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On nonstationary solutions of the Navier-Stokes equations in an exterior domain

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Introduction

Let us consider a moving body in a viscous incompressible fluid filling the whole space R^3. If we describe the fluid motion by using a coordinate system attached to the body, we obtain the exterior problem for the Navier-Stokes equations:

$$
(*) \qquad \begin{array}{rll}
\partial u/\partial t - \Delta u + (u, \nabla)u = f - \nabla p & \text{in} & D \times (0, T), \\
\operatorname{div} u = 0 & \text{in} & D \times (0, T), \\
u(x, t) = u^*(x, t) & \text{on} & S \times (0, T), \\
u(x, t) \to u_\infty(t) & \text{as} & |x| \to \infty, \\
u(x, 0) = a(x) & \text{in} & D.
\end{array}
$$

Here D is the exterior to the body with the boundary S which we assume to be smooth; $u = \{u^j(x, t)\}_{j=1}^3$ and $p = p(x, t)$ denote, respectively, the unknown velocity and pressure, while $f = \{f^j(x, t)\}_{j=1}^3$ and $a = \{a^j(x)\}_{j=1}^3$ denote, respectively, the given external force and initial velocity. u^* and u_∞ are given boundary data. For this problem, Hopf [16] proved the existence of a square-summable weak solution, when $u^* = u_\infty = 0$, for an arbitrary square-summable initial velocity.

On the other hand, in the case of stationary flow, i.e., when $\partial u/\partial t = 0$, $u^* = 0$ and $u_\infty = $ const., Finn [4], [5], [6] proved the existence of a solution, called a physically reasonable solution, which exhibits a phenomenon of wake. Moreover, in [3] he showed that if $u(x)$ is such a solution and if the force exerted to the body by the flow does not vanish, then $u(x) - u_\infty$ is not square-summable over D.

In view of the above result, it seems reasonable to seek a solution of the problem $(*)$ in a class of functions $u(x, t)$ such that $u(x, t) - u_\infty(t)$ is not square-summable over D. This problem was discussed by Heywood in a series of papers [12], [13], [14], [15]. He showed a local existence result in the class of functions with finite Dirichlet integral by using a variant of the Faedo-Galerkin approximation developed by Hopf [16], Kiselev and Ladyzhenskaya [18] and Prodi [28]. However, he assumes that the initial function a be square-summable in proving the existence of a global solution; see [15, Th. 6].

The purpose of this paper is to weaken the assumptions on the initial data imposed by Heywood by using the semigroup approach which was developed by Kato and Fujita [7], [17] in the case of a bounded domain. In particular, under the homogeneous boundary condition, we shall show the existence of a non-square-summable global solution for non-square-summable initial data when they are sufficiently small. Further, we shall give a rate of pointwise decay for our global solutions. This decay result improves the result of Heywood [15] since the global solutions obtained in [15] are necessarily square-summable.

In Section 1 we first prove the direct sum decomposition of the Banach space $(L_r(D))^3(1<r<\infty)$ into its solenoidal and potential parts for an arbitrary exterior domain D in R^3. In the case of a bounded domain, the corresponding result is given in Fujiwara and Morimoto [8]. Using this decomposition, we then define the Stokes operator in L_r spaces and discuss some of its basic properties.

Section 2 deals with fractional powers of the Stokes operator in the L_2 space over a three-dimensional exterior domain. Our purpose is to establish several imbedding theorems for spaces related to the domains of fractional powers, which will be needed in Section 3 in estimating the nonlinear term of the Navier-Stokes equations.

Using the results in Section 2, we discuss in Section 3 the problem $(*)$ under the homogeneous boundary condition. We shall prove the existence of a unique solution, local or global in time, which is not in general square-summable. In addition, we show that our global solutions decay uniformly in $x \in D$ like $t^{-1/4}$ as $t \to \infty$. This extends the decay results of Heywood [15] and Masuda [24] to the case of non-square-summable solutions.

Section 4 is devoted to the investigation of the Oseen operator in L_r spaces over an exterior domain in R^3. Using the fact that the Oseen operator generates a holomorphic semigroup, we give a result concerning the domains of its fractional powers, which is needed in Section 5 in discussing the existence of the solutions for the problem $(*)$ under a nonhomogeneous boundary condition. In fact, in Section 5, we discuss the problem $(*)$ with $u^*=0$, $u_\infty(t)=u_\infty=\text{const}$. With the aid of the usual technique of extending the boundary data to the whole of D, we reduce the problem to a homogeneous case. Because of the additional term of the form: $(b, \nabla)u+(u, \nabla)b$ which appears in the resulting homogeneous equations, our result in this section is only local in time.

In [7], Fujita and Kato suggested the use of semigroup theory for the Stokes operator in L_2 to discuss the exterior nonstationary problem under the homogeneous boundary condition. However, it seems that one cannot prove the existence of a global solution by the method suggested there; moreover, they consider only square-summable solutions. We can avoid this difficulty in proving the existence of a global solution if we use function spaces defined in Section 2.

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1. The Helmholtz decomposition of vector fields

Throughout this paper we denote by D an exterior domain in R^3 with smooth boundary S. It is well known (see [32]) that the Hilbert space $(L_2(D))^3$ admits the following orthogonal decomposition, the Helmholtz decomposition:

$$(L_2(D))^3 = X_2 \oplus G_2.$$

Here X_2 is the closure in $(L_2(D))^3$ of the space

$$(C_0^\infty(D))_\sigma^3 = \{u \in (C_0^\infty(D))^3;\ \operatorname{div} u = 0 \text{ in } D\},$$

and $G_2 = \{\nabla p \in (L_2(D))^3;\ p \in L_2^{\mathrm{loc}}(\bar{D})\}$ with $\bar{D}$ the closure of D.

Our purpose in this section is to give a similar decomposition for the Banach spaces $(L_r(D))^3$ $(1<r<\infty)$. Set

$$E_r(D) = \{u \in (L_r(D))^3;\ \operatorname{div} u \in L_r(D)\},\ 1<r<\infty.$$

It is easy to see that $E_r(D)$ is a Banach space with the norm: $\|u\|_{0,r} + \|\operatorname{div} u\|_{0,r}$ where $\|\cdot\|_{0,r}$ denotes the usual L_r-norm. Let $(C_0^\infty(\bar{D}))^3$ be the space of all the restrictions to $\bar{D}$ of the functions in $(C_0^\infty(R^3))^3$.

LEMMA 1.1. *$(C_0^\infty(\bar{D}))^3$ is dense in $E_r(D)$.*

For the proof of this lemma, we refer to Temam [32].

In what follows, we denote by $W_r^s(D)$ (resp. $W_r^s(S)$) the usual Sobolev space of order $s \in R$ such that $W_r^0(D) = L_r(D)$ (resp. $W_r^0(S) = L_r(S)$) with the norm $\|\cdot\|_{s,r}$ (resp. $\|\cdot\|_{s,r,S}$); see [21].

PROPOSITION 1.2. *Let ν be the unit exterior normal vector to S. Then, there exists a unique bounded linear operator, γ_ν: $E_r(D) \to W_r^{-1/r}(S)$ such that $\gamma_\nu u = u \cdot \nu = \sum_j u^j \nu^j$ if u is smooth near S, and*

$$\langle \gamma_\nu u, p|_S \rangle = (\operatorname{div} u, p) + (u, \nabla p), \quad p \in W_{r'}^1(D),\ r' = r/(r-1).$$

Here and hereafter, $(\cdot, \cdot)$ (resp. $\langle \cdot, \cdot \rangle$) denotes the duality pairing of functions on D (resp. S).

PROOF. Fix $u \in E_r(D)$ and consider the linear form:

$$T_u(p) = (\operatorname{div} u, q) + (u, \nabla q), \quad q \in W_{r'}^1(D),\ q|_S = p. \tag{1.1}$$

Since $C_0^\infty(D)$ is dense in $\{q \in W_{r'}^1(D);\ q|_S = 0\}$, it follows by an integration by parts that $T_u(p)$ is independent of the choice of q. By the surjectivity of the trace operator: $W^1_{r'}(D) \to W^{1-1/r'}_{r'}(S)$ we can find for each $p \in W^{1-1/r'}_{r'}(S)$ an element $q \in W^1_{r'}(D)$ so that

$$q|_S = p, \qquad \|q\|_{1,r'} \leqq C\|p\|_{1-1/r',r',S}$$

with a constant $C>0$ independent of p; see [21]. Therefore, from (1.1) we have

$$\begin{aligned} |T_u(p)| &\leqq (\|u\|_{0,r} + \|\operatorname{div} u\|_{0,r})\|q\|_{1,r'} \\ &\leqq C(\|u\|_{0,r} + \|\operatorname{div} u\|_{0,r})\|p\|_{1-1/r',r',S}. \end{aligned}$$

This implies the existence of an element $\gamma_\nu u$ in $(W^{1-1/r'}_{r'}(S))^* = W^{-1/r}_r(S)$ such that

$$\langle \gamma_\nu u, p\rangle = T_u(p) \qquad \text{for} \quad p \in W^{1-1/r'}_{r'}(S),$$
$$\|\gamma_\nu u\|_{-1/r,r,S} \leqq C(\|u\|_{0,r} + \|\operatorname{div} u\|_{0,r}).$$

Thus we have proved the existence of an operator γ_ν. The uniqueness of γ_ν and the fact that $\gamma_\nu u = u \cdot \nu$ for smooth u follow from Lemma 1.1 and Green's formula. This completes the proof.

Let X_r be the closure of $(C_0^\infty(D))^3_\sigma$ in $(L_r(D))^3$. By the above proposition, we see that

$$(1.2) \qquad X_r \subset Y_r \equiv \{u \in (L_r(D))^3;\ \operatorname{div} u = 0 \text{ in } D,\ \gamma_\nu u = 0\},$$

and that Y_r is a closed subspace of $(L_r(D))^3$. Set $G_r = X^\perp_{r'}$, the annihilator of $X_{r'}$.

