \alpha))=b$, then we have $\phi(b)=\psi(\alpha)$. This means that $w_i(\alpha)=\pi_i(b)$ for each i, and hence b belongs to A' by Lemma 1. q.e.d.

In the following we may identify A' with D by the injection i.

Now let B be a noetherian ring and let $\mathfrak{p}_1,\dots, \mathfrak{p}_n$ be prime ideals of B. Then we say that $(B; \mathfrak{p}_1,\dots, \mathfrak{p}_n)$ *can be glued over a subring* A *of* B, if B is a finite A-module and if there is a prime ideal $\mathfrak{p}$ of A such that $\mathfrak{p}_1,\dots, \mathfrak{p}_n$ are the prime ideals of B lying over $\mathfrak{p}$. By Nagata-Eakin's Theorem (cf. [3]) A is necessarily noetherian. The following proposition gives a condition for $(B; \mathfrak{p}_1,\dots, \mathfrak{p}_n)$ to be glued.

PROPOSITION 1. *Let B be a noetherian ring of finite dimension and let $\mathfrak{p}_1,\dots, \mathfrak{p}_n$ be prime ideals of B. Then $(B; \mathfrak{p}_1,\dots, \mathfrak{p}_n)$ can be glued over a subring of B if and only if the followings are satisfied:*
(i) $\dim B/\mathfrak{p}_i=\dim B/\mathfrak{p}_j$ *for any* i *and* j.
(ii) *There is an injective homomorphism w from an integral domain R to the residue ring $B/\cap_{i=1}^n\mathfrak{p}_i$ such that $B/\cap_{i=1}^n\mathfrak{p}_i$ is a finite R-module.*

PROOF. First assume that $(B; \mathfrak{p}_1,\dots, \mathfrak{p}_n)$ can be glued over a subring A of B. If $\mathfrak{p}=A\cap \mathfrak{p}_i$, then we see $\mathfrak{p}=(\cap_{i=1}^n\mathfrak{p}_i)\cap A$. Put $R=A/\mathfrak{p}$ and let w be the canonical homomorphism $R=A/\mathfrak{p}\to B/\cap_{i=1}^n\mathfrak{p}_i$. Then it is clear that R and w satisfy the assertion (ii) of Proposition 1. On the other hand, we see $\dim B/\mathfrak{p}_i =\dim R$, because we have $\mathfrak{p}_i\cap A=\mathfrak{p}$ and B is a finite A-module. This means that the assertion (i) of Proposition 1 is satisfied. Conversely assume that the assertions (i) and (ii) are satisfied. Let π be the canonical homomorphism $B\to B/\cap_{i=1}^n\mathfrak{p}_i$. Denote by A the inverse image $\pi^{-1}(w(R))$ of $w(R)$. Then it is easy to see that A is a subring of B with a prime ideal $\cap_{i=1}^n\mathfrak{p}_i$, and that $A/\cap_{i=1}^n\mathfrak{p}_i$ is isomorphic to R. Since $B/\cap_{i=1}^n\mathfrak{p}_i$ is a finite $A/\cap_{i=1}^n\mathfrak{p}_i$-module, B is a finite A-module. If $\mathfrak{q}$ is a prime ideal of B such that $\mathfrak{q}\cap A=\cap_{i=1}^n\mathfrak{p}_i$, we have $\mathfrak{q}\supset\mathfrak{p}_j$ for some j. Since $\cap_{i=1}^n\mathfrak{p}_i=\mathfrak{q}\cap A\supset\mathfrak{p}_j\cap A\supset \cap_{i=1}^n\mathfrak{p}_i$, we see $\mathfrak{q}\supset A=\mathfrak{p}_j\cap A$. This means $\mathfrak{q}=\mathfrak{p}_j$, because B is integral over A. Since $B/\cap_{i=1}^n\mathfrak{p}_i$ is integral over $A/\cap_{i=1}^n\mathfrak{p}_i$, we have $\dim A/\cap_{i=1}^n\mathfrak{p}_i=\dim B/\mathfrak{p}_j$ for some j, and hence $\dim A/\cap_{i=1}^n\mathfrak{p}_i=\dim B/\mathfrak{p}_h$ for any h by our assumption (i). Therefore we see $\dim A/A\cap\mathfrak{p}_h=\dim B/\mathfrak{p}_h=\dim A/\cap_{i=1}^n\mathfrak{p}_i$ and hence we have $\mathfrak{p}_h\cap A=\cap_{i=1}^n\mathfrak{p}_i$ for any h. This means that $\mathfrak{p}_1,\dots, \mathfrak{p}_n$ are the prime ideals of B lying over the prime ideal $\cap_{i=1}^n\mathfrak{p}_i$ of A. q.e.d.

REMARK 1. Let B, $\mathfrak{p}_1,\dots, \mathfrak{p}_n$, R, w and π be as above and consider the pull-back diagram of commutative rings with respect to π and w:

$$\begin{array}{ccc} B & \xleftarrow{\quad f \quad} & S \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle g} \\ B/\bigcap_{i=1}^{n}\mathfrak{p}_i & \xleftarrow[\quad w \quad]{} & R \end{array}$$

Then we see easily that $A=\pi^{-1}(w(R))$ is isomorphic to S by the homomorphism f.

REMARK 2. If there is a subring C of B such that $\pi(C)=w(R)$, then $(B;\mathfrak{p}_1,\dots,\mathfrak{p}_n)$ can be glued over $C+\bigcap_{i=1}^{n}\mathfrak{p}_i$ of B. In fact it is clear that $A=\pi^{-1}(w(R))$ is equal to $C+\bigcap_{i=1}^{n}\mathfrak{p}_i$. In particular if $(B;\mathfrak{p}_1,\dots,\mathfrak{p}_n)$ can be glued over a subring C of B, then it can be also glued over $C+\bigcap_{i=1}^{n}\mathfrak{p}_i$.

