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Finitely generated subalgebras of generalized solvable Lie algebras

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Introduction

Recently many authors considered several conditions under which a subalgebra of a Lie algebra is a subideal or an ascendant subalgebra. Such conditions have been also investigated for groups. Especially Peng [4] and Whitehead [5] presented some criteria for a finitely generated subgroup to be subnormal.

In this paper we shall give several conditions which ensure that a finitely generated subalgebra of a Lie algebra is a subideal or an ascendant subalgebra. The following is our main result: When L is a solvable Lie algebra of not necessarily finite dimension over a field of characteristic zero, any subalgebra H of L generated by $\{h_1,\dots, h_n\}$ is a subideal of L if and only if there exists an integer $m\geq 0$ such that $L(\mathrm{ad}\, h_i)^m\subseteq H$ for $1\leq i\leq n$ (Theorem 1(a)). Conditions for a finitely generated subalgebra to be an ascendant subalgebra are also given (Theorem 1(b) and Theorem 2).

1. Preliminaries

Throughout this paper L will denote a Lie algebra of not necessarily finite dimension over a field $\mathfrak{k}$ of characteristic zero. We shall follow [1] for notation and terminology. In particular, $H \text{ si } L$, $H \text{ asc } L$, and $H \lhd^{\omega} L$ mean respectively that H is a subideal, an ascendant subalgebra, and an ω-step ascendant subalgebra of L, where ω is the first infinite ordinal. Triangular brackets $\langle\ \ \rangle$ denote the subalgebra of L generated by elements inside them.

$\mathfrak{F}$, $\mathfrak{N}$, $\mathrm{E}\mathfrak{A}$ denote respectively the classes of finite dimensional, nilpotent, and solvable Lie algebras. A Lie algebra L belongs to the class $\acute{\mathrm{E}}\mathfrak{A}$ if there is an ordinal λ and an ascending series $(L_\alpha)_{\alpha\leq\lambda}$ of L whose factors $L_{\alpha+1}/L_\alpha$ are abelian. If in addition each L_α is an ideal of L, then L belongs to the class $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$.

For $x, y\in L$ and an integer $n\geq 0$, we write $[x, {}_n y]=x(\mathrm{ad}\, y)^n$. The similar notation is used for subspaces. A derivation d of L is nil if for any finite dimensional subspace M of L there is an integer $n=n(M)\geq 0$ such that $Md^n=0$. We denote by $[\mathrm{End}(V)]$ the Lie algebra of all linear endomorphisms of a vector space V over $\mathfrak{k}$.

We begin with the following

LEMMA 1. *Let* $H=\langle h_1,\dots, h_n\rangle$ *be a subalgebra of a Lie algebra* L.

(a) *Suppose that* L *is solvable. If* $\mathrm{ad}\, h_1,\dots, \mathrm{ad}\, h_n$ *are nilpotent derivations of* L, *then* H *is a subideal of* L.

(b) *Suppose that* $L\in \acute{\mathrm{E}}\mathfrak{A}$. *If* $\mathrm{ad}\, h_1,\dots, \mathrm{ad}\, h_n$ *are nil derivations of* L, *then* H *is an ascendant subalgebra of* L.

PROOF. (a) Let $m\geq 0$ be an integer such that $L(\mathrm{ad}\, h_i)^m=0$ for $1\leq i\leq n$. Then by Amayo and Stewart [1, Theorem 16.4.2(b)] $\langle h_i\rangle$ are subideals of L. Since the class $\mathfrak{N}\cap\mathfrak{F}$ is coalescent (see [1, Theorem 3.2.4]), $H=\langle\langle h_1\rangle,\dots, \langle h_n\rangle\rangle$ is a subideal of L.

(b) Since $\mathrm{ad}\, h_1,\dots, \mathrm{ad}\, h_n$ are nil, for any element $x\in L$ there exists an integer $m=m(x)\geq 0$ such that $x(\mathrm{ad}\, h_i)^m=0$ for $1\leq i\leq n$. Then it follows by an argument similar to the above that $H=\langle h_1,\dots, h_n\rangle$ asc L.

The following lemma is well-known [2, p. 38] and we omit its proof.

