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Construction of one-dimensional classical dynamical system of infinitely many particles with nearest neighbor interaction

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§1. Introduction

In the investigation of the time evolution of a system of infinitely many particles which can be described by Newton's equations of motion, the first problem is to construct a dynamical system, more precisely, to determine a class of initial configurations for which equations of motion have solutions; the next problem is to investigate statistical mechanical properties of the dynamical system such as ergodicity. As for the construction of dynamical systems many results were obtained ([1], [2], [4]–[7]); especially in [5] and [6] ν-dimensional systems with long range interactions were treated. However, an explicit description of a class of initial configurations for which equations of motion have solutions was given only in the works of Dobrushin and Fritz ([1], [2]) in 1977.

We consider a system of infinitely many classical particles moving on the real line $\mathbf{R}$ in such a way that each particle is under interaction (repulsive force) only with its two right and left neighboring particles (the precise description of our model is given in §2). In this paper we construct the dynamical system for our model starting with a class $\mathscr{X}_\gamma$ of initial configurations, $0 \leqq \gamma < 1$. The class $\mathscr{X}_\gamma$ can be described as in [1]; in fact, it is given by (2.8) in §2. The uniqueness problem is also considered. The Gibbs states for our model become renewal measures ([3]), and from this fact it will follow that the class $\mathscr{X}_\gamma$ has full measure with respect to the Gibbs states. In this sense $\mathscr{X}_\gamma$ may be considered sufficiently wide.

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§2. Definitions and results

In this section we give the definitions and notations used throughout this paper and state the theorems.

Given a potential function $\Phi(r)$, $r > 0$, we consider the one-dimensional system of infinitely many (indistinguishable) particles moving according to the classical law of mechanics under the nearest neighbor interaction caused by $\Phi(r)$. We assume that

(2.1) $\Phi(r) \geqq 0$ and $-\Phi'(r) \geqq 0$ for $r > 0$,

(2.2) $-\Phi'(r)$ is nonincreasing and

$$\lim_{r\to 0+} \Phi(r) = \lim_{r\to 0+} -\Phi'(r) = \infty, \qquad \lim_{r\to\infty} \Phi(r) = \lim_{r\to\infty} -\Phi'(r) = 0.$$

As the phase space of our system we adopt the set of all locally finite configurations, that is, the set $\hat{\mathscr{X}}$ of all equivalence classes of (possibly finite or even empty) sequences $x=(q_i, p_i)_i$, $q_i, p_i \in \mathbf{R}$, such that the q_i's are different and $N(x; \Delta) \equiv \sharp\{i \mid q_i \in \Delta\} < \infty$ for any compact interval Δ. Here two sequences are said to be equivalent if they are the same as subsets of $\mathbf{R} \times \mathbf{R}$. The q_i's and p_i's represent the position and momenta of particles. We have included finite configurations in $\hat{\mathscr{X}}$ only for mathematical convention; in what follows we restrict our attention to the set $\mathscr{X} = \{x \in \hat{\mathscr{X}} \mid N(x; (-\infty, 0)) = N(x; [0, \infty)) = \infty\}$.

The precise description of our system is given as follows. Take an initial configuration $x \in \mathscr{X}$, label it in such a way that

$$x = (q_i, p_i)_i, \qquad \cdots < q_{-1} < 0 \leqq q_0 < q_1 < \cdots, \tag{2.3}$$

and consider the equations of motion

$$\begin{cases} \dfrac{dq_i(t)}{dt} = p_i(t) \\[2ex] \dfrac{dp_i(t)}{dt} = -\Phi'(q_i(t) - q_{i-1}(t)) + \Phi'(q_{i+1}(t) - q_i(t)) \end{cases} \tag{2.4}$$

with the initial condition

$$(q_i(0), p_i(0))_i = (q_i, p_i)_i. \tag{2.5}$$

For simplicity, we are taking the particles to be identical and to have mass one. From (2.3) and assumption (2.2), it will follow that the solution $x(t) = (q_i(t), p_i(t))_i$ of (2.4) and (2.5) (if exists) satisfies

$$\cdots < q_{-1}(t) < q_0(t) < q_1(t) < \cdots. \tag{2.6}$$

Forgetting the labels of $x(t) = (q_i(t), p_i(t))_i$, we then obtain the configuration at time t, which is still denoted by $x(t)$ with confusion.

Take $x \in \mathscr{X}$, label it as in (2.3) and set

$$H(x; \Delta) = 2^{-1} \textstyle\sum_{q_i \in \Delta} p_i^2 + \sum_{q_i \text{ or } q_{i+1} \in \Delta} \Phi(q_{i+1} - q_i). \tag{2.7}$$

We also set

$$\mathscr{X}_\gamma = \{x \in \mathscr{X} \mid \sup_{n\in\mathbf{N}} (2n)^{-1} N(x; \Delta_n) < \infty,\ \sup_{n\in\mathbf{N}} (2n)^{-1-\gamma} H(x; \Delta_n) < \infty\} \tag{2.8}$$

for each γ with $0 \leqq \gamma < 1$, where $\Delta_n = [-n, n]$.

THEOREM 1. *Let $\Phi(r)$ satisfy* (2.1) *and* (2.2). *Then for each $x = (q_i, p_i)_i \in \mathscr{X}_\gamma$ for some γ with $0 \leqq \gamma < 1$ there exists a solution $x(t) = (q_i(t), p_i(t))_i$, $t \in \mathbf{R}$, of* (2.4) *with initial condition* (2.5) *satisfying*

(i) $\cdots < q_{i-1}(t) < q_i(t) < q_{i+1}(t) < \cdots$,

(ii) $x(t) \in \mathscr{X}_\gamma$,

(iii) *there is a constant $\delta > 0$ such that for any i and $t \in \mathbf{R}$*

$$(2.9) \qquad \begin{cases} \lim_{k\to\infty} \sigma_{i+k} \circ \cdots \circ \sigma_{i+1} \circ \sigma_i(t) = \infty \\ \lim_{k\to\infty} \sigma_{i-k} \circ \cdots \circ \sigma_{i-1} \circ \sigma_i(t) = \infty, \end{cases}$$

where

$$(2.10) \qquad \sigma_i(t) = \inf \{ s \geqq t \mid q_{i+1}(s) - q_i(s) \leqq \delta \}^{1)}.$$

The condition (iii) implies that the solution is not "being driven at infinity" ([5]). A solution of (2.4) is said to be regular if it satisfies the condition (iii) for some $\delta > 0$. When we want to stress δ, it will be called a δ-regular solution.

To discuss the uniqueness of the solutions, we further assume the following condition on $\Phi(r)$:

$$(2.11) \qquad \lim_{n\to\infty} n^{-2} G(n^{1+\gamma}) = 0$$

for some γ with $0 \leqq \gamma < 1$, here

$$(2.12) \qquad G(u) = \sup\{ |(\Phi'(r) - \Phi'(s))/(r-s)| \mid r, s > 0,\ r \neq s,\ \Phi(r) \leqq u,\ \Phi(s) \leqq u \}.$$

As an example satisfying (2.11), we can take $\Phi(r) = r^{-\alpha}$, $\alpha > 2$; in this case γ must be in $[0, (\alpha-2)/(\alpha+2))$.

THEOREM 2. *Let $\Phi(r)$ satisfy* (2.1), (2.2) *and* (2.11) *for some γ with $0 \leqq \gamma < 1$. Then for any initial configuration $x \in \mathscr{X}_\gamma$ satisfying*

$$(2.13) \qquad \limsup_{n\to\infty} n^{-1} \{ N(x; [-n, 0)) \wedge N(x; [0, n]) \} > 0^{2)},$$

a regular solution of (2.4) *and* (2.5) *is unique.*

From the equilibrium statistical mechanical viewpoint it is desirable that the initial configuration space $\mathscr{X}_\gamma$ has full measure with respect to the Gibbs states. Before giving the definition of Gibbs states we summarize the topology and the Borel structure on $\hat{\mathscr{X}}$ briefly; for details see Lanford [5]. Let $\mathscr{K}$ be the set of all continuous functions $\psi(q, p)$ on $\mathbf{R} \times \mathbf{R}$ vanishing for sufficiently large $|q|$, and put

1) We adopt the convention $\inf \phi = \infty$ and $\sup \phi = -\infty$.