In the proof of Proposition 1 we have obtained a subring $A=\pi^{-1}(w(R))$ of B from given R and w, which we shall denote by A_R in the following. Moreover let A'_R be the ring obtained from B by glueing over $\mathfrak{p}_i\cap A_R$. We call A'_R *a glueing of* $(B;\mathfrak{p}_1,\dots,\mathfrak{p}_n)$. Assuming that the assertions (i) and (ii) of Proposition 1 are satisfied, we denote by σ the composition of canonical homomorphisms

$$B\xrightarrow{\pi}B/\textstyle\bigcap_{i=1}^{n}\mathfrak{p}_i\longrightarrow\prod_{i=1}^{n}B/\mathfrak{p}_i\hookrightarrow\prod_{i=1}^{n}\kappa(\mathfrak{p}_i)\,.$$

On the other hand we may identify the quotient field $Q(R)$ of R with a subfield of the total quotient ring $Q(B/\bigcap_{i=1}^{n}\mathfrak{p}_i)=\prod_{i=1}^{n}\kappa(\mathfrak{p}_i)$ by a canonical homomorphism w_1 obtained from w, because the image of a non-zero element of R by w is not a zero-divisor of $B/\bigcap_{i=1}^{n}\mathfrak{p}_i$.

PROPOSITION 2. *Assume that the assertions* (i) *and* (ii) *of Proposition* 1 *are satisfied and let the following diagram*

$$\begin{array}{ccc} B & \xleftarrow{\quad f_1 \quad} & S_1 \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle g_1} \\ \prod_{i=1}^{n}\kappa(\mathfrak{p}_i) & \xleftarrow[\quad w_1 \quad]{} & Q(R) \end{array}$$

be the pullback one with respect to σ *and* w_1. *Then* A'_R *is isomorphic to* S_1 *by the homomorphism* f_1.

PROOF. Since R is isomorphic to $\pi(A_R)=A_R/\bigcap_{i=1}^{n}\mathfrak{p}_i$ as seen in the proof of Proposition 1, this is a direct consequence of Lemma 2. q.e.d.

THEOREM 1. *Let the notations be as above and assume that the assertions* (i) *and* (ii) *of Proposition* 1 *are satisfied. Then* A_R *coincides with* A'_R *if and only if we have* $w_1(R)=\sigma(B)\cap w_1(Q(R))$. *In particular if* R *is a normal domain,* A_R *is a glueing of* $(B;\mathfrak{p}_1,\dots,\mathfrak{p}_n)$.

PROOF. First assume that $A_R=A'_R$. Then it is sufficient to show that $w_1(R)$

$\supset \sigma(B) \cap w_1(Q(R))$. Let x be an element of $\sigma(B) \cap w_1(Q(R))$. Since S_1 is isomorphic to the subring of $B \times Q(R)$ consisting of the elements (b, r) satisfying $\sigma(b) = w_1(r)$, there exists an element y in S_1 such that $x = \sigma f_1(y) = w_1 g_1(y)$. By our assumption, $f_1(y)$ is an element of $A_R = A'_R = f_1(S_1)$ and hence we see easily by Remark 1 that $g_1(y) = g(y)$ is in R. This means that $x = w_1 g_1(y)$ is contained in $w_1(R)$.

Conversely assume that $w_1(R) = \sigma(B) \cap w_1(Q(R))$. If y is an element of S_1, $x = \sigma f_1(y) = w_1 g_1(y)$ is in $\sigma(B) \cap w_1(Q(R))$ and hence in $w_1(R)$. Therefore from definition of π, σ, w and w_1 it is easy to see that there is an element z in $B/\bigcap_{i=1}^n \mathfrak{p}_i$ whose image in $\prod_{i=1}^n \kappa(\mathfrak{p}_i)$ by the canonical injection $B/\bigcap_{i=1}^n \mathfrak{p}_i \to \prod_{i=1}^n \kappa(\mathfrak{p}_i)$ is x, and that z is in $\pi(B) \cap w(R)$. This means that there is an element s in S such that $wg(s) = \pi f(s) = z$ in the diagram of Remark 1. Since $\pi f(s) = \pi f_1(y) = z$, $f(s) - f_1(y)$ is in $\bigcap_{i=1}^n \mathfrak{p}_i$. Since A_R contains $\bigcap_{i=1}^n \mathfrak{p}_i$, this means, from Lemma 2 and Proposition 2, that A_R coincides with A'_R. Lastly assume that R is an integral domain. Since $B/\bigcap_{i=1}^n \mathfrak{p}_i$ is integral over $w(R)$, $\sigma(B)$ is integral over $w_1(R)$. Therefore $\sigma(B) \cap w_1(Q(R))$ must be $w_1(R)$ and hence we have $A_R = A'_R$. q.e.d.

§2. Examples

(1) *Let k be a field and let B be a finitely generated k-algebra. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be prime ideals of B. Then $(B; \mathfrak{p}_1, \dots, \mathfrak{p}_n)$ can be glued over a subring of B if and only if we have* $\dim B/\mathfrak{p}_i = \dim B/\mathfrak{p}_j$ *for any i and j.* In fact $B/\bigcap_{i=1}^n \mathfrak{p}_i$ is also a finitely generated k-algebra, and hence, from Noether's normalization lemma (cf. [1], Chap. 5, § 3, Th. 1), there is a subring R of $B/\bigcap_{i=1}^n \mathfrak{p}_i$ isomorphic to a polynomial ring over k such that $B/\bigcap_{i=1}^n \mathfrak{p}_i$ is a finite R-module. Therefore $(B; \mathfrak{p}_1, \dots, \mathfrak{p}_n)$ can be glued over a subring A_R of B by Proposition 1, and moreover we have $A_R = A'_R$ by Theorem 1. In particular *if we choose algebraically independent elements $z_1, \dots, z_d$ of B over k such that B is integral over $k[z_1, \dots, z_d]$ and that we have $(\bigcap_{i=1}^n \mathfrak{p}_i) \cap k[z_1, \dots, z_d] = (z_1, \dots, z_s)$, then $A = k'[z_1, \dots, z_s] + \bigcap_{i=1}^n \mathfrak{p}_i$ for any subfield k' of k such that $[k: k'] < \infty$ is a glueing of $(B; \mathfrak{p}_1, \dots, \mathfrak{p}_n)$.* This is the main result of Tamone [10] and follows easily from Proposition 1, Remark 2 and Theorem 1 by using the fact that $B/\bigcap_{i=1}^n \mathfrak{p}_i$ is integral over a subring isomorphic to $k'[z_1, \dots, z_d]/(z_1, \dots, z_s) \cong k'[z_{s+1}, \dots, z_d]$. Moreover A is a finitely generated k'-algebra by Lemma 9, in [1], Chap. 5, § 1, no. 9.