LEMMA 1.3. $G_r = \{\nabla p \in (L_r(D))^3;\ p \in L^{\mathrm{loc}}_r(\bar{D})\}$.

PROOF. Let $f = \nabla p \in (L_r(D))^3$, $p \in L^{\mathrm{loc}}_r(\bar{D})$. By an integration by parts we see easily that

$$(f, u) = (\nabla p, u) = -(p, \operatorname{div} u) = 0, \quad u \in (C_0^\infty(D))^3_\sigma.$$

Therefore, by the definition of $X_{r'}$ we have $(f, u) = 0$ for any $u \in X_{r'}$. This implies $f \in G_r$.

Conversely, suppose that $f = \{f^j\}^3_{j=1} \in (L_r(D))^3$ satisfies

$$(f, u) = 0, \qquad \text{for any} \quad u \in (C_0^\infty(D))^3_\sigma.$$

By a theorem of de Rham ([30, Th. 17′]), there exists a distribution p such that $f = \nabla p$. Since $f \in (L_r(D))^3$, we have $\Delta p = \operatorname{div} f \in W^{-1}_r(D)$, hence $p \in L^{\mathrm{loc}}_r(D)$. Now, let D' be a bounded domain with smooth boundary containing the complement of D in its interior and set $D'' = D \cap D'$. By our assumption, $(f, u) = 0$ for any $u \in (C_0^\infty(D''))^3_\sigma$. So by [8, Lemma 7], $f = \nabla q \in (L_r(D''))^3$ for some $q \in W^1_r(D'')$, which implies $p \in L_r(D'')$. This completes the proof.

Let us now construct a projection P_r from $(L_r(D))^3$ onto the closed subspace Y_r which is needed in proving our decomposition theorem. We start with the following lemma.

LEMMA 1.4. *Let p satisfy*

$$\Delta p = 0 \quad in \quad D, \qquad \nabla p \in (L_r(D))^3, \qquad \partial p/\partial \nu (= \gamma_\nu \nabla p) = 0.$$

Then, $\nabla p = 0$.

PROOF. Let $B_\rho(x)$ be the closed ball with radius ρ centered at $x \in R^3$. Since each component of ∇p is harmonic in D, the mean value theorem for harmonic functions yields

$$\begin{aligned} |\nabla p(x)| &\leqq V^{-1} \int_{B_1(x)} |\nabla p(y)| dy \\ &\leqq V^{-1+1/r'} \left\{ \int_{B_1(x)} |\nabla p(y)|^r dy \right\}^{1/r} \longrightarrow 0, \quad \text{as} \quad |x| \longrightarrow \infty, \end{aligned}$$

where V denotes the volume of the unit ball. In view of the expansion theorem for harmonic functions at infinity (see [27]), this implies

$$|\nabla p(x)| = O(|x|^{-1}), \quad \text{as} \quad |x| \longrightarrow \infty. \tag{1.3}$$

Let S_ρ be the sphere with radius ρ centered at the origin. Then, (1.3) implies

$$\rho^{-3} \int_{S_\rho} |p(y)| \, dS(y) \longrightarrow 0, \quad \text{as} \quad \rho \longrightarrow \infty,$$

where dS is the surface element on S_ρ. From this and the expansion theorem for harmonic functions at infinity we obtain

$$p(x) = p_0 + O(|x|^{-1}), \quad \text{as} \quad |x| \longrightarrow \infty$$

where p_0 is a constant. Since $q(x) = p(x) - p_0$ satisfies the assumptions imposed on p and $q(x) \to 0$ as $|x| \to \infty$, it follows from the uniqueness of solutions of the exterior Neumann problem that $p(x) = p_0$ and so $\nabla p = 0$. This completes the proof.

COROLLARY. $Y_r \cap G_r = 0$.

PROPOSITION 1.5. *There exists a bounded operator P_r from $(L_r(D))^3$ onto Y_r such that $P_r u = u$ for $u \in Y_r$.*

PROOF. For $u \in (C_0^\infty(\bar{D}))^3$ we define

(1.4) $$P_r u = u - \nabla(p_1+p_2),$$

where p_1 and p_2 are chosen to satisfy

(1.5) $$\text{(i) } \Delta p_1 = \operatorname{div} \tilde{u} \text{ in } R^3, \quad \text{(ii) } \begin{cases} \Delta p_2 = 0 \text{ in } D, \\ \partial p_2/\partial \nu = \gamma_\nu(u - \nabla p_1) \text{ on } S, \end{cases}$$

(1.6) $$\nabla p_1 \in (L_r(R^3))^3, \quad \nabla p_2 \in (L_r(D))^3.$$

Here $\tilde{u}$ denotes a C^1-extension of u to R^3 such that

$$\|\tilde{u}\|_{0,r,R^3} \leqq C\|u\|_{0,r}$$

with a constant $C>0$ independent of u. Obviously $P_r u \in Y_r$. Applying the Fourier inversion formula to (1.5) (i), we see that ∇p_1 is determined uniquely by

(1.7) $$\partial p_1/\partial x_j = R_j \textstyle\sum_{k=1}^3 R_k \tilde{u}^k, \quad 1 \leqq j \leqq 3,$$

where $R_j\,(1 \leqq j \leqq 3)$ denotes the Riesz transform; see [31]. Since R_j is a bounded operator in $L_r(R^3)$, $1<r<\infty$, we have

(1.8) $$\|\nabla p_1\|_{0,r} \leqq \|\nabla p_1\|_{0,r,R^3} \leqq C\|\tilde{u}\|_{0,r,R^3} \leqq C\|u\|_{0,r}.$$

Let us now turn to the problem (1.5) (ii). Consider the exterior Neumann problem:

(1.9) $$\Delta q = 0 \text{ in } D, \quad \partial q/\partial \nu = h \in W_r^{-1/r}(S), \quad q(x) \longrightarrow 0\,(|x| \to \infty).$$

As is well known, (1.9) can be solved by means of a single-layer potential, and ∇q is determined uniquely by Lemma 1.4 if $\nabla q \in (L_r(D))^3$. Assuming

(1.10) $$\langle h, 1 \rangle = 0,$$

we shall show that $\nabla q \in (L_r(D))^3$. In fact, from (1.10) and the expansion theorem for harmonic functions at infinity we have

(1.11) $$q(x) = O(|x|^{-2}), \quad |\nabla q(x)| = O(|x|^{-3}) \quad \text{as} \quad |x| \longrightarrow \infty.$$

Since $r>1$, it follows from (1.11) that $\nabla q \in L_r$ near the infinity. On the other hand, ∇q is in L_r near the boundary S in view of the well-known elliptic theory in a bounded domain ([22]). Thus we have proved $\nabla q \in (L_r(D))^3$.

We shall now show that the solution of (1.9) satisfies the estimate:

(1.12) $$\|\nabla q\|_{0,r} \leqq C\|h\|_{-1/r,r,S} \quad \text{if} \quad \langle h, 1 \rangle = 0.$$

By virtue of the closed graph theorem, we have only to show that the map: $h \mapsto \nabla q$ defines a closed operator. Suppose that

(1.13) $\quad h_n \longrightarrow h$ in $W_r^{-1/r}(S)$, $\quad \nabla q_n \longrightarrow f$ in $(L_r(D))^3$, $\quad$ as $n \longrightarrow \infty$,

and that q_n is the solution of (1.9) with $h=h_n$. It is easy to see that $(f, u)=0$ for any $u \in (C_0^\infty(D))_\sigma^3$ and so $f=\nabla q \in G_r$ for some $q \in L_r^{\mathrm{loc}}(\bar{D})$. Since $\Delta q_n=0$, we have $\Delta q=0$. Thus, by Proposition 1.2, $\partial q/\partial \nu \equiv \tilde{h}$ is well defined and belongs to $W_r^{-1/r}(S)$. Since, by Proposition 1.2,

$$\begin{aligned}\|h_n - \tilde{h}\|_{-1/r,r,S} &\leqq C(\|\nabla q_n - \nabla q\|_{0,r} + \|\Delta q_n - \Delta q\|_{0,r}) \\ &= C\|\nabla q_n - \nabla q\|_{0,r} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty,\end{aligned}$$

we see that $h=\tilde{h}$. Thus we have proved (1.12).

Let p_2 be the solution of (1.5) (ii). Since obviously $\langle \gamma_\nu(u-\nabla p_1), 1\rangle=0$, it follows from (1.8), (1.12) and Proposition 1.2 that

$$\begin{aligned}\|\nabla p_2\|_{0,r} &\leqq C\|\gamma_\nu(u-\nabla p_1)\|_{-1/r,r,S} \\ &\leqq C(\|u-\nabla p_1\|_{0,r} + \|\mathrm{div}\,(u-\nabla p_1)\|_{0,r}) \\ &= C\|u-\nabla p_1\|_{0,r} \leqq C\|u\|_{0,r}.\end{aligned}$$

From this and (1.8) we obtain

$$\|P_r u\|_{0,r} \leqq C\|u\|_{0,r} \qquad \text{for} \quad u \in (C_0^\infty(\bar{D}))^3 \tag{1.14}$$

with a $C>0$ independent of u. Since $(C_0^\infty(\bar{D}))^3$ is dense in $(L_r(D))^3$, P_r is extended uniquely to a bounded operator from $(L_r(D))^3$ into Y_r. The fact that $P_r u=u$ for any $u \in Y_r$ follows easily from Lemma 1.1 and the corollary to Lemma 1.4. This completes the proof.

THEOREM 1.6. (i) $(L_r(D))^3 = Y_r \oplus G_r$ $(1<r<\infty$, *direct sum*).
(ii) $X_r=Y_r$, $X_r^*=X_{r'}$, $r'=r/(r-1)$, *where* X_r^* *is the dual space of* X_r.

