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On a certain class of irreducible unitary representations of the infinite dimensional rotation group I

Dedicated to Professor Y. Matsushima for his 60th birthday

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Introduction

The purpose of this paper is to show that the McKean's conjecture in [2] is valid for the set of all equivalence classes of irreducible unitary representations of class one.

§ 1. Spherical functions

Let $\boldsymbol{H}$ be a separable Hilbert space over $\boldsymbol{R}$ (or $\boldsymbol{C}$). In this paper, we fix, once for all, an orthonomal basis $\{\xi_j; j \in \boldsymbol{N}\}$ of $\boldsymbol{H}$, where $\boldsymbol{N}$ is the set of all positive integers. Let $\boldsymbol{E}$ be the space algebraically spanned by the basis $\{\xi_j; j \in \boldsymbol{N}\}$. We denote by $\boldsymbol{E}_m$ the space spanned by the set $\{\xi_j; j=1,\dots, m\}$. Then we have $\boldsymbol{E}=\bigcup_{m=1}^{\infty} \boldsymbol{E}_m$. Since a countable inductive limit of nuclear spaces is nuclear, $\boldsymbol{E}$ is a nuclear space. Let G be the group of all isometries g of $\boldsymbol{H}$ such that $g\xi_j=\xi_j$ except finitely many j in $\boldsymbol{N}$. We denote by G_m the group of all elements g in G such that $g\xi_j=\xi_j$ $(j=m+1, m+2,\dots)$. Then we have $G=\bigcup_{m=1}^{\infty} G_m$. By the inductive limit topology G is a topological group. For a g in G_m, putting $g\xi_j=\sum_{i=1}^{m} g_{ij}\xi_i$ $(j=1,\dots, m)$, we can identify g with the matrix (g_{ij}) in $O(m)$ (or $U(m)$).

We denote by $\boldsymbol{E}^*$ the dual space of $\boldsymbol{E}$, then we have a triple

$$\boldsymbol{E} \subset \boldsymbol{H} \subset \boldsymbol{E}^*.$$

By the Bochner-Minlos theorem, there exists a probability measure μ on $\boldsymbol{E}^*$ such that for any ξ in $\boldsymbol{E}$ we have

$$e^{-\|\xi\|^2/2} = \int_{\boldsymbol{E}^*} e^{i\langle x, \xi\rangle} d\mu(x). \tag{1.1}$$

We use the same notation for the dual action of g on $\boldsymbol{E}^*$. Clearly μ is G-invariant. For any g in G and f in $L^2(\boldsymbol{E}^*, \mu)$ we define

$$(\pi_*(g)f)(x) = f(g^{-1}x) \quad \text{for a.e. } x \text{ in } \boldsymbol{E}^*.$$

Then it is easy to see that π_* is a unitary representation of G on $L^2(\boldsymbol{E}^*, \mu)$. For any finite dimensional unitary representation π of G_m let $d\pi$ be the infinitesimal representation of π. Then it is well known that $d\pi(C_m)$ is a symmetric operator, where C_m denote the Casimir operator of G_m (for the definition of the Casimir operator see § 3 and § 5).

Now we put $K=\{g\in G;\ g\xi_1=\xi_1\}$. Let $(\pi, \mathfrak{H})$ be an irreducible unitary representation of G on $\mathfrak{H}$. We call π a class one representation (with respect to K) if the following (A.1) and (A.2) hold.

(A.1) The space of all $\pi(K)$-fixed vectors is of one dimension.
(A.2) Let v_0 be a $\pi(K)$-fixed vector. Then v_0 is $\pi(G_m)$-finite ($m\in \boldsymbol{N}$) and $\lim_{m\to\infty} d\pi(C_m)v_0$ is convergent in $\mathfrak{H}$.

Let $(\pi, \mathfrak{H})$ be a class one repersentation of G. We pick a $\pi(K)$-fixed unit vector v_0 and define a function ϕ_π on G by $\phi_\pi(g)=(v_0, \pi(g)v_0)$ ($g\in G$). Then by (A.1) ϕ_π is independent of the choice of the unit vector v_0. ϕ_π is called the spherical function on G.

PROPOSITION 1. *Let $(\pi, \mathfrak{H})$ and $(\pi', \mathfrak{H}')$ be class one representations. Then π is equivalent to π' if and only if $\phi_\pi=\phi_{\pi'}$.*

PROOF. Assume that π is equivalent to π', then we have an isometry U of $\mathfrak{H}$ onto $\mathfrak{H}'$ such that $\pi'(g)U=U\pi(g)$ ($g\in G$). As U maps the space of $\pi(K)$-fixed vectors onto the space of $\pi'(K)$-fixed vectors, by (A.1) we have $\phi_\pi=\phi_{\pi'}$.

Conversely assume that $\phi_\pi=\phi_{\pi'}$. We define U as follows;

$$U(\textstyle\sum_i c_i\pi(g_i)v_0) = \sum_i c_i\pi'(g_i)v_0'.$$

If we put $v=\sum_i a_i\pi(g_i)v_0$ and $w=\sum_j b_j\pi(h_j)v_0$, then we have

$$\begin{aligned}(Uv, Uw) &= (\textstyle\sum_i a_i\pi'(g_i)v_0', \sum_j b_j\pi'(h_j)v_0') \\ &= \textstyle\sum_{i,j} a_i\bar{b}_j\phi_{\pi'}(g_i^{-1}h_j) = \sum_{i,j} a_i\bar{b}_j\phi_\pi(g_i^{-1}h_j) \\ &= (v, w).\end{aligned}$$

It follows that U is well-defined and preserves the inner product. From the fact that $(\pi, \mathfrak{H})$ and $(\pi', \mathfrak{H}')$ are irreducible, U can be extended to an isometry of $\mathfrak{H}$ onto $\mathfrak{H}'$, so that π is equivalent to π'.