2) $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$.

$$S_\psi(x) = \sum_i \psi(q_i, p_i), \qquad x = (q_i, p_i)_i \in \hat{\mathscr{X}}.$$

We give $\hat{\mathscr{X}}$ the weakest topology which makes the mapping S_ψ continuous for all $\psi \in \mathscr{K}$. Then $\hat{\mathscr{X}}$ is a Polish space, and $\mathscr{X}$ is a G_δ-set of $\hat{\mathscr{X}}$ ([3]). Denote by $\mathscr{B}(\hat{\mathscr{X}})$ the topological Borel field of $\hat{\mathscr{X}}$ and by $\mathscr{B}(\mathscr{X})$ the restriction of $\mathscr{B}(\hat{\mathscr{X}})$ to $\mathscr{X}$. For any Borel set $M \subset \mathbf{R}$ let $\Pi_M(x)$ be the restriction of $x=(q_i, p_i)_i$ to M, that is, $\Pi_M(x)=(q_i, p_i)_{i:q_i \in M}$; denote by $B(\mathscr{X}, M)$ the set of bounded measurable functions φ on $\mathscr{X}$ such that $\varphi(x)=\varphi(y)$ for all $x, y \in \mathscr{X}$ satisfying $\Pi_M(x)=\Pi_M(y)$, and by $\tilde{\mathscr{B}}^M$ the smallest σ-algebra on $\mathscr{X}$ for which every element of $B(\mathscr{B}, M^c)$ is measurable.

A probability measure μ on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ is called a Gibbs state associated with the nearest neighbor interaction caused by Φ, the inverse temperature β and the chemical potential u if it satisfies the following condition: For every compact interval $\Delta=[a, b]$ the conditional expectation $E\{\varphi \mid \tilde{\mathscr{B}}^\Delta\}(x)$ of $\varphi(x) \in L^1(\mathscr{X}, \mu)$ given $\tilde{\mathscr{B}}^\Delta$ is equal to

$$(2.14) \quad \frac{1}{\Xi_\Delta(x)}\Big[\varphi(y) + \exp\{\beta\Phi(q^*-q_*)\} \sum_{k=1}^\infty \frac{1}{k!}\exp(\beta u k) \times \int\cdots\int_{(\Delta\times\mathbf{R})^k} dz\,\varphi(y\cdot z)\exp\{-\beta H(y\cdot z;\Delta)\}\Big],$$

where $\Xi_\Delta(x)$ is the normalizing factor, $y=\Pi_{\Delta^c}(x)$, $q^*=\min\{q_i \mid q_i>b\}$, $q_*=\max\{q_i \mid q_i<a\}$ for $y=(q_i, p_i)_i$, and $y\cdot z$ is the configuration in $\mathscr{X}$ defined by $\Pi_{\Delta^c}(y\cdot z)=y$ and $\Pi_\Delta(y\cdot z)=z$.

Note that the above condition is equivalent to the following equilibrium equation: For any compact interval $\Delta=[a, b]$ and $\varphi(x) \in L^1(\mathscr{X}, \mu)$

$$(2.15) \quad \int_{\mathscr{X}} \mu(dx)\varphi(x) = \int_{\mathscr{X}(\Delta^c)} \mu(dy)\Big[\varphi(y) + \exp\{\beta\Phi(q^*-q_*)\} \times \sum_{k=1}^\infty \frac{1}{k!}\exp(\beta u k)\int\cdots\int_{(\Delta\times\mathbf{R})^k} dz\,\varphi(y\cdot z)\exp\{-\beta H(y\cdot z;\Delta)\}\Big],$$

where $\mathscr{X}(\Delta^c)=\Pi_{\Delta^c}(\mathscr{X})$.

The set of all Gibbs states associated with Φ, β and u is denoted by $\mathscr{G}_{\beta,u}(\Phi)$. For our potential Φ it can be seen from §6 of [3] that $\sharp\mathscr{G}_{\beta,u}(\Phi)=1$, and we have

THEOREM 3. *Let $\Phi(r)$ satisfy (2.1) and (2.2). For $\mu \in \mathscr{G}_{\beta,u}(\Phi)$ with $\beta>0$ and real u, $\mu(\mathscr{X}_0)=1$, and hence $\mu(\mathscr{X}_\gamma)=1$ for $\gamma \in [0, 1)$.*

§3. Basic Lemma

In this section we will prove Basic Lemma concerning the fluctuation of energy of finitely many particles for the motion of time interval $[-1, 0]$, which plays an essential role in the proof of our results.

Suppose an initial configuration $x=(q_i, p_i)_i$ (labelled as in (2.3)) is given. For each $K\in\mathbf{N}$ we denote by $x^K(t)=(q_i^K(t), p_i^K(t))_i$, $t\in\mathbf{R}$, the (unique) solution of equations of motion (3.1):

$$(3.1a)\qquad \begin{cases} \dfrac{dq_i^K(t)}{dt} = p_i^K(t) \\ \dfrac{dp_i^K(t)}{dt} = -\Phi'(q_i^K(t)-q_{i-1}^K(t)) + \Phi'(q_{i+1}^K(t)-q_i^K(t)) \\ q_i^K(0) = q_i, \quad p_i^K(0) = p_i \end{cases}$$

for i with $q_i\in\Delta_K$ and

$$(3.1b)\qquad q_i^K(t)\equiv q_i, \quad p_i^K(t)\equiv 0$$

for i with $q_i\notin\Delta_K$. Set

$$(3.2)\qquad H_{i,j}^K(t) = 2^{-1}\textstyle\sum_{l=i}^{j} p_l^K(t)^2 + \sum_{l=i}^{j+1}\Phi(q_l^K(t)-q_{l-1}^K(t)), \quad i\leqq j.$$

Basic Lemma. *Suppose an initial configuration $x=(q_i, p_i)_i$ (labelled as in (2.3)) belongs to $\mathscr{X}_\gamma$ for some $\gamma\in[0, 1)$. Then for each i, j with $i\leqq j$ there exists a constant $M_{i,j}\geqq 0$ such that $H_{i,j}^K(t)\leqq M_{i,j}$ for any $t\in[-1, 0]$ and $K\in\mathbf{N}$.*

We devide the proof of Basic Lemma into several steps. For s, $t\in\mathbf{R}$ set

$$(3.3)\quad \Delta H_{i,j}^K(s, t) = -\int_s^t du\Phi'(q_i^K(u)-q_{i-1}^K(u))p_{i-1}^K(u) + \int_s^t du\Phi'(q_{j+1}^K(u)-q_j^K(u))p_{j+1}^K(u).$$

Then we have

Lemma 1. $H_{i,j}^K(t) = H_{i,j}^K(s) + \Delta H_{i,j}^K(s, t)$, $\quad s, t\in\mathbf{R}$.

For the proof, recall that $p_i^K(u)\equiv 0$ for i with $q_i\notin\Delta_K$, and differentiate $H_{i,j}^K(t) - H_{i,j}^K(s)$ with respect to t.

Let $\delta>0$ and put

$$(3.4)\quad \Delta H_{i,j}^K(s, t)^* = 2\Phi(\delta) + |\Phi'(\delta)|(s-t)\{\max_{t\leqq u\leqq s}|p_i^K(u)| + \max_{t\leqq u\leqq s}|p_j^K(u)|\}, \quad t\leqq s.$$

Lemma 2. *Suppose that $q_i^K(t)-q_{i-1}^K(t)\geqq\delta$ and $q_{j+1}^K(t)-q_j^K(t)\geqq\delta$ hold for all $t\in[\tau_0, \tau_1]$ $(-\infty<\tau_0<\tau_1<\infty)$. Then*

$$\Delta H_{i,j}^K(\tau_1, t)\leqq\Delta H_{i,j}^K(\tau_1, t)^*, \qquad t\in[\tau_0, \tau_1].$$

Proof. Since

$$\int_{\tau_1}^{t} du\Phi'(q_i^K(u)-q_{i-1}^K(u))(p_i^K(u)-p_{i-1}^K(u))$$
$$= \Phi(q_i^K(t)-q_{i-1}^K(t)) - \Phi(q_i^K(\tau_1)-q_{i-1}^K(\tau_1)),$$

it follows from the assumption that

$$-\int_{\tau_1}^{t} du\Phi'(q_i^K(u)-q_{i-1}^K(u))p_{i-1}^K(u) \leqq \Phi(\delta) + |\Phi'(\delta)|(\tau_1-t)\max_{t\leqq u\leqq\tau_1}|p_i^K(u)|$$

for $t\in[\tau_0, \tau_1]$. Analogously we have

$$\int_{\tau_1}^{t} du\Phi'(q_{j+1}^K(u)-q_j^K(u))p_{j+1}^K(u) \leqq \Phi(\delta) + |\Phi'(\delta)|(\tau_1-t)\max_{t\leqq u\leqq\tau_1}|p_j^K(u)|$$

for $t\in[\tau_0, \tau_1]$. These inequalities prove the lemma. □

Since $x\in\mathscr{X}_\gamma$, we can take two positive numbers ρ and θ such that