(2) *Let k be a valued field and let B be an analytic k-algebra. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be prime ideals of B. Then $(B; \mathfrak{p}_1, \dots, \mathfrak{p}_n)$ can be glued over a subring of B if and only if we have* $\dim B/\mathfrak{p}_i = \dim B/\mathfrak{p}_j$ *for any i and j.* This follows also from an analogous result for analytic k-algebras to Noether's normalization lemma for finitely generated algebras over a field. Precisely, $B/\bigcap_{i=1}^n \mathfrak{p}_i$ contains a regular local ring over which it is a finite module (cf. Sätze 3, 4 in Kap. II, § 5

of [5]).

(3) *Let B be a noetherian ring, and let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be prime ideals of B which may be equal to each other. Let π_i be the canonical homomorphism $B \to B/\mathfrak{p}_i$ for $i=1, 2$. Let ϕ be an isomorphism of $B/\mathfrak{p}_1$ onto $B/\mathfrak{p}_2$ and let A be the subring of B consisting of the elements b in B such that $\phi\pi_1(b)=\pi_2(b)$. Assume that $B/\mathfrak{p}_1$ is a finite A-module by the canonical homomorphism $A \hookrightarrow B \to B/\mathfrak{p}_1$. Then A is a glueing of $(B; \mathfrak{p}_1, \mathfrak{p}_2)$.*

PROOF. It is easy to see $A \cap \mathfrak{p}_1 = A \cap \mathfrak{p}_2 = \mathfrak{p}_1 \cap \mathfrak{p}_2$, which we denote by $\mathfrak{p}$. Since $B/\mathfrak{p}_1$ is a finite A-module, so is $B/\mathfrak{p}_2$. Therefore we have

$$B = Ab_1 + \cdots + Ab_s + \mathfrak{p}_1 = Ab'_1 + \cdots + Ab'_s + \mathfrak{p}_2$$

for some elements b_i and b'_j in B. Now let $\mathfrak{p}_1$ be generated by $x_1, \ldots, x_t$. If b is an element of B, we see

$$b = a_1 b_1 + \cdots + a_s b_s + c_1 x_1 + \cdots + c_t x_t$$

for some elements a_i in A and c_j in B. Moreover there are elements a_{ij} in A and y_i in $\mathfrak{p}_2$ for $1 \leqq i \leqq t$ and $1 \leqq j \leqq s$ such that

$$c_i = a_{i1} b'_1 + \cdots + a_{is} b'_s + y_i.$$

This means that we have

$$b = a_1 b_1 + \cdots + a_s b_s + \textstyle\sum_{i,j} a_{ij} b'_j x_i + \sum_i y_i x_i$$

and hence

$$B = A + Ab_1 + \cdots + Ab_s + \textstyle\sum_{i,j} Ab'_j x_i,$$

because $\sum_i y_i x_i$ is an element of $\mathfrak{p}_1 \cap \mathfrak{p}_2$ which is contained in A. In particular if we put $R = A/\mathfrak{p}$, then $B/\mathfrak{p}_1 \cap \mathfrak{p}_2$ contains a subring isomorphic to R over which it is a finite module. Therefore $(B; \mathfrak{p}_1, \mathfrak{p}_2)$ can be glued over A by Proposition 1 and Remark 2. Next let ϕ^* be the isomorphism of $\kappa(\mathfrak{p}_1)$ onto $\kappa(\mathfrak{p}_2)$ obtained from ϕ naturally and h_i the canonical injection of $\kappa(\mathfrak{p}) = Q(A/\mathfrak{p}) = Q(R)$ into $\kappa(\mathfrak{p}_i)$ for $i=1, 2$. Let σ and w_1 be as in Proposition 2 and Theorem 1, and let ρ_i be the canonical projection of $\kappa(\mathfrak{p}_1) \times \kappa(\mathfrak{p}_2)$ for $i=1, 2$. Then if we have $\sigma(b) = w_1(x)$ for $b \in B$ and $x \in Q(R)$, we see, from $h_2 = \phi^* h_1$, $\pi_i = \rho_i \sigma$ and $h_i = \rho_i w_1$,

$$\begin{aligned} \phi\pi_1(b) &= \phi^*\pi_1(b) = \phi^*\pi_1\sigma(b) = \phi^*\rho_1 w_1(x) \\ &= \phi^* h_1(x) = h_2(x) = \rho_2 w_1(x) = \rho_2\sigma(b) = \pi_2(b) \end{aligned}$$

and hence b is an element of A. This means that $\sigma(b) = w_1(x)$ belongs to $\sigma(A) = w_1(R)$ and hence that we have $w_1(R) = \sigma(B) \cap w_1(Q(R))$. By Theorem 1 and

Remark 2 our assertion follows from this and the fact that A contains $\mathfrak{p}=\mathfrak{p}_1 \cap \mathfrak{p}_2$.
q.e.d.

REMARK 3. Pedrini treated some special cases of the above example (3) in Theorems 1 and 3 of his paper [8].

§3. Property (S_2) and glueings

Let A be a noetherian ring. Then we recall that *A has Serre's property (S_2) if and only if* Ass(A) *and* Ass(A/fA) *for any regular element f of A have no embedded prime ideals.* First we show a lemma giving a condition for a noetherian ring with (S_1) to have (S_2), which is due to S. Itoh and essential in later discussion.