PROOF. (i) From the corollary to Lemma 1.4 it follows that Y_r+G_r is a direct sum. On the other hand, by our construction of P_r we have $(C_0^\infty(\bar{D}))^3 \subset Y_r \oplus G_r \subset (L_r(D))^3$, in particular, $Y_r \oplus G_r$ is dense in $(L_r(D))^3$. Since $P_r\colon (L_r(D))^3 \to Y_r$ is a bounded operator and Y_r, G_r are closed, $Y_r \oplus G_r$ is closed. Hence $Y_r \oplus G_r = (L_r(D))^3$. This proves (i).

(ii) By (i), $Y_r=(L_r(D))^3/G_r$. Hence

$$Y_r^* = \{(L_r(D))^3/G_r\}^* = G_r^\perp = X_{r'}.$$

On the other hand, $Y_{r'}$ is regarded as a subset of Y_r^*. Indeed, let v_1, v_2 be in $Y_{r'}$ and suppose $(v_1-v_2, u)=0$ for any $u \in Y_r$. Since $X_r \subset Y_r$ we see $v_1-v_2 \in X_r^\perp = G_{r'}$. Thus we have $v_1-v_2=0$ because $Y_{r'} \cap G_{r'}=0$. This implies $Y_{r'} \subset X_{r'}$ so that we have $X_r=Y_r$ and $X_r^*=X_{r'}$ for any r, $1<r<\infty$. This completes the proof.

COROLLARY. *The dual operator* P_r^* *of* P_r: $(L_r(D))^3 \to (L_r(D))^3$ *is identical with* $P_{r'}$.

This is easily verified by using Theorem 1.6 if we note that, for each $1<r<\infty$, $P_r u=u$ if and only if $u \in X_r$; $P_r u=0$ if and only if $u \in G_r$.

Using the above results, we now define the Stokes operator A_r and discuss some of its basic properties. Let us consider the Stokes boundary value problem:

$$
(1.15) \qquad \begin{aligned}
(\lambda-\Delta)w + \nabla p &= f && \text{in } D,\\
\operatorname{div} w &= 0 && \text{in } D,\\
w &= 0 && \text{on } S,\\
w &\to 0 && \text{as } |x|\to\infty.
\end{aligned}
$$

The following result is due to Giga ([9, Th. 1′]).

THEOREM 1.7. *Let* $1<r<\infty$. *Then, one finds a constant* $M>0$ *such that for each complex number* λ *with* $\operatorname{Re}\lambda \geqq M$ *and each* $f \in X_r$ *there exists a unique* $w \in X_r \cap \{v \in (W_r^2(D))^3;\ v|_S=0\}$ *which satisfies* (1.15) *with some* $\nabla p \in G_r$. *Moreover, if we write* $w=G_\lambda f$, G_λ *is a bounded injective operator on* X_r *such that* $\|G_\lambda\| \leqq \text{const.}\,|\lambda|^{-1}$, *where* $\|G_\lambda\|$ *is the operator norm of* G_λ.

Let us denote $G_\lambda^{-1}=\lambda+A_r$. From Theorem 1.7 we see easily that A_r is a closed operator in X_r independent of λ such that

$$
(1.16) \qquad \begin{aligned}
D(A_r) &= X_r \cap \{v \in (W_r^2(D))^3;\ v|_S = 0\},\\
A_r w &= -P_r \Delta w, \qquad \text{for} \quad w \in D(A_r).
\end{aligned}
$$

Since Theorem 1.7 implies $\|(\lambda+A_r)^{-1}\| \leqq \text{const.}\,|\lambda|^{-1}$ for $\operatorname{Re}\lambda \geqq M$, and since $(\lambda+A_r)^{-1}$ defines a bounded operator from X_r onto $D(A_r)$, we obtain

COROLLARY. (i) $-A_r$ *generates in* X_r *a holomorphic semigroup, which we denote by* $\{e^{-tA_r};\ t\geqq 0\}$.

(ii) *There exists a constant* $C_r>0$ *such that*

$$\|w\|_{2,r} \leqq C_r(\|A_r w\|_{0,r} + \|w\|_{0,r}) \qquad \textit{for} \quad w \in D(A_r).$$

THEOREM 1.8. *Let* A_r^* *be the dual operator of* A_r. *Then,*

$$A_r^* = A_{r'},\ r' = r/(r-1).$$

The proof of this theorem is the same as that of the corresponding assertion in Fujiwara and Morimoto [8], and so omitted.

REMARK 1.9. Theorem 1.8 means in particular that A_2 is a self-adjoint operator in the real Hilbert space X_2. Since $(A_2 w, w)=(\nabla w, \nabla w)$, A_2 is non-negative. Moreover, from this and the corollary to Theorem 1.7, we see that A_2 coincides with the Friedrichs extension of $-P_2\Delta$ restricted to the smooth elements in $D(A_2)$.

2. The Stokes operator in X_2 over an exterior domain

Let us consider the Stokes boundary value problem (1.15) in the space X_2 under the form:

$$Aw = f. \tag{2.1}$$

Here and hereafter, we denote A_2 simply by A. As is noted in Remark 1.9, A is a non-negative self-adjoint operator in X_2 and $(Aw, v)=(\nabla w, \nabla v)$ if $v, w \in D(A)$. So we shall define the weak form of the equation (2.1) by

$$(\nabla w, \nabla v) = (f, v), \qquad \text{for} \quad v \in (C_0^\infty(D))_\sigma^3. \tag{2.2}$$

Since A is the Friedrichs extension (see Remark 1.9) attached to the bilinear form $(\nabla w, \nabla v)$, it follows immediately that

$$D(A^{1/2}) = X_2 \cap (\mathring{W}_2^1(D))^3, \quad \|A^{1/2}w\|_{0,2}^2 = \|\nabla w\|_{0,2}^2, \tag{2.3}$$

where $\mathring{W}_2^1(D)=\{v \in W_2^1(D);\ v\,|_S=0\}$.

Let $\{e^{-tA};\ t\geqq 0\}$ be the semigroup in X_2 generated by $-A$. Using the spectral representation for A, we can prove the following result; see [24].

PROPOSITION 2.1. (i) $\{e^{-tA};\ t\geqq 0\}$ *is a holomorphic contraction semigroup of non-negative self-adjoint operators.*

(ii) $\|A^\alpha e^{-tA}w\|_{0,2} \leqq t^{-\alpha}\|w\|_{0,2}$ *for* $w \in X_2,\ t>0,\ 0\leqq\alpha\leqq 1$.

(iii) $\|(I-e^{-tA})w\|_{0,2} \leqq (t^\alpha/\alpha)\|A^\alpha w\|_{0,2}$ *for* $w\in D(A^\alpha),\ t>0,\ 0<\alpha\leqq 1$.

As is noted in the introduction, our aim is to study the Navier-Stokes equations in the spaces of non-square-summable functions. For this purpose we introduce some function spaces.

DEFINITION 2.2. For $0\leqq\alpha\leqq 1/2$, we denote by H^α the completion of $D(A^\alpha)$ with respect to the norm $|w|_\alpha=\|A^\alpha w\|_{0,2}$.

REMARK 2.3. (i) In view of (2.2), we see that $Aw=0$ implies $w=0$. Hence $|\cdot|_\alpha$ defines a norm on $D(A^\alpha)$.

(ii) When D is bounded, the norm of $D(A^\alpha)$, i.e., the graph-norm of A^α, is equivalent with $|\cdot|_\alpha$, because A is invertible in X_2. Hence, in this case, H^α coincides with $D(A^\alpha)$. However, when D is an exterior domain, the space H^α is larger than $D(A^\alpha)$.

From (2.3), $|w|_{1/2}=\|\nabla w\|_{0,2}$; hence by Lemma 7 in [20, Chap. 1], we have

$$H^{1/2} \subset X_6, \qquad \|w\|_{0,6} \leqq (48)^{1/6}|w|_{1/2} \qquad (w \in H^{1/2}). \tag{2.4}$$

THEOREM 2.4. *The space* $H^{\theta/2}$, $0 \leqq \theta \leqq 1$, *is equal to the complex interpolation space*: $[H^0, H^{1/2}]_\theta$. *Moreover, we have*

$$\|w\|_{0,r(\theta)} \leqq C_\theta |w|_{\theta/2}, \qquad \text{for any} \quad w \in H^{\theta/2}, \tag{2.5}$$

with $r(\theta)^{-1}=(1-\theta)/2+\theta/6$, $C_\theta=(48)^{\theta/6}$.

PROOF. Since (2.5) follows easily from (2.4), the Riesz-Thorin theorem and interpolation theory for linear operators, we need only to show the first assertion. Since $D(A^{1/2})$ is dense in both of $H^{1/2}$ and $H^0=X_2$, it follows that $D(A^{1/2})$ is dense in $[H^0, H^{1/2}]_\theta$, so that $D(A^{\theta/2})$ is also dense in $[H^0, H^{1/2}]_\theta$; see [33, § 1.9.3]. Hence we have only to show that the norm of $w \in D(A^{1/2})$ in $[H^0, H^{1/2}]_\theta$ is equivalent with $\|A^{\theta/2}w\|_{0,2}$.

