Hiroshima Math. J.
11 (1981), 285–297

Generally induced modules in Lie algebras

Fujio Kubo

(Received September 19, 1980)

Introduction

Throughout this paper let K denote a field of characteristic zero and $U(M)$ the enveloping algebra of a Lie algebra M over K. Now let G be a Lie algebra over K. Let H be a subalgebra of G and W be an H-module. Regarding $U(G)$ as a right $U(H)$-module, we can form the left G-module $U(G)\otimes_{U(H)} W$. This module is called the G-module induced by W and discussed in [1, pp. 169–189].

In this paper we generalize the construction of the induced G-module to define the generally induced G-module by taking a subalgebra R of $U(G)$ instead of taking a subalgebra of G. We mainly investigate the generally induced G-module in the case that $U(G)$ has a good basis, namely a regular basis, as a right R-module.

For $u \in U(G)$ we say that u is permutable with R if $Ru=uR$. Then we have an automorphism $p(u)$ of R such that $ru=up(u)(r)$ for any $r\in R$. We call it the permuting map of R associated with u. The permuting map will play an important role to investigate the generally induced module.

In §3 we give several conditions under which every R-endomorphism of an irreducible R-module W is algebraic over K. Such conditions enable us to have a central character of the R-module W when K is algebraically closed. We then give criteria of the homogeneity of an R-submodule of $U(G)\otimes_R W$ by using the central character and the permuting map of R given in §2.

In §4 we discuss the structure and the classification of $R[u_\lambda, u_\tau]\otimes_R W$ in the case that $u_\lambda u_\tau\in R$, where u_λ and u_τ belong to a regular basis of $U(G)$, $u_\lambda\neq 1$ and $u_\tau\neq 1$.

In §5 and §6 we apply the results given in §§1–4 to the case that G is $sl(2, K)$ or the Heisenberg algebra. Generally induced modules given in these sections cannot be constructed as any modules induced by modules over their proper subalgebras.

§1. Definition of generally induced modules

Definition. Let G be a Lie algebra over K and R be a subalgebra of $U(G)$. For an R-module W we can form the left G-module

$$U(G)\otimes_R W,$$

regarding $U(G)$ as a right R-module. We shall term it the *G-module induced by the R-module W.* We shall also call such a G-module a *generally induced G-module.*

If $R=U(H)$ for some subalgebra H of G, the G-modules induced by R-modules coincide with the G-modules induced by H-modules.

EXAMPLE. Let $G=\langle g\rangle$ be a one-dimensional Lie algebra over K. Take the subalgebra R of $U(G)$ generated by g^2. Let $W=Kw$ be an R-module with the action of R on W defined by $g^2w=0$ and $1w=w$. Since $\{1, g\}$ is a basis for a right R-module $U(G)$, the generally induced G-module V is $K(1\otimes w)+K(g\otimes w)$. Therefore V is two-dimensional. On the other hand let H be a proper subalgebra of G and W_1 be a non-trivial $U(H)$-module. Since $H=0$, the induced module $U(G)\otimes_K W_1$ is infinite-dimensional. Hence V cannot be any G-module induced by a module over a proper subalgebra of G.

§2. Pre-regular bases and regular bases

Let G be a Lie algebra over K and R be a subalgebra of $U(G)$. If $Ru=uR$ for $u\in U(G)$, we say that u is permutable with R. We assume that u is permutable with R. Since $U(G)$ is integral, for any $r\in R$ there exists a unique element $r'\in R$ such that $ru=ur'$. Therefore we have a map $p(u)$ of R into itself such that $ru=up(u)(r)$ for any $r\in R$. We call $p(u)$ the permuting map of R associated with u. $p(u)$ is obviously an automorphism of R. Now we state the following

DEFINITION. Let G be a Lie algebra over K and R be a subalgebra of $U(G)$. $U(G)$ is said to have a *pre-regular basis as a right R-module* if $U(G)$ has a basis including 1 as a right R-module such that every member of the basis is permutable with R. $U(G)$ is said to have a *regular basis as a right R-module* if $U(G)$ has a pre-regular basis as a right R-module such that all permuting maps of R associated with members of the basis are distinct each other.

EXAMPLE 1. Let G be a Lie algebra over K with a basis $\{x, y\}$ and with multiplication $[x, y]=y$. Let $R=K[x]$. Then $R=U(\langle x\rangle)$ and $U(G)$ has a basis $\{1, y, y^2,\dots\}$ as a right R-module. Since $x^iy^j=y^j(x+j)^i$, every member of this basis is permutable with R. Furthermore since $p(y^j)(x)=x+j$, $p(y^j)\neq p(y^i)$ for $i\neq j$. Therefore $U(G)$ has a regular basis $\{1, y, y^2,\dots\}$ as a right R-module.

EXAMPLE 2. Let G be as above. Let $R=K[xy, x]$. Let w be a non-zero element of $U(G)$ and we write $w=\sum_{i=0}^{n} f_i(x)y^i$ where $f_i(x)\in K[x]$. We claim that $wr\in R$ for some non-zero element r of R. If $n=0$, our assertion is obvious. If $n\geqq 1$, by using the formula

$$y^m(x+1)\cdots(x+m) = (xy)^m$$

we have

$$\begin{aligned} w(x+1)\cdots(x+n) &= \textstyle\sum_{i=0}^{n} f_i(x)y^i(x+1)\cdots(x+i)(x+i+1)\cdots(x+n) \\ &= \textstyle\sum_{i=0}^{n} f_i(x)(xy)^i(x+i+1)\cdots(x+n) \in R. \end{aligned}$$

This claim shows that 1 is linearly dependent on any non-zero element of $U(G)$ over R. Therefore $U(G)$ does not have a pre-regular basis as a right R-module.