(3.5) $\limsup_{n\to\infty}(2n)^{-1}N(x; \Delta_n) < \rho, \qquad \limsup_{n\to\infty}(2n)^{-1-\gamma}H(x; \Delta_n) < \theta.$

Choose $\delta>0$ so that

(3.6) $$1-2\rho\delta > 0$$

and define

$$\sigma_i^K(t) = \inf\{s \geqq t \mid q_{i+1}^K(s)-q_i^K(s) \leqq \delta\}.$$

LEMMA 3. *$x^K(t)$, $t\in\mathbf{R}$, is δ-regular; namely,* (2.9) *holds for any i and $t\in\mathbf{R}$ (replacing $x(t)$ and $\sigma_i(t)$ by $x^K(t)$ and $\sigma_i^K(t)$, respectively).*

PROOF. Suppose $x^K(t)$ is not δ-regular. Then there exists a number i such that

$$q_{j+1}-q_j \leqq \delta \qquad \text{for all } j \text{ with } q_j > K\vee q_i$$

or

$$q_j-q_{j-1} \leqq \delta \qquad \text{for all } j \text{ with } q_j < (-K)\wedge q_i,$$

because the particles located outside Δ_K are fixed. This implies that

$$\limsup_{n\to\infty}(2n)^{-1}N(x; \Delta_n) \geqq (2\delta)^{-1} > \rho \qquad \text{(by (3.6))},$$

which contradicts (3.5). □

Put

$$i(-1) = i, \qquad j(-1) = j, \qquad t_0 = -1,$$

and define for $k=0, 1, 2,\ldots$

$$\begin{cases} i(k) = \min\{l \leqq i(k-1) \mid \sigma^K_{l-1}\circ\cdots\circ\sigma^K_{i(k-1)-2}\circ\sigma^K_{i(k-1)-1}(t_k) > t_k\}, \\ j(k) = \max\{l \geqq j(k-1) \mid \sigma^K_{l}\circ\cdots\circ\sigma^K_{j(k-1)+1}\circ\sigma^K_{j(k-1)}(t_k) > t_k\}, \\ t_{k+1} = \sigma^K_{i(k)-1}(t_k) \wedge \sigma^K_{j(k)}(t_k), \end{cases}$$

inductively. By virtue of Lemma 3 we can choose a nonnegative integer m such that

$$t_0 \equiv -1 < t_1 < \cdots < t_m < 0 \leqq t_{m+1}.$$

The followings are immediate from the definition:

(3.7) $\quad i(m) \leqq i(m-1) \leqq \cdots \leqq i(0) \leqq i(-1) = i \leqq j = j(-1) \leqq j(0) \leqq \cdots \leqq j(m);$

(3.8) $\quad (i(k), j(k)) \neq (i(k+1), j(k+1)), \qquad k=0, 1,\ldots, m-1;$

(3.9) $$q^K_{i(k)}(t) - q^K_{i(k)-1}(t) > \delta, \qquad q^K_{j(k)+1}(t) - q^K_{j(k)}(t) > \delta$$
$$\text{for all } t \in [t_k, t_{k+1}), \qquad k = 0, 1,\ldots, m;$$

(3.10) $$q^K_{i(k-1)}(t_k) - q^K_{i(k)}(t_k) \leqq \{i(k-1) - i(k)\}\delta,$$
$$q^K_{j(k)}(t_k) - q^K_{j(k-1)}(t_k) \leqq \{j(k) - j(k-1)\}\delta, \qquad k = 0, 1,\ldots, m.$$

Using these notations we define a function $\hat{H}^K_{i,j}\colon [-1, 0] \to [0, \infty)$, $i \leqq j$, by

(3.11) $$\hat{H}^K_{i,j}(t) = H_{i(m),j(m)} + \textstyle\sum_{l=k+1}^m \Delta H^K_{i(l),j(l)}(t_{l+1} \wedge 0, t_l)^* + \Delta H^K_{i(k),j(k)}(t_{k+1} \wedge 0, t)^*$$
$$\text{for } t \in [t_k, t_{k+1}) \cap [-1, 0], \qquad k = 0, 1,\ldots, m,$$

where

$$H_{i,j} = 2^{-1} \textstyle\sum_{l=i}^j p_l^2 + \sum_{l=i}^{j+1} \Phi(q_l - q_{l-1}).$$

(Notice that the definition of $\hat{H}^K_{i,j}$ depends on x and δ.)

LEMMA 4. *$\hat{H}^K_{i,j}(t)$ is nonincreasing in $t \in [-1, 0]$, and*

(3.12) $$H^K_{i,j}(t) \leqq H^K_{i(k),j(k)}(t) \leqq \hat{H}^K_{i,j}(t), \qquad t \in [t_k, t_{k+1}) \cap [-1, 0],$$

for $k=0, 1,\ldots, m$. In particular

(3.13) $$H^K_{i,j}(t) \leqq \hat{H}^K_{i,j}(-1), \qquad t \in [-1, 0].$$

PROOF. We prove only (3.12). The rest is obvious. If (3.12) holds for $k=l$ $(1 \leqq l \leqq m)$, so does for $k=l-1$. In fact we have

(3.14) $$\Delta H^K_{i(l-1),j(l-1)}(t_l, t) \leqq \Delta H^K_{i(l-1),j(l-1)}(t_l, t)^*, \qquad t \in [t_{l-1}, t_l)$$

by (3.9) and Lemma 2, and then

$$
\begin{aligned}
H^K_{i,j}(t) &\leqq H^K_{i(l-1),j(l-1)}(t) && \text{(by (3.7))}\\
&= H^K_{i(l-1),j(l-1)}(t_l) + \Delta H^K_{i(l-1),j(l-1)}(t_l, t) && \text{(by Lemma 1)}\\
&\leqq H^K_{i(l),j(l)}(t_l) + \Delta H^K_{i(l-1),j(l-1)}(t_l, t)^* && \text{(by (3.7) and (3.14))}\\
&\leqq \hat{H}^K_{i,j}(t_l) + \Delta H^K_{i(l-1),j(l-1)}(t_l, t)^* && \text{(by (3.12) with } k=l)\\
&= \hat{H}^K_{i,j}(t) && \text{(by (3.11))}
\end{aligned}
$$

for $t \in [t_{l-1}, t_l)$. (3.12) for $k=m$ is verified in a similar way to the above:

$$
\begin{aligned}
H^K_{i,j}(t) &\leqq H^K_{i(m),j(m)}(t) = H^K_{i(m),j(m)}(0) + \Delta H^K_{i(m),j(m)}(0, t)\\
&\leqq H_{i(m),j(m)} + \Delta H^K_{i(m),j(m)}(0, t)^*\\
&= \hat{H}^K_{i,j}(t), \qquad t \in [t_m, t_{m+1}) \cap [-1, 0]. \quad \square
\end{aligned}
$$

Let

(3.15) $$P(\delta; i,j) = 2|\Phi'(\delta)| + [4|\Phi'(\delta)|^2 + 2\{H_{i,j}+2(j-i+1)\Phi(\delta)\}]^{1/2}.$$

LEMMA 5. $\hat{H}^K_{i,j}(-1) \leqq 2^{-1}P(\delta; i(m), j(m))^2$.