Set $A_\varepsilon = A+\varepsilon$, $\varepsilon>0$. Since A_ε is invertible in X_2, $f(z)=A_\varepsilon^{-(z-\theta)/2}w$ is an X_2-valued function which is continuous for $0 \leqq \operatorname{Re} z \leqq 1$, and analytic for $0< \operatorname{Re} z<1$. Furthermore,

$$f(iy) \in X_2 = H^0, \qquad f(1+iy) \in D(A^{1/2}) \subset H^{1/2}, \qquad \text{for} \quad y \in R,$$

and $f(\theta)=w$. Hence, if we denote the norm of $[H^0, H^{1/2}]_\theta$ by $|||\cdot|||_\theta$, we have ([33, § 1.9])

$$|||w|||_\theta \leqq \max\{\sup \|f(iy)\|_{0,2}, \sup \|A^{1/2}f(1+iy)\|_{0,2}\}.$$

Since A_ε^{iy} is unitary and $\|A^{1/2}A_\varepsilon^{-1/2}\| \leqq 1$, we see easily that the right hand side of the above inequality is dominated by $\|A_\varepsilon^{\theta/2}w\|_{0,2}$. Letting $\varepsilon \to 0$, we obtain

$$|||w|||_\theta \leqq \|A^{\theta/2}w\|_{0,2}. \tag{2.6}$$

Let us now prove the converse of (2.6). We denote by $g(z)$ an arbitrary function expressed as a finite linear combination of functions of the form: $\exp(\delta z^2+\gamma z)b$, $\delta>0$, $\gamma \in R$, $b \in D(A^{1/2})$. Since $A_\varepsilon^{z/2}g(z)$ is continuous for $0 \leqq \operatorname{Re} z \leqq 1$, and analytic for $0<\operatorname{Re} z<1$, it follows from the three-line theorem that

$$\|A^{\theta/2}w\|_{0,2} \leqq \|A_\varepsilon^{\theta/2}w\|_{0,2} \leqq \inf_{g(\theta)=w} \max_{j=0,1} \sup \|A_\varepsilon^{(j+iy)/2}g(j+iy)\|_{0,2}.$$

Noting that A_ε^{iy} is unitary and letting $\varepsilon \to 0$, we obtain

$$\|A^{\theta/2}w\|_{0,2} \leqq \inf_{g(\theta)=w} \max_{j=0,1} \sup \|A^{j/2}g(j+iy)\|_{0,2}.$$

Since $D(A^{1/2})$ is dense in both of $H^{1/2}$ and H^0, it follows from the Theorem in [33, § 1.9.1] that

(2.7) $$\|A^{\theta/2}w\|_{0,2} \leqq |||w|||_{\theta}.$$

By (2.6) and (2.7) the proof is completed.

Let us define the space $H^{-\alpha}$, $0 \leqq \alpha \leqq 1/2$, by

(2.8) $$H^{-\alpha} = \text{the dual space of } H^{\alpha}.$$

By Theorem 2.4 and interpolation theory, we have

(2.9) $$H^{-\theta/2} = [H^0, H^{-1/2}]_{\theta}, \quad 0 \leqq \theta \leqq 1.$$

Here we identify $H^0 = X_2$ with its dual by the usual L_2-inner product; see Theorem 1.6.

Next we define the spaces H^{α} for $1/2 < \alpha \leqq 1$. Our definition is based on the following result.

THEOREM 2.5 ([23, Th. 2]). *Let $w \in H^{1/2}$ and $f \in H^{-1/2}$ satisfy* (2.2). *Then $f \in H^0$ if and only if $-\Delta w \in (L_2(D))^3$. Moreover, in this case we have*

$$f = -P_2 \Delta w, \qquad \|D^2 w\|_{0,2} \leqq C(|f|_0 + |w|_{1/2}),$$

where $D^2 w$ stands for an arbitrary second-order derivative of w.

Let us now define H^{α}, $1/2 < \alpha \leqq 1$, by

(2.10) H^{α} = the completion of $D(A^{\alpha})$ with respect to the norm:

$$|w|_{\alpha} = (\|A^{\alpha}w\|_{0,2}^2 + \|A^{1/2}w\|_{0,2}^2)^{1/2} \qquad (1/2 < \alpha \leqq 1).$$

THEOREM 2.6. *We have*

$$\|\nabla u\|_{0,r(\alpha)} \leqq C_{\alpha}|u|_{\alpha}, \qquad \text{for} \quad u \in H^{\alpha}, \quad 1/2 \leqq \alpha \leqq 1,$$

with $r(\alpha)^{-1} = (5-4\alpha)/6$ and a constant $C_{\alpha} > 0$ independent of u.

PROOF. By the Sobolev imbedding theorem we have $W_2^1(D) \subset L_6(D)$ with the continuous injection. On the other hand, Theorem 2.5 implies that $\nabla u \in (W_2^1(D))^9$ for any $u \in H^1$. Thus we obtain the desired result with $\alpha = 1$. Since our assertion is clear when $\alpha = 1/2$, it is enough to show that

$$H^{\alpha} = [H^{1/2}, H^1]_{2\alpha-1} \qquad (1/2 \leqq \alpha \leqq 1).$$

To see this, we note that H^{α}, $1/2 \leqq \alpha \leqq 1$, is equal to the completion of $D(A^{\alpha})$ with respect to the norm: $\|A^{1/2}(A+1)^{\theta/2}u\|_{0,2}$ with $\alpha = (1+\theta)/2$, which is easily verified by using the spectral representation for A. Using this, we can prove (2.11) in just the same way as in the proof of Theorem 2.4, and so we omit the details.

The following result is needed in Section 3 in constructing the solutions of the problem $(*)$ with $u^*=u_\infty=0$.

PROPOSITION 2.7. (i) *$\{e^{-tA}; t\geqq 0\}$ defines uniquely a holomorphic contraction semigroup on each of the spaces H^α, $-1/2\leqq\alpha\leqq 1$.*

(ii) *For each $\alpha<\beta$, $-1/2\leqq\alpha\leqq 1/2$, $\beta\geqq 0$, and each $t>0$, we have the estimate*

$$\|A^\beta e^{-tA}w\|_{0,2} \leqq t^{\alpha-\beta}|w|_\alpha, \qquad \text{for} \quad w\in H^\alpha.$$

In particular, e^{-tA} defines a bounded operator from H^α to H^β.

(iii) *For each $\alpha<\beta$, $-1/2\leqq\alpha$, $\beta<0$, and each $t>0$, e^{-tA} defines a bounded operator from H^α to H^β such that*

$$|e^{-tA}w|_\beta \leqq t^{\alpha-\beta}|w|_\alpha, \qquad \text{for} \quad w\in H^\alpha.$$

PROOF. (i) When $\alpha\geqq 0$, the assertion is obvious since $e^{-tA}A^\alpha=A^\alpha e^{-tA}$ on $D(A^\alpha)$. So we have only to consider the case $-1/2\leqq\alpha<0$. First we shall show that $(C_0^\infty(D))_\sigma^3$ is dense in H^α, $-1/2\leqq\alpha<0$. Since $(C_0^\infty(D))_\sigma^3$ is dense in X_r and $(C_0^\infty(D))_\sigma^3\subset H^{-\alpha}\subset X_r$, $1/r=1/2+2\alpha/3$, we see that $H^{-\alpha}$ is dense in the X_r. Hence, by duality, $X_{r'}$ $(r'=r/(r-1))$ is dense in H^α, which implies that $(C_0^\infty(D))_\sigma^3$ is also dense in H^α.

Now, we define e^{-tA}: $H^\alpha\to H^\alpha$ as the dual of e^{-tA}: $H^{-\alpha}\to H^{-\alpha}$. Since e^{-tA} is self-adjoint in X_2, it follows that e^{-tA}: $H^\alpha\to H^\alpha$ $(-1/2\leqq\alpha<0)$ defined above coincides with the original one on $(C_0^\infty(D))_\sigma^3$.

(ii) When $\alpha\geqq 0$, the assertion follows easily from Proposition 2.1 since $A^\beta e^{-tA}=A^{\beta-\alpha}e^{-tA}A^\alpha$ on $D(A^\alpha)$. Therefore we assume $-1/2\leqq\alpha<0$. For v, $w\in(C_0^\infty(D))_\sigma^3$, we have, again by Proposition 2.1,

$$\begin{aligned}|(A^\beta e^{-tA}w, v)| &= |(w, A^\beta e^{-tA}v)| \leqq |w|_\alpha|A^\beta e^{-tA}v|_{-\alpha} \\ &= |w|_\alpha\|A^{\beta-\alpha}e^{-tA}v\|_{0,2} \leqq t^{\alpha-\beta}|w|_\alpha\|v\|_{0,2}.\end{aligned}$$

Since $(C_0^\infty(D))_\sigma^3$ is dense in both of H^α and $H^0=X_2$, we obtain the desired result.

(iii) follows immediately from (ii) and a duality argument. This completes the proof.

3. The exterior nonstationary problem with the homogeneous boundary condition

The purpose of this section is to give an existence and uniqueness result for the problem $(*)$ under the homogeneous boundary condition: $u^*=u_\infty(t)=0$, in the function space $H^{1/2}$ defined in the preceding section. We shall prove the existence of a unique solution, local or global in time, for an arbitrary initial function

$a \in H^{1/4}$ and an arbitrary forcing term $P_2 f \in C((0, T]; H^{-\delta})$, $0 \leqq \delta \leqq 1/4$, such that $|P_2 f(t)|_{-\delta} = o(t^{\delta-3/4})$ as $t \to 0$. Moreover, we shall show that, when $P_2 f$ is defined on $(0, \infty)$, and $0 \leqq \delta < 1/4$, our solution $u(x, t)$ satisfies

$$\sup_{x \in D} |u(x, t)| = O(t^{-1/4}) \quad \text{as} \quad t \longrightarrow \infty,$$

if it exists globally. This is a generalization of a result in Heywood [15], which gives the same decay result for a solution belonging to X_2 with respect to $x \in D$.

In this section, we denote P_2 simply by P and consider the problem (*) with $u^* = u_\infty(t) = 0$ under the form:

$$u(t) = e^{-tA}a + \int_0^t e^{-(t-s)A} Fu(s) ds + \int_0^t e^{-(t-s)A} Pf(s) ds, \tag{3.1}$$

where $Fu = -P(u, \nabla)u$; see [7], [11], [17], [25], [26]. To solve this equation we employ the iteration argument which was developed by Kato and Fujita ([7], [17]) in the case of a bounded domain; see also [11], [25], [26]. First we prepare estimates for the nonlinear term Fu.