LEMMA 2.1. *Let R be a commutative subalgebra of $U(G)$ and $u_1,\dots, u_n$ be non-zero elements of $U(G)$. Assume that each u_i is permutable with R. If $p(u_1),\dots, p(u_n)$ are distinct, then $u_1,\dots, u_n$ are linearly independent over R.*

PROOF. If $n=1$, our assertion follows from the fact that $U(G)$ is integral. Assume true for n and let $\sum_{i=1}^{n+1} u_ir_i=0$ for some $r_i \in R$. Since $r(\sum_{i=1}^{n+1} u_ir_i)=(\sum_{i=1}^{n+1} u_ir_i)p(u_{n+1})(r)=0$, we have

$$\textstyle\sum_{i=1}^{n} u_i(p(u_i)(r) - p(u_{n+1})(r))r_i = 0 \quad \text{for any} \quad r \in R.$$

By induction hypothesis, $(p(u_i)(r)-p(u_{n+1})(r))r_i=0$ for any $r \in R$. Since $p(u_i)\neq p(u_{n+1})$ for $i=1,\dots, n$, we conclude that $r_1=\cdots=r_n=0$. Therefore $u_{n+1}r_{n+1}=0$. Since $U(G)$ is integral, we have $r_{n+1}=0$. Q.E.D.

PROPOSITION 2.2. *Let R be a commutative subalgebra of $U(G)$ and let $U(G)$ have a regular basis $\{1, u_\lambda\colon \lambda \in \Lambda\}$ as a right R-module. Assume that $p(u_\lambda)$ has an order q. Then $u_\lambda^q \in R$.*

PROOF. Let $u_\lambda^q=\sum_{i=1}^{n} u_{\lambda(i)}r_i+r_0$ where $\lambda(i)\in\Lambda$ and $r_i \in R$. Since $ru_\lambda^q= u_\lambda^q p(u_\lambda)^q(r)=u_\lambda^q r$ and $ru_\lambda^q=\sum_{i=1}^{n} u_{\lambda(i)}p(u_{\lambda(i)})(r)r_i+rr_0$, we have $p(u_{\lambda(i)})(r)r_i=rr_i$ for any $r \in R$. Observing that $p(u_{\lambda(i)})\neq 1$ for $i=1,\dots, n$, we have $r_1=\cdots=r_n=0$. Hence $u_\lambda^q=r_0 \in R$. Q.E.D.

Let us denote $R_\lambda=\sum_{i=0}^{\infty} u_\lambda^i R$. Then we have

COROLLARY 2.3. *Under the same assumption as in the previous proposition we have*

$$R_\lambda = \textstyle\sum_{i=o}^{q-1} u_\lambda^i R.$$

PROOF. Since $1, p(u_\lambda),\dots, p(u_\lambda)^{q-1}$ are distinct, $\{1, u_\lambda,\dots, u_\lambda^{q-1}\}$ is a linearly independent set over R by Lemma 2.1. On the other hand since $u_\lambda^q \in R$ by Proposition 2.2, $\{1, u_\lambda,\dots, u_\lambda^{q-1}\}$ generates R_λ. Q.E.D.

COROLLARY 2.4. *Let R be a commutative subalgebra of $U(G)$ and let $U(G)$ have a regular basis $\{1, u_\lambda\colon \lambda \in \Lambda\}$. Let $\lambda, \tau \in \Lambda$ with $\lambda\neq\tau$. If $\{p(u_\lambda)^n\colon n=0,1,\dots\}$ $\cap\, \{p(u_\tau)^n\colon n=0, 1,\dots\}=\{1\}$, then $R_\lambda \cap R_\tau=R$.*

PROOF. We prove our assertion in the case that $p(u_\lambda)$, $p(u_\tau)$ has an order n, m respectively. The other cases will be done similarly. Now let $\sum_{i=1}^{n-1} u_\lambda^i r_i + r_0 = \sum_{j=1}^{m-1} u_\tau^j r'_j + r'_0$ for some r_i, $r'_j \in R$. Since $\{p(u_\lambda)^n : n=0, 1,\dots\} \cap \{p(u_\tau)^n : n=0, 1,\dots\} = \{1\}$, $\{1, u_\lambda,\dots, u_\lambda^{n-1}, u_\tau,\dots, u_\tau^{m-1}\}$ is a linearly independent set over R by Lemma 2.1. Therefore we have $r_1 = \cdots = r_{n-1} = r'_1 = \cdots = r'_{m-1} = 0$ and $r_0 = r'_0$.

Q.E.D.

§3. Central characters

LEMMA 3.1. *Let R be an algebra over K and W be a non-trivial irreducible R-module. If one of the following conditions is satisfied, every R-endomorphism of W is algebraic over K.*

(1) *R has an increasing filtration such that the graded algebra of R associated with this filtration is finitely generated and commutative.*

(2) *R is finitely generated and $R[R, R]RW=0$.*

PROOF. (1) is Lemma 2.6.4 in [1, p. 87].

(2): Let $B = R/R[R, R]R$ and let x be an R-endomorphism of W. Since $R[R, R]RW=0$, we can regard W as a B-module and x as a B-endomorphism. Since B is finitely generated and commutative, x is algebraic over K by the first assertion. Q.E.D.

If R satisfies one of the conditions (1) and (2) in Lemma 3.1 and if K is algebraically closed, then every element of $Z(R)$ acts on W as a scalar where $Z(R)$ is the center of R. Therefore we have the central character χ of R on W as

$$rw = \chi(r)w \quad \text{for} \quad r \in Z(R) \quad \text{and} \quad w \in W.$$

From now to the end of this paper let K be an algebraically closed field of characteristic zero. We are now ready to investigate the homogeneity of generally induced modules.

THEOREM 3.2. *Let R be a subalgebra of $U(G)$ and W be a non-trivial irreducible R-module such that the condition* (1) *or* (2) *in Lemma* 3.1 *holds. Let χ be the central character of R on W. Assume that $U(G)$ has a pre-regular basis $\{u_\lambda : \lambda \in \Lambda\}$ as a right R-module. Then*

(1) *If $\chi p(u_\lambda) \neq \chi p(u_\tau)$ on $Z(R)$ for any $\lambda, \tau \in \Lambda$ with $\lambda \neq \tau$, then every R-submodule of $U(G) \otimes_R W$ is R-homogeneous.*

(2) *If every R-submodule of $U(G) \otimes_R W$ is R-homogeneous, then $p(u_\lambda) \neq p(u_\tau)$ for any $\lambda, \tau \in \Lambda$ with $\lambda \neq \tau$.*