LEMMA 2. *Let* x, y *be elements of a Lie algebra* L. *Then for any integer* $n\geq 0$,

$$(\mathrm{ad}\, x)^n\, \mathrm{ad}\, y = \textstyle\sum_{i=0}^{n} (-1)^i \binom{n}{i}(\mathrm{ad}\, [y, {}_i x])(\mathrm{ad}\, x)^{n-i}.$$

2. Subideals

We consider finitely generated subalgebras of a Lie algebra in the class $\mathrm{E}\mathfrak{A}$ or $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$. To this end we consider nilpotent endomorphisms of a vector space in the following

PROPOSITION 1. *Let* V *be a not necessarily finite dimensional vector space over* $\mathfrak{k}$. *Let* $f_1,\dots,f_n$ *be nilpotent endomorphisms of* V *and* $H=\langle f_1,\dots,f_n\rangle$ *a subalgebra of* $[\mathrm{End}(V)]$.

(a) *If* H *is solvable, then there exists an integer* $k>0$ *such that* $g_1\cdots g_k=0$ *for any* $g_1,\dots, g_k\in H$.

(b) *If* $H\in\acute{\mathrm{E}}\mathfrak{A}$, *then for any* $x\in V$ *there exists an integer* $k=k(x)>0$ *such that* $xg_1\cdots g_k=0$ *for any* $g_1,\dots, g_k\in H$.

PROOF. We consider V as an abelian Lie algebra. Then $[\mathrm{End}(V)]=\mathrm{Der}(V)$ and H is a subalgebra of $\mathrm{Der}(V)$. Hence we can form the split extension.

$$L = V \dotplus H, \quad V\lhd L.$$

By hypothesis there exists an integer $l>0$ such that $f_i^l=0$ for $1\leq i\leq n$, so that

$$V(\mathrm{ad} f_i)^l = Vf_i^l= 0.$$

Let $g \in H$. Then by induction on m we have

$$g(\operatorname{ad} f_i)^m = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f_i^j g f_i^{m-j} \qquad (1 \leq i \leq n).$$

Put $m=2l-1$ so that $f_i^j=0$ or $f_i^{m-j}=0$, whence $g(\operatorname{ad} f_i)^m=0$. Thus $H(\operatorname{ad} f_i)^m=0$, and therefore

$$L(\operatorname{ad} f_i)^m = 0 \qquad (1 \leq i \leq n). \tag{$*$}$$

(a) Since L is solvable, by Lemma 1(a) it follows from ($*$) that H si L. Hence there exists an integer $k>0$ such that $[L, {}_kH] \subseteq H$, and therefore

$$Vg_1 \cdots g_k = V(\operatorname{ad} g_1) \cdots (\operatorname{ad} g_k) \subseteq [V, {}_kH] \subseteq V \cap H = 0$$

for any $g_1, \ldots, g_k \in H$. Thus $g_1 \cdots g_k=0$.

(b) Clearly $L \in \acute{\mathrm{E}}\mathfrak{A}$. Hence by ($*$) and Lemma 1(b) we have H asc L. Now the argument before Theorem 3.2.5 of [1] shows that for any $x \in V$ there exists an integer $k=k(x)>0$ such that

$$xg_1 \cdots g_k \in [x, {}_kH] \subseteq V \cap H = 0$$

for any $g_1, \ldots, g_k \in H$.

We consider some special cases which will be useful to use induction later.

Lemma 3. *Let $H=\langle h_1, \ldots, h_n \rangle$ be a subalgebra of a Lie algebra L and A an abelian ideal of L. Suppose that there exists an integer $m \geq 0$ such that $A(\operatorname{ad} h_i)^m \subseteq H$ for $1 \leq i \leq n$.*

(a) *If H is solvable, then H si $A+H$.*

(b) *If $H \in \acute{\mathrm{E}}\mathfrak{A}$, then $H \lhd^{\omega} A+H$.*

Proof. Let $m>0$. Since $A \cap H \lhd A+H$, we may assume that $A \cap H=0$. Then

$$A(\operatorname{ad} h_i)^m \subseteq A \cap H = 0 \qquad (1 \leq i \leq n),$$

whence $\operatorname{ad}_A h_i$ are nilpotent derivations of A. Let $\varphi\colon H \to \operatorname{Der}(A)$ be a homomorphism such that $\varphi(h)=\operatorname{ad}_A h$ for $h \in H$. Then

$$\varphi(H) = \langle \operatorname{ad}_A h_i \mid i = 1, \ldots, n \rangle$$

is a subalgebra of $\operatorname{Der}(A)=[\operatorname{End}(A)]$.