Lemma 3.1. *The estimate*

$$|P(u, \nabla)v|_{-\gamma} \leqq M|u|_\theta |v|_\rho, \quad u \in H^\theta, \quad v \in H^\rho, \tag{3.2}$$

holds whenever $0 \leqq \gamma \leqq 1/4$, $0 < \theta \leqq 1/2$, $\rho \geqq 1/2$, $\theta + \rho + \gamma = 5/4$. *Here* $M > 0$ *is a constant depending on* γ, θ, ρ.

Proof. By Hölder's inequality we have

$$|(P(u, \nabla)v, w)| = |((u, \nabla)v, w)| \leqq \|u\|_{0,q} \|\nabla v\|_{0,r} \|w\|_{0,s}$$

for any $w \in (C_0^\infty(D))_\sigma^3$, where $1/q + 1/r + 1/s = 1$. Since we can choose q, r and s so that $1/q = 1/2 - 2\theta/3$, $1/r = (5 - 4\rho)/6$, $1/s = 1/2 - 2\gamma/3$, the estimate (3.2) follows from Theorems 2.4 and 2.6. This completes the proof.

Let us now discuss the existence problem for the equation (3.1) by means of the iteration scheme:

$$\begin{aligned} u_0(t) &= e^{-tA}a + \int_0^t e^{-(t-s)A} Pf(s) ds, \\ u_{m+1}(t) &= u_0(t) + \int_0^t e^{-(t-s)A} Fu_m(s) ds, \quad m \geqq 0. \end{aligned} \tag{3.3}$$

In what follows, we denote the norm $\|\cdot\|_{0,2}$ simply by $\|\cdot\|$. Fix $a \in H^{1/4}$ and $Pf \in C((0, T]; H^{-\delta})$ such that $|Pf(t)|_{-\delta} = o(t^{\delta-3/4})$ for some δ, $0 \leqq \delta \leqq 1/4$. By Proposition 2.7 we have

$$(3.4)\qquad \begin{aligned}\|A^\alpha u_0(t)\| &\leqq \|A^\alpha e^{-tA}a\| + \int_0^t (t-s)^{-\alpha-\delta}|Pf(s)|_{-\delta}ds \\ &\leqq K_{\alpha 0}t^{1/4-\alpha}, \quad 1/4 \leqq \alpha < 1-\delta,\end{aligned}$$

where

$$(3.5)\qquad \begin{aligned}K_{\alpha 0} &= \sup\nolimits_{t>0} t^{\alpha-1/4}\|A^\alpha e^{-tA}a\| + NB(1-\delta-\alpha,\ \delta+1/4), \\ N &= \sup\nolimits_{0<t\leqq T} t^{3/4-\delta}|Pf(t)|_{-\delta}.\end{aligned}$$

Here $B(p, q)$ is the beta function. For each $m \geqq 0$ we set

$$(3.6)\qquad K_m(t) = \sup\nolimits_{0<s\leqq t} s^{1/4}\|A^{1/2}u_m(s)\|.$$

Applying Proposition 2.7 and Lemma 3.1 with $\gamma=1/4$, $\theta=\rho=1/2$, we see by induction on m that

$$(3.7)\qquad K_{m+1}(t) \leqq K_0(t) + M_1B_1K_m(t)^2, \quad 0 < t \leqq T,$$

where $B_1 = B(1/4, 1/2)$ and M_1 is the constant in (3.2) with $\gamma=1/4$, $\theta=\rho=1/2$. Further we have, again by induction on m,

$$(3.8)\qquad \|A^\alpha u_{m+1}(t)\| \leqq \{K_{\alpha 0} + M_1K_m(T)^2B(3/4-\alpha,\ 1/2)\}^{1/4-\alpha},$$

for $1/4 \leqq \alpha < 3/4$, $0 < t \leqq T$. Now, assume that

$$(3.9)\qquad K_0(T) < 1/4M_1B_1.$$

By an elementary calculation we obtain, from (3.7),

$$(3.10)\qquad K_m(T) \leqq K \equiv \{1-(1-4M_1B_1K_0(T))^{1/2}\}/2M_1B_1 < 1/2M_1B_1,$$

so that, from (3.6) and (3.8),

$$(3.11)\qquad \begin{aligned}&\|A^{1/2}u_m(t)\| \leqq Kt^{-1/4}, \\ &\|A^\alpha u_{m+1}(t)\| \leqq \{K_{\alpha 0} + M_1K^2B(3/4-\alpha,\ 1/2)\}t^{1/4-\alpha}, \\ &\qquad\qquad t\in(0,\ T],\ 1/4 \leqq \alpha < 3/4.\end{aligned}$$

Assuming (3.9), we shall show that $\{u_m(t)\}$ converges. Set

$$w_m(t) \equiv u_{m+1}(t) - u_m(t) = \int_0^t e^{-(t-s)A}\{Fu_m(s) - Fu_{m-1}(s)\}ds, \quad m \geqq 1.$$

By (3.2) we have

$$|Fu_m(s) - Fu_{m-1}(s)|_{-1/4} \leqq M_1|w_{m-1}(s)|_{1/2}\{|u_m(s)|_{1/2} + |u_{m-1}(s)|_{1/2}\}.$$

Hence, denoting $W_m = \sup_{0<t\leqq T} t^{1/4}\|A^{1/2}w_m(t)\|$ and taking (3.11) into account, we obtain, by induction on m,

$$(3.12)\quad \begin{aligned} &W_m \leqq (2M_1B_1K)^m W_0, \quad m \geqq 1, \\ &\|A^\alpha w_m(t)\| \leqq (2M_1B_1K)^{m-1} 2M_1KB(3/4-\alpha, 1/2)t^{1/4-\alpha}, \quad m \geqq 2, \end{aligned}$$

for $1/4 \leqq \alpha < 3/4$. Since $2M_1B_1K < 1$ (see (3.10)), we see from (3.12) that $\{u_m(t)\}$ converges in $C([0, T]; H^{1/4}) \cap C((0, T]; H^\alpha)$, $1/4 < \alpha < 3/4$, to an element $u(t)$ such that

$$(3.11)'\quad \begin{aligned} &\|A^{1/2}u(t)\| \leqq Kt^{-1/4}, \\ &\|A^\alpha u(t)\| \leqq \{K_{\alpha 0} + M_1K^2B(3/4-\alpha, 1/2)\}t^{1/4-\alpha}, \end{aligned}$$

for $1/4 \leqq \alpha < 3/4$. Since

$$\begin{aligned} |Fu_m(s) - Fu(s)|_{-1/4} &\leqq M_1|u_m(s) - u(s)|_{1/2}\{|u_m(s)|_{1/2} + |u(s)|_{1/2}\} \\ &\longrightarrow 0, \quad \text{as} \quad m \longrightarrow \infty, \\ |Fu_m(s)|_{-1/4} &\leqq M_1K^2s^{-1/2}, \end{aligned}$$

we can apply the dominated convergence theorem to (3.3) and see that $u(t)$ is a solution of (3.1). This proves the existence of a solution of (3.1) under the assumption (3.9). On the other hand, from [7, Lemma 2.10] we can deduce that $t^{1/4}\|A^{1/2}e^{-tA}a\| \to 0$ as $t \to 0$, if $a \in H^{1/4}$. Since $t^{3/4-\delta}|Pf(t)|_{-\delta} \to 0$ as $t \to 0$, we see from (3.5) and (3.6) that (3.9) holds if we choose $T > 0$ small enough. Hence we have proved the existence part of the following theorem.

THEOREM 3.2. (i) *For each* $a \in H^{1/4}$ *and each* $Pf \in C((0, T]; H^{-\delta})$ *such that* $|Pf(t)|_{-\delta} = o(t^{\delta-3/4})$ $(t \to 0)$ *for some* δ, $0 \leqq \delta \leqq 1/4$, *there exist a* T_*, $0 < T_* \leqq T$, *and a solution* $u(t)$ *of* (3.1) *such that, for any* α, $1/4 < \alpha < 3/4$,

(*a*) $u \in C([0, T_*]; H^{1/4}) \cap C((0, T_*]; H^\alpha)$,
(*b*) $\|A^\alpha u(t)\| = o(t^{1/4-\alpha})$ *as* $t \longrightarrow 0$.

(ii) *The solution is unique within the class of functions* $w(t) \in C([0, T_*]; H^{1/4}) \cap C((0, T_*]; H^{1/2})$ *such that* $|w(t)|_{1/2} = o(t^{-1/4})$ *as* $t \to 0$.

We note that (*b*) can be seen from the fact that the constants $K_{\alpha 0}$ in (3.5) and K in (3.10) can be made arbitrarily small if we take $T > 0$ small. The uniqueness of the solution can be shown in the same way as in the proof of the corresponding result in [7], and so we omit the details; see also [11], [25].

In view of (3.9), the following result is obvious.

THEOREM 3.3. *Let* $a \in H^{1/4}$ *and let* $Pf \in C((0, \infty); H^{-\delta})$, $0 \leqq \delta \leqq 1/4$, *satisfy* $|Pf(t)|_{-\delta} = o(t^{\delta-3/4})$ *as* $t \to 0$. *Then there exists a solution* $u(t)$ *of* (3.1) *in* $C([0, \infty); H^{1/4}) \cap C((0, \infty); H^\alpha)$, $1/4 < \alpha < 3/4$, *which satisfies* (*b*) *in Theorem* 3.2, *if*

(3.13) $\quad |a|_{1/4} + B(1/2-\delta, 1/4+\delta)\sup_{t>0} t^{3/4-\delta}|Pf(t)|_{-\delta} < 1/4M_1B_1.$

Here M_1 *is the constant in* (3.1) *with* $\gamma=1/4$, $\theta=\rho=1/2$.