PROOF. (1): Let M be a non-trivial R-submodule of $U(G) \otimes_R W$. Take a non-zero element $v = \sum_{i=1}^{n} u_{\lambda(i)} \otimes w_i$ of M with $w_i \neq 0$. We show by induction on

n that $u_{\lambda(i)}\otimes w_i\in M$ for $i=1,\dots,n$. If $n=1$, it is trivial. Let $n\geqq 2$. Since $\chi p(u_{\lambda(1)})\neq\chi p(u_{\lambda(n)})$, there exists an element r_0 in $Z(R)$ such that $\chi p(u_{\lambda(1)})(r_0)\neq \chi p(u_{\lambda(n)})(r_0)$. Then we have

$$r_0v-\chi p(u_{\lambda(1)})(r_0)v=\sum_{i=2}^{n}(\chi p(u_{\lambda(i)})(r_0)-\chi p(u_{\lambda(1)})(r_0))u_{\lambda(i)}\otimes w_i\in M$$

and $(\chi p(u_{\lambda(n)})(r_0)-\chi p(u_{\lambda(1)})(r_0))u_{\lambda(n)}\otimes w_n\neq 0$. By our induction hypothesis, $u_{\lambda(n)}\otimes w_n\in M$. Therefore $\sum_{i=1}^{n-1}u_{\lambda(i)}\otimes w_i\in M$. By induction hypothesis again we have $u_{\lambda(i)}\otimes w_i\in M$ for $i=1,\dots,n-1$.

(2): For any $\lambda,\tau\in\Lambda$ with $\lambda\neq\tau$ we consider an R-submodule M of $U(G)\otimes_R W$ generated by $u_\lambda\otimes w-u_\tau\otimes w$ where w is a generator of the R-module W. Since M is R-homogeneous, $M=u_\lambda\otimes W+u_\tau\otimes W$. Therefore there exists an element r in R such that

$$r(u_\lambda\otimes w-u_\tau\otimes w)=u_\lambda\otimes w.$$

Hence $u_\lambda\otimes p(u_\lambda)(r)w=u_\lambda\otimes w$ and $u_\tau\otimes p(u_\tau)(r)w=0$. So we have $p(u_\lambda)(r)w=w$ and $p(u_\tau)(r)w=0$, which implies our assertion. Q.E.D.

COROLLARY 3.3. *Let $U(G)$, R, W, χ be as in the previous theorem. If R is commutative, then the following two statements are equivalent*:

(1) $\chi p(u_\lambda)\neq\chi p(u_\tau)$ *for any* $\lambda,\tau\in\Lambda$ *with* $\lambda\neq\tau$.

(2) *Every R-submodule of $U(G)\otimes_R W$ is R-homogeneous.*

PROOF. Assume (2). For any $\lambda,\tau\in\Lambda$ with $\lambda\neq\tau$ we consider an R-submodule M of $U(G)\otimes_R W$ generated by $u_\lambda\otimes w-u_\tau\otimes w$ where w is a generator of W. Since M is R-homogeneous, $M=u_\lambda\otimes W+u_\tau\otimes W$. Therefore there exists an element r in R such that $r(u_\lambda\otimes w-u_\tau\otimes w)=u_\lambda\otimes w$. Hence

$$\chi p(u_\lambda)(r)u_\lambda\otimes w-\chi p(u_\tau)(r)u_\tau\otimes w=u_\lambda\otimes w,$$

and we have $\chi p(u_\lambda)(r)=1$ and $\chi p(u_\tau)(r)=0$. Hence (1) holds.

The converse is shown in Theorem 3.2. Q.E.D.

Let R be a subalgebra of $U(G)$ and W be a non-trivial irreducible R-module such that the condition (1) or (2) in Lemma 3.1 holds. Let χ be the central character of R on W. Then we can regard $U(G)\otimes_R W$ as a $Z(R)$-module. The set of all $Z(R)$-endomorphisms of $U(G)\otimes_R W$ is called the centralizer of the $Z(R)$-module $U(G)\otimes_R W$. We give a characterization of the condition (1) in Theorem 3.2.

PROPOSITION 3.4. *Let $U(G)$, R, W, χ be as in Theorem 3.2. Then the following two statements are equivalent*:

(1) *The centralizer of a $Z(R)$-module $U(G)\otimes_R W$ is*

$$\{g\in\mathrm{End}_K(U(G)\otimes_R W): g(u_\lambda\otimes W)\subseteqq u_\lambda\otimes W \quad \textit{for any}\quad \lambda\in\Lambda\}.$$

(2) $\chi p(u_\lambda) \neq \chi p(u_\tau)$ *on* $Z(R)$ *for any* $\lambda, \tau \in \Lambda$ *with* $\lambda \neq \tau$.

PROOF. If $\chi p(u_\lambda)=\chi p(u_\tau)$ on $Z(R)$ for some $\lambda, \tau \in \Lambda$ with $\lambda \neq \tau$, then we can construct a $Z(R)$-endomorphism f of $U(G)\otimes_R W$ defined by $f(u_\lambda \otimes w)=u_\tau \otimes w$ and $f(u_\gamma \otimes w)=0$ for $\gamma \neq \lambda$ and $w \in W$. Then f does not belong to the set given in the statement (1). Conversely, let g be a $Z(R)$-endomorphism of $U(G)\otimes_R W$. Set $g(u_\lambda \otimes w)=\sum_{i=1}^n u_{\lambda(i)} \otimes w_i + u_\lambda \otimes w_0$ where $\lambda(i) \in \Lambda$ and $u_{\lambda(i)} \neq u_\lambda$. Since $g(r(u_\lambda \otimes w)) = rg(u_\lambda \otimes w)$ for any $r \in Z(R)$, we have

$$\chi p(u_\lambda)(r) g(u_\lambda \otimes w) = \sum_{i=1}^n \chi p(u_{\lambda(i)})(r) u_{\lambda(i)} \otimes w_i + \chi p(u_\lambda)(r) u_\lambda \otimes w_0.$$

Therefore we have $\chi p(u_\lambda)(r) w_i = \chi p(u_{\lambda(i)})(r) w_i$ for $i=1,\dots, n$. Since $\chi p(u_\lambda) \neq \chi p(u_{\lambda(i)})$ on $Z(R)$, we have $w_1=\cdots=w_n=0$. Hence $g(u_\lambda \otimes w)=u_\lambda \otimes w_0$. Q. E. D.

We also give a following criterion about the irreducibility of generally induced modules.