(a) Clearly $\varphi(H)$ is solvable. Hence by Proposition 1(a) there is an integer $k>0$ such that

$$[A, {}_kH] = A\varphi(H)^k = 0.$$

Therefore by [3, Lemma 3(a)] we have $H \operatorname{si} A+H$.

(b) It is clear that $\varphi(H)$ is an $\acute{\mathrm{E}}\mathfrak{A}$-subalgebra of $[\mathrm{End}(A)]$. Hence by Proposition 1(b) for any $a \in A$ there is an integer $k=k(a)>0$ such that

$$[a, {}_kH] = a\varphi(H)^k = 0.$$

By [3, Lemma 3(b)] we have $H \lhd^{\omega} A+H$.

Now we obtain the following

THEOREM 1. *Let L be a Lie algebra and $H=\langle h_1,\dots, h_n\rangle$ a finitely generated subalgebra of L.*

(a) *Suppose that L is solvable. Then H is a subideal of L if and only if there exists an integer $m\geq 0$ such that $L(\mathrm{ad}\, h_i)^m \subseteq H$ for $1\leq i\leq n$.*

(b) *Suppose that L belongs to the class $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$. If there exists an integer $m\geq 0$ such that $L(\mathrm{ad}\, h_i)^m \subseteq H$ for $1\leq i\leq n$, then H is an ascendant subalgebra of L.*

PROOF. (a) Let $m\geq 0$ be an integer such that $L(\mathrm{ad}\, h_i)^m \subseteq H$ for $1\leq i\leq n$. Since L is solvable, there is a finite abelian series $(L_j)_{0\leq j\leq k}$ of ideals of L. Let $\bar{L}=L/L_j$ and put bars for images under the natural homomorphism $L\rightarrow L/L_j$ $(0\leq j<k)$. Then $\bar{H}=\langle \bar{h}_1,\dots, \bar{h}_n\rangle$ is a solvable subalgebra of $\bar{L}$ and $\bar{L}_{j+1}$ is an abelian ideal of $\bar{L}$. Clearly

$$\bar{L}_{j+1}(\mathrm{ad}\, \bar{h}_i)^m \subseteq \bar{H} \qquad (1\leq i\leq n).$$

Hence by Lemma 3(a) we have

$$\bar{H} \operatorname{si} \bar{L}_{j+1} + \bar{H} \qquad (0\leq j<k).$$

Thus we conclude that

$$H = L_0 + H \operatorname{si} L_k + H = L.$$

The converse is clear.

(b) Let $(L_\alpha)_{\alpha\leq\lambda}$ be an ascending abelian series of ideals of L, where λ is an ordinal. Then by using Lemma 3(b) we have

$$L_\alpha + H \lhd^{\omega} L_{\alpha+1} + H$$

for any $\alpha<\lambda$. Therefore

$$H = L_0 + H \operatorname{asc} L_\lambda + H = L.$$

3. Ascendant subalgebras

In this section we consider finitely generated subalgebras of a Lie algebra which is in the class $\acute{\mathrm{E}}\mathfrak{A}$.

PROPOSITION 2. *Let x be an element of a Lie algebra L. If there exists an integer $n \geq 0$ such that $\langle L(\mathrm{ad}\, x)^n \rangle = \langle L(\mathrm{ad}\, x)^{n+r} \rangle$ for any integer $r \geq 0$, then $\langle L(\mathrm{ad}\, x)^n \rangle$ is an ideal of L.*

PROOF. Take any element $a \in L(\mathrm{ad}\, x)^{3n}$. Then $a = b(\mathrm{ad}\, x)^{3n}$ for some $b \in L$. By using Lemma 2 we have for any $y \in L$,

$$a \,\mathrm{ad}\, y = [b, {}_n x](\mathrm{ad}\, x)^{2n} \mathrm{ad}\, y$$

$$= \sum_{i=0}^{2n} (-1)^i \binom{2n}{i} [b, {}_n x](\mathrm{ad}\, [y, {}_i x])(\mathrm{ad}\, x)^{2n-i}.$$