PROOF. Put $P=\{2\hat{H}^K_{i,j}(-1)\}^{1/2}$. Lemma 4 gives us

$$
\begin{aligned}
\max\{p^K_{i(k)}(t)^2, p^K_{j(k)}(t)^2\} &\leqq 2H^K_{i(k),j(k)}(t)\\
&\leqq 2\hat{H}^K_{i,j}(-1) = P^2, \qquad t \in [t_k, t_{k+1}) \cap [-1, 0],
\end{aligned}
$$

for $k=0, 1, 2,\dots, m$. Hence by (3.4)

$$\Delta H^K_{i(k),j(k)}(t_{k+1}\wedge 0, t_k)^* \leqq 2\{\Phi(\delta) + |\Phi'(\delta)|(t_{k+1}\wedge 0 - t_k)P\},$$

$k=0, 1,\dots, m$. It then follows from (3.11) and (3.8) that

(3.16) $$
\begin{aligned}
2^{-1}P^2 = \hat{H}^K_{i,j}(-1) &\leqq H_{i(m),j(m)} + 2\{(m+1)\Phi(\delta)+|\Phi'(\delta)|P\}\\
&\leqq H_{i(m),j(m)} + 2\{(j(m)-i(m)+1)\Phi(\delta) + |\Phi'(\delta)|P\}.
\end{aligned}
$$

This inequality implies that $P \leqq P(\delta; i(m), j(m))$. $\square$

Let $A_{i,j}$ be the set of all pairs $(i(m), j(m))$ which appears in (3.7) when K varies in $\mathbf{N}$, and let $\xi(i,j)$ be the maximum solution of

(3.17) $$
\begin{aligned}
&\{(1-2\rho\delta)\xi - |q_i| \vee |q_j| - 2\rho\delta\}/2\\
&\quad = 2|\Phi'(\delta)| + [4|\Phi'(\delta)|^2 + 4\{2^\gamma\theta(\xi+1)^{1+\gamma} + 2\rho\Phi(\delta)(\xi+1)\}]^{1/2}.
\end{aligned}
$$

Note that $A_{i,j}$ depends on x, δ and that the left-hand side of (3.17) is greater than the right for $\xi > \xi(i,j)$.

LEMMA 6. *Let N_1 be a positive number such that*

(3.18) $(2n)^{-1}N(x; \Delta_n) < \rho, \qquad (2n)^{-1-\gamma}H(x; \Delta_n) < \theta \qquad \textit{for all} \quad n \geqq N_1.$

Then the pair (I, J) *with* $I \leqq i \leqq j \leqq J$ *and* $|q_I| \vee |q_J| > N_1 \vee \xi(i,j)$ *does not belong to* $A_{i,j}$. *In particular* $\sharp A_{i,j} < \infty$.

PROOF. Assume that there exists a pair (I, J) such that

$$I \leqq i \leqq j \leqq J, \quad |q_I| \vee |q_J| > N_1 \vee \xi(i,j) \quad \text{and} \quad (I, J) \in A_{i,j}.$$

Since $(I, J) \in A_{i,j}$, there exists a $K \in \mathbf{N}$ and then a nonnegative integer m such that $i(m) = I$ and $j(m) = J$. Note that $|q_I| \vee |q_J| > \xi(i,j) > |q_i| \vee |q_j|$ implies $q_I \neq q_J$. Without loss of generality we can assume that $|q_I| \vee |q_J| = |q_J|$. Then $|q_J| = q_J > N_1$, and we have

$$\begin{aligned} (3.19) \qquad J - I + 1 &\leqq N(x; [-q_J, q_J]) \leqq 2(q_J + 1)\rho, \\ H_{I,J} &\leqq H(x; [-q_J, q_J]) \leqq \{2(q_J + 1)\}^{1+\gamma}\theta. \end{aligned}$$

Therefore if we put $P = \{2\hat{H}^K_{i,j}(-1)\}^{1/2}$, it follows from Lemma 5 that

$$\begin{aligned} (3.20) \quad P &\leqq P(\delta; I, J) \\ &\leqq 2|\Phi'(\delta)| + [4|\Phi'(\delta)|^2 + 4\{2^\gamma\theta(q_J + 1)^{1+\gamma} + 2\rho\Phi(\delta)(q_J + 1)\}]^{1/2}. \end{aligned}$$

On the other hand, since $p^K_{j(k)}(t) \leqq P$ for $t \in [t_k, t_{k+1}) \cap [-1, 0]$ from Lemma 4, we have

$$|q^K_{j(k)}(t_k) - q^K_{j(k)}(t_{k+1} \wedge 0)| \leqq (t_{k+1} \wedge 0 - t_k)P, \qquad k = 0, 1, \dots, m,$$

and hence by (3.10)

$$\begin{aligned} q^K_{j(k-1)}(t_k) &\geqq q^K_{j(k)}(t_k) - \{j(k) - j(k-1)\}\delta \\ &\geqq q^K_{j(k)}(t_{k+1} \wedge 0) - (t_{k+1} \wedge 0 - t_k)P - \{j(k) - j(k-1)\}\delta \end{aligned}$$

for $k = 0, 1, \dots, m$. Summing up these inequalities for $k = 0, 1, \dots, m$ and using (3.19), we get

$$q^K_j(-1) \geqq q_J - P - (J - j)\delta \geqq q_J - P - 2(q_J + 1)\rho\delta.$$

Since $q^K_j(-1) \leqq q_j + P$ by Lemma 4, we then have

$$(3.21) \qquad 2P \geqq (1 - 2\rho\delta)q_J - q_j - 2\rho\delta \geqq (1 - 2\rho\delta)q_J - |q_i| \vee |q_j| - 2\rho\delta.$$

By the choice of $\xi(i,j)$, (3.20) and (3.21) imply that $q_J \leqq \xi(i,j)$. This is a contradiction. □

PROOF OF BASIC LEMMA. Choose ρ, θ as in (3.5), and δ as in (3.6). Then Lemmas 4, 5 and 6 give us

$$H^K_{i,j}(t) \leqq \hat{H}^K_{i,j}(-1) \leqq \max\{2^{-1}P(\delta;\, I, J)^2 \,|\, (I, J) \in A_{i,j}\} < \infty,$$
$$t \in [-1, 0], \quad K \in \mathbf{N}.$$

Thus we may take

$$M_{i,j} = \max\{2^{-1}P(\delta;\, I, J)^2 \,|\, (I, J) \in A_{i,j}\}. \quad \square \tag{3.22}$$

Concluding this section we will state some remarks which will be used later.

REMARKS. 1. Given $x \in \mathscr{X}_\gamma$, the right-hand side of (3.22) defines a function $M_{i,j}(\delta)$, $0<\delta<(2\rho)^{-1}$. What we have proved is that Basic Lemma holds with $M_{i,j}=M_{i,j}(\delta)$ for *each* positive δ satisfying (3.6).

2. The whole argument of this section also holds for a δ-regular solution $x(t)$ of (2.4) and (2.5) (whenever δ satisfies (3.6)); in this case the suffix "K" must be neglected.

§ 4. Proof of Theorems 1 and 2

In this section, using results obtained in § 3, we will prove Theorems 1 and 2.

PROOF OF THEOREM 1. For $x=(q_i, p_i)_i \in \mathscr{X}_\gamma$ (labelled as in (2.3)) and $K \in \mathbf{N}$, let $x^K(t)=(q^K_i(t), p^K_i(t))_i$ be the solution of (3.1). Take positive numbers ρ_0, θ_0, δ_0 such that (3.5) and (3.6) hold for $\rho=\rho_0$, $\theta=\theta_0$, $\delta=\delta_0$, and fix them. Then Basic Lemma and (3.2) give us that for each i there exists a constant $M_{i,i} \geqq 0$ (independent of $K \in \mathbf{N}$) satisfying

$$\max_{-1\leqq t\leqq 0} |p^K_i(t)| \leqq (2M_{i,i})^{1/2}; \tag{4.1}$$

$$\min_{k=\pm 1} \min_{-1\leqq t\leqq 0} |q^K_i(t) - q^K_{i+k}(t)| \geqq \min\{r \,|\, \Phi(r) = M_{i,i}\}. \tag{4.2}$$

It then follows from (4.1) that $\{q^K_i(t)\}_{K\in\mathbf{N}}$ is uniformly bounded and equicontinuous on $[-1, 0]$ for each i:

$$\begin{cases} \max_{-1\leqq t\leqq 0} |q^K_i(t)| \leqq |q_i| + (2M_{i,i})^{1/2}, & K \in \mathbf{N}; \\ |q^K_i(t) - q^K_i(t')| \leqq (2M_{i,i})^{1/2}|t-t'|, & t, t' \in [-1, 0], \quad K \in \mathbf{N}. \end{cases}$$