LEMMA 3. *Let A be a noetherian ring whose prime ideals of height ≥ 1 contain regular elements. Let $A^{(1)}$ be the set of elements z in the total quotient ring $Q(A)$ of A such that any prime ideal of A containing the ideal $A:_A z$ is of height ≥ 2. Then A has (S_2) if any only if $A=A^{(1)}$.*

PROOF. First assume that A has (S_2). By definition A is contained in $A^{(1)}$. If $A^{(1)}$ contains an element $z=a/b$ not belonging to A where a and b are in A and b is regular, then any minimal prime divisor $\mathfrak{p}$ of $A:_A z=bA:_A a$ is of height ≥ 2. Moreover we see depth $A_\mathfrak{p}=1$, because $\mathfrak{p}A_\mathfrak{p}$ is the radical ideal of $(bA:_A a)A_\mathfrak{p}$. On the other hand we have depth $A_\mathfrak{p} \geq \inf(2, \operatorname{ht} A_\mathfrak{p}) \geq 2$ from (S_2). This is a contradiction. Therefore we see that $A=A^{(1)}$. Conversely assume that A does not have (S_2). Then there exists a prime ideal $\mathfrak{p}$ of A such that depth $A_\mathfrak{p} \leq 1$ and ht $A_\mathfrak{p} \geq 2$, because A has (S_1). If b is a regular element in $\mathfrak{p}$, the image $\bar{b}$ of b in $A_\mathfrak{p}$ is also regular in $A_\mathfrak{p}$. Therefore we see depth $A_\mathfrak{p}=1$. This means easily that $\mathfrak{p}A_\mathfrak{p}$ is an element of $\operatorname{Ass}_{A_\mathfrak{p}}(A_\mathfrak{p}/\bar{b}A_\mathfrak{p})$ and hence that $\mathfrak{p}$ is an element of $\operatorname{Ass}_A(A/bA)$ by Lemma (7.C) in [6]. Therefore there exists an element a in A such that $bA:_A a=A:_A a/b=\mathfrak{p}$. Then a/b belongs to $A^{(1)}$ from the fact that $\operatorname{ht}(\mathfrak{p}) \geq 2$. Moreover a/b is not an element of A by $\mathfrak{p}=A:_A a/b$. So A does not coincides with $A^{(1)}$. q.e.d.

Now we give the following theorem which includes Theorems 2 and 4 of Pedrini [8] as special cases.

THEOREM 2. *Let B be a noetherian ring and let $\mathfrak{p}_1,\dots, \mathfrak{p}_n$ be prime ideals of B. Let A be a glueing of $(B; \mathfrak{p}_1,\dots, \mathfrak{p}_n)$ and put $\mathfrak{p}=A \cap \mathfrak{p}_i$. Then we have the followings:*

(i) *If any $\mathfrak{p}_i$ is of height 1 for $i=1,\dots, n$ and if B has (S_2), then A has also (S_2).*

(ii) *Assume that A is not equal to B. If any $\mathfrak{p}_i$ contains a regular element of B for $i=1,\dots, n$, then $A_\mathfrak{p}$ is of depth 1. Furthermore if some $\mathfrak{p}_i$ is of height >1,* *then A does not have* (S_2).

PROOF. (i) First we show that $\mathrm{ht}(\mathfrak{p})=1$. It is clear that $\mathrm{ht}(\mathfrak{p})\geq 1$, because $\mathrm{ht}(\mathfrak{p}_i)=1$. If $\mathrm{ht}(\mathfrak{p})\geq 2$, there are prime ideals $\mathfrak{q}_1$ and $\mathfrak{q}_2$ of A such that $\mathfrak{p}\supsetneqq\mathfrak{q}_1\supsetneqq\mathfrak{q}_2$. Then, by Going-up theorem, there are three prime ideals $\mathfrak{P}$, $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ of B such that $\mathfrak{P}\supsetneqq\mathfrak{Q}_1\subsetneqq\mathfrak{Q}_2$, $\mathfrak{P}\cap A=\mathfrak{p}$, $\mathfrak{Q}_1\cap A=\mathfrak{q}_1$ and $\mathfrak{Q}_2\cap A=\mathfrak{q}_2$. Since $\mathfrak{P}$ is one of $\mathfrak{p}_1,\dots,\mathfrak{p}_n$, $\mathfrak{P}$ is of height 1. This is a contradiction. Now let $\mathfrak{Q}$ be a prime ideal of B and put $\mathfrak{q}=\mathfrak{Q}\cap A$. Then we see that $\mathrm{ht}(\mathfrak{Q})\geq 2$ if any only if $\mathrm{ht}(\mathfrak{q})\geq 2$. In fact if $\mathfrak{Q}$ contains $\mathfrak{p}=\bigcap_{i=1}^n\mathfrak{p}_i$, so does $\mathfrak{q}$. Then we see that $\mathrm{ht}(\mathfrak{Q})\geq 2\Leftrightarrow Q\supsetneqq\mathfrak{p}_i$ for some $i\Leftrightarrow\mathfrak{q}\supsetneqq\mathfrak{p}\Leftrightarrow\mathrm{ht}(\mathfrak{q})\geq 2$. If $\mathfrak{Q}$ does not contains $\mathfrak{p}$, we have $\mathfrak{q}\not\supseteq\mathfrak{p}$. Then we see $B_{\mathfrak{Q}}=A_{\mathfrak{q}}$ by Lemma 1.6 of [11] and hence $\mathrm{ht}(\mathfrak{Q})=\mathrm{ht}(\mathfrak{q})$. Since we have $\mathrm{ht}(\mathfrak{p})=\mathrm{ht}(\mathfrak{p}_i)=1$ for $i=1,\dots,n$, and $\mathfrak{p}=\bigcap_{i=1}^n\mathfrak{p}_i$ contains a regular element of B, we see easily that the total quotient ring of A may be considered to coincides with that of B. Let z be an element of $A^{(1)}$ and let $\mathfrak{a}$ be the ideal $A:_A z$. Then $\mathfrak{a}B$ is contained in $B:_B z$. Therefore if $\mathfrak{P}$ is any prime ideal of B containing $B:_B z$, $\mathfrak{P}\cap A$ is a prime ideal of A containing $\mathfrak{a}$ and hence is of height ≥ 2. This means from the above that $\mathfrak{P}$ is of height ≥ 2, and so z is an element of $B^{(1)}$. Therefore we see that $A^{(1)}\subset B^{(1)}=B$ by Lemma 3, because B has (S_2) and hence (S_1). Let x be an element of $A^{(1)}$. Then x belongs to B, but $\mathfrak{p}$ does not contains $A:_A x$, because $\mathrm{ht}(\mathfrak{p})=1$. If s is an element of $(A:_A x)\setminus\mathfrak{p}$, $a=sx$ is an element of A. Let w_i be the canonical homomorphism $\kappa(\mathfrak{p})\to\kappa(\mathfrak{p}_i)$, and π_i the composition of the canonical homomorphisms $B\to B/\mathfrak{p}_i\to\kappa(\mathfrak{p}_i)$ for $i=1,\dots,n$. Then we see easily that $\pi_i(x)\in w_i(\kappa(\mathfrak{p}))$ and $w_i^{-1}(\pi_i(x))=w_j^{-1}(\pi_j(x))=\bar{a}/\bar{s}$ for $i, j=1,\dots,n$, where $\bar{a}$ and $\bar{s}$ are the classes of a and s in $A/\mathfrak{p}\subset\kappa(\mathfrak{p})$. This means by Lemma 1 that x belongs to A, since A is a glueing of $(B;\mathfrak{p}_1,\dots,\mathfrak{p}_n)$. Therefore we see $A=A^{(1)}$. Since B has (S_2) and we see easily $\mathrm{ht}(\mathfrak{Q}\cap A)=0$ for any prime ideal $\mathfrak{Q}$ of B of height 0, A has also (S_1). Therefore A has (S_2) by Lemma 3.