THEOREM 3.5. *Let* $U(G), R, W, \chi$ *be as in Theorem* 3.2. *Assume that* $\chi p(u_\lambda) \neq \chi p(u_\tau)$ *on* $Z(R)$ *for any* $\lambda, \tau \in \Lambda$ *with* $\lambda \neq \tau$. *Then the following two statements are equivalent*:

(1) $U(G)\otimes_R W$ *is an irreducible* $U(G)$*-module.*

(2) *For each* u_λ, u_τ *there exists* $a_{\lambda\tau} \in U(G)$ *such that*

$$a_{\lambda\tau} u_\lambda \in u_\tau + \sum_{\gamma \in \Lambda} u_\gamma \operatorname{Ann}_R(w)$$

where w *is a generator of* W.

PROOF. Assume (1). Then there exists $a_{\lambda\tau} \in U(G)$ such that $a_{\lambda\tau}(u_\lambda \otimes w) = u_\tau \otimes w$. Therefore $a_{\lambda\tau} u_\lambda - u_\tau \in \operatorname{Ann}_{U(G)}(1 \otimes w) = \sum_{\gamma \in \Lambda} u_\gamma \operatorname{Ann}_R(w)$. Conversely, assume (2). Let M be a non-trivial submodule of the $U(G)$-module $U(G)\otimes_R W$. Since $\chi p(u_\lambda) \neq \chi p(u_\tau)$ on $Z(R)$ for any $\lambda, \tau \in \Lambda$ with $\lambda \neq \tau$, M is R-homogeneous by Theorem 3.2. Therefore $u_\lambda \otimes W \subseteqq M$ for some $\lambda \in \Lambda$. Now for each $\tau \in \Lambda$ we select $a_{\lambda\tau} \in U(G)$ such that $a_{\lambda\tau} u_\lambda \in u_\tau + \sum_{\gamma \in \Lambda} u_\gamma \operatorname{Ann}_R(w)$. Then we have $a_{\lambda\tau}(u_\lambda \otimes w) = u_\tau \otimes w$, and $u_\tau \otimes W \subseteqq M$. Hence $U(G)\otimes_R W = \sum_{\tau \in \Lambda} u_\tau \otimes W = M$. Q. E. D.

§4. $R[u_\lambda, u_\tau]\otimes_R W$

Let R be a finitely generated commutative subalgebra of $U(G)$ and assume that $U(G)$ has a pre-regular basis $\{1, u_\lambda : \lambda \in \Lambda\}$ as a right R-module. We investigate the structure of $R[u_\lambda, u_\tau]\otimes_R W$ in the case that $u_\lambda u_\tau \in R$. In this section we use the notation p_λ instead of $p(u_\lambda)$ and we denote by $P_\lambda = \{p_\lambda^n : n=0, 1, 2,\dots\}$. Let $u_\lambda u_\tau = s \in R$. Then $u_\lambda(u_\tau u_\lambda) = s u_\lambda = u_\lambda p_\lambda(s)$. Therefore $u_\tau u_\lambda = p_\lambda(s)$. Furthermore since R is commutative, $r u_\lambda u_\tau = u_\lambda u_\tau r$ and $r u_\tau u_\lambda = u_\tau u_\lambda r$ for any $r \in R$. Then we have $p_\tau(p_\lambda(r)) = r$ and $p_\lambda(p_\tau(r)) = r$ for any $r \in R$. Therefore $p_\tau = p_\lambda^{-1}$. Hence

p_λ has a finite order if and only if p_τ has a finite order. We remark that $P_\lambda=P_\tau$ if p_λ or p_τ has a finite order.

PROPOSITION 4.1. *Let R be a finitely generated commutative subalgebra of $U(G)$ and let $U(G)$ have a regular basis $\{1, u_\lambda: \lambda \in \Lambda\}$ as a right R-module. Let W be a non-trivial irreducible R-module with the central character χ. Hence W is one-dimensional and we write $W=Kw$. Let $\lambda, \tau \in \Lambda$ with $\lambda \neq \tau$. Assume that $u_\lambda u_\tau = s$ for some $s \in R$ and that $P_\lambda \cap P_\tau=\{1\}$. Then*

(1) $R[u_\lambda, u_\tau] \otimes_R W = \sum_{i=1}^{\infty} Ku_\lambda^i \otimes w + K(1\otimes w) + \sum_{j=1}^{\infty} Ku_\tau^j \otimes w$

(direct sum of K-vector spaces).

(2) $R[u_\lambda, u_\tau] \otimes_R W$ *has the following structure;*

$$u_\lambda(u_\tau^n \otimes w) = \chi p_\lambda^{1-n}(s)(u_\tau^{n-1}\otimes w),\ u_\tau(u_\lambda^n \otimes w) = \chi p_\lambda^n(s)(u_\lambda^{n-1}\otimes w) \quad \text{for} \quad n \geqq 1.$$

(3) $R[u_\lambda, u_\tau]\otimes_R W$ *is* $R[u_\lambda, u_\tau]$*-irreducible if* $\chi p_\lambda^i \neq \chi p_\lambda^j$ *for* $i \neq j \geqq 0$ *and* $\chi p_\lambda^n(s) \neq 0$ *for any integer n.*

PROOF. (1): $1, p_\lambda, p_\lambda^2,\dots, p_\tau, p_\tau^2,\dots$ are distinct by the remark above. Then $\{1, u_\lambda, u_\lambda^2,\dots, u_\tau, u_\tau^2,\dots\}$ is a linearly independent set over R by Lemma 2.1. It is easy to see by induction on n that $u_\lambda^n u_\tau^n = sp_\lambda^{-1}(s)\cdots p_\lambda^{1-n}(s)$ and $u_\tau^n u_\lambda^n = p_\lambda(s)p_\lambda^2(s)\cdots p_\lambda^n(s)$ for $n \geqq 1$. Then we have

$$u_\lambda^i u_\tau^j = \begin{cases} u_\lambda^{i-j} sp_\lambda^{-1}(s)\cdots p_\lambda^{1-j}(s) & \text{if} \quad i \geqq j \geqq 1 \\ u_\tau^{j-i} p_\lambda^{i-j}(s) p_\lambda^{i-j-1}(s)\cdots p_\lambda^{1-j}(s) & \text{if} \quad j > i \geqq 1, \end{cases}$$

$$u_\tau^i u_\lambda^j = \begin{cases} u_\tau^{i-j} p_\lambda(s)p_\lambda^2(s)\cdots p_\lambda^j(s) & \text{if} \quad i \geqq j \geqq 1 \\ u_\lambda^{j-i} p_\lambda^{j-i+1}(s)\cdots p_\lambda^j(s) & \text{if} \quad j > i \geqq 1. \end{cases}$$

Therefore $\{1, u_\lambda, u_\lambda^2,\dots, u_\tau, u_\tau^2,\dots\}$ generates the R-module $R[u_\lambda, u_\tau]$. Hence we have the first assertion.