§ 2. Casimir operator

Let $(\pi, \mathfrak{H})$ be a class one representation of G. Then by (A.1) there exists a $\pi(K)$-fixed unit vector v_0. We denote by $\mathfrak{H}_m$ the smallest $\pi(G_m)$-invariant subspace of $\mathfrak{H}$ which contains v_0. Then by (A.2) $\mathfrak{H}_m$ is finite dimensional. Clearly $d\pi(C_m)$ is self-adjoint on $\mathfrak{H}_m$. Let $D_{d\pi(C)}$ denote the space of all elements v in $\mathfrak{H}$ such that

$\lim_{m\to\infty} d\pi(C_m)P_m v$ is convergent where P_m is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_m$. For any v in $D_{d\pi(C)}$ we put

$$d\pi(C)v = \lim_{m\to\infty} d\pi(C_m)P_m v.$$

Then it is easy to see that $d\pi(C)$ defines an unbounded linear operator with domain $D_{d\pi(C)}$. It follows from (A.2) that v_0 is contained in $D_{d\pi(C)}$. Since π is irreducible, $D_{d\pi(C)}$ is dense in $\mathfrak{H}$. For any v and w in $D_{d\pi(C)}$ we have

$$\begin{aligned}(d\pi(C)v, w) &= \lim_{m\to\infty} (d\pi(C_m)P_m v, w) = \lim_{m\to\infty} (d\pi(C_m)P_m v, P_m w)\\ &= \lim_{m\to\infty} (P_m v, d\pi(C_m)P_m w) = (v, d\pi(C)w).\end{aligned}$$

This implies that $d\pi(C)\subset d\pi(C)^*$ where $d\pi(C)^*$ denotes the adjoint operator of $d\pi(C)$. Now suppose that w be any element of the domian of $d\pi(C)^*$. Then there exists a u in $\mathfrak{H}$ such that

$$(d\pi(C)v, w) = (v, u) \quad \text{for all } v \text{ in } D_{d\pi(C)}.$$

For any m in $\boldsymbol{N}$ and for any v in $\mathfrak{H}_m$ we have

$$\begin{aligned}(d\pi(C)v, w) = (d\pi(C_m)P_m v, w) &= (v, d\pi(C_m)P_m w),\\ (v, u) &= (v, P_m u).\end{aligned}$$

This shows that $d\pi(C_m)P_m w = P_m u$ $(m \in \boldsymbol{N})$. Thus we get

$$\lim_{m\to\infty} d\pi(C_m)P_m w = \lim_{m\to\infty} P_m u = u.$$

This implies that $w \in D_{d\pi(C)}$. It follows that $d\pi(C)$ is self-adjoint.

PROPOSITION 2. $\pi(g)d\pi(C) = d\pi(C)\pi(g) \quad (g \in G)$.

PROOF. Let v be any vector in $D_{d\pi(C)}$. Then by (A.2) $\lim_{m\to\infty} d\pi(C_m)P_m v$ is convergent. There exists an m_0 such that $g \in G_{m_0}$. We remark that $g \in G_m$ for any m such that $m \geqq m_0$. Thus we have

$$\pi(g)d\pi(C)v = \pi(g)\lim_{m\to\infty} d\pi(C_m)P_m v = \lim_{m\to\infty} d\pi(C_m)\pi(g)P_m v.$$

Since $\mathfrak{H}_m$ is $\pi(G_m)$-invariant we have

$$\pi(g)d\pi(C)v = \lim_{m\to\infty} d\pi(C_m)P_m(\pi(g)v).$$

This implies that

$$\pi(g)D_{d\pi(C)} = D_{d\pi(C)}, \quad \pi(g)d\pi(C) = d\pi(C)\pi(g) \qquad (g \in G).$$

§3. Wiener-Itô decomposition (real case)

In §3 and §4 we assume that $\boldsymbol{E}$ and $\boldsymbol{H}$ are real vector spaces. For each non-negative integer k we consider the Hermite polynomial;

$$H_k(t) = (-1)^k e^{t^2} \frac{d^k}{dt^k} e^{-t^2} \qquad (t \in \boldsymbol{R}).$$

It satisfies the following equations;

$$(3.1) \qquad H_k''(t) - 2tH_k'(t) + 2kH_k(t) = 0,$$

$$(3.2) \qquad H_k'(t) = 2kH_{k-1}(t),$$

$$(3.3) \quad H_k(c_1 t_1 + \cdots + c_i t_i) = k! \sum_{k_1+\cdots+k_i=k} \prod_j (k_j!)^{-1}(c_j)^{k_j} H_{k_j}(t_j),$$

$$\text{where} \quad c_1^2 + \cdots + c_i^2 = 1.$$

For any non-negative integer n we put

$$\mathfrak{B}_n = \{(\prod_{j=1}^{\infty} n_j! 2^{n_j})^{-1/2} \prod_{j=1}^{\infty} H_{n_j}(\langle x, \xi_j \rangle / 2^{1/2});\ \sum_{j=1}^{\infty} n_j = n,\ n_j \geqq 0\}.$$

Then it is known that $\bigcup_{n=0}^{\infty} \mathfrak{B}_n$ is an orthonomal basis of $L^2(\boldsymbol{E}^*, \mu)$. We denote by $\mathscr{H}_n$ the closed subspace spanned by $\mathfrak{B}_n$. Then we have

$$L^2(\boldsymbol{E}^*, \mu) = \sum_{n=0}^{\infty} \oplus \mathscr{H}_n \quad \text{(Wiener-Itô decomposition)}, \quad \text{(see [1])}.$$

From (3.3) we see that $\mathscr{H}_n$ is $\pi_*(G)$-invariant so that we have the subrepresentation π_n of G on $\mathscr{H}_n$. For any i in $\boldsymbol{N}$ we put

$$\Phi_i^n(x) = (n! 2^n)^{-1/2} H_n(\langle x, \xi_i \rangle / 2^{1/2}) \qquad (x \in \boldsymbol{E}^*).$$

The following Lemma 1 ~ Lemma 4 are well known, but for the sake of completeness, we give a brief outline of the proof of them.