(ii) Let S be the multiplicatively closed subset $A\setminus\mathfrak{p}$ of B. Then $A_{\mathfrak{p}}=A_S$ is seminormal in B_S by Corollary 2.2 in [11] and the conductor $A_S:_{B_S}B_S$ is $\mathfrak{p}A_{\mathfrak{p}}=\bigcap_{i=1}^n\mathfrak{p}_iB_S$, because the conductor $A:_B B$ is equal to $\mathfrak{p}=\bigcap_{i=1}^n\mathfrak{p}_i$ by Proposition 1.2 in [10]. Then we see easily from the proof of Theorem 2.1 in [11] that $A_{\mathfrak{p}}$ is a glueing of $(B_S;\mathfrak{p}_1B_S,\dots,\mathfrak{p}_nB_S)$. Moreover if an element a in $\mathfrak{p}_i$ is regular in B, the image of a in B_S is regular in B_S. Therefore we may assume that B is a semilocal ring with the maximal ideals $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ and that A is a local ring with the maximal ideal $\mathfrak{p}$. Since $\mathfrak{p}$ and $\mathfrak{p}_i$ contain regular elements of B and we see easily from Lemma 1.6 in [11] that any regular element of A is also regular in B, we may consider that the total quotient ring $Q(A)$ of A is a subring of the total quotient ring $Q(B)$ of B. Then we see that $Q(A)=Q(B)$, because $\mathfrak{p}=A:_B B$ contains a regular element of B. Therefore if z is an element of B which does not belongs to A, there are a regular element b of A and an element a of A such that $z=a/b$

in $Q(A)=Q(B)$. Then we see that $z\mathfrak{p}\subset B(A:_B B)\subset A$ and hence that $\mathfrak{p}=A:_A z$, since $zA \not\subset A$. This means that $bA:_A a=\mathfrak{p}$, and so $\mathfrak{p}$ is contained in $\mathrm{Ass}_A(A/bA)$. In other words we have $\operatorname{depth} A=1$. In particular if some $\mathfrak{p}_i$ is of height >1, so is $\mathfrak{p}$. Therefore A does not have (S_2). q.e.d.

COROLLARY. *Let B be a noetherian ring, and let A be a subring of B which is seminormal in B. Assume that B is a finite A-module. If a prime divisor in A of the conductor $\mathfrak{c}$ of A in B is of height >1 and if $\mathfrak{c}$ contains a regular element of B, then A does not have (S_2).*

PROOF. If $\mathfrak{p}$ is a prime divisor in A of $\mathfrak{c}$ of height >1, let S be the multiplicatively closed subset $A\setminus\mathfrak{p}$ of A. Then A_S is seminormal in B_S and $\mathfrak{c}_S$ is the conductor of A_S in B_S as seen in the proof of Theorem 2, (ii). Since $\mathfrak{c}$ is a radical ideal by Lemma 1.3 in [11], $\mathfrak{c}_S$ is the maximal ideal of $A_S=A_{\mathfrak{p}}$. Therefore we see, from the proof of Theorem 2.1 in [11], that A_S is obtained from B_S by glueing over the prime ideal $\mathfrak{p}A_S$ of A_S of height >1. This means by Theorem 2, (ii) that A_S does not have (S_2). Therefore, from the definition of (S_2), it is easy to see that A does not have (S_2). q.e.d.

REMARK 4. Let B and A be as in Corollary to Theorem 2. Assume that B has property (S_2). Then even if any prime divisor in B of the conductor of A in B is of height 1, A does not necessarily have property (S_2). In fact let B be *a* finitely generated algebra over an algebraically closed field k. Assume that B is a normal domain and the dimension of B is larger than 1, and let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two prime ideals of B such that $\operatorname{ht}(\mathfrak{p}_1)=\operatorname{ht}(\mathfrak{p}_2)=1$. Then we see $\dim B/\mathfrak{p}_1=\dim B/\mathfrak{p}_2=\dim B-1\geq 1$ and hence there is a glueing A_1 of $(B; \mathfrak{p}_1, \mathfrak{p}_2)$ as seen in § 2, (1). If $\mathfrak{p}$ is the prime ideal $A_1\cap\mathfrak{p}_1=A_1\cap\mathfrak{p}_2$ of A_1, we have $\dim A_1/\mathfrak{p}=\dim B/\mathfrak{p}_i\geq 1$ and hence $\mathfrak{p}$ is not a maximal ideal of A_1. Since A_1 is also a finitely generated k-algebra as seen in § 2, (1), there are infinitely many maximal ideals of A_1 containing $\mathfrak{p}$. Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be such two different maximal ideals of A_1 and let A be a glueing of $(A_1; \mathfrak{m}_1, \mathfrak{m}_2)$. Then we see easily that A is seminormal in B and that the conductor of A in B is the ideal $\mathfrak{p}_1\cap\mathfrak{p}_2$. Then A does not have property (S_2), although B does.