If $i = 0, \dots, n$, since $2n - i \geq n$ we have

$$[b, {}_n x](\mathrm{ad}\, [y, {}_i x])(\mathrm{ad}\, x)^{2n-i} \in L(\mathrm{ad}\, x)^n,$$

and if $i = n+1, \dots, 2n$, since $[y, {}_i x] \in L(\mathrm{ad}\, x)^n$ we obtain

$$\begin{aligned} &[b, {}_n x](\mathrm{ad}\, [y, {}_i x])(\mathrm{ad}\, x)^{2n-i} \\ &\in [L(\mathrm{ad}\, x)^n, L(\mathrm{ad}\, x)^n](\mathrm{ad}\, x)^{2n-i} \\ &\subseteq (L(\mathrm{ad}\, x)^n)^{(1)}. \end{aligned}$$

Conseqeuntly $a \,\mathrm{ad}\, y \in \langle L(\mathrm{ad}\, x)^n \rangle$, and therefore

$$\langle L(\mathrm{ad}\, x)^n \rangle \mathrm{ad}\, y = \langle L(\mathrm{ad}\, x)^{3n} \rangle \mathrm{ad}\, y \subseteq \langle L(\mathrm{ad}\, x)^n \rangle.$$

For any element $x \in L$ let $L_0(x)$ and $L_1(x)$ be Fitting zero and one components of L with respect to $\mathrm{ad}\, x$. In the above proposition $\langle L_1(x) \rangle$ is not necessarily an ideal of L, which will be shown later in Example 2. However if there is an integer $n \geq 0$ such that $L(\mathrm{ad}\, x)^n = L(\mathrm{ad}\, x)^{n+1}$, then it is known that $L = L_0(x) + L_1(x)$ ([1, Lemma 12.2.6]). In this case we have the following

COROLLARY. *Let x be an element of a Lie algebra L. If there exists an integer $n \geq 0$ such that $L(\mathrm{ad}\, x)^n = L(\mathrm{ad}\, x)^{n+1}$, then $\langle L_1(x) \rangle$ is an ideal of L.*

It is to be noted that Proposition 2 and its corollary hold for Lie algebras over a field of characteristic $p > 0$.

Now we obtain the following

THEOREM 2. *Let L be a Lie algebra in the class $\acute{\mathrm{E}}\mathfrak{A}$ and $H=\langle h_1,\dots, h_n\rangle$ a finitely generated subalgebra of L. Suppose that there exists an integer $k\geq 0$ such that $\langle H(\mathrm{ad}\, h_i)^k\rangle=\langle H(\mathrm{ad}\, h_i)^{k+r}\rangle$ for any integers $r\geq 0$ and $1\leq i\leq n$. If there exists an integer $m\geq 0$ such that $L(\mathrm{ad}\, h_i)^m\subseteq H$ for $1\leq i\leq n$, then H is an ascendant subalgebra of L.*

PROOF. Let $m\geq 0$ be an integer such that $L(\mathrm{ad}\, h_i)^m\subseteq H$ for $1\leq i\leq n$. Then we have

$$\begin{aligned}\cdots\subseteq H(\mathrm{ad}\, h_i)^{2m+k} &\subseteq L(\mathrm{ad}\, h_i)^{2m+k} \subseteq H(\mathrm{ad}\, h_i)^{m+k} \\ &\subseteq L(\mathrm{ad}\, h_i)^{m+k} \subseteq H(\mathrm{ad}\, h_i)^{k} \subseteq\cdots.\end{aligned}$$

Since $\langle H(\mathrm{ad}\, h_i)^k\rangle=\langle H(\mathrm{ad}\, h_i)^{k+r}\rangle$ for $r\geq 0$, it follows that

$$\langle L(\mathrm{ad}\, h_i)^{m+k}\rangle = \langle L(\mathrm{ad}\, h_i)^{m+k+r}\rangle$$

for $r\geq 0$. Put

$$I = \textstyle\sum_{i=1}^n \langle L(\mathrm{ad}\, h_i)^{m+k}\rangle$$

Then I is an ideal of L by Proposition 2. Since $I\subseteq H$, we may assume that $I=0$. Then we have

$$L(\mathrm{ad}\, h_i)^{m+k} = 0 \qquad (1\leq i\leq n).$$

Therefore by Lemma 1(b) we obtain that $H \operatorname{asc} L$.

COROLLARY. *Let L be a Lie algebra in the class $\acute{\mathrm{E}}\mathfrak{A}$ and $H=\langle h_1,\dots, h_n\rangle$ a finite dimensional subalgebra of L. If there exists an integer $m\geq 0$ such that $L(\mathrm{ad}\, h_i)^m\subseteq H$ for $1\leq i\leq n$, then $H \operatorname{asc} L$.*

4. Examples

In this section we give some remarks and examples.