(2) is immediate from the facts that $u_\lambda u_\tau^n = u_\tau^{n-1} p_\lambda^{1-n}(s)$ and $u_\tau u_\lambda^n = u_\lambda^{n-1} p_\lambda^n(s)$.

(3): Let $N=K(1\otimes w)+\sum_{i=1}^{\infty} Ku_\lambda^i \otimes w$. Then every R-submodule of N is R-homogeneous by the similar proof to that of Theorem 3.2 (1). Now let M be a non-zero $R[u_\lambda, u_\tau]$-submodule of $R[u_\lambda, u_\tau]\otimes_R W$. Let $v=\sum_{j=1}^{m} a_j u_\tau^j \otimes w + n$ where $n \in N$. If $a_m \neq 0$, then $u_\lambda^m v \in a_m \chi(sp_\lambda^{-1}(s)\cdots p_\lambda^{1-m}(s))(1\otimes w)+\sum_{i=1}^{\infty} Ku_\lambda^i \otimes w$. Therefore $u_\lambda^m v \neq 0$ and $u_\lambda^m v \in N$. Hence $M \cap N \neq 0$. Since $M \cap N$ is R-homogeneous, in $M \cap N$ there exists an element $u_\lambda^m \otimes w$ for some $m \geqq 0$. If $m \geqq 1$, $u_\tau^m(u_\lambda^m \otimes w) = \chi(p_\lambda(s)\cdots p_\lambda^m(s))(1\otimes w) \in M$. Therefore $1\otimes w \in M$, which implies $R[u_\lambda, u_\tau]\otimes_R W = R[u_\lambda, u_\tau](1\otimes w) \subseteqq M$. Q.E.D.

COROLLARY 4.2. *Let $U(G)$, R, W, χ be as in Proposition 4.1. Assume that every R-submodule of $R[u_\lambda, u_\tau]\otimes_R W$ is R-homogeneous. Then $R[u_\lambda, u_\tau]\otimes_R W$*

is $R[u_\lambda, u_\tau]$*-irreducible if and only if* $\chi p_\lambda^n(s)\neq 0$ *for any integer* n.

PROOF. Assume that $\chi p_\lambda^n(s)\neq 0$ for any integer n. Since every R-submodule of $R[u_\lambda, u_\tau]\otimes_R W$ is R-homogeneous, we can prove the irreducibility of $R[u_\lambda, u_\tau]\otimes_R W$ as in the proof of the previous proposition. Conversely, if $\chi p_\lambda^n(s)=0$ for some $n\geqq 0$, then $\sum_{i=n}^{\infty} Ku_\lambda^i\otimes w$ is a non-zero proper submodule of $R[u_\lambda, u_\tau]\otimes_R W$. If $\chi p_\lambda^n(s)=0$ for some $n<0$, then $\sum_{j=1-n}^{\infty} Ku_\tau^j\otimes w$ is also a non-zero proper submodule of $R[u_\lambda, u_\tau]\otimes_R W$. Q.E.D.

We now classify the module $R[u_\lambda, u_\tau]\otimes_R W$.

THEOREM 4.3. *Let* R *be a finitely generated commutative subalgebra of* $U(G)$. *Let* W, W' *be irreducible* R*-modules with the central characters* χ, χ' *and with generators* w, w' *respectively. Assume that* $U(G)$ *has a regular basis* $\{1, u_\lambda: \lambda\in\Lambda\}$ *and* $P_\lambda\cap P_\tau=\{1\}$ *for* $\lambda, \tau\in\Lambda$ *with* $\lambda\neq\tau$. *If* $u_\lambda u_\tau=s$ *for some* $s\in R$ *and if* $\chi p_\lambda^n(s)\neq 0$ *for any integer* n, *then the following two statements are equivalent*:

(1) $R[u_\lambda, u_\tau]\otimes_R W'$ *is isomorphic to* $R[u_\lambda, u_\tau]\otimes_R W$ *as an* $R[u_\lambda, u_\tau]$*-module.*

(2) $\chi'=\chi p_\lambda^q$ *for some integer* q.

PROOF. Assume that $\chi'\neq\chi p_\lambda^q$ for any integer q. Let f be an $R[u_\lambda, u_\tau]$-homomorphism of $R[u_\lambda, u_\tau]\otimes_R W'$ into $R[u_\lambda, u_\tau]\otimes_R W$. Let us write $f(1\otimes w')=\sum_{i=1}^{n} a_iu_\lambda^i\otimes w+b(1\otimes w)+\sum_{j=1}^{m} c_ju_\tau^j\otimes w$ (a_i, b, $c_j\in K$). Since $f(r(1\otimes w'))=rf(1\otimes w')$ for any $r\in R$, we have

$$\chi'(r)f(1\otimes w') = \textstyle\sum_{i=1}^{n} a_i\chi p_\lambda^i(r)u_\lambda^i\otimes w + b\chi(r)(1\otimes w) + \sum_{j=1}^{m} c_j\chi p_\lambda^{-j}(r)u_\tau^j\otimes w.$$

Therefore $(\chi'(r)-\chi p_\lambda^i(r))a_i=(\chi'(r)-\chi(r))b=(\chi'(r)-\chi p_\lambda^{-j}(r))c_j=0$ for any $r\in R$. Since $\chi'\neq\chi p_\lambda^q$ for any integer q, we have $a_1=\cdots=a_n=b=c_1=\cdots=c_m=0$. Hence $f=0$.

