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Generalized hypergeometric equations of non-Fuchsian type

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1. Introduction

We consider a connection problem for the differential equation

$$z^n y^{(n)} = \sum_{l=1}^{n} (a_l + b_l z^q) z^{n-l} y^{(n-l)}, \tag{1.1}$$

where q is a complex number and the a_l, b_l are complex constants. This differential equation has only two singularities at the origin and infinity in the whole complex z-plane, and so, it may be assumed without loss of generality that $\mathrm{Re}\, q \geqq 0$. In [3] we dealt with a case in which $b_l = 0$ $(l=0, 1,\dots, n-1)$. In this case (1.1) is of just the extended form of the classical Bessel equation. By solving the connection problem for it and investigating global behaviors of such solutions, we could obtain the extension of the Airy function and the Bessel function. In this paper we shall treat a general case in which $b_l = 0$ $(l=1, 2,\dots, \nu-1)$ and $b_\nu \neq 0$ for $1 \leqq \nu < n$. As is well-known, (1.1) can be reduced to a generalized hypergeometric equation. In fact, let us denote

$$[\rho]_n - \sum_{l=1}^{n} a_l [\rho]_{n-l} = \prod_{j=1}^{n} (\rho - \hat{\rho}_j),$$
$$\sum_{l=\nu}^{n} b_l [\gamma]_{n-l} = b_\nu \prod_{j=1}^{n-\nu} (\gamma - \hat{\gamma}_j),$$

where brackets imply the Pochhammer notation, i.e.,

$$[\rho]_p = \rho(\rho-1)\cdots(\rho-p+1), \quad [\rho]_0 = 1.$$

Then (1.1) can be written in the form

$$\left[\prod_{j=1}^{n} (D - \hat{\rho}_j)\right] y = b_\nu z^q \left[\prod_{j=1}^{n-\nu} (D - \hat{\gamma}_j)\right] y \qquad (D = zd/dz).$$

The change of variables $z = t^\alpha$ yields

$$\left[\prod_{j=1}^{n} (\mathscr{D} - \alpha\hat{\rho}_j)\right] y = b_\nu \alpha^\nu t^{q\alpha} \left[\prod_{j=1}^{n-\nu} (\mathscr{D} - \alpha\hat{\gamma}_j)\right] y \qquad (\mathscr{D} = td/dt).$$

Putting

$$\alpha = \nu/q, \quad b_\nu \alpha^\nu = \beta,$$
$$\rho_j = \alpha\hat{\rho}_j \quad (j=1, 2,\dots, n),\ \gamma_j = \alpha\hat{\gamma}_j \quad (j=1, 2,\dots, n-\nu),$$

we thus obtain a general form of non-Fuchsian generalized hypergeometric equation

$$(1.2) \qquad [\prod_{j=1}^{n}(\mathscr{D}-\rho_j)]y = \beta t^{\nu}[\prod_{j=1}^{n-\nu}(\mathscr{D}-\gamma_j)]y,$$

which has a regular singularity at the origin and an irregular singularity of rank 1 at infinity in the whole complex t-plane.

For reference for the comparison with the Bessel system stated in [3] we here write down the system of differential equations which is equivalent to (1.2) as follows:

$$tY' = (A_0+tA_1)Y,$$

where

$$A_0 = \begin{pmatrix} \hat{\rho}_1 & 1 & & & & & & \\ & \hat{\rho}_2 & 1 & & & & 0 & \\ & & \ddots & \ddots & & & & \\ & & & \hat{\rho}_{n-\nu} & 1 & & & \\ & & & & \hat{\rho}_{n-\nu+1} & 0 & & \\ & 0 & & & & \ddots & \ddots & \\ & & & & & & \ddots & 0 \\ & & & & & & & \hat{\rho}_n \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & & & & & \\ & \ddots & \ddots & & & 0 & \\ & & 0 & 0 & & & \\ & & & 0 & 1 & & \\ & 0 & & & \ddots & \ddots & \\ & & & & & \ddots & 1 \\ \eta_{n-\nu} & \cdots & \eta_1 & \beta & 0 & \cdots & 0 \end{pmatrix}.$$

This can be easily obtained by putting $x_1=y$, $x_i=[\prod_{j=1}^{i-1}(\mathscr{D}-\rho_j)]y$ $(i=2, 3,\dots, n)$ and applying the shearing transformation $Y=S(t)X$, where $S(t)=t^{-\nu+1}I_{n-\nu+1}\oplus \operatorname{diag}(t^{-\nu+2}, t^{-\nu+3},\dots, t^{-1}, 1)$, I_p being the p by p identity matrix.