Therefore using the Ascoli-Arzelà theorem and the diagonal method, we can extract a subsequence $\{x^{K(l)}(t)\}_{l\in\mathbf{N}}$ of $\{x^K(t)\}_{K\in\mathbf{N}}$ such that for each i $\{q^{K(l)}_i(t)\}_{l\in\mathbf{N}}$ converges uniformly on $[-1, 0]$ as $l\to\infty$; put $q_i(t)=\lim_{l\to\infty} q^{K(l)}_i(t)$, $t \in [-1, 0]$. Since $q^{K(l)}_i(t)$, $q_i \in \Delta_{K(l)}$, satisfies the (integral form of) equations of motion

$$q^{K(l)}_i(t) = q_i + p_i t + \int_0^t ds(t-s)\{-\Phi'(q^{K(l)}_i(s) - q^{K(l)}_{i-1}(s)) + \Phi'(q^{K(l)}_{i+1}(s) - q^{K(l)}_i(s))\}, \tag{4.3}$$

it follows from (4.2) that

(4.4) $q_i(t) = q_i + p_i t + \int_0^t ds(t-s)\{-\Phi'(q_i(s)-q_{i-1}(s)) + \Phi'(q_{i+1}(s)-q_i(s))\}$

for $t \in [-1, 0]$. It is easy to see that, by (4.3) and (4.4), $p_i^{\tilde{K}(l)}(t)$ also converges uniformly on $[-1, 0]$ to $p_i(t) = \dot{q}_i(t)$ for each i as $l \to \infty$. Thus we have constructed a solution $x(t) = (q_i(t), p_i(t))_i$ of (2.4) and (2.5) on the time interval $[-1, 0]$. We now state several properties of this $x(t)$ in Proposition 1, which will be proved later.

PROPOSITION 1. (i) $\cdots < q_{i-1}(t) < q_i(t) < q_{i+1}(t) < \cdots, \quad t \in [-1, 0]$.
(ii) $x(t) \in \mathscr{X}_\gamma$ *for every* $t \in [-1, 0]$; *more precisely,*

a) $\limsup_{n\to\infty} (2n)^{-1} N(x(t); \Delta_n) \leqq \limsup_{n\to\infty} (2n)^{-1} N(x; \Delta_n)$,

b) $\limsup_{n\to\infty} (2n)^{-1-\gamma} H(x(t); \Delta_n) \begin{cases} < \infty & \text{if } \gamma = 0, \\ \leqq \limsup_{n\to\infty} (2n)^{-1-\gamma} H(x; \Delta_n) & \text{if } 0 < \gamma < 1. \end{cases}$

(iii) $\lim_{n\to\infty} n^{-1} \sup_{q_i \in \Delta_n} \max_{-1 \leqq t \leqq 0} |p_i(t)| = 0$.

Consider $\tilde{x} = (q_i, -p_i)_i \in \mathscr{X}_\gamma$ as an initial configuration, and apply the preceding argument to $x^{K(l)}(t) = (q_i^{K(l)}(-t), -p_i^{K(l)}(-t))_i$, $t \in [-1, 0]$. Then there exists a subsequence $\{\tilde{K}(l)\}_{l \in \mathbf{N}}$ of $\{K(l)\}_{l \in \mathbf{N}}$ such that for each i $(q_i^{\tilde{K}(l)}(-t), -p_i^{\tilde{K}(l)}(-t))$ converges uniformly on $[-1, 0]$ to some $(\tilde{q}_i(t), \tilde{p}_i(t))$ as $l \to \infty$. If we put $x(t) = (\tilde{q}_i(-t), -\tilde{p}_i(-t))_i$ for $t \in [0, 1]$, $x(t)$ satisfies (4.4) and Proposition 1 (replacing $[-1, 0]$ by $[0, 1]$). In this manner we have a solution $x(t)$ of (2.4) and (2.5) on the time interval $[-1, 1]$. Since $x(-1), x(1) \in \mathscr{X}_\gamma$, we can continue the above procedure and have a solution $x(t)$, $t \in \mathbf{R}$, of (2.4) and (2.5) satisfying (i), (ii) of Theorem 1 and

(4.5) $$\lim_{n\to\infty} n^{-1} \sup_{q_i \in \Delta_n} \max_{t \in \Delta_T} |p_i(t)| = 0, \qquad 0 < T < \infty.$$

Now we prove (iii) of Theorem 1 for this $x(t)$, $t \in \mathbf{R}$. Let ρ be a positive number defined by (3.5) for $x \in \mathscr{X}_\gamma$ $(0 \leqq \gamma < 1)$, that is,

(4.6) $$\limsup_{n\to\infty} (2n)^{-1} N(x; \Delta_n) < \rho.$$

Then take any $\delta > 0$ satisfying (3.6). Notice that ρ and δ may be different from ρ_0 and δ_0. We can prove the δ-regularity of $x(t)$ in the following way. Assume that $x(t)$ is not δ-regular. Then there exist an integer i and $\tau_0, \tau_1 \in \mathbf{R}$ $(\tau_0 \leqq \tau_1)$ such that

$$\lim_{k\to\infty} \sigma_{i+k} \circ \cdots \circ \sigma_{i+1} \circ \sigma_i(\tau_0) = \tau_1$$

or

$$\lim_{k\to\infty} \sigma_{i-k} \circ \cdots \circ \sigma_{i-1} \circ \sigma_i(\tau_0) = \tau_1.$$

We may also assume that the first case occurs. Write

$$T = |\tau_0| \vee |\tau_1|,$$
$$s_k = q_{i+k}\circ\cdots\circ\sigma_{i+1}\circ\sigma_i(\tau_0), \qquad k = 0, 1, 2,\dots,$$
$$V(n) = \sup_{q_j\in\Delta_n} \max_{t\in\Delta_T} |p_j(t)|, \qquad n\in\mathbf{N},$$
$$m(q) = \min\{m\in\mathbf{N} \mid |q|\leqq m\}. \tag{4.7}$$

Notice that

$$-T\leqq s_0 \leqq s_1 \leqq\cdots\leqq s_k \leqq\cdots\leqq T, \tag{4.8}$$
$$0 < q_{i+k}(s_{k-1}) - q_{i+(k-1)}(s_{k-1}) \leqq \delta, \qquad k = 1, 2,\dots, \tag{4.9}$$
$$|q_i(t) - q_i(t')| \leqq V(n)|t-t'|, \qquad q_i\in\Delta_n, \quad t, t'\in\Delta_T, \quad n\in\mathbf{N}. \tag{4.10}$$

Then for each positive integer k we have

$$\begin{aligned} q_{i+k}(s_k) &\leqq q_{i+k}(s_{k-1}) + V(m(q_{i+k}))(s_k - s_{k-1}) && \text{(by (4.10))} \\ &\leqq q_{i+(k-1)}(s_{k-1}) + \delta + V(m(q_{i+k}))(s_k - s_{k-1}) && \text{(by (4.9))} \\ &\cdots\cdots \\ &\leqq q_i(s_0) + k\delta + \textstyle\sum_{l=1}^k V(m(q_{i+l}))(s_l - s_{l-1}). \end{aligned}$$

For every k with $q_{i+k}>|q_i|$ it holds that $m(q_{i+l})\leqq m(q_{i+k})$, $l=0, 1,\dots, k$, and hence the above inequalities yield that

$$\begin{aligned} q_{i+k}(s_k) &\leqq q_i(s_0) + k\delta + V(m(q_{i+k}))(s_k - s_0) \\ &\leqq q_i(0) + k\delta + 3V(m(q_{i+k}))T \qquad \text{(by (4.8) and (4.10))}. \end{aligned}$$

On the other hand

$$q_{i+k}(s_k) \geqq q_{i+k}(0) - V(m(q_{i+k}))T$$

by (4.8) and (4.10). Thus we have

$$q_{i+k} \leqq q_i + k\delta + 4V(m(q_{i+k}))T$$

for every k with $q_{i+k}>|q_i|$. Then

$$\begin{aligned} 1 = \limsup_{k\to\infty} q_{i+k}/m(q_{i+k}) &\leqq \limsup_{k\to\infty} k\delta/m(q_{i+k}) && \text{(by (4.5))} \\ &= \limsup_{k\to\infty} (i+k)\delta/m(q_{i+k}) < 2\rho\delta && \text{(by (4.6))}, \end{aligned}$$

which contradicts (3.6). □

PROOF OF PROPOSITION 1. (i) is obvious from (4.2). We devide the proof of (ii) into four steps.