Conversely, assume (2). Then we can construct an $R[u_\lambda, u_\tau]$-isomorphism g of $R[u_\lambda, u_\tau]\otimes_R W'$ onto $R[u_\lambda, u_\tau]\otimes_R W$ as follows; if $\chi'=\chi p_\lambda^q$ for some $q\geqq 0$, then g is given by

$$\begin{aligned}
&g(u_\lambda^i\otimes w') = u_\lambda^{q+i}\otimes w \quad (i\geqq 1) \qquad g(1\otimes w') = u_\lambda^q\otimes w,\\
&g(u_\tau^j\otimes w') = \chi(p_\lambda^q(s)p_\lambda^{q-1}(s)\cdots p_\lambda^{q-j+1}(s))u_\lambda^{q-j}\otimes w \qquad (1\leqq j\leqq q-1),\\
&g(u_\tau^q\otimes w') = \chi(p_\lambda^q(s)p_\lambda^{q-1}(s)\cdots p_\lambda^2(s)p_\lambda(s))(1\otimes w),\\
&g(u_\tau^{q+j}\otimes w') = \chi(p_\lambda^q(s)p_\lambda^{q-1}(s)\cdots p_\lambda^2(s)p_\lambda(s))u_\tau^j\otimes w \qquad (j\geqq 1).
\end{aligned}$$

If $\chi'=\chi p_\lambda^{-q}$ for some $q>0$, then g is given by

$$g(u_\tau^j\otimes w') = u_\tau^{q+j}\otimes w \quad (j\geqq 1), \qquad g(1\otimes w') = u_\tau^q\otimes w,$$

$$g(u_\lambda^i \otimes w') = \chi(p_\lambda^{1-q}(s)p_\lambda^{2-q}(s)\cdots p_\lambda^{i-q}(s))u_\tau^{q-i} \otimes w \qquad (1 \leqq i \leqq q-1),$$
$$g(u_\lambda^q \otimes w') = \chi(p_\lambda^{1-q}(s)p_\lambda^{2-q}(s)\cdots p_\lambda^{-1}(s)s)(1 \otimes w),$$
$$g(u_\lambda^{q+i} \otimes w') = \chi(p_\lambda^{1-q}(s)p_\lambda^{2-q}(s)\cdots p_\lambda^{-1}(s)s)u_\lambda^i \otimes w \qquad (i \geqq 1). \qquad \text{Q.E.D.}$$

Let R be a finitely generated commutative subalgebra of $U(G)$. Then R can be identified with the quotient of the algebra of polynominals in r indeterminates $T_1,\dots, T_r$ by a certain ideal I of $K[T_1,\dots, T_r]$. Let t_i be a canonical image of T_i in R. Let $X(R)$ be the set of all characters on R. Then the map η of $X(R)$ into the variety E in K^r vanished by I which is defined by $\eta(\chi)=(\chi(t_1),\dots, \chi(t_r))$ is bijective by Hilbert's Nullstellensatz.

Let g be an endomorphism of R, then g induces an endomorphism g_* of $X(R)$ defined by $g_*(\chi)=\chi\circ g$ for $\chi \in X(R)$. Then we have a morphism $\varepsilon(g)=\eta\circ g_*\circ\eta^{-1}$ of E into itself. If g is an automorphism of R, then $\varepsilon(g)$ is an automorphism of the variety E.

Now we assume that $U(G)$ has a pre-regular basis $\{1, u_\lambda: \lambda \in \Lambda\}$. For $\lambda, \tau \in \Lambda$ with $\lambda \neq \tau$, assume that $u_\lambda u_\tau = s$ for some $s \in R$. Since p_λ is an automorphism of R, $\varepsilon(p_\lambda)$ is an automorphism of the variety E. Let $X_\lambda^*(R)$ be the set of all characters χ on R which satisfy $\chi p_\lambda^q(s) \neq 0$ for any integer q. Since $(p_\lambda^q)_*$ is a bijective map of $X_\lambda^*(R)$ into itself, we can define an equivalence relation $\sim$ on $\eta(X_\lambda^*(R))$ as follows: For $a, b \in \eta(X_\lambda^*(R))$, $a \sim b$ if and only if $b=\varepsilon(p_\lambda)^q(a)$ for some integer q. Then we have the following

Corollary 4.4. *Under the same assumption as in Theorem* 4.3, *for any* $\chi, \chi' \in X_\lambda^*(R)$ *the following two statements are equivalent*:

(1) $R[u_\lambda, u_\tau]\otimes_R W'$ *is isomorphic to* $R[u_\lambda, u_\tau]\otimes_R W$.

(2) $\eta(\chi)\sim\eta(\chi')$.

§5. $sl(2, K)$

Let S be a Lie algebra over K with a basis $\{x, y, h\}$ and with multiplication $[x, y]=h$, $[x, h]=2x$ and $[y, h]=-2y$. The following formulas in $U(S)$ are easily seen but useful.

Lemma 5.1. *For any positive integer n we have*

$$hx^n = x^n h - 2nx^n, \quad hy^n = y^n h + 2ny^n,$$
$$yx^n = x^n y - nx^{n-1}h + n(n-1)x^{n-1}, \quad xy^n = y^n x + ny^{n-1}h + n(n-1)y^{n-1}.$$

Let R be a subalgebra of $U(S)$ generated by xy and h. Then R is a finitely generated commutative subalgebra of $U(S)$, and we have the following

Proposition 5.2. *Let* $B=\{1, x^i, y^j;\ i, j \geqq 1\}$. *Then*

(1) *B is a regular basis for a right R-module U(S).*

(2) *The permuting maps $p(x^i)$, $p(1)$ and $p(y^j)$ are given as follows:*

$$p(x^i)(xy) = xy - ih + i(i-1), \qquad p(x^i)(h) = h - 2i,$$
$$p(1)(xy) = xy, \qquad p(1)(h) = h,$$
$$p(y^j)(xy) = xy + jh + j(j+1), \qquad p(y^j)(h) = h + 2j.$$

In this section we simply denote $p=p(x)$. Then $p^i=p(x^i)$ and $p^{-i}=p(y^i)$ for $i\geqq 1$.

PROOF OF PROPOSITION 5.2. Let U^n be the subspace of $U(S)$ spanned by the elements whose degree is less than or equal to n. Let us prove by induction on $i+j$ that $x^iy^j \in \sum_{i=1}^{\infty} x^iR+R+\sum_{j=1}^{\infty} y^jR$.