LEMMA 1. *Φ_1^n is a cyclic vector of π_n.*

PROOF. Let V be a space spanned by all elements of the form $\pi_n(g)\Phi_1^n$ ($g \in G$). Pick any w in $V^{\perp}$ and let

$$w = \sum_{n_1+\cdots=n} c_{n_1,\ldots} \prod_j H_{n_j}(\langle x, \xi_j \rangle / 2^{1/2}).$$

Fix any m in $\boldsymbol{N}$ and any non-zero vector $(t_1, \ldots, t_m)$ in $\boldsymbol{R}^m$ and put

$$a_i = (t_1^2 + \cdots + t_m^2)^{-1/2} t_i \qquad (i = 1, \ldots, m).$$

Then there exists a g in G_m such that $g\xi_1 = \sum_{i=1}^m a_i \xi_i$. By (3.3) we have

$$(\pi_n(g)\Phi_1^n)(x) = n! \sum_{n_1+\cdots+n_m=n} \prod_j (n_j!)^{-1}(a_j)^{n_j} H_{n_j}(\langle x, \xi_j \rangle / 2^{1/2}).$$

It follows that

$$0 = (w, \pi_n(g)\Phi_1^n) = \sum_{n_1+\cdots+n_m=n} n! 2^n c_{n_1,\ldots,n_m} a_1^{n_1} \cdots a_m^{n_m}.$$

Hence we have $\sum_{n_1+\cdots+n_m=n} c_{n_1,\ldots,n_m} (t_1)^{n_1} \cdots (t_m)^{n_m} = 0$.

It follows that all coefficients of w are equal to zero. This implies that V is dense in $\mathscr{H}_n$.

LEMMA 2. *Any $\pi_n(G)$-fixed vector in $\mathscr{H}_n$ is equal to zero if $n \neq 0$.*

PROOF. We assume that $n \neq 0$. For any j in $\boldsymbol{N}$, there exists a g in G such that $\pi_n(g)\Phi_j^n = \Phi_1^n$. Let v be any $\pi_n(G)$-fixed vector in $\mathscr{H}_n$. Then we have

$$(v, \Phi_j^n) = (\pi_n(g)v, \pi_n(g)\Phi_j^n) = (v, \Phi_1^n).$$

This implies that $(v, \Phi_1^n) = 0$. Since v is a $\pi_n(G)$-fixed vector, from Lemma 1 we conclude that $v = 0$.

LEMMA 3. *For any $\pi_n(K)$-fixed vector v in $\mathscr{H}_n$, there exists a constant c such that $v = c\Phi_1^n$.*

PROOF. Let v be a $\pi_n(K)$-fixed vector, then v is written as follows;

$$v = \sum_{n_1+\cdots=n} c_{n_1,\ldots} \prod_j H_{n_j}(\langle x, \xi_j\rangle/2^{1/2}) = f_0 + \sum_{l=1}^n f_l \Phi_1^l,$$

where f_l $(l=0,\ldots, n)$ are independent of $\langle x, \xi_1\rangle$. As Φ_1^l $(l=1,\ldots, n)$ are $\pi_n(K)$-fixed vectors, for any k in K, we have

$$f_0 + \sum_{l=1}^n f_l \Phi_1^l = v = \pi_n(k)v = \pi_n(k)f_0 + \sum_{l=1}^n (\pi_n(k)f_l)\Phi_1^l.$$

This implies that f_l $(l=0,\ldots, n)$ are $\pi_n(K)$-fixed vectors. By Lemma 2, we have $f_l = 0$ if $l \neq n$. Thus we obtain $v = c\Phi_1^n$ where c is a constant.

LEMMA 4. *$(\pi_n, \mathscr{H}_n)$ is an irreducible unitary representation of G.*

PROOF. Let W be a $\pi_n(G)$-invariant closed subspace in $\mathscr{H}_n$, and let P_W be the orthogonal projection of $\mathscr{H}_n$ onto W. Since $W^\perp$ is again $\pi_n(G)$-invariant for any g in G and v in $\mathscr{H}_n$, we have

$$\pi_n(g)P_W v = P_W \pi_n(g)v. \tag{3.4}$$

It follows that for any k in K

$$P_W \Phi_1^n = P_W \pi_n(k)\Phi_1^n = \pi_n(k)P_W \Phi_1^n.$$

By Lemma 3, there exists a constant c such that $P_W \Phi_1^n = c\Phi_1^n$. From Lemma 1 and (3.4) we have $P_W = cI$ where I is the identity operator on $\mathscr{H}_n$. Thus we conclude that $W = \{0\}$ or $W = \mathscr{H}_n$.