We shall now study the rate of decay of our global solutions in the case $0\leqq\delta<1/4$. The following proposition shows that we can get more regularity for our solutions when $0\leqq\delta<1/4$.

PROPOSITION 3.4. *When* $0\leqq\delta<1/4$, *the solution* $u(t)$ *given in Theorem* 3.2 (*resp. Theorem* 3.3) *belongs to* $C((0, T_*]; H^\alpha)$, (*resp.* $C((0, \infty); H^\alpha)$), *for any* α, $3/4\leqq\alpha<1-\delta$. *Further, we have*

$$\begin{aligned} &\|A^{3/4}u(t)\| \leqq C_1(t^{-1/2} + t^{-3/8}), \\ &\|A^\alpha u(t)\| \leqq C_2(t^{1/4-\alpha} + t^{1/2-\alpha}), \quad 3/4 < \alpha < 1-\delta, \end{aligned} \tag{3.14}$$

with C_1 *and* C_2 *independent of* t.

PROOF. We set $u(t)=u_0(t) + v(t)$, where

$$u_0(t)=e^{-tA}a + \int_0^t e^{-(t-s)A}Pf(s)ds,$$

$$v(t) = \int_0^t e^{-(t-s)A}Fu(s)ds.$$

By Proposition 2.7 we have

$$\begin{aligned} \|A^\alpha u_0(t)\| &\leqq t^{1/4-\alpha}|a|_{1/4} + N\int_0^t (t-s)^{-\alpha-\delta}s^{\delta-3/4}ds \\ &= t^{1/4-\alpha}\{|a|_{1/4} + NB(1-\delta-\alpha, 1/4+\delta)\}, \quad 3/4\leqq\alpha<1-\delta. \end{aligned}$$

Hence, we have only to estimate $v(t)$. Applying Lemma 3.1 with $\gamma=1/8$, $\theta=1/2$, $\rho=5/8<3/4$, we obtain

$$\begin{aligned} \|A^{3/4}v(t)\| &\leqq \int_0^t (t-s)^{-7/8}M|u(s)|_{1/2}|u(s)|_{5/8}ds \\ &\leqq MC\int_0^t (t-s)^{-7/8}s^{-1/4}(s^{-1/4} + s^{-3/8})ds \\ &= MC\{B(1/8, 1/2)t^{-3/8} + B(1/8, 3/8)t^{-1/2}\}, \end{aligned}$$

where M is the constant in (3.2) with $\gamma=1/8$, $\theta=1/2$, $\rho=5/8$. Note that here we have used the fact that $u(t)$ satisfies (3.11)′.

Suppose now $3/4<\alpha<1-\delta$. Applying Lemma 3.1 with $\gamma=0$, $\theta=1/2$, $\rho=3/4$, we have

$$\begin{aligned}\|A^\alpha v(t)\| &\leqq \int_0^t (t-s)^{-\alpha} M' |u(s)|_{1/2} |u(s)|_{3/4} ds \\ &\leqq M'C \int_0^t (t-s)^{-\alpha} s^{-1/4} (s^{-1/4} + s^{-3/8} + s^{-1/2}) ds \\ &= M'C\{B(1-\alpha, 1/2)t^{1/2-\alpha} + B(1-\alpha, 3/8)t^{3/8-\alpha} + B(1-\alpha, 1/4)t^{1/4-\alpha}\} \\ &\leqq M'C(t^{1/4-\alpha} + t^{1/2-\alpha}),\end{aligned}$$

where M' is the constant in (3.2) with $\gamma=0$, $\theta=1/2$, $\rho=3/4$. This completes the proof.

Using the above result, we can now prove the following

THEOREM 3.5. *Suppose that* (3.13) *holds with* $0 \leqq \delta < 1/4$. *Then the solution* $u(t)$ *given in Theorem* 3.3 *is continuous and bounded in* $(x, t) \in D \times [\eta, \infty)$ *for any* $\eta > 0$. *Further, we have*

$$\sup_{x\in D} |u(x, t)| = O(t^{-1/4}) \quad as \quad t \longrightarrow \infty. \tag{3.15}$$

PROOF. Fix α, $3/4 < \alpha < 1-\delta$. By Theorem 2.6 and (3.14), we see that $\nabla u \in (L_r(D))^9$, $1/r = (5-4\alpha)/6$, and

$$\begin{aligned}\|\nabla u(t)\|_{0,r} &\leqq C|u(t)|_\alpha \leqq C(t^{1/4-\alpha} + t^{1/2-\alpha} + t^{-1/4}) \\ &= O(t^{-1/4}) \quad \text{as} \quad t \longrightarrow \infty.\end{aligned} \tag{3.16}$$

Since $u(t)$ satisfies (3.11)$'$ for all $t>0$, it follows from Theorem 2.4 that

$$\|u(t)\|_{0,6} \leqq C|u(t)|_{1/2} = O(t^{-1/4}) \quad \text{as} \quad t \longrightarrow \infty. \tag{3.17}$$

Now, choose an open cube Q_0 whose sides are parallel to the coordinate axes such that the complement of D is contained in its interior. Since $3<r<6$, it follows from (3.16) and (3.17) that $u(t) \in (W_r^1(D'))^3$, $D' = Q_0 \cap D$, and hence by the Sobolev imbedding theorem,

$$\begin{aligned}\sup_{x\in D'} |u(x, t)| &\leqq C(\|\nabla u(t)\|_{0,r,D'} + \|u(t)\|_{0,r,D'}) \\ &\leqq C(\|\nabla u(t)\|_{0,r} + |Q_0|^{1/r-1/6} \|u(t)\|_{0,6}).\end{aligned} \tag{3.18}$$

Let us now divide $R^3 \setminus Q_0$ into a countable number of open cubes Q_j, $j \geqq 1$, which are mutually congruent so that

$$R^3 \setminus Q_0 = \cup_j \bar{Q}_j, \quad Q_j \cap Q_k = \phi \qquad \text{if} \quad j \neq k.$$

Since $u(t) \in (W_r^1(Q_j))^3$ for each $j \geqq 1$, we see as above that

$$\begin{aligned}\sup_{x\in Q_j} |u(x, t)| &\leqq C'(\|\nabla u(t)\|_{0,r,Q_j} + \|u(t)\|_{0,r,Q_j}) \\ &\leqq C'(\|\nabla u(t)\|_{0,r} + |Q_j|^{1/r-1/6} \|u(t)\|_{0,6}).\end{aligned} \tag{3.19}$$

By our choice of Q_j, the constant C' and the volume $|Q_j|$ are independent of j. Hence, from (3.16)–(3.19), we have

$$\sup_{x\in D}|u(x, t)| \leqq C(\|\nabla u(t)\|_{0,r} + \|u(t)\|_{0,6}) = O(t^{-1/4})$$

as $t\to\infty$, which completes the proof.

In the following lemma, we denote by $C^{\gamma}((0, T]; E)$ the set of functions which are Hölder continuous on each $[\varepsilon, T]\,(0<\varepsilon<T)$ with exponent γ and with values in a Banach space E.

LEMMA 3.6. *When* $0\leqq\delta<1/4$, *the solution* $u(t)$ *of* (3.1) *given in Theorem* 3.2 *belongs to* $C^{\alpha}((0, T_*]; H^{1/2})\cap C^{\beta}((0, T_*]; H^{3/4})$ *for any* α, β, *such that* $0<\alpha<1/2-\delta$, $0<\beta<1/4-\delta$.

PROOF. We have only to consider

$$v(t)\equiv\int_0^t e^{-(t-s)A}\,Fu(s)ds+\int_0^t e^{-(t-s)A}Pf(s)ds,$$

because $e^{-tA}a$, $a\in H^{1/4}$, is a smooth function of $t>0$ with values in both of $H^{3/4}$ and $H^{1/2}$. Fix $0<\varepsilon<T_*$. Then, for $\varepsilon\leqq t<t+h\leqq T_*$, we have

$$\begin{aligned}\|A^{1/2}v(t+h)-A^{1/2}v(t)\| &\leqq \int_0^t \|(e^{-hA}-I)A^{1/2}e^{-(t-s)A}Fu(s)\|ds\\ &\quad+\int_t^{t+h}\|A^{1/2}e^{-(t+h-s)A}Fu(s)\|ds\\ &\quad+\int_0^t \|(e^{-hA}-I)A^{1/2}e^{-(t-s)A}Pf(s)\|ds\\ &\quad+\int_t^{t+h}\|A^{1/2}e^{-(t+h-s)A}Pf(s)\|ds\\ &\equiv I_1+I_2+I_3+I_4.\end{aligned}$$

Using Propositions 2.1, 2.7, (3.11)′ and Lemma 3.1 with $\gamma=0$, $\theta=1/2$, $\rho=3/4$, we obtain

$$\begin{aligned}I_1+I_3 &\leqq (h^{\alpha}/\alpha)\int_0^t \|A^{\alpha+1/2}e^{-(t-s)A}Fu(s)\|ds\\ &\quad+(h^{\alpha}/\alpha)\int_0^t \|A^{\alpha+1/2}e^{-(t-s)A}Pf(s)\|ds\\ &\leqq (h^{\alpha}/\alpha)C\int_0^t (t-s)^{-1/2-\alpha}s^{-1/4}(s^{-1/4}+s^{-1/2})ds\\ &\quad+(h^{\alpha}/\alpha)C\int_0^t (t-s)^{-1/2-\alpha-\delta}s^{\delta-3/4}ds\\ &\leqq C_{\alpha\varepsilon}h^{\alpha},\end{aligned}$$

and

$$I_2 + I_4 \leqq C_\varepsilon \int_t^{t+h} (t+h-s)^{-1/2} ds + C_\varepsilon \int_t^{t+h} (t+h-s)^{-1/2-\delta} ds$$
$$\leqq C_\varepsilon (h^{1/2} + h^{1/2-\delta}).$$

Hence we have shown $u \in C^\alpha((0, T_*]; H^{1/2})$. That $u \in C^\beta((0, T_*]; H^{3/4})$ can be proved similarly. This completes the proof.