1°. Take *any* positive numbers ρ, θ and δ satisfying (3.5) and (3.6), and let $\xi(i,j)$ be the maximum solution of (3.17). Then

(4.11) $$\lim_{|q_i|\vee|q_j|\to\infty} \xi(i,j)/\{|q_i|\vee|q_j|\} = (1-2\rho\delta)^{-1}.$$

In fact, let α be any accumulation point of $\{\xi(i,j)/(|q_i|\vee|q_j|)\,|\,i\leqq j\}$ (we permit the case $\alpha=\infty$) and choose a sequence $\{(q_{i_k}, q_{j_k})\}_{k\in\mathbf{N}}$ satisfying

$$\lim_{k\to\infty} |q_{i_k}|\vee|q_{j_k}| = \infty, \qquad \lim_{k\to\infty} \xi(i_k,j_k)/\{|q_{i_k}|\vee|q_{j_k}|\} = \alpha.$$

Setting $i=i_k$, $j=j_k$ in (3.17) and letting $k\to\infty$, we have $0<\alpha<\infty$ and $(1-2\rho\delta\alpha-1=0$.

2°. Let $N_1>0$ satisfy (3.18). Then for i, j with $i\leqq j$ and $|q_i|\vee|q_j|\geqq N_1$,

(4.12) $$H_{i,j}(t) \leqq \hat{M}_{i,j}, \qquad t\in[-1,0],$$

where

$$H_{i,j}(t) = 2^{-1}\textstyle\sum_{l=i}^{j} p_l(t)^2 + \sum_{l=i}^{j+1} \Phi(q_l(t)-q_{l-1}(t)),$$

(4.13) $$\hat{M}_{i,j} = 2^{-1}\{2|\Phi'(\delta)| + [4|\Phi'(\delta)|^2 + 4\{2^\gamma\theta(\xi(i,j)+1)^{1+\gamma} + 2\rho\Phi(\delta)(\xi(i,j)+1)\}]^{1/2}\}^2.$$

Indeed, let $i\leqq j$ and $|q_i|\vee|q_j|\geqq N_1$. Then Lemma 6 implies that

$$N_1 \leqq |q_i|\vee|q_j| \leqq |q_I|\vee|q_J| \leqq N_1\vee\xi(i,j)$$

for any $(I, J)\in A_{i,j}$. On the other hand $|q_i|\vee|q_j|<\xi(i,j)$ by the definition of $\xi(i,j)$ in (3.17). Therefore we have

$$|q_I|\vee|q_J| \leqq \xi(i,j), \qquad N_1 \leqq \xi(i,j),$$

and hence

$$J-I+1 \leqq N(x; [-\xi(i,j), \xi(i,j)]) \leqq 2(\xi(i,j)+1)\rho,$$
$$H_{I,J} \leqq H(x; [-\xi(i,j), \xi(i,j)]) \leqq \{2(\xi(i,j)+1)\}^{1+\gamma}\theta$$

for all $(I, J)\in A_{i,j}$. Thus, if we take $M_{i,j}$ as in (3.22), we have by (3.15)

(4.14) $$M_{i,j} \leqq \hat{M}_{i,j} \quad \text{for } i, j \text{ with } i\leqq j \text{ and } |q_i|\vee|q_j|\geqq N_1.$$

The solution $x(t)$, $t\in[-1,0]$, constructed in the above may depend on ρ_0, θ_0 and δ_0. But $H_{i,j}^K(t)\leqq M_{i,j}$, $t\in[-1,0]$, $K\in\mathbf{N}$ (Basic Lemma and Remark 1). Therefore $H_{i,j}(t)\leqq M_{i,j}$, $t\in[-1,0]$, and (4.12) follows from (4.14).

3°. For any $\varepsilon\in(0,1)$, there exists a positive number N_2 such that for every $n\geqq N_2$ $|q_i|\leqq m((1+\varepsilon)n)$ holds for all i with $\min_{-1\leqq t\leqq 0}|q_i(t)|\leqq n$, where $m(\cdot)$ is the function defined by (4.7).

To prove 3° it is sufficient to show that there exists N_2 such that for every $n\geqq N_2$ $q_i>m((1+\varepsilon)n)$ implies $q_i(t)>n$, $t\in[-1,0]$. Set

$$\hat{V}(n) = \sup_{q_i\in\Delta_n}\max_{-1\leqq t\leqq 0}|p_i(t)|,$$

$$L(n) = \min\{i \mid -n \leqq q_i\}, \quad R(n) = \max\{i \mid q_i \leqq n\}, \qquad n \in \mathbf{N}.$$

Then we have $\hat{V}(n) \leqq \max_{-1 \leqq t \leqq 0} \{2H_{L(n),R(n)}(t)\}^{1/2}$. Therefore we obtain

$$0 \leqq \limsup_{n\to\infty} n^{-1}\hat{V}(n) \leqq \limsup_{n\to\infty} \frac{\{2\hat{M}_{L(n),R(n)}\}^{1/2}}{|q_{L(n)}| \vee |q_{R(n)}|} \quad \text{(by } 2^\circ)$$
$$= 0 \qquad \text{(by (4.13) and } 1^\circ),$$

and so

$$\lim_{n\to\infty} n^{-1}\hat{V}(n) = 0. \tag{4.15}$$

Choose $N_2 > 0$ so that $\hat{V}(n)/(n-1) < \varepsilon/2$ holds for all $n \geqq N_2$. Suppose that $n \geqq N_2$ and $q_i > m((1+\varepsilon)n)$; take $k \in \mathbf{N}$ such that

$$m((1+\varepsilon)n) + k - 1 < q_i \leqq m((1+\varepsilon)n) + k.$$

Then $\max_{-1 \leqq t \leqq 0} |p_i(t)| \leqq \hat{V}(m((1+\varepsilon)n)+k)$, and hence for $t \in [-1, 0]$ we have

$$q_i(t) \geqq q_i - \hat{V}(m((1+\varepsilon)n)+k)|t| \geqq m((1+\varepsilon)n) + k - 1 - \hat{V}(m((1+\varepsilon)n)+k)$$
$$\geqq \{m((1+\varepsilon)n)+k-1\}(1-\varepsilon/2) \geqq (1+\varepsilon)n(1-\varepsilon/2) > n.$$

Therefore 3° is proved.

4°. (Proof of (a).) Let $\varepsilon \in (0, 1)$ and take $N_2 > 0$ as in 3°. Then for each $t \in [-1, 0]$ we have

$$N(x(t); \Delta_n) \leqq N(x; \Delta_{m((1+\varepsilon)n)}), \qquad n > N_2.$$

Therefore

$$\limsup_{n\to\infty} (2n)^{-1}N(x(t); \Delta_n) \leqq \limsup_{n\to\infty} (2n)^{-1}N(x; \Delta_{m((1+\varepsilon)n)})$$
$$\leqq (1+\varepsilon) \limsup_{n\to\infty} (2n)^{-1}N(x; \Delta_n), \qquad t \in [-1, 0],$$

which implies (a).

(Proof of (b).) For $\varepsilon \in (0, 1)$ take $N_2 > 0$ as in 3°. Then

$$H(x(t); \Delta_n) \leqq H_{L(m((1+\varepsilon)n)),R(m((1+\varepsilon)n))}(t), \qquad n > N_2$$

for $t \in [-1, 0]$. Hence we have

$$\limsup_{n\to\infty} (2n)^{-1-\gamma}H(x(t); \Delta_n)$$
$$\leqq \limsup_{n\to\infty} (2n)^{-1-\gamma}H_{L(m((1+\varepsilon)n)),R(m((1+\varepsilon)n))}(t)$$
$$\leqq \limsup_{n\to\infty} (2n)^{-1-\gamma}\hat{M}_{L(m((1+\varepsilon)n)),R(m((1+\varepsilon)n))} \qquad \text{(by } 2^\circ)$$
$$\leqq \begin{cases} (1+\varepsilon)(\theta+2\rho\Phi(\delta))(1-2\rho\delta)^{-1} & \text{if } \gamma = 0 \\ (1+\varepsilon)^{1+\gamma}\theta(1-2\rho\delta)^{-1-\gamma} & \text{if } 0 < \gamma < 1 \end{cases} \qquad \text{(by (4.13) and } 1^\circ)$$

for $t\in[-1, 0]$. Therefore (b) for $\gamma=0$ is proved; in case $0<\gamma<1$, letting $\varepsilon\downarrow 0$, $\delta\downarrow 0$, we get

$$\limsup_{n\to\infty}(2n)^{-1-\gamma}H(x(t);\Delta_n)\leqq\theta,\qquad t\in[-1, 0].$$