If $i+j=1$, the assertion is obvious. We may assume that $i, j\geqq 1$. Then

$$\begin{aligned} x^iy^j &= x^{i-1}(xy^{j-1}) = x^{i-1}(y^jx + jy^{j-1}h + j(j-1)y^{j-1}) \\ &= x^{i-1}y^{j-1}(xy-h) + jx^{i-1}y^{j-1}h + j(j-1)x^{i-1}y^{j-1} \\ &\in \textstyle\sum_{i=1}^{\infty} x^iR + R + \sum_{j=1}^{\infty} y^jR \quad \text{by assumption.} \end{aligned}$$

Therefore B generates the right R-module $U(S)$. Let

$$\textstyle\sum_{ijk} a_{ijk}x^i(xy)^jh^k + \sum_{emn} b_{emn}y^e(xy)^mh^n + \sum_{st} c_{st}(xy)^sh^t = 0,$$

where i, $e\geqq 1$. Since $(xy)^n - x^ny^n \in U^{2n-1}$ and $y^mx^n - x^ny^m \in U^{n+m-1}$, we have

$$x^i(xy)^jh^k \in x^{i+j}y^jh^k + U^{i+2j+k-1},$$
$$y^e(xy)^mh^n \in x^my^{e+m}h^n + U^{m+2e+n-1},$$
$$(xy)^sh^t \in x^sy^sh^t + U^{2s+t-1}.$$

Therefore by the Poincaré-Birkhoff-Witt Theorem we may assume that $i+2j+k=m+2e+n=2s+t$ and that

$$\textstyle\sum_{ijk}a_{ijk}x^{i+j}y^jh^k + \sum_{emn} b_{emn}x^my^{e+m}h^n + \sum_{st} c_{st}x^sy^sh^t = 0.$$

Since $i+j>j$ and $m<e+m$, we see that $x^{i+j}y^jh^k$, $x^my^{e+m}h^n$ and $x^sy^sh^t$ are linearly independent over K. Then we have $a_{ijk}=b_{emn}=c_{st}=0$. Hence B is a linearly independent set over R.

For $i, j\geqq 1$ we have

$$hx^i = x^i(h - 2i), \quad (xy)x^i = x^i(xy - ih + i(i-1)),$$
$$hy^j = y^j(h + 2j) \quad \text{and} \quad (xy)y^j = y^j(xy + jh + j(j+1)). \qquad \text{Q.E.D.}$$

PROPOSITION 5.3. *Let R be the subalgebra of U(S) given above and W be* *an irreducible R-module with the central character χ and a generator w. Let $\chi(h)=\alpha$ and $\chi(xy)=\beta$. Then*

(1) *$U(S)\otimes_R W=\sum_{i=1}^{\infty} Kx^i\otimes w+K(1\otimes w)+\sum_{j=1}^{\infty} Ky^j\otimes w$, where the right side is a direct sum of K-vector spaces.*

(2) *$U(S)\otimes_R W$ has the following structure: For $n\geqq 1$*

$$x(y^n\otimes w) = (n^2 + (\alpha-1)n + \beta-\alpha)y^{n-1} \otimes w,$$

$$y(x^n\otimes w) = (n^2 - (\alpha+1)n + \beta)x^{n-1} \otimes w.$$

(3) *Every R-submodule of $U(S)\otimes_R W$ is R-homogeneous.*

(4) *$U(S)\otimes_R W$ is $U(S)$-irreducible if and only if $\zeta^2-(\alpha+1)\zeta+\beta=0$ has no integral solution.*

PROOF. By Proposition 5.2 (2), $\{p^n: n=0, 1,\dots\}\cap\{p^{-n}: n=0, 1,\dots\}=\{1\}$. Since $xy\in R$, the assertion (1) follows from Proposition 4.1 (1). Combining Proposition 4.1 (2) and Proposition 5.2 (2) we have our assertion (2).

(3): Since $(\chi p^n-\chi p^m)(h)=2(m-n)$ for any integers m and n, we have our assertion by Corollary 3.3.

(4): By Corollary 4.2, $U(S)\otimes_R W$ is $U(S)$-irreducible if and only if $\chi p^n(xy)\neq 0$ for any integer n. Since $\chi p^n(xy)=\beta-n\alpha+n(n-1)$, we complete the proof. Q.E.D.

We now classify some irreducible S-modules induced by non-trivial one-dimensional R-modules. Let $X(R)$, E and η be the set of all characters on R, the variety corresponding to R and the bijective map of $X(R)$ onto E defined in §4 respectively. Let χ be a character in $X^*(R)=\{\chi\in X(R): \chi p^n(xy)\neq 0$ for any integer $n\}$. Then we can regard K as an R-module by $rz=\chi(r)z$ for $r\in R$ and $z\in K$. Then we can construct a generally induced S-module $U(S)\otimes_R K$. We denote this by $V(\chi)$. It is not hard to see that $V(\chi)$ is not isomorphic to $U(S)\otimes_{U(H)}W$ for any proper subalgebra H of S and an H-module W. By Corollary 4.4, $V(\chi)$ is isomorphic to $V(\chi')$ if and only if $\eta(\chi)\sim\eta(\chi')$. Since R is a transcendental extension of K generated by the transcendental basis $\{xy, h\}$, $E=K^2$. Since $X^*(R)=\{\chi\in X(R): \zeta^2+(\chi(h)+1)\zeta+\chi(xy)=0$ has no integral solution$\}$ by Proposition 5.3 (2), $\eta(X^*(R))=\{(\alpha, \beta)\in K^2: \zeta^2+(\alpha+1)\zeta+\beta=0$ has no integral solution$\}$. Since $\chi p^q(h)=\chi(h)-2q$ and $\chi p^q(xy)=\chi(xy)-q\chi(h)+q(q-1)$ for any integer q, we have

$$\varepsilon(p)^q(\alpha, \beta) = (\alpha-2q,\ q^2 - (\alpha+1)q+\beta) \quad \text{for} \quad (\alpha, \beta)\in\eta(X^*(R)).$$

Therefore $(\alpha, \beta)\sim(\alpha', \beta')$ in $\eta(X^*(R))$ if and only if there exists an integer q such that $\alpha'=\alpha-2q$ and $4(\beta-\beta')=(\alpha-\alpha')(\alpha+\alpha'+2)$.