Now we consider the differential equation (1.2) and deal with, for simplicity, a non-logarithmic case in which it is assumed that the characteristic constants ρ_j are mutually distinct and their differences are not ingegers. We can then find a fundamental set of convergent power series solutions of the form

$$(1.3) \qquad y_i(t) = t^{\rho_i}\sum_{m=0}^{\infty} G_i(m)t^m \qquad (i = 1, 2,\dots, n),$$

where the coefficients $G_i(m)$ are given by particular solutions of the linear difference equations

$$(1.4) \qquad [\prod_{j=1}^{n}(m+\rho_i-\rho_j)]G_i(m) = \beta[\prod_{j=1}^{n-\nu}(m-\nu+\rho_i-\gamma_j)]G_i(m-\nu)$$

with the initial conditions

$$(1.5) \qquad G_i(0) \neq 0, \quad G_i(r) = 0 \qquad (r = -1, -2,\dots, -\nu+1).$$

From (1.4), when the $G_i(m)$ are regarded as functions of integral variable m, it can be immediately seen that $G_i(m)=0$ for $m\neq m'\nu$, m' being a positive integer and for all negative integers m. In later considerations we need explicit forms of the $G_i(m)$ as functions of complex variable m. To seek them, we put $\hat{G}_i(m)=G_i(m\nu)$ in (1.4) and then obtain

(1.6) $$\hat{G}_i(m) = \frac{\beta}{\nu^\nu} \frac{\prod_{j=1}^{n-\nu}\left(m-1+\frac{\rho_i-\gamma_j}{\nu}\right)}{\prod_{j=1}^{n}\left(m+\frac{\rho_i-\rho_j}{\nu}\right)} \hat{G}_i(m-1)$$

$$= \left(\frac{\beta}{\nu^\nu}\right)^m \frac{\prod_{j=1}^{n-\nu}\Gamma\left(m+\frac{\rho_i-\gamma_j}{\nu}\right)}{\prod_{j=1}^{n}\Gamma\left(m+1+\frac{\rho_i-\rho_j}{\nu}\right)}.$$

We also see that the $G_i(m)=\hat{G}_i(m/\nu)p(m)$, $p(m)$ being a periodic function with period ν, satisfy (1.4). So, in order to obtain the desired solutions $G_i(m)$ with properties stated above, we have only to determine $p(m)$. From the initila conditions (1.5) we can take $p(m)$ as

(1.7) $$p(m) = \prod_{j=1}^{\nu-1}\left\{e^{i\pi m/\nu} \sin \pi\left(\frac{m+j}{\nu}\right)\right\}.$$

In the above and hereafter it is also assumed that for each i, $\rho_i-\gamma_j \neq integer$ $(j=1, 2,\dots, n-\nu)$.

We have thus obtained a fundamental set of solutions expressed in terms of so-called generalized hypergeometric series:

(1.8) $$y_i(t) = \{\prod_{j=1}^{\nu-1} \sin (j/\nu)\} t^{\rho_i} \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{n-\nu}\Gamma\left(m+\frac{\rho_i-\gamma_j}{\nu}\right)}{\prod_{j=1}^{n}\Gamma\left(m+1+\frac{\rho_i-\rho_j}{\nu}\right)} \left\{\frac{\beta^{1/\nu}}{\nu}t\right\}^{m\nu}$$

$$(i = 1, 2,\dots, n).$$

In the paper [1] B. J. L. Braaksma has investigated asymptotic behaviors of such generalized hypergeometric series by means of Barnes-integrals in a great detail. The purpose of this paper is to obtain more explicit results than those of [1], that is, to clarify the global relations between the solutions $y_i(t)$ and n formal solutions derived in §2.

2. Formal solutions and Stokes multipliers

We shall now seek formal solutions of (1.2) at the irregular singularity. For simplicity in later consideration, we assume in (1.2) that the γ_j are mutually distinct and $\gamma_j \neq \gamma_k \bmod \nu$ $(j \neq k)$. Then we can easily verify that there exist $(n-\nu)$ linearly independent formal solutions of the algebraic form

$$\hat{y}^k(t) = t^{\gamma_k} \sum_{s=0}^{\infty} \hat{H}^k(s)t^{-s} \qquad (k = 1, 2,\dots, n-\nu).$$