Let $\mathfrak{g}_m$ be the Lie algebra of G_m, and let exp be the exponential mapping of $\mathfrak{g}_m$ to G_m as usual. We denote by E_{ij} the $m \times m$ matrix with 1 in the i, jth position and zeros elsewhere. And we put $X_{ij} = E_{ij} - E_{ji}$. Then $\mathfrak{g}_m$ is canonically identified with the linear Lie algebra generated by $\{X_{ij};\ 1 \leqq i < j \leqq m\}$. We define a bilinear form B: $\mathfrak{g}_m \times \mathfrak{g}_m \rightarrow \boldsymbol{R}$ by $(X, Y) \longrightarrow (m-2)\,\mathrm{tr}XY$. Then B is non-degenerate. We denote by C_m the element of the universal enveloping algebra of $\mathfrak{g}_m$ by the formula

$$C_m = -c_m \sum_{1 \leq i < j \leq m} X_{ij}^2, \quad c_m = 1/(2m-4), \tag{3.5}$$

C_m is called the Casimir operator associated to B.

PROPOSITION 3. *$(\pi_n, \mathscr{H}_n)$ is a class one representation of G.*

PROOF. From Lemma 1 ~ Lemma 4, we have only to show that Φ_1^n satisfies (A.2). It is clear that Φ_1^n is $\pi_n(G_m)$-finite $(m \in \boldsymbol{N})$. Put $x_j = \langle x, \xi_j \rangle$ $(j \in \boldsymbol{N})$. Then any element of the space spanned by $\pi_n(G_m)\Phi_1^n$ can be regarded as a function only of $x_1, \ldots, x_m$. Using this identification we get

$$d\pi_n(X_{ij})^2 = \left(x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}\right)^2.$$

As Φ_1^n is a function only of x_1, we have

$$d\pi_n(C_m)\Phi_1^n(x) = -c_m \left\{ \left(\sum_{j=2}^m x_j^2\right) \frac{\partial^2}{\partial x_1^2} - (m-2)x_1 \frac{\partial}{\partial x_1} \right\} \Phi_1^n(x). \tag{3.6}$$

By the strong law of large numbers we have

$$\lim_{m\to\infty} m^{-1} \sum_{j=1}^m \langle x, \xi_j \rangle^2 = 1 \quad \text{a.e. } x \text{ in } \boldsymbol{E}^*. \tag{3.7}$$

Since Φ_1^n does not depend on m, from (3.6) and (3.7) it follows that

$$\lim_{m\to\infty} d\pi_n(C_m)\Phi_1^n(x) = -2^{-1}\left(\frac{\partial^2}{\partial x_1^2} - x_1 \frac{\partial}{\partial x_1}\right)\Phi_1^n(x).$$

Using the formulas (3.1) and (3.2) we have

$$\lim_{m\to\infty} d\pi_n(C_m)\Phi_1^n = 2^{-1} n \Phi_1^n.$$

Finally we calculate the spherical function ϕ_{π_n}.

PROPOSITION 4. $\phi_{\pi_n}(g) = \langle \xi_1, g\xi_1 \rangle^n \qquad (g \in G)$.

PROOF. Let $g \in G$. Then there exists an m in $\boldsymbol{N}$ such that $g \in G_m$. We put $g\xi_1 = \sum_{j=1}^m g_{j1}\xi_j$. Using (3.3) we have

$$\begin{aligned} \phi_{\pi_n}(g) &= (\Phi_1^n, \pi(g)\Phi_1^n) \\ &= (n!2^n)^{-1}(H_n(\langle \cdot, \xi_1 \rangle/2^{1/2}), \pi(g)H_n(\langle \cdot, \xi_1 \rangle/2^{1/2})) \\ &= (n!2^n)^{-1}(H_n(\langle \cdot, \xi_1 \rangle/2^{1/2}), H_n(\textstyle\sum_{j=1}^m g_{j1}\langle \cdot, \xi_j \rangle/2^{1/2})) \\ &= g_{11}^n = \langle \xi_1, g\xi_1 \rangle^n. \end{aligned}$$

§ 4. McKean's conjecture (real case)

We denote by A the group of all elements g in G_2 such that $\det g=1$. Then we have "the Cartan decomposition"; $G=KAK$. We can identify A with $SO(2)$, and we denote by a_θ the element of A defined by

$$a_\theta\xi_1 = \cos\theta\xi_1 - \sin\theta\xi_2, \quad a_\theta\xi_2 = \sin\theta\xi_1 + \cos\theta\xi_2. \tag{4.1}$$

Let $(\pi, \mathfrak{H})$ be a class one representation of G, and let v_0 be a $\pi(K)$-fixed unit vector. As the spherical function ϕ_π is K-biinvariant, ϕ_π can be considered as a function on A. We define the function F_π on A by $F_\pi(\theta)=\phi_\pi(a_\theta)$ $(a_\theta \in A)$. From Proposition 2 we can use the Schur's Lemma, and conclude that $d\pi(C)$ is a scalar operator; $d\pi(C)=\chi_\pi(C)I$ where $\chi_\pi(C)$ is a constant and I is the identity operator on $\mathfrak{H}$.