COROLLARY. *Under the assumption in Lemma* 3.6, *$Fu(t)$ is Hölder continuous on each $[\varepsilon, T_*]$ $(0<\varepsilon<T_*)$ with values in X_2.*

THEOREM 3.7. *If $Pf \in C((0, T]; X_2)$ and is Hölder continuous on each compact subinterval of* $(0, T]$, *then the solution $u(t)$ given in Theorem* 3.2 *satisfies $du/dt - P\Delta u = Pf + Fu$ on* $(0, T_*]$. *In particular, $du/dt \in C((0, T_*]; X_2)$.*

PROOF. We write

$$u(t) = e^{-(t-)\eta A} u(\eta) + \int_\eta^t e^{-(t-s)A}\{Fu(s) + Pf(s)\} ds$$
$$\equiv w(t) + v(t), \quad t \geqq \eta > 0.$$

By the above corollary it is clear that $v(t)$ satisfies

$$dv/dt + Av = Fu + Pf \quad \text{in} \quad X_2, \qquad t > \eta. \tag{3.20}$$

Hence, it is enough to show

$$dw/dt - P\Delta w = 0 \quad \text{for} \quad t > \eta. \tag{3.21}$$

By Proposition 2.7, $w(t)$ is in $C([\eta, T_*]; H^1)$. So by the estimate in Theorem 2.5, $\Delta w(t)$ is in $(L_2(D))^3$. Hence, $P\Delta w$ is in $C([\eta, T_*]; X_2)$. On the other hand, since $u(\eta) \in H^{1/2} \subset X_6$, we see that $w(t)$ belongs to $C^1((\eta, T_*]; X_6) \cap C((\eta, T_*]; D(A_6))$ and

$$dw/dt + A_6 w = 0 \quad \text{for} \quad t > \eta.$$

Since $A_6 w = -P\Delta w$, $dw/dt \in C((\eta, T^*]; X_2)$ and (3,21) holds. By (3.20) and (3.21) the proof is completed.

REMARK 3.8. Bemelmans [1] discussed the exterior problem in Hölder spaces and proved the existence of a global solution for small data. But, he did not give a rate of time-decay. Heywood [15] showed a result similar to ours under the assumption that $a \in D(A^{1/2})$. This assumption implies that the global solution obtained in [15] is necessarily in X_2. It is also proved in [15] that if $Pf=0$, the global solution obtained in [15] decays uniformly in $x \in D$ like $t^{-1/2}$.

For more discussions on the decay properties of solutions to the exterior problem, we refer to an interesting paper [19] of Knightly. If our solutions were identical with those discussed in [19], they would decay like $t^{-1/2}$ even when $Pf \neq 0$.

4. The Oseen operator

As a preparation for the study of a nonhomogeneous problem in the next section, we consider in this section some properties of the Oseen operator in X_r, $1<r<\infty$:

$$L_r = A_r + P_r(u_\infty, \nabla),$$

where u_∞ is a constant vector. This operator is obtained if we linearize the Navier-Stokes equations around the velocity at infinity u_∞. Our main purpose is to show that $-L_r$ generates in X_r a holomorphic semigroup, which we denote by $\{e^{-tL_r};\, t \geqq 0\}$. Using this fact, we then discuss fractional powers of L_r.

LEMMA 4.1. (i) *There exists for each* $\varepsilon>0$ *a constant* $C_\varepsilon>0$ *such that*

$$\|P_r(u_\infty, \nabla)w\|_{0,r} \leqq \varepsilon\|A_r w\|_{0,r} + C_\varepsilon\|w\|_{0,r}, \quad w \in D(A_r).$$

(ii) *There exists a constant* $C_r>0$ *such that*

$$\|w\|_{2,r} \leqq C_r(\|L_r w\|_{0,r} + \|w\|_{0,r}), \quad w \in D(A_r).$$

Hence, L_r *defines a closed operator in* X_r *such that* $D(L_r)=D(A_r)$.

PROOF. (i) In what follows we denote by C_ε various constants depending on $\varepsilon>0$. Using the corollary to Theorem 1.7, we have

$$\begin{aligned}\|P_r(u_\infty, \nabla)w\|_{0,r} &\leqq C\|w\|_{1,r} \leqq C(\varepsilon\|w\|_{2,r} + C_\varepsilon\|w\|_{0,r}) \\ &\leqq C\varepsilon(\|A_r w\|_{0,r} + \|w\|_{0,r}) + CC_\varepsilon\|w\|_{0,r}.\end{aligned}$$

Since the constant C in the above estimate is independent of $\varepsilon>0$, this proves (i).

(ii) By (i) and the corollary to Theorem 1.7 we have

$$\begin{aligned}\|w\|_{2,r} &\leqq C_r(\|A_r w\|_{0,r} + \|w\|_{0,r}) = C_r(\|L_r w - P_r(u_\infty, \nabla)w\|_{0,r} + \|w\|_{0,r}) \\ &\leqq C_r(\|L_r w\|_{0,r} + \|w\|_{0,r}) + \varepsilon\|A_r w\|_{0,r} + C_\varepsilon\|w\|_{0,r} \\ &\leqq C_r(\|L_r w\|_{0,r} + \|w\|_{0,r}) + \varepsilon C\|w\|_{2,r} + C_\varepsilon\|w\|_{0,r}.\end{aligned}$$

Choosing $\varepsilon>0$ sufficiently small, we obtain (ii). This completes the proof.

By (i) in the above lemma and the discussion in [29, p. 253], we obtain the following result.

THEOREM 4.2. $\{e^{-tL_r};\, t \geqq 0\}$ *is a holomorphic semigroup in* X_r.

REMARK 4.3. When $r=2$, we have

$$(L_2 w, w) = (Aw, w) + (P(u_\infty, \nabla)w, w) = \|\nabla w\|_{0,2}^2, \quad w \in D(A_2),$$

because an integration by parts yields $(P(u_\infty, \nabla)w, w)=((u_\infty, \nabla)w, w)=0$. Hence, $-L_2$ generates in X_2 a contraction semigroup. For $r \neq 2$, we have no such boundedness result for the semigroup $\{e^{-tL_r}; t \geqq 0\}$ even when $u_\infty=0$. We note that properties of the fundamental solution for the Oseen operator are discussed in detail in Bemelmans [1] and Faxén [2].

Theorem 4.2 and the corollary to Theorem 1.7 assure the existence of fractional powers of $L_r+\lambda$ and $A_r+\lambda$ if $\lambda>0$ is sufficiently large. For simplicity, we fix such a λ and denote $L_r+\lambda$ and $A_r+\lambda$ by L_r and A_r respectively.

THEOREM 4.4. *There exists for each $\beta>\alpha$, $0 \leqq \alpha$, $\beta \leqq 1$, a constant $C_{\alpha\beta}>0$ such that*

$$\|A_r^\alpha w\|_{0,r} \leqq C_{\alpha\beta}\|L_r^\beta w\|_{0,r}, \quad w \in D(L_r^\beta). \tag{4.1}$$

The proof of this theorem is the same as that of [26, Lemma 2.6] and so omitted.

REMARK 4.5. Actually, we can show

$$D(L_r^\alpha) = D(A_r^\alpha), \quad 0 \leqq \alpha \leqq 1,$$

by a method developed in Giga [10]. But we shall not enter into the details here, since we do not use (4.2) in this paper.

According to [10], $D(A_r^\alpha)$ is continuously imbedded into the space of Bessel potentials $(H_r^{2\alpha}(D))^3$. Hence, from Theorem 4.4 we obtain the following result.

PROPOSITION 4.6. *For each $\varepsilon>0$ and each α, $0 \leqq \alpha < 1$, the space $D(L_r^{\alpha+\varepsilon})$ is continuously imbedded into $(H_r^{2\alpha}(D))^3$.*

5. A nonhomogeneous case

In this section we discuss the solvability of the problem (*) under the assumption:

$$u^* = 0, \quad u_\infty(t) = u_\infty = \text{const.}$$

As is shown in [4, p. 369], we can choose a C^2 vector field $b(x)$, $x \in \bar{D}$, which vanishes for large $|x|$ and satisfies

$$\operatorname{div} b = 0 \quad \text{in} \quad D, \quad b = -u_\infty \quad \text{on} \quad S.$$

Substituting $u=u_\infty+b+w$ into (*) we obtain

$$(*)'\qquad \begin{aligned} \partial w/\partial t - \Delta w + (u_\infty, \nabla)w + (b, \nabla)w + (w, \nabla)b & \\ = g - \nabla p - (w, \nabla)w \quad &\text{in} \quad D \times (0, T), \\ \operatorname{div} w = 0 \quad &\text{in} \quad D \times (0, T), \\ w = 0 \quad &\text{on} \quad S \times (0, T), \\ w \to 0 \quad &\text{as} \quad |x| \to \infty, \\ w(x, 0) = \tilde{a}(x) \quad &\text{in } D, \end{aligned}$$

where $g=f-(b, \nabla)b+\Delta b-(u_\infty, \nabla)b$; $\tilde{a}=a-u_\infty-b$. Applying the projection P_r formally to $(*)'$, we obtain the following evolution equation in X_r:

$$(5.1)\qquad \begin{aligned} dw/dt + L_r w &= Fw + Gw + P_r g, \quad t > 0, \\ w(0) &= \tilde{a}, \end{aligned}$$

where $Fw=-P_r(w, \nabla)w$; $Gw=-P_r(b, \nabla)w-P_r(w, \nabla)b$, and L_r is the Oseen operator in X_r defined in the preceding section. We shall consider (5.1) in X_r, $r>3$, under the form:

$$(5.2)\qquad w(t) = e^{-tL}\tilde{a} + \int_0^t e^{-(t-s)L}\{Fw+Gw\}(s)ds + \int_0^t e^{-(t-s)L}Pg(s)ds.$$

Here and hereafter we drop the subscript r in L_r and P_r. Since our consideration is local in time, we may assume without loss of generality that L is invertible and the fractional power L^α is defined; see [25, p. 527]. First we prepare estimates for Fw and Gw.