REMARK 5. The author proved originally Theorem 2 and its Corollary for noetherian domains using a well known criterion for a noetherian domain A to have (S_2): A has (S_2) if and only if $A=\bigcap_{\mathfrak{p}}A_{\mathfrak{p}}$ where $\mathfrak{p}$ runs over all prime ideals of A of height 1. However S. Itoh pointed that if we use Lemma 3 instead of this criterion, the same proof can work in the more general cases stated in the above results. On the other hand S. Goto obtained in [4] the following result holding a close similarity to ours: Let B be a semilocal Cohen-Macaulay ring with maximal ideals $\mathfrak{n}_1,\dots,\mathfrak{n}_n$. Assume that we have $\dim B=\dim B_{\mathfrak{n}_i}$ for any i, and let A be a glueing of $(B; \mathfrak{n}_1,\dots, \mathfrak{n}_n)$ different from B. Then A is a Cohen-Macaulay ring if and only if $\dim B = 1$.

§4. Glueings of semilocal rings

Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Denote the multiplicity $e_A(\mathfrak{m})$ of $\mathfrak{m}$, the embedded dimension $\dim_k \mathfrak{m}/\mathfrak{m}^2$ and the graded ring $\sum_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ associated to A by $e(A)$, embdim(A) and $G(A)$, respectively. Furthermore let $G(A)^+$ be the ideal $\sum_{i=1}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ of $G(A)$. Then we have the following

LEMMA 4. *Let B be a noetherian semilocal local ring, and let $\mathfrak{n}_1,\dots, \mathfrak{n}_n$ be the maximal ideals of B. If A is a glueing of $(B; \mathfrak{n}_1,\dots, \mathfrak{n}_n)$, then A is a local ring and $G(A)$ is isomorphic to a graded subring G of $G(B_{\mathfrak{n}_1})+\cdots+G(B_{\mathfrak{n}_n})$ such that $G^+ = G(B_{\mathfrak{n}_1})^+ +\cdots+ G(B_{\mathfrak{n}_n})^+$. In particular we have*

$$\text{embdim}(A) = \sum_{i=1}^{n} [\kappa(\mathfrak{n}_i)\colon \kappa(\mathfrak{m})]\, \text{embdim}(B_{\mathfrak{n}_i}),$$

where $\mathfrak{m}$ is the maximal ideal $\mathfrak{n}_i \cap \cdots \cap \mathfrak{n}_n$ of A. Moreover suppose that $\text{ht}(\mathfrak{n}_i) = \text{ht}(\mathfrak{m})$ *for* $1 \leqq i \leqq s$ *and* $\text{ht}(\mathfrak{n}_j) < \text{ht}(\mathfrak{m})$ *for* $s+1 \leqq j \leqq n$. *Then we have*

$$e(A) = \sum_{i=1}^{s} [\kappa(\mathfrak{n}_i)\colon \kappa(\mathfrak{m})] e(B_{\mathfrak{n}_i}).$$

PROOF. It is easy to see that A is a local ring and that $\mathfrak{m}$ coincides with the Jacobson radical $\mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_n = \mathfrak{n}_1 \cdots \mathfrak{n}_n$ of B, because A is a glueing of $(B; \mathfrak{n}_1,\dots, \mathfrak{n}_n)$. Hence $\mathfrak{m}^s$ is equal to $\mathfrak{n}_1^s \cap \cdots \cap \mathfrak{n}_n^s$ for $s = 1, 2,\dots$. Let f_s be the canonical homomorphism $B/\mathfrak{n}_1^s \cap \cdots \cap \mathfrak{n}_n^s \to B/\mathfrak{n}_1^s \oplus \cdots \oplus B/\mathfrak{n}_n^s$ for each $s > 1$. Since $\mathfrak{n}_i^s$ and $\mathfrak{n}_j^s$ are coprime for any $i \neq j$, f_s is an isomorphism by Chinese remainder theorem. Therefore we see easily that f_s induces an isomorphism between $\mathfrak{m}^s/\mathfrak{m}^{s+1} = \mathfrak{n}_1^s \cap \cdots \cap \mathfrak{n}_n^s/\mathfrak{n}_1^{s+1} \cap \cdots \cap \mathfrak{n}_n^{s+1}$ and $\mathfrak{n}_1^s/\mathfrak{n}_1^{s+1} \oplus \cdots \oplus \mathfrak{n}_n^s/\mathfrak{n}_n^{s+1}$ for each $s \geqq 1$, and hence that $G(A) \simeq A/\mathfrak{m} \oplus G(B_{\mathfrak{n}_1})^+ \oplus \cdots \oplus G(B_{\mathfrak{n}_n})^+$ as graded rings. Since $A/\mathfrak{m}$ is isomorphic to a subring of $B/\mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_n = B/\mathfrak{n}_1 \oplus \cdots \oplus B/\mathfrak{n}_n$, the first assertion is proved. The second one is a direct consequence of the above isomorphism. Lastly we see also from the above that

$$\text{length}_A(B/\mathfrak{m}^s) = \sum_{i=1}^{n} \text{length}_A(B/\mathfrak{n}_i^s) = \sum_{i=1}^{n} \text{length}_B(B/\mathfrak{n}_i^s)[\kappa(\mathfrak{n}_i)\colon \kappa(\mathfrak{m})].$$

Since we have $\text{length}_A(A/\mathfrak{m}^s) = \text{length}_A(B/\mathfrak{m}^s) - \text{length}_A(B/A)$ and $\text{length}_A(B/A) = \text{length}_A((B/\cap_{i=1}^{n} \mathfrak{n}_i)/(A/\mathfrak{m})) < \infty$, we see, from the definition of multiplicity and our assumption on heights of maximal ideals $\mathfrak{n}_1,\dots, \mathfrak{n}_n$,

$$e(A) = \sum_{i=1}^{s} [\kappa(\mathfrak{n}_i)\colon \kappa(\mathfrak{m})] e(B_{\mathfrak{n}_i}). \qquad \text{q.e.d.}$$

PROPOSITION 3. *Let B be a noetherian semilocal ring with maximal* *ideals* $\mathfrak{n}_1,\ldots,\mathfrak{n}_n$, *and let A be a local subring of B with maximal ideal* $\mathfrak{m}$ *such that B is a finite A-module. Then the followings are equivalent if B is not equal to A*:

(i) *A is seminormal in B and the conductor of A in B is* $\mathfrak{m}$*-primary.*

(ii) *A is a glueing of* $(B;\ \mathfrak{n}_1,\ldots,\mathfrak{n}_n)$.