THEOREM 1. *Let $(\pi, \mathfrak{H})$ be a class one representation of G with respect to K. Then $2\chi_\pi(C)$ is a non-negative integer, and $(\pi, \mathfrak{H})$ is equivalent to $(\pi_n, \mathscr{H}_n)$ where $n=2\chi_\pi(C)$.*

PROOF. By (A.2) there exists a $\pi(K)$-fixed unit vector v_0 such that $\lim_{m\to\infty} d\pi(C_m)v_0$ is convergent. From the above remark we have

$$\chi_\pi(C)F_\pi(\theta) = (v_0, \pi(a_\theta)d\pi(C)v_0). \tag{4.2}$$

On the other hand we have $(v_0, \pi(a_\theta)d\pi(C)v_0)=\lim_{m\to\infty}(v_0, \pi(a_\theta)d\pi(C_m)v_0)$. Using the formula (3.5) and the fact that $\exp tX_{ij} \in K$ $(i=2,\dots, m)$, we get

$$(v_0, \pi(a_\theta)d\pi(C_m)v_0) = -c_m \textstyle\sum_{j=2}^m (v_0, \pi(a_\theta)d\pi(X_{1j})^2v_0). \tag{4.3}$$

The following formulas are easily checked.

$$\mathrm{Ad}(a_\theta)^{-1}X_{2j} = \cos\theta X_{2j} - \sin\theta X_{1j} \qquad (j=3,\dots, m), \tag{4.4}$$

$$[\mathrm{Ad}(a_\theta)^{-1}X_{2j}, X_{2j}] = \sin\theta X_{12} \qquad (j=3,\dots, m). \tag{4.5}$$

Using (4.4) and (4.5) we have

$$\begin{aligned} X_{1j}^2 &= \operatorname{cosec}^2\theta(\mathrm{Ad}(A_\theta)^{-1}X_{2j})^2 \\ &\quad - \cot\theta\operatorname{cosec}\theta\{2(\mathrm{Ad}(a_\theta)^{-1}X_{2j} - \sin\theta X_{12}\} + \cot^2\theta X_{2j}^2 \\ &\qquad\qquad (j=3,\dots, m). \end{aligned} \tag{4.6}$$

We note that

$$\begin{aligned} &\textstyle\sum_{j=2}^m (v_0, \pi(a_\theta)d\pi(X_{1j})^2v_0) \\ &\quad = (v_0, \pi(a_\theta)d\pi(X_{12})^2v_0) + \textstyle\sum_{j=3}^m (v_0, \pi(a_\theta)d\pi(X_{1j})^2v_0). \end{aligned} \tag{4.7}$$

Clearly the first term is $\frac{d^2}{d\theta^2}F_\pi(\theta)$. Substituting (4.6) into the second term of (4.7), and after some calculations we obtain

$$(4.8)\quad (v_0, \pi(a_\theta)d\pi(C)v_0) = -\lim_{m\to\infty} c_m \left\{\frac{d^2}{d\theta^2}F_\pi(\theta) + (m-2)\cot\theta\frac{d}{d\theta}F_\pi(\theta)\right\}$$
$$= -2^{-1}\cot\theta\frac{d}{d\theta}F_\pi(\theta).$$

Thus by (4.2) and (4.8) we have

$$\chi_\pi(C)F_\pi(\theta) = -2^{-1}\cot\theta\frac{d^2}{d\theta^2}F_\pi(\theta).$$

Since F_π is C^∞ and $F_\pi(0)=1$, we conclude that $2\chi_\pi(C)$ is a non-negative integer and that if we put $2\chi_\pi(C)=n$ we have

$$F_\pi(\theta) = \cos^n\theta.$$

On the other hand, from Proposition 4, putting $g=k'a_\theta k$ we can compute the spherical function of the representation $(\pi_n, \mathscr{H}_n)$ as follows;

$$\phi_{\pi_n}(g) = \langle\xi_1, g\xi_1\rangle^n = \cos^n\theta.$$

Thus we have $\phi_\pi=\phi_{\pi_n}$. It follows from Proposition 1 that $(\pi, \mathfrak{H})$ is equivalent to $(\pi_n, \mathscr{H}_n)$.

§5. Wiener-Itô decomposition (complex case)

In §5 and §6 we assume that $\boldsymbol{E}$ and $\boldsymbol{H}$ are complex vector spaces. For any non-negative integers p and q, we consider the complex Hermite polynomial;

$$H_{p,q}(t,\bar{t}) = (-1)^{p+q}e^{t\bar{t}}\frac{\partial^{p+q}}{\partial\bar{t}^p\partial t^q}e^{-t\bar{t}} \qquad (t\in\boldsymbol{C}).$$

It satisfies the following equations;

$$(5.1)\quad \begin{cases} \dfrac{\partial^2}{\partial t\partial\bar{t}}H_{p,q}(t,\bar{t}) - \bar{t}\dfrac{\partial}{\partial\bar{t}}H_{p,q}(t,\bar{t}) + qH_{p,q}(t,\bar{t}) = 0, \\ \dfrac{\partial^2}{\partial\bar{t}\partial t}H_{p,q}(t,\bar{t}) - t\dfrac{\partial}{\partial t}H_{p,q}(t,\bar{t}) + pH_{p,q}(t,\bar{t}) = 0. \end{cases}$$

$$(5.2)\quad \frac{\partial}{\partial t}H_{p,q}(t,\bar{t}) = pH_{p-1,q}(t,\bar{t}), \quad \frac{\partial}{\partial\bar{t}}H_{p,q}(t,\bar{t}) = qH_{p,q-1}(t,\bar{t}).$$

(5.3) If $t=\sum_{j=1}^m a_jt_j$ with $|a_1|^2+\cdots+|a_m|^2=1$, then

$$H_{p,q}(t,\bar{t}) = p!q!\sum\prod_j(p_j!q_j!)^{-1}(a_j)^{p_j}(\bar{a}_j)^{q_j}H_{p_j,q_j}(t_j,\bar{t}_j),$$

where $\sum$ is taken over all non-negative integers p_j, q_j $(j=1,\dots,m)$ with $\sum_j p_j=p$,

$\sum_j q_j = q$.