At first we notice that Theorems 1 and 2 do not hold for Lie algebras over a field of characteristic $p>0$. This will be shown by Hartley's example [1, Lemma 3.1.1].

Secondly we cannot expect that Theorems 1 and 2 hold for residually solvable Lie algebras. This is shown by the following

EXAMPLE 1. Let A be the vector space over $\mathfrak{k}$ with basis $\{a_i, b_i \mid i\in\mathbf{N}\}$, where $\mathbf{N}$ is the set of nonnegative integers. We define linear endomorphisms x_n, y_n, z_n $(n\in\mathbf{N})$ of A by the following: For any n, $i\in\mathbf{N}$,

$$x_n\colon a_i \longmapsto (-2)^n b_{i+n}, \quad b_i \longmapsto 0,$$
$$y_n\colon a_i \longmapsto 0, \quad b_i \longmapsto 2^n a_{i+n+1},$$
$$z_n\colon a_i \longmapsto (-2)^n a_{i+n+1}, \quad b_i \longmapsto -(-2)^n b_{i+n+1}.$$

Then it is easy to verify that

$$[x_n, y_m] = (-1)^m z_{n+m}, \quad [x_n, z_m] = x_{n+m+1}, \quad [y_n, z_m] = (-1)^m y_{n+m+1},$$
$$[x_n, x_m] = [y_n, y_m] = [z_n, z_m] = 0,$$

for any $n, m \in \mathbf{N}$. Let H be the subspace of $\mathrm{End}(A)$ spanned by $\{x_n, y_n, z_n \mid n \in \mathbf{N}\}$. Clearly H is a subalgebra of $[\mathrm{End}(A)]$. Consider A as an abelian Lie algebra, so that $H \subseteq \mathrm{Der}(A)$. Hence we can form the split extension

$$L = A \dotplus H, \quad A \lhd L.$$

It is not hard to see that

$$H^{(\omega)} = \bigcap_{n \in \mathbf{N}} H^{(n)} = 0, \quad L^{(\omega)} = 0,$$

whence L is residually solvable. Clearly H is generated by x_0, y_0 and

$$L(\mathrm{ad}\, x_0)^2 \subseteq H, \quad L(\mathrm{ad}\, y_0)^2 \subseteq H,$$
$$L(\mathrm{ad}\, x_0)^3 = 0, \quad L(\mathrm{ad}\, y_0)^3 = 0.$$

However for any $n \in \mathbf{N}$

$$A(\mathrm{ad}\, z_0)^n = (a_{n+i}, b_{n+i} \mid i \in \mathbf{N}) \nsubseteq H.$$

Therefore $L_L(H) = H$, and H is neither a subideal nor an ascendant subalgebra of L, as desired.

Finally we show in the following that $\langle L_1(x) \rangle$ is not necessarily an ideal of L even if $\langle L(\mathrm{ad}\, x) \rangle = \langle L(\mathrm{ad}\, x)^n \rangle$ for any integer $n > 0$.

EXAMPLE 2. Let A be the vector space over $\mathfrak{k}$ with basis $\{a_i \mid i \in \mathbf{N}\}$, and let x, y_n $(n \in \mathbf{N})$ be linear endomorphisms of A defined as follows: For any $n, i \in \mathbf{N}$,

$$x\colon a_i \longmapsto (i+1) a_{i+1},$$
$$y_n\colon a_i \longmapsto \begin{cases} a_{i-n} & \text{if} \quad i - n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Consider A as an abelial Lie algebra, then x and y_n $(n \in \mathbf{N})$ are derivations of A. It is easily seen that for any $n, m \in \mathbf{N}$,

$$[x, y_n] = n y_{n-1} \ (n > 0), \quad [x, y_0] = [y_n, y_m] = 0.$$

Let Y be the subspace of $\mathrm{Der}(A)$ spanned by $\{y_n \mid n \in \mathbf{N}\}$, and form the split extension

$$L = A \dotplus (Y + \langle x \rangle), \quad A \lhd L.$$

Then we clearly have $\langle L(\mathrm{ad}\, x)^n \rangle = A + Y$ for $n > 0$. However

$$\bigcap_{n \in \mathbf{N}} L(\mathrm{ad}\, x)^n = Y.$$

Thus $\langle L_1(x) \rangle = Y$ is not an ideal of L.

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