(iii) has been already proved in (4.15). □

REMARK 3. The inequality in (ii–a) of Proposition 1 can be replaced by the equality. In fact, we choose N_2 as in the proof of 3° (immediately after (4.15)). Then, if $n>N_2/(1-\varepsilon)$, $\varepsilon\in(0, 1)$, and $|q_i|\leqq[(1-\varepsilon)n]$[3], we have for $t\in[-1, 0]$

$$\begin{aligned} q_i(t) &\leqq q_i+\hat{V}([(1-\varepsilon)n])|t|\leqq[(1-\varepsilon)n]+\hat{V}([(1-\varepsilon)n])\\ &\leqq[(1-\varepsilon)n](1+\varepsilon/2)\leqq n,\\ q_i(t) &\geqq q_i-\hat{V}([(1-\varepsilon)n])|t|\geqq -n. \end{aligned}$$

This implies that

$$N(x(t);\Delta_n)\geqq N(x;\Delta_{[(1-\varepsilon)n]}),\qquad t\in[-1, 0],$$

for $n>N_2/(1-\varepsilon)$. Therefore

$$\begin{aligned} &\limsup\,(2[(1-\varepsilon)n])^{-1}N(x(t);\Delta_n)\\ &\quad\geqq\limsup_{n\to\infty}(2[(1-\varepsilon)n])^{-1}N(x;\Delta_{[(1-\varepsilon)n]})=\limsup_{n\to\infty}(2n)^{-1}N(x;\Delta_n), \end{aligned}$$

which proves the opposite inequality of (ii–a).

PROOF OF THEOREM 2. Let $\Phi(r)$, γ and $x=(q_i, p_i)_i$ satisfy the conditions of Theorem 2, and let $\bar{x}(t)=(\bar{q}_i(t), \bar{p}_i(t))_i$, $\tilde{x}(t)=(\tilde{q}_i(t), \tilde{p}_i(t))_i$ be two regular solutions of (2.4) and (2.5). It is sufficient for us to prove

(4.16) $\qquad \bar{x}(t)=\tilde{x}(t),\qquad t\in[-1, 0],$

(4.17) $\qquad \bar{x}(-1)\in\mathscr{X}_\gamma$ and

$$\begin{aligned} &\limsup_{n\to\infty}n^{-1}\{N(\bar{x}(-1);[-n, 0))\wedge N(\bar{x}(-1);[0, n])\}\\ &\qquad\qquad\geqq\limsup_{n\to\infty}n^{-1}\{N(x;[-n, 0))\wedge N(x;[0, n])\}. \end{aligned}$$

Take ρ, θ as in (3.5); choose $\delta>0$ so small that $1-2\rho\delta>0$ and that both $\bar{x}(t)$ and $\tilde{x}(t)$ are δ-regular (notice that if $x(t)$ is δ-regular and if $0<\delta'\leqq\delta$, then $x(t)$ is also δ'-regular); define $\bar{M}_{i,j}$ [resp. $\tilde{M}_{i,j}$] as in (3.22) for $\bar{x}(t)$ [resp. $\tilde{x}(t)$]. Then Remark 2 implies that Basic Lemma as well as (i), (ii) of Theorem 1 holds for both $\bar{x}(t)$ and $\tilde{x}(t)$. Set

$$\begin{aligned} &\bar{r}_i(t)=\bar{q}_i(t)-\bar{q}_{i-1}(t),\qquad \tilde{r}_i(t)=\tilde{q}_i(t)-\tilde{q}_{i-1}(t),\\ &\Delta r_i(t)=|\bar{r}_i(t)-\tilde{r}_i(t)|, \end{aligned}$$

3) $[p]$ denotes the largest integer not exceeding p.

$$D_{i,j} = \min_{i\leqq l\leqq j+1} \min_{-1\leqq t\leqq 0} \{\bar{r}_l(t) \wedge \tilde{r}_l(t)\},$$
$$M^*_{i,j} = \bar{M}_{i,j} \vee \tilde{M}_{i,j}.$$

Then the followings are immediate:

(4.18) $D_{k,l} \leqq D_{i,j}, \qquad k \leqq i \leqq j \leqq l,$

(4.19) $|\Phi'(\bar{r}_l(t)) - \Phi'(\tilde{r}_l(t))| \leqq G(\Phi(D_{i,j}))\Delta r_l(t), \quad t \in [-1, 0], \quad i \leqq l \leqq j+1,$

(the function G is defined by (2.12)),

(4.20) $D_{i,j} \geqq \min\{r \mid \Phi(r) = M^*_{i,j}\}$ (by Basic Lemma).

Since

$$\max_{-1\leqq t\leqq 0} \{|\bar{p}_l(t)| \vee |\tilde{p}_l(t)|\} \leqq (2M^*_{i,j})^{1/2}, \qquad i \leqq l \leqq j,$$

by Basic Lemma, we have

(4.21)
$$\begin{aligned} \Delta r_l(t) &\leqq \textstyle\sum_{k=0,-1} \{|\bar{q}_{l+k}(t) - q_{l+k}| + |\tilde{q}_{l+k}(t) - q_{l+k}|\} \\ &\leqq 4|t|(2M^*_{i,j})^{1/2}, \qquad t \in [-1, 0], \quad i+1 \leqq l \leqq j. \end{aligned}$$

On the other hand, (4.4) implies that $\bar{r}_i(t)$ [resp. $\tilde{r}_i(t)$] satisfies

$$\begin{aligned} \bar{r}_i(t) &= (q_i - q_{i-1}) + (p_i - p_{i-1})t \\ &\quad + \int_0^t ds(t-s)\{-2\Phi'(\bar{r}_i(s)) + \Phi'(\bar{r}_{i-1}(s)) + \Phi'(\bar{r}_{i+1}(s))\}. \end{aligned}$$

Therefore we have for $t \in [-1, 0]$

(4.22)
$$\begin{aligned} \Delta r_i(t) &\leqq 4G(\Phi(D_{i-1,i})) \int_0^t dt_1(t-t_1) \max_{i-1\leqq l\leqq i+1} \Delta r_l(t_1) \qquad \text{(by (4.19))} \\ &\leqq \{\textstyle\prod_{k=1}^{n-1} 4G(\Phi(D_{i-k,i+k-1}))\} \int_0^t dt_1(t-t_1) \int_0^{t_1} dt_2(t_1-t_2) \cdots \\ &\qquad \cdots \int_0^{t_{n-2}} dt_{n-1}(t_{n-2}-t_{n-1}) \max_{i-(n-1)\leqq l\leqq i+n-1} \Delta r_l(t_{n-1}) \\ &\leqq \{4G(M^*_{i-n,i+n})\}^{n-1} \int_0^t dt_1(t-t_1) \int_0^{t_1} dt_2(t_1-t_2) \cdots \\ &\qquad \cdots \int_0^{t_{n-2}} dt_{n-1}(t_{n-2}-t_{n-1}) 4|t_{n-1}|(2M^*_{i-n,i+n})^{1/2} \\ &\qquad\qquad \text{(by (4.18), (4.20) and (4.21))} \\ &\leqq \{4G(\hat{M}_{i-n,i+n})\}^{n-1} 4(2\hat{M}_{i-n,i+n})^{1/2} \frac{1}{(2n-1)!} \qquad \text{(by (4.14))} \\ &\leqq \tilde{c} \frac{4e}{(2\pi)^{1/2}} \left\{\frac{4e^2 G(\hat{M}_{i-n,i+n})}{(2n-1)^2}\right\}^{n-1} \frac{(2\hat{M}_{i-n,i+n})^{1/2}}{(2n-1)^{3/2}} \\ &\qquad\qquad \text{(by Stirling's formula)} \end{aligned}$$

for all sufficiently large n, where $\tilde{c}$ is a constant independent of n. Since

$$c \equiv \limsup_{n\to\infty} n^{-1}\{N(x; [-n, 0)) \wedge N(x; [0, n])\} > 0 \qquad \text{(by (2.13))},$$

it is easy to find an increasing sequence $\{n(l)\}_{l\in\mathbf{N}}$ of positive integers satisfying

$$\limsup_{l\to\infty} \frac{1}{n(l)} \{|q_{i-n(l)}| \vee q_{i+n(l)}\} \leqq c^{-1}$$

(the sequence $\{n(l)\}_l$ may depend on i). Then

$$\begin{aligned}\limsup_{l\to\infty} \frac{1}{n(l)^2} G(\hat{M}_{i-n(l),i+n(l)}) &\leqq \limsup_{l\to\infty} \frac{1}{c^2} \frac{G(\hat{M}_{i-n(l),i+n(l)})}{\{|q_{i-n(l)}| \vee q_{i+n(l)}\}^2} \\ &= 0 \qquad \text{(by (4.13), (4.11) and (2.11))}\end{aligned}$$

and also

$$\begin{aligned}\limsup_{l\to\infty} \frac{1}{n(l)^3} \hat{M}_{i-n(l),i+n(l)} &\leqq \limsup_{l\to\infty} \frac{\hat{M}_{i-n(l),i+n(l)}}{[c\{|q_{i-n(l)}| \vee q_{i+n(l)}\}]^3} \\ &= 0, \qquad 0 \leqq \gamma < 1.\end{aligned}$$

Therefore letting $n\to\infty$ in (4.22) via the subsequence $\{n(l)\}_{l\in\mathbf{N}}$, we have $\Delta r_i(t)=0$, $t\in[-1, 0]$. Thus (4.16) follows from (4.4). $\bar{x}(-1)\in\mathscr{X}_\gamma$ is clear from Theorem 1; the rest of (4.17) is proved analogously to that of Remark 3. □

§5. Proof of Theorem 3

In this section we will prove Theorem 3. The proof relies essentially on the results of [3; §6]; we also use the terminology *locally bounded* (l.b.), *locally positive* (l.p.),... as in [3].