We put

$$\mathfrak{B}_{p,q} = \{\prod_{j=1}^{\infty} (p_j! q_j!)^{-1/2} H_{p_j,q_j}(\langle z, \xi_j \rangle, \overline{\langle z, \xi_j \rangle}); \quad p_1 + p_2 + \cdots = p,\ q_1 + q_2 + \cdots = q,\ p_j,\ q_j \geqq 0\}.$$

Then it is known that $\bigcup_{n=0}^{\infty} (\bigcup_{p+q=n} \mathfrak{B}_{p,q})$ is an orthonomal basis of $L^2(\boldsymbol{E}^*, \mu)$, (see [1]). We denote by $\mathscr{H}_{p,q}$ the closed subspace spanned by $\mathfrak{B}_{p,q}$. Then we have

$$L^2(\boldsymbol{E}^*, \mu) = \sum_{n=0}^{\infty} \oplus \sum_{p+q=n} \oplus \mathscr{H}_{p,q} \quad \text{(Wiener-Itô decomposition)}.$$

From (5.3) we see that $\mathscr{H}_{p,q}$ is $\pi_*(G)$-invariant, so that we have the subrepresentation $\pi_{p,q}$ of G on $\mathscr{H}_{p,q}$. For any i in $\boldsymbol{N}$ we put

$$\Phi_i^{p,q}(z, \bar{z}) = (p! q!)^{-1/2} H_{p,q}(\langle z, \xi_i \rangle, \overline{\langle z, \xi_i \rangle}).$$

The following Lemma 5$\sim$Lemma 8 can be proved similarly to the real case.

LEMMA 5. *$\Phi_1^{p,q}$ is a cyclic vector of $\mathscr{H}_{p,q}$.*

LEMMA 6. *Any $\pi_{p,q}(G)$-fixed vector in $\mathscr{H}_{p,q}$ is equal to zero if $(p, q) \neq (0, 0)$.*

LEMMA 7. *For any $\pi_{p,q}(K)$-fixed vector v in $\mathscr{H}_{p,q}$, there exists a constant c such that $v = c\Phi_1^{p,q}$.*

LEMMA 8. *$(\pi_{p,q}, \mathscr{H}_{p,q})$ is an irreducible unitary representation of G.*

Let $\mathfrak{g}_m$ be the Lie algebra of G_m, and let E_{ij} be the $m \times m$ matrix defined in § 3. We put $X_{ij} = E_{ij} - E_{ji}$, $Y_{ij} = i(E_{ij} + E_{ji})$ for $i < j$ and $Y_{ii} = iE_{ii}$. Then $\mathfrak{g}_m$ is canonically identified with the linear Lie algebra generated by $\{X_{ij}, Y_{ij}, Y_{ii};\ 1 \leqq i < j \leqq m\}$. We define a bilinear form B: $\mathfrak{g}_m \times \mathfrak{g}_m \to \boldsymbol{C}$ by $(X, Y) \longmapsto 2m\, \mathrm{tr} XY$. Then B is non-degenerate, so we define the Casimir operator C_m associated to B by the formula;

$$(5.4) \qquad C_m = -c_m \sum_{1 \leqq i < j \leqq m} (X_{ij}^2 + Y_{ij}^2) - 2c_m \sum_{i=1}^{m} Y_{ii}^2, \quad c_m = 1/4m.$$

PROPOSITION 5. *$(\pi_{p,q}, \mathscr{H}_{p,q})$ is a class one representation of G.*

PROOF. From Lemma 5$\sim$Lemma 8, we have only to show that $\Phi_1^{p,q}$ satisfies (A.2). It is clear that $\Phi_1^{p,q}$ is $\pi_{p,q}(G_m)$-finite $(m \in \boldsymbol{N})$. Let $z_i = \langle z, \xi_i \rangle$ $(i \in \boldsymbol{N}$, $z \in \boldsymbol{E}^*)$. Then any element of the space spanned by $\pi_{p,q}(G_m)\Phi_1^{p,q}$ can be regarded as a function only of $z_1, \ldots, z_m, \bar{z}_1, \ldots, \bar{z}_m$. Using this identification we get

$$(5.5) \qquad d\pi_{p,q}(X_{ij})^2 = \left(z_i \frac{\partial}{\partial z_j} - z_j \frac{\partial}{\partial z_i} + \bar{z}_i \frac{\partial}{\partial \bar{z}_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_i}\right)^2,$$

$$d\pi_{p,q}(Y_{ij})^2 = -\left(z_i \frac{\partial}{\partial z_j} + z_j \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_i}\right)^2, \tag{5.6}$$

$$d\pi_{p,q}(Y_{ii})^2 = -\left(z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i}\right)^2. \tag{5.7}$$

As $\Phi_1^{p,q}$ is a function only of z_1 and $\bar{z}_1$, using (5.5), (5.6) and (5.7), we have

$$\begin{aligned} d\pi_{p,q}(C_m)\Phi_1^{p,q} = \Big\{ & 2^{-1}\left(z_1 \frac{\partial}{\partial z_1} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1}\right) \\ & + 2c_m\left(z_1^2 \frac{\partial^2}{\partial z_1^2} + \bar{z}_1^2 \frac{\partial^2}{\partial \bar{z}_1^2}\right) - 4c_m \textstyle\sum_{j=1}^m z_j \bar{z}_j \dfrac{\partial^2}{\partial z_1 \partial \bar{z}_1}\Big\} \Phi_1^{p,q}. \end{aligned} \tag{5.8}$$

By the strong law of large numbers we have

$$\lim_{m\to\infty} m^{-1} \textstyle\sum_{j=1}^m |\langle z, \xi_j\rangle|^2 = 1 \quad \text{a.e. } z \text{ in } \boldsymbol{E}^*. \tag{5.9}$$