(iii) $\mathfrak{m}$ *is equal to the Jacobson radical* $\mathfrak{n}=\mathfrak{n}_1\cap\cdots\cap\mathfrak{n}_n$ *of B.*

(iv) *The canonical homomorphism* $f\colon \mathfrak{m}/\mathfrak{m}^2\to\mathfrak{n}/\mathfrak{n}^2$ *induced by the injection* $A\hookrightarrow B$ *is surjective.*

(v) *The homomorphism f is bijective.*

(vi) *We have* $\mathfrak{n}^2\cap\mathfrak{m}=\mathfrak{m}^2$ *and* $\mathrm{embdim}\,(A)=\sum_{i=1}^{n}[\kappa(\mathfrak{n}_i)\colon\kappa(\mathfrak{m})]\,\mathrm{embdim}\,(B_{\mathfrak{n}_i})$.

(vii) *The homomorphism f induces a bijection between the ideals* $G(A)^+$ *and* $G(B_{\mathfrak{n}_1})^+\oplus\cdots\oplus G(B_{\mathfrak{n}_n})^+$.

Proof. (i)⇒(ii). Assume that A is seminormal in B. Then the conductor $\mathfrak{c}$ of A in B must be a radical ideal by Lemma 1.3 of [11]. Therefore if $\mathfrak{c}$ is $\mathfrak{m}$-primary, we see $\mathfrak{c}=\mathfrak{m}$. Then we see from the proof of Theorem 2.1 of [11] that A is the ring obtained from B by glueing over $\mathfrak{m}$. In other words A is a glueing of $(B;\ \mathfrak{n}_1,\ldots,\mathfrak{n}_n)$.

(ii)⇒(vii). This is a direct consequence of Lemma 4.

(vii)⇒(v)⇒(iv). Trivial.

(iv)⇒(iii). Let A' be the ring obtained from B by glueing over $\mathfrak{m}$. Then A' is a local ring with maximal ideal $\mathfrak{n}=\mathfrak{n}_1\cap\cdots\cap\mathfrak{n}_n$. If f is surjective, we have $\mathfrak{n}=\mathfrak{m}+\mathfrak{n}^2$ and hence $\mathfrak{n}=\mathfrak{m}A'+\mathfrak{n}\mathfrak{n}$. Therefore we have $\mathfrak{n}=\mathfrak{m}A'$ by Nakayama's lemma. This means that we have $\mathfrak{m}A'=\mathfrak{m}+(\mathfrak{m}A')^2=\mathfrak{m}+\mathfrak{m}(\mathfrak{m}A')$. Since B is noetherian and a finite A-module, A is also noetherian by Nagata-Eakin's theorem (cf. [3]). Therefore A' and $\mathfrak{m}'$ are both finite A-modules and so we see $\mathfrak{m}=\mathfrak{m}A'$ again by Nakayama's lemma. This means that $\mathfrak{m}=\mathfrak{n}$.

(iii)⇒(ii). Let A' be as above. Then we see $A'/\mathfrak{n}$ is isomorphic to $A/\mathfrak{m}$ by the induced homomorphism of the injection $A\hookrightarrow A'$. Therefore we have $A'=A+\mathfrak{n}$ $=A+\mathfrak{m}=A$ from the assumption that $\mathfrak{m}=\mathfrak{n}$.

(ii)⇒(i). This is well known. (Cf. § 1 in [11].)

(v)⇔(vi). Since the kernel of f is $\mathfrak{n}^2\cap\mathfrak{m}/\mathfrak{m}^2$, f is injective if and only if $\mathfrak{n}^2\cap\mathfrak{m}$ $=\mathfrak{m}^2$. Moreover the right hand side of the second equality in (vi) is equal to the dimension of the vector space $\mathfrak{n}/\mathfrak{n}^2$ over $k=A/\mathfrak{m}$ as seen in the proof of Lemma 4. From these facts our assertion follows easily. q.e.d.

Proposition 4. *Let k be an algebraically closed field, and let A be a local ring of a closed point of an algebraic variety defined over k. Let B be the integral closure of A in the quotient field of A, and let* $\mathfrak{n}_1,\ldots,\mathfrak{n}_n$ *be the maximal ideals of B. Then the followings are equivalent*:

(i) *B is regular and A is a glueing of* $(B;\ \mathfrak{n}_1,\ldots,\mathfrak{n}_n)$.

(ii) *The multiplicity* $e(A)$ *of* A *coincides with the number* n *of the maximal ideals of* B *and the conductor of* A *in* B *is the maximal ideal* $\mathfrak{m}$ *if* $e(A)>1$.