Let f be a nonnegative l.b. measurable function on $(0, \infty)$, and let g be a probability density function (p.d.f.) on $\mathbf{R}$. For any compact interval $\Delta=[a, b]$ and $x\in\mathscr{X}$, we define a probability measure $m^{(\Delta,x)}$ on $\{y\in\mathscr{X} \mid \Pi_{\Delta^c}(y)=\Pi_{\Delta^c}(x)\}$ by

(5.1) $$\begin{aligned} & u(q_*, a, b, q^*)\int \varphi(y)m^{(\Delta,x)}(dy) \\ &= \varphi(\phi)f(q^*-q_*) + \int_a^b dq\int_{\mathbf{R}} dp\varphi((q, p))f(q-q_*)f(q^*-q)g(p) \\ &\quad + \sum_{k=2}^\infty \int\cdots\int_{a\leqq q_1<\cdots<q_k\leqq b} dq_1\cdots dq_k \int\cdots\int_{\mathbf{R}^k} dp_1\cdots dp_k\varphi((q_l, p_l)_{l=1}^k) \\ &\quad\quad \times f(q_1-q_*)\prod_{i=2}^k f(q_i-q_{i-1})f(q^*-q_k)\prod_{j=1}^k g(p_j), \\ &\qquad\qquad \varphi\in B(\mathscr{X}, [a, b]).\end{aligned}$$

Here $u(q_*, a, b, q^*)$ is the normalizing factor, and $(q_l, p_l)_{l=1}^k$ denotes any element of $\mathscr{X}$ whose restriction to Δ is $(q_l, p_l)_{l=1}^k$. (For other notations, see § 2.) Denote by $\mathscr{G}_{f,g}$ the set of probability measures m on $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$ such that for every compact interval $\Delta=[a, b]$ the regular conditional probability distributions (r.c.p.d.) of $m \mid \tilde{\mathscr{B}}^{\Delta}$ evaluated at $x \in \mathscr{X}$ coincide with the measure $m^{(\Delta,x)}$ given by (5.1). Then we can prove the following facts by arguments similar to [3].

(a) If f is a l.b. p.d.f. on $(0, \infty)$ with a finite first moment, then $\mathscr{G}_{f,g} \neq \phi$.

(b) If f is l.b. and l.p. and if $\mathscr{G}_{f,g} \neq \phi$, then there is a λ such that $\hat{f}(r)=e^{\lambda r}f(r)$ is a p.d.f. on $(0, \infty)$ with a finite first moment ρ^{-1}, and $\mathscr{G}_{f,g}$ consists of exactly one element $m_{\hat{f}}$ whose marginals are given by

$$
\begin{aligned}
(5.2)\quad & \int \varphi(x) m_{\hat{f}}(dx) \\
&= \varphi(\phi)\int_{b-a}^{\infty} \rho(1-\hat{F}(r))dr \\
&+\int_a^b dq \int_{\mathbf{R}} dp\varphi((q, p))\rho(1-\hat{F}(q-a))(1-\hat{F}(b-q))g(p) \\
&+ \textstyle\sum_{k=2}^{\infty} \displaystyle\int\cdots\int_{a \leqq q_1<\cdots<q_k \leqq b} dq_1\cdots dq_k \int\cdots\int_{\mathbf{R}^k} dp_1\cdots dp_k\varphi((q_l, p_l)_{l=1}^k) \\
&\quad \times \rho(1-\hat{F}(q_1-a))\textstyle\prod_{i=2}^k \hat{f}(q_i-q_{i-1})(1-\hat{F}(b-q_k))\prod_{j=1}^k g(p_j), \\
&\qquad\qquad \varphi \in B(\mathscr{X}, [a, b]),
\end{aligned}
$$

where $\hat{F}(t)=\int_0^t \hat{f}(s)ds$.

PROOF OF THEOREM 3. Set

$$f(r) = \exp\{-\beta\Phi(r) + \beta u + \log(2\pi/\beta)^{1/2}\}, \quad g(p) = (\beta/2\pi)^{1/2} \exp\{-\beta p^2/2\}.$$

Then $\mathscr{G}_{f,g}=\mathscr{G}_{\beta,u}(\Phi)$. Since $f(r)$ is bounded on $(0, \infty)$, there is a $\lambda>0$ such that $\int_0^{\infty} \hat{f}(r)dr=1$ for $\hat{f}(r)=e^{-\lambda r} f(r)$. In this case $\hat{f}$ has a finite first moment ρ^{-1}. It then follows from (a) that $\mathscr{G}_{\hat{f},g}=\mathscr{G}_{f,g}=\mathscr{G}_{\beta,u}(\Phi)\neq\phi$. Unfortunately $f(r)$ is not l.p., and so we cannot use (b) directly. However, $f(r)$ is bounded, strictly positive and continuous on $(0, \infty)$, and hence Lemma 6.9 (replacing $N(\beta-x)$ by $N(s_1-x, \beta-x)\equiv\inf_{s_1-x\leqq t\leqq\beta-x} f(t)$ in the proof), Lemma 6.22 and Lemma 6.23 of [3] still hold. Therefore the conclusion of (b) is also valid, which implies $\#\mathscr{G}_{\beta,u}(\Phi)=1$. Let μ be the unique element of $\mathscr{G}_{\beta,u}(\Phi)$, which satisfies (5.2) with $m_{\hat{f}}$ replaced by μ. Regarding q_0, q_i-q_{i-1} $(i\geqq 1)$, p_i $(i\geqq 0)$ as random variables on the probability space $(\mathscr{X}, \mu)$, we can easily see from (5.2) that they are mutually independent, and their p.d.f.'s are $\rho(1-\hat{F}(r))$, $\hat{f}(r)$ and $g(p)$, respectively. Then the law of large numbers implies that

$$(5.3) \qquad \begin{cases} \lim_{k\to\infty} k^{-1}\{q_0 + \sum_{i=1}^{k}(q_i - q_{i-1})\} = \rho^{-1} & \mu\text{-a.e.,} \\ \lim_{k\to\infty} k^{-1}\sum_{i=1}^{k} 2^{-1}p_i^2 = (2\beta)^{-1} & \mu\text{-a.e.;} \end{cases}$$

hence we have

$$(5.4) \qquad \lim_{n\to\infty} n^{-1}N(x; [0, n]) = \rho \qquad \mu\text{-a.e.}$$

Let $\tilde{\theta}$ be a positive number such that

$$E(\Phi(q_i - q_{i-1})) = \int_0^\infty \Phi(r)\hat{f}(r)dr = \tilde{\theta} \qquad i = 1, 2, \ldots .$$

The law of large numbers also gives us

$$\lim_{k\to\infty} k^{-1}\sum_{i=1}^{k} \Phi(q_i - q_{i-1}) = \tilde{\theta} \qquad \mu\text{-a.e.}$$

Therefore using (5.4) we get

$$\lim_{n\to\infty} n^{-1}H(x; [0, n]) = \rho\{(2\beta)^{-1} + \tilde{\theta}\} \qquad \mu\text{-a.e.,}$$

which implies our assertion that $\mu(\mathscr{X}_0)=1$. □

REMARK 4. From (5.4) it follows that the condition (2.13) of Theorem 2 is satisfied for μ-a.e. x.

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