LEMMA 5.1. *Let $3<r<\infty$. Then the estimate*

$$(5.3)\qquad \|P(u, \nabla)v\|_{0,r} \leqq M\|u\|_{0,r}\|L^\gamma v\|_{0,r}, \quad u \in X_r, \quad v \in D(L^\gamma),$$

holds for each γ, $3/2r+1/2<\gamma\leqq 1$. Here M is a constant depending on r and γ.

PROOF. By the Sobolev imbedding theorem, the space $H_r^s(D)$ is continuously imbedded into the space of bounded continuous functions on $\bar{D}$ if $s>3/r$. On the other hand, we see from Proposition 4.6 that the gradient operator ∇ sends $D(L^\gamma)$ continuously into $(H_r^{2\alpha-1}(D))^9$ where α is an arbitrary number smaller than γ. Since $\gamma>3/2r+1/2$, we may assume that $2\alpha-1>3/r$. Hence,

$$\begin{aligned} \|P(u, \nabla)v\|_{0,r} &\leqq C\| |u|\cdot|\nabla v| \|_{0,r} \leqq C\|u\|_{0,r} \sup|\nabla v| \\ &\leqq C\|u\|_{0,r}\|L^\gamma v\|_{0,r}, \end{aligned}$$

which completes the proof.

COROLLARY. *There exists a constant* M_1 *which depends on* $r>3$, $3/2r+1/2<\gamma\leqq 1$ *and* b *such that*

$$\|Gu\|_{0,r} \leqq M_1\|L^\gamma u\|_{0,r}, \quad u \in D(L^\gamma). \tag{5.4}$$

PROOF. By Lemma 5.1 we need only to consider $P(u, \nabla)b$. By Proposition 4.6 and the Sobolev theorem we obtain

$$\|P(u, \nabla)b\|_{0,r} \leqq C\|b\|_{1,r} \sup|u| \leqq C\|b\|_{1,r}\|L^\alpha u\|_{0,r}$$

for any $\alpha>3/2r$. This completes the proof.

Let us now prove our main result in this section. In what follows, we denote the norm of X_r simply by $\|\cdot\|$.

THEOREM 5.2. *Fix* $3<r<\infty$ *and* $3/2r+1/2<\gamma<1$. *Then, for each* $\tilde{a}\in X_r$ *and each* $Pg\in C((0, T]; X_r)$ *such that* $\|Pg(t)\|=O(t^{-\gamma})$ *as* $t\to 0$, *there exist a* T_*, $0<T_*\leqq T$, *and a unique solution* $w(t)$ *of* (5.2) *such that*

(a) $w\in C([0, T_*]; X_r)\cap C((0, T_*]; D(L^\gamma))$,

(b) $\|L^\gamma w(t)\|=o(t^{-\gamma})$ *as* $t\to 0$.

PROOF. Let $E(T_*; N_1, N_2)$ be the set of all $v(t)$ in $C([0, T_*]; X_r)\cap C((0, T_*]; D(L^\gamma))$ such that

$$\sup_{0\leqq t\leqq T_*}\|v(t)\| \leqq N_1, \quad \sup_{0<t\leqq T_*} t^\gamma\|L^\gamma v(t)\| \leqq N_2,$$

where N_1, N_2 and $T_*\leqq T$ are arbitrary positive numbers. It is easy to see that $E(T_*; N_1, N_2)$ is a complete metric space with respect to the distance function induced by the norm:

$$|||v||| = \sup_{0\leqq t\leqq T_*}\|v(t)\| + \sup_{0<t\leqq T_*} t^\gamma\|L^\gamma v(t)\|.$$

(i) Let C_1, C_2 be constants such that

$$\|e^{-tL}\| \leqq C_1, \quad t^\gamma\|L^\gamma e^{-tL}\| \leqq C_2 \qquad \text{for all} \quad t\in(0, T].$$

We shall show that the nonlinear operator K defined by

$$Kv(t) = e^{-tL}\tilde{a} + \int_0^t e^{-(t-s)L}\{Fv+Gv\}(s)ds + \int_0^t e^{-(t-s)L}Pg(s)ds$$

leaves $E(T_*; N_1, N_2)$ invariant if we choose T_*, N_1 and N_2 appropriately. In fact, using (5.3) and (5.4) we have

$$\begin{aligned}\|Kv(t)\| &\leqq C_1\|\tilde{a}\| + C_1\int_0^t \|L^\gamma v(s)\|\{M_1+M\|v(s)\|\}ds + C_1\int_0^t \|Pg(s)\|ds \\ &\leqq C_1\|\tilde{a}\| + C_1N_2(M_1+N_1M)t^{1-\gamma}/(1-\gamma) + C_1\int_0^t \|Pg(s)\|ds,\end{aligned} \tag{5.5}$$

$$
\begin{aligned}
&\|L^{\gamma}Kv(t)\| \\
&\leqq \|L^{\gamma}e^{-tL}\tilde{a}\| + C_2\int_0^t (t-s)^{-\gamma}\|L^{\gamma}v(s)\|\{M_1+M\|v(s)\|\}ds \\
&\quad + C_2\int_0^t (t-s)^{-\gamma}\|Pg(s)\|ds \qquad (5.6)\\
&\leqq \|L^{\gamma}e^{-tL}\tilde{a}\| + C_2N_2B(1-\gamma,\, 1-\gamma)(M_1+N_1M)t^{1-2\gamma} \\
&\quad + \{\sup\nolimits_{0<s\leqq t} s^{\gamma}\|Pg(s)\|\}B(1-\gamma,\, 1-\gamma)t^{1-2\gamma}.
\end{aligned}
$$

From (5.6) we obtain

$$
\begin{aligned}
(5.7)\quad t^{\gamma}\|L^{\gamma}Kv(t)\| &\leqq t^{\gamma}\|L^{\gamma}e^{-tL}\tilde{a}\| + C_2N_2B(1-\gamma,\, 1-\gamma)(M_1+N_1M)t^{1-\gamma} \\
&\quad + \{\sup\nolimits_{0<s\leqq t} s^{\gamma}\|Pg(s)\|\}B(1-\gamma,\, 1-\gamma)t^{1-\gamma}.
\end{aligned}
$$

Since $\gamma<1$, we see from (5.5) and (5.7) that T_*, N_1 and N_2 can be chosen in such a way that K leaves $E(T_*;\, N_1,\, N_2)$ invariant.

(ii) Choosing T_*, N_1 and N_2 as above, we shall show that K defines a contraction map on $E(T_*;\, N_1,\, N_2)$ if we take $T_*>0$ sufficiently small. Indeed we have, again by (5.3) and (5.4),

$$
\begin{aligned}
&\|Kv(t) - Kw(t)\| \\
&\leqq C_1M_1\int_0^t \|L^{\gamma}(v-w)(s)\|ds \\
&\quad + C_1M\int_0^t \{\|v(s)\|\cdot\|L^{\gamma}(v-w)(s)\| + \|(v-w)(s)\|\cdot\|L^{\gamma}w(s)\|\}ds \qquad (5.8)\\
&\leqq C_1(M_1+N_1M+N_2M)|||v-w|||t^{1-\gamma}/(1-\gamma),
\end{aligned}
$$

$$
\begin{aligned}
&\|L^{\gamma}Kv(t) - L^{\gamma}Kw(t)\| \\
&\leqq C_2M_1\int_0^t (t-s)^{-\gamma}\|L^{\gamma}(v-w)(s)\|ds \\
&\quad + C_2M\int_0^t (t-s)^{-\gamma}\{\|v(s)\|\cdot\|L^{\gamma}(v-w)(s)\| + \|(v-w)(s)\|\cdot\|L^{\gamma}w(s)\|\}ds \qquad (5.9)\\
&\leqq C_2(M_1+N_1M+N_2M)B(1-\gamma,\, 1-\gamma)|||v-w|||t^{1-2\gamma}.
\end{aligned}
$$

Hence, from (5.8) and (5.9) we obtain

$$
\begin{aligned}
(5.10)\quad &|||Kv-Kw||| \\
&\leqq (M_1+N_1M+N_2M)T_*^{1-\gamma}\{C_1/(1-\gamma) + C_2B(1-\gamma,\, 1-\gamma)\}|||v-w|||.
\end{aligned}
$$

Choosing $T_*>0$ small, we obtain the desired result.

The existence and uniqueness of the solution follows from (i), (ii) and the

Banach fixed point theorem. Since $t^{\gamma}\|L^{\gamma}e^{-tL}\tilde{a}\|\to 0$ as $t\to 0$ (see the proof of [7, Lemma 2.10]), the assertion (b) follows from (5.7). This completes the proof.

REMARK 5.3. (i) By an analogous method we can show the existence of a unique local solution for the problem (∗) under general boundary condition if we use A_r instead of L_r. This result contains the local existence result of [15] in view of (2.4).

(ii) In [15] Heywood established the existence of a unique global solution for the problem (∗) when $\tilde{a}\in D(A_2^{1/2})$, in our notation, is sufficiently small. In doing so, he used in an essential way an energy inequality in X_2, and so his solution necessarily belongs to X_2. We do not know any global existence result for the problem (5.2) for $r\neq 2$ even when $u_\infty=0$. This is the reason why we discussed the homogeneous case separately in Section 3.

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