Since $\Phi_1^{p,q}$ does not depend on m, it follows from (5.8) and (5.9) that

$$\lim_{m\to\infty} d\pi_{p,q}(C_m)\Phi_1^{p,q} = 2^{-1}\left(z_1 \frac{\partial}{\partial z_1} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} - 2\frac{\partial^2}{\partial z_1 \partial \bar{z}_1}\right)\Phi_1^{p,q}.$$

Using the formula (5.2) we obtain

$$\lim_{m\to\infty} d\pi_{p,q}(C_m)\Phi_1^{p,q} = 2^{-1}(p+q)\Phi_1^{p,q}.$$

PROPOSITION 6. $\phi_{\pi_{p,q}}(g) = \langle \xi_1, g\xi_1\rangle^p \overline{\langle \xi_1, g\xi_1\rangle}^q \qquad (g \in G)$.

PROOF. Let $g \in G$. Then we have an m in $\boldsymbol{N}$ such that $g \in G_m$. We put $g\xi_1 = \sum_{j=1}^m g_{j1}\xi_j$. Using the formula (5.3), we have

$$\begin{aligned} \phi_{\pi_{p,q}}(g) &= (\Phi_1^{p,q}, \pi_{p,q}(g)\phi_1^{p,q}) \\ &= (p!q!)^{-1}(H_{p,q}(\langle \cdot, \xi_1\rangle, \overline{\langle \cdot, \xi_1\rangle}), H_{p,q}(\langle \cdot, g\xi_1\rangle, \overline{\langle \cdot, g\xi_1\rangle})) \\ &= (p!q!)^{-1}(H_{p,q}(\langle \cdot, \xi_1\rangle, \overline{\langle \cdot, \xi_1\rangle}), \\ &\qquad p!q! \textstyle\sum \prod_j (p_j!q_j!)^{-1}(g_{j1})^{p_j}(\bar{g}_{j1})^{q_j} H_{p_j,q_j}(\langle \cdot, \xi_j\rangle, \overline{\langle \cdot, \xi_j\rangle})) \\ &= \bar{g}_{11}^p g_{11}^q = \langle \xi_1, g\xi_1\rangle^p \overline{\langle \xi_1, g\xi_1\rangle}^q, \end{aligned}$$

where $\sum$ is the same as in (5.3).

§ 6. McKean's conjecture (complex case)

We put $T = G_1$. And we denote by a_θ the element of G_2 defined by (4.1). Let A be the group of all elements a_θ. Then we have "the Cartan decomposition"; $G = KTAK$. We note that $kt = tk$ ($t \in T$, $k \in K$). We denote by t_φ the element of T defined by $t_\varphi \xi_1 = e^{i\varphi}\xi_1$. Then T is isomorphic to $U(1)$, so that the character group $\hat{T}$ of T is isomorphic to $\boldsymbol{Z}$ where $\boldsymbol{Z}$ is the additive group of all integers. We denote by σ the canonical isomorphism of $\hat{T}$ to $\boldsymbol{Z}$ defined by $\sigma(\eta)=l$ where $\eta\in\hat{T}$ and $\eta(t_\varphi)=e^{il\varphi}$ $(t_\varphi\in T)$.

Let $(\pi, \mathfrak{H})$ be a class one representation of G and let v_0 be a $\pi(K)$-fixed unit vector. For any t_φ in T and k in K, it follows that

$$\pi(k)\pi(t_\varphi)v_0 = \pi(t_\varphi)\pi(k)v_0 = \pi(t_\varphi)v_0.$$

Thus $\pi(t_\varphi)v_0$ is a $\pi(K)$-fixed vector. By (A.1) there exists a constant $\eta_\pi(t_\varphi)$ such that $\pi(t_\varphi)v_0=\eta_\pi(t_\varphi)v_0$. Then we have

$$|\eta_\pi(t_\varphi)| = 1, \qquad \eta_\pi(t_\varphi t_{\varphi'}) = \eta_\pi(t_\varphi)\eta_\pi(t_{\varphi'}).$$

Thus η_π is a character of T.

From Proposition 2 $d\pi(C)$ is a scalar operator, so that we put $d\pi(C)=\chi_\pi(C)I$.

THEOREM 2. *Let $(\pi, \mathfrak{H})$ be a class one representation of G with respect to K. Then $2\chi_\pi(C)$ is a non-negative integer, and if $|\sigma(\eta_\pi)|\leqq 2\chi_\pi(C)$ $(\pi, \mathfrak{H})$ is equivalent to $(\pi_{p,q}, \mathscr{H}_{p,q})$ where $p+q=2\chi_\pi(C)$ and $p-q=\sigma(\chi_\pi)$.*

PROOF. By (A.2) there exists a $\pi(K)$-fixed unit vector v_0 such that $\lim_{m\to\infty} d\pi(C_m)v_0$ is convergent. As in the real case, we denote by F_π the function on A such that $F_\pi(\theta)=\phi_\pi(a_\theta)$. Since ϕ_π is K-biinvariant, putting $g=k't_\varphi a_\theta k$, we have

$$\phi_\pi(g) = e^{-il\varphi}F_\pi(\theta) \quad \text{where} \quad l=\sigma(\eta_\pi).$$

Now we note that

$$\chi_\pi(C)F_\pi(\theta) = (v_0, \pi(a_\theta)d\pi(C)v_0) = \lim_{m\to\infty}(v_0, \pi(a_\theta)d\pi(C_m)v_0). \tag{6.1}$$