PROOF. First notice that we have $\mathrm{ht}(\mathfrak{n}_i)=\dim B=\dim A$ for any i and that $B_{\mathfrak{n}_i}$ is unmixed in the sense of §25 in [7] as seen easily from analytically unramifiedness of $B_{\mathfrak{n}_i}$ (cf. (36.4) in [7] or Theorem 31 of Chap. VIII in [12]). Now assume that the assertion (i) is true. Then $B_{\mathfrak{n}_i}$ is a regular local ring and hence we see $e(B_{\mathfrak{n}_i})=1$. If $\mathfrak{m}$ is the maximal ideal of A, then $B/\mathfrak{n}_j$ is isomorphic to $A/\mathfrak{m}$ for each i by the assumption that k is algebraically closed. Therefore we have $e(A)=\sum_{j=1}^{n} e(B_{\mathfrak{n}_j})=n$ by Lemma 4. On the other hand we see $\mathfrak{m}=\mathfrak{n}_1\cap\cdots\cap\mathfrak{n}_n$ by Proposition 3. This means that the conductor of A in B is equal to $\mathfrak{m}$ if $e(A)=n>1$. Conversely assume that $e(A)$ is equal to n and that $B\supsetneqq A$. Then, since $\mathrm{ht}(\mathfrak{n}_i)=\dim A$ for each i, we see $n=\sum_{i=1}^{n} e(\mathfrak{m}B_{\mathfrak{n}_i})$ by a similar argument in the proof of Lemma 4 or by Corollary 1 to Theorem 24 of Chap. VIII in [12], where we denote by $e(\mathfrak{m}B_{\mathfrak{n}_i})$ the multiplicity of an $\mathfrak{n}_iB_{\mathfrak{n}_i}$-primary ideal $\mathfrak{m}B_{\mathfrak{n}_i}$. This means that $e(\mathfrak{m}B_{\mathfrak{n}_i})=1$ for any i. Since $\mathfrak{m}B_{\mathfrak{n}_i}$ is $\mathfrak{n}_iB_{\mathfrak{n}_i}$-primary, we see easily $1\leqq e(\mathfrak{n}_iB_{\mathfrak{n}_i})\leqq e(\mathfrak{m}B_{\mathfrak{n}_i})=1$ by the definition of multiplicities of $\mathfrak{n}_iB_{\mathfrak{n}_i}$-primary ideals. Then we have $e(B_{\mathfrak{n}_i})=e(\mathfrak{n}_iB_{\mathfrak{n}_i})=1$ and hence $B_{\mathfrak{n}_i}$ is a regular local ring by Theorem (40.6) in [7], because $B_{\mathfrak{n}_i}$ is unmixed as noticed in the above. Therefore B is regular. Moreover there is a system of parameters $x_1,\dots, x_d$ of $\mathfrak{m}B_{\mathfrak{n}_i}$ such that $e(\mathfrak{m}B_{\mathfrak{n}_i})=e((x_1,\dots, x_d)B_{\mathfrak{n}_i})$ by Theorem (24.1) in [7], because $B_{\mathfrak{n}_i}/\mathfrak{n}_iB_{\mathfrak{n}_i}\cong B/\mathfrak{n}_i$ is an algebraically closed field. Since $B_{\mathfrak{n}_i}$ is regular, any system of parameters of $B_{\mathfrak{n}_i}$ is distinct by Theorem (25.7) in [7] and hence we see length $B_{\mathfrak{n}_i}/(x_1,\dots, x_d)B_{\mathfrak{n}_i}$ $=e((x_1,\dots, x_d)B_{\mathfrak{n}_i})=e(\mathfrak{m}B_{\mathfrak{n}_i})=1$. This means that $(x_1,\dots, x_d)B_{\mathfrak{n}_i}$ coincides with the maximal ideal $\mathfrak{n}_iB_{\mathfrak{n}_i}$ and hence we have $\mathfrak{m}B_{\mathfrak{n}_i}=\mathfrak{n}_iB_{\mathfrak{n}_i}$. Since $\mathfrak{n}_1,\dots, \mathfrak{n}_n$ are the prime divisors of $\mathfrak{m}B$, we see $\mathfrak{m}B=\mathfrak{n}_1\cap\cdots\cap\mathfrak{n}_n$. On the other hand if $\mathfrak{m}$ is the conductor of A in B, then $\mathfrak{m}B$ is an ideal of A containing $\mathfrak{m}$ and so coincides with $\mathfrak{m}$. Therefore A is a glueing of $(B; \mathfrak{n}_1,\dots, \mathfrak{n}_n)$ by the equivalence (ii)⇔(iii) in Proposition 3. q.e.d.

REMARK 6. Let A and B be as in Proposition 4. From the proof of Proposition 4 we see that if $e(A)$ coincides with the number of the maximal ideals of B, then B is regular. However A is not necessarily a glueing of $(B; \mathfrak{n}_1,\dots, \mathfrak{n}_n)$. For example let A be a local ring of an ordinary triple point of a plane curve. Then we have $e(A)=3$ and the derived normal ring B of A in the quotient field of A has exactly three maximal ideals $\mathfrak{n}_1$, $\mathfrak{n}_2$, $\mathfrak{n}_3$. But A is not seminormal and hence not a glueing of $(B; \mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3)$, because singular seminormal points on a plane curve are only ordinary double points (cf. [9]). Therefore the condition that the conductor of A in B is the maximal ideal of A is indispensable in the assertion (ii) of Proposition 4.

REMARK 7. Let A and B be as above, and assume that they satisfy the conditions (i) and (ii) of Proposition 4. Then we see $\mathrm{embdim}(A)=e(A)\times\dim A$ by Lemma 4. On the other hand $G(B_{\mathfrak{n}_i})$ is isomorphic to a polynomial ring over k for each i and hence $G(A)$ is reduced, because it is isomorphic to a subring of a reduced ring $G(B_{\mathfrak{n}_1})+\cdots+G(B_{\mathfrak{n}_n})$ again by Lemma 4. If $\dim A=1$, these two properties means, conversely, that A is seminormal (cf. Theorem 1 in [2]). However the author does not know whether a similar result holds for higher dimensional cases.

REMARK 8. Let A be a reduced noetherian local ring with maximal ideal $\mathfrak{m}$ and infinite residue field $A/\mathfrak{m}$, and let B be the integral closure of A in its total quotient ring. Assume that B is a Cohen-Macaulay ring and a finite A-module, and that we have $\mathrm{ht}(\mathfrak{n}_i)=\dim B$ for each maximal ideal $\mathfrak{n}_i$ of B $(i=1,\dots,n)$. Then if we have $e(A)=\sum_{i=1}^{n}[B/\mathfrak{n}_i: A/\mathfrak{m}]$ and $\mathfrak{m}$ is the conductor of A in B, B is regular and A is a glueing of $(B;\ \mathfrak{n}_1,\dots,\mathfrak{n}_n)$. In fact this can be shown in a similar way to the proof of Proposition 4. But we omit the proof.

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