Let $C(\chi)$ be the Casimir operator of $V(\chi)$. Then $8C(\chi)v=(-2xy-2yx+hh)v$ for $v\in V(\chi)$. Since $-2xy-2yx+hh\in R$ and $\chi p^q(-2xy-2yx+hh)=\alpha^2+2\alpha-4\beta$

where $\chi(h)=\alpha$ and $\chi(xy)=\beta$, $C(\chi)$ acts on $V(\chi)$ as a scalar $(\alpha^2+2\alpha-4\beta)/8$. Then we have the following

Corollary 5.5. *$V(\chi)$ is isomorphic to $V(\chi')$ if and only if $\alpha-\alpha'$ is an even integer and $C(\chi)=C(\chi')$.*

Proof. Since $C(\chi)=C(\chi')$ if and only if $4(\beta-\beta')=(\alpha-\alpha')(\alpha+\alpha'+2)$, our assertion follows from Proposition 5.4. Q.E.D.

§6. Heisenberg algebra A_1

Let A be a Lie algebra over K with a basis $\{x, y, z\}$ and multiplication $[x, y]=z$, $[x, z]=[y, z]=0$. Then we have the following

Lemma 6.1. *For any positive integer n we have*

$$xy^n = y^n x + nzy^{n-1} \quad \textit{and} \quad yx^n = x^n y - nzx^{n-1}.$$

Let R be a subalgebra of $U(A)$ generated by xy and z. Then R is a finitely generated commutative subalgebra of $U(A)$. By the similar proof to that of Proposition 5.2 we have the following

Proposition 6.2. *Let $B=\{1, x^i, y^j: i, j\geqq 1\}$. Then*

(1) *B is a regular basis for the right R-module $U(A)$.*

(2) *The permuting maps $p(x^i)$, $p(1)$ and $p(y^j)$ are given by*

$$p(x^i)(xy) = xy - iz, \qquad p(1)(xy) = xy, \qquad p(y^j)(xy) = xy + jz,$$
$$p(x^i)(z) = z, \qquad p(1)(z) = z, \qquad p(y^j)(z) = z.$$

In this section we simply denote $p=p(x)$. Then $p^i=p(x^i)$ and $p^{-i}=p(y^i)$ for $i\geqq 1$.

We have the following result which corresponds to Proposition 5.3.

Proposition 6.3. *Let R be the subalgebra of $U(A)$ given above and W be an irreducible R-module with the central character χ and a generator w. Let $\chi(x)=\alpha$ and $\chi(xy)=\beta$. Then*

(1) *$U(A)\otimes_R W=\sum_{i=1}^{\infty} Kx^i\otimes w+K(1\otimes w)+\sum_{j=1}^{\infty} Ky^j\otimes w$, where the right side is a direct sum of K-vector spaces.*

(2) *$U(A)\otimes_R W$ has the following structure: For $n\geqq 1$,*

$$x(y^n\otimes w) = (\beta+(n-1)\alpha)y^{n-1} \otimes w, \ y(x^n\otimes w) = (\beta-n\alpha)x^{n-1} \otimes w.$$

(3) *Every R-submodule of $U(A)\otimes_R W$ is R-homogeneous if and only if $\alpha\neq 0$.*

(4) *Assume that* $\alpha \neq 0$. *Then* $U(A) \otimes_R W$ *is* $U(A)$*-irreducible if and only if* $\alpha\zeta + \beta = 0$ *has no integral solution.*

PROOF. We can prove (1) and (2) as in the proof of Proposition 5.3.

(3): If $\alpha = 0$, then $\chi p^q(xy) = \beta$ for any integer q. Therefore $K(x \otimes w + 1 \otimes w)$ is a non-homogeneous R-submodule of $U(A) \otimes_R W$. If $\alpha \neq 0$, then $(\chi p^n - \chi p^m)(xy) = m - n$ for any integers m and n. Therefore applying Corollary 3.3 we have our assertion (3).

(4): By Corollary 4.2, $U(A) \otimes_R W$ is $U(A)$-irreducible if and only if $\chi p^n(xy) \neq 0$ for any integer n. Since $\chi p^n(xy) = \beta - n\alpha$, we complete the proof. Q. E. D.

Let $X(R)$, E and η be the set of all characters on R, the variety corresponding to R and the bijective map of $X(R)$ onto E defined in §4 respectively. Let χ be a character in $X^*(R) = \{\chi \in X(R)\colon \chi p^n(xy) \neq 0$ for any integer $n\}$. Then we can regard K as an R-module by $rz = \chi(r)z$ for $r \in R$ and $z \in K$. Then we can construct a generally induced A-module (not necessarily irreducible) $U(A) \otimes_R K$. We denote this by $A(\chi)$. It is not hard to see that $A(\chi)$ is not isomorphic to $U(A) \otimes_{U(H)} W$ for any proper subalgebra H of A and an H-module W. By Corollary 4.4, $A(\chi)$ is isomorphic to $A(\chi')$ if and only if $\eta(\chi) \sim \eta(\chi')$. As in §5 we have $E = K^2$, $X^*(R) = \{\chi \in X(R)\colon \chi(z)\zeta + \chi(xy) = 0$ has no integral solution$\}$ and $\eta(X^*(R)) = \{(\alpha, \beta) \in K^2\colon \alpha\zeta + \beta = 0$ has no integral solution$\}$. Since $\chi p^q(xy) = \chi(xy) - q\chi(z)$ and $\chi p^q(z) = \chi(z)$, we have

$$\varepsilon(p)^q(\alpha, \beta) = (\alpha, \beta - q\alpha) \quad \text{for} \quad (\alpha, \beta) \in \eta(X^*(R)).$$

Therefore $(\alpha, \beta) \sim (\alpha', \beta')$ if and only if $\alpha = \alpha'$ and $\beta - \beta'$ is an integral multiple of α. We have then the following

PROPOSITION 4. *$A(\chi)$ is isomorphic to $A(\chi')$ if and only if $\alpha = \alpha'$ and $\beta - \beta'$ is an integral multiple of α.*

Reference

[1] J. Dixmier, Enveloping Algebras, North-Holland, New York, 1975.

Department of Mathematics,
Faculty of Science,
Hiroshima University