Using the fact that $\exp tX_{ij}$, $\exp t_{ij}$ and $\exp tY_{ii}$ are in K if $i\geqq 2$, we have

$$(v_0, \pi(a_\theta)d\pi(C_m)v_0) = -c_m\sum_{j=2}^m(v_0, \pi(a_\theta)d\pi(X_{1j})^2v_0) - c_m\sum_{j=2}^m(v_0, \pi(a_\theta)d\pi(Y_{1j})^2v_0) - 2c_m(v_0, \pi(a_\theta)d\pi(Y_{11})^2v_0). \tag{6.2}$$

As in the real case, the first term of (6.2) is

$$-c_m\left\{\frac{d^2}{d\theta^2}F_\pi(\theta) + (m-2)\cot\theta\frac{d}{d\theta}F_\pi(\theta)\right\}. \tag{6.3}$$

It is easy to get the followings;

$$\mathrm{Ad}(a_\theta)^{-1}Y_{2j} = \cos\theta Y_{2j} - \sin\theta Y_{1j} \qquad (j=3, 4,\dots),$$
$$[\mathrm{Ad}(a_\theta)^{-1}Y_{2j}, Y_{2j}] = \sin\theta X_{12} \qquad (j=3, 4,\dots).$$

Then we have

(6.4) $$Y_{1j}^2 = \cot^2\theta Y_{2j}^2 + \cot\theta X_{12} - 2\cot\theta\operatorname{cosec}\theta\,\mathrm{Ad}(a_\theta)^{-1}Y_{2j}Y_{2j} + \operatorname{cosec}^2\theta\,(\mathrm{Ad}(a_\theta)^{-1}Y_{2j})^2 \qquad (j=3, 4,\dots).$$

We substitute (6.4) into the second term of (6.2), and after some calculations we get

(6.5) $$-c_m\{(v_0, \pi(a_\theta)d\pi(Y_{12})^2v_0) + (m-2)\cot\theta(v_0, \pi(a_\theta)d\pi(X_{12})v_0)\}.$$

To calculate the first term of (6.5), we use the following formula;

$$\mathrm{Ad}(a_\theta)^{-1}Y_{11} = \cos^2\theta Y_{11} + \cos\theta\sin\theta\, Y_{12} + \sin^2\theta Y_{22}.$$

Then we have

$$\begin{aligned}Y_{12}^2 = \sec^2\theta\operatorname{cosec}^2\theta\{&(\mathrm{Ad}(a_\theta)^{-1}Y_{11})^2 + \cos^4\theta Y_{11}^2 + \sin^4\theta Y_{22}^2\\ &- \cos^2\theta(\mathrm{Ad}(a_\theta)^{-1}Y_{11}Y_{11} + Y_{11}\,\mathrm{Ad}(a_\theta)^{-1}Y_{11})\\ &- \sin^2\theta(\mathrm{Ad}(a_\theta)^{-1}Y_{11}Y_{22} + Y_{22}\,\mathrm{Ad}(a_\theta)^{-1}Y_{11})\\ &+ \sin^2\theta\cos^2\theta(Y_{11}Y_{22} + Y_{22}Y_{11})\}\end{aligned}$$

Since $\exp tY_{11} \in T\,(t\in\boldsymbol{R})$, we have

$$(v_0, \pi(a_\theta)d\pi(\mathrm{Ad}(a_\theta)^{-1}Y_{11})^2v_0) = -l^2F_\pi(\theta),$$
$$(v_0, \pi(a_\theta)d\pi(\mathrm{Ad}(a_\theta)^{-1}Y_{11})d\pi(Y_{11})v_0) = -l^2F_\pi(\theta),$$

where $l=\sigma(\eta_\pi)$. It follows from these equations that the first term of (6.5) is

$$2\cot 2\theta\frac{d}{d\theta}F_\pi(\theta) - l^2\tan^2\theta F_\pi(\theta).$$

Thus the second term of (6.2) becomes

(6.6) $$-c_m\left\{2\cot 2\theta\frac{d}{d\theta}F_\pi(\theta) - l^2\tan^2\theta F_\pi(\theta) + (m-2)\cot\theta\frac{d}{d\theta}F_\pi(\theta)\right\}.$$

It is easy to see that the third term of (6.2) is

(6.7) $$2c_ml^2F_\pi(\theta).$$

Finally, substituting (6.3), (6.6) and (6.7) in (6.1), we obtain

$$\chi_\pi(C)F_\pi(\theta) = -2^{-1}\cot\theta\frac{d}{d\theta}F_\pi(\theta).$$

Since F_π is C^∞ and $F_\pi(0)=1$, we conclude that $2\chi_\pi(C)$ is a non-negative integer. Putting $2\chi_\pi(C)=n$, we have $F_\pi(\theta)=\cos^n\theta$. Thus we get $\phi_\pi(g)=e^{-il\varphi}\cos^n\theta$ where $g=k't_\varphi a_\theta k$.

If $|\sigma(\eta_\pi)|\leqq 2\chi_\pi(C)$, then there exist non-negative integers p and q such that

$p+q=n$ and $p-q=l$. From Proposition 6, putting $g=k't_\varphi a_\theta k$, we can compute the spherical function of representation $(\pi_{p,q}, \mathscr{H}_{p,q})$ as follows;

$$\phi_{\pi_{p,q}}(g) = \langle \xi_1, g\xi_1 \rangle^p \overline{\langle \xi_1, g\xi_1 \rangle}^q = e^{-i(p-q)\varphi} \cos^{p+q} \theta.$$

Thus we have $\phi_\pi = \phi_{\pi_{p,q}}$. From Proposition 1 we see that $(\pi, \mathfrak{H})$ is equivalent to $(\pi_{p,q}, \mathscr{H}_{p,q})$.

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