In fact, substituting above series into (1.2) and euqating coefficients of like powers of t in both sides, we have

$$\beta[\prod_{j=1}^{n-\nu}(\gamma_k-s-\nu-\gamma_j)]\hat{H}^k(s+\nu)=[\prod_{j=1}^{n}(\gamma_k-s-\rho_j)]\hat{H}^k(s)$$

with the initial considions

$$\hat{H}^k(0)\neq 0,\quad \hat{H}^k(r)=0 \qquad (r<0).$$

From this it follows that $\hat{H}^k(s)=0$ for $s\neq s'\nu$, s' being a positive integer and $\hat{H}^k(s\nu)$ can be expressed in terms of gamma functions as follows:

$$\begin{aligned}\hat{H}^k(s\nu) &= \frac{(-\nu)^\nu}{\beta}\frac{\prod_{j=1}^{n}\left(s-1-\frac{\gamma_k-\rho_j}{\nu}\right)}{\prod_{j=1}^{n-\nu}\left(s-\frac{\gamma_k-\gamma_j}{\nu}\right)}\hat{H}^k((s-1)\nu)\\ &= \left\{\frac{(-\nu)^\nu}{\beta}\right\}^s\frac{\prod_{j=1}^{n}\Gamma\left(s+\frac{\rho_j-\gamma_k}{\nu}\right)}{\prod_{j=1}^{n-\nu}\Gamma\left(s+1+\frac{\gamma_j-\gamma_k}{\nu}\right)},\end{aligned}$$

where we have taken the initial value

$$\hat{H}^k(0)=\frac{\prod_{j=1}^{n}\Gamma\left(\frac{\rho_j-\gamma_k}{\nu}\right)}{\prod_{j=1}^{n-\nu}\Gamma\left(1+\frac{\gamma_j-\gamma_k}{\nu}\right)}.$$

We have thus determined $(n-\nu)$ algebraic formal solutions

$$\hat{y}^k(t)=t^{\gamma_k}\sum_{s=0}^{\infty}\frac{\prod_{j=1}^{n}\Gamma\left(s+\frac{\rho_j-\gamma_k}{\nu}\right)}{\prod_{j=1}^{n-\nu}\Gamma\left(s+1+\frac{\gamma_j-\gamma_k}{\nu}\right)}\left\{-\frac{\beta^{1/\nu}}{\nu}t\right\}^{-\nu s} \tag{2.1}$$

$$(k=1,2,\dots,n-\nu).$$

On the other hand, there exist ν formal solutions of the exponential type

$$y^k(t)=e^{\lambda_k t}t^{\mu_k}\sum_{s=0}^{\infty}H^k(s)t^{-s} \qquad (H^k(0)=1;\ k=1,2,\dots,\nu). \tag{2.2}$$

To see this, putting

$$\begin{cases}\prod_{j=1}^{n}(\mathscr{D}-\rho_j)=[\mathscr{D}]_n+\sum_{l=1}^{n}\alpha_l[\mathscr{D}]_{n-l},\\ \beta\prod_{j=1}^{n-\nu}(\mathscr{D}-\gamma_j)=\beta[\mathscr{D}]_{n-\nu}+\sum_{l=1}^{n-\nu}\beta_l[\mathscr{D}]_{n-\nu-l},\end{cases} \tag{2.3}$$

we rewrite (1.2) in the form

$$y^{(n)}+\sum_{l=1}^{n}\alpha_l t^{-l}y^{(n-l)}=\beta y^{(n-\nu)}+\sum_{l=1}^{n}\beta_l t^{-l}y^{(n-\nu-l)}. \tag{2.4}$$

As explained in [2], we here use the following device:

$$y^{(p)}(t) = e^{\lambda t} t^{\mu} \sum_{s=0}^{\infty} H_p(s) t^{-s} \qquad (p = 0, 1, \dots, n),$$

where $H_0(s) = H(s)$. Substituting these series into (2.4) and identifying coefficients of like powers of t in both sides, we have the recurrence formula

$$\text{(2.5)} \qquad H_n(s) + \sum_{l=1}^{n} \alpha_l H_{n-l}(s-l) = \beta H_{n-\nu}(s) + \sum_{l=1}^{n-\nu} \beta_l H_{n-\nu-l}(s-l).$$

Moreover, from the relation of differentiation $y^{(p)} = (y^{(p-1)})'$ we obtain another recurrence formulas

$$\text{(2.6)} \qquad H_p(s) = \lambda H_{p-1}(s) + (\mu - s + 1) H_{p-1} \qquad (p = 1, 2, \dots, n)$$

which in turn yield

$$\text{(2.7)} \quad H_p(s) = \lambda^p H(s) + p\lambda^{p-1}(\mu - s + 1) H(s-1) + \sum_{l=2}^{p} M(p\colon l\colon s) H(s-l)$$
$$(p = 1, 2, \dots, n),$$

where the $M(p\colon l\colon s)$ are functions of s. From (2.5) and (2.7) we therefore obtain the formula satisfied by $H(s)$:

$$(\lambda^n - \lambda^{n-\nu}\beta) H(s) + \{(n\lambda^{n-1} - (n-\nu)\lambda^{n-\nu-1}\beta)(\mu - s + 1) + \alpha_1 \lambda^{n-1} - \beta_1 \lambda^{n-\nu-1}\} H(s-1)$$
$$= R(s\colon H(s-2), \dots, H(s-n))$$

the right hand member of which is linear in $H(s-2), \dots, H(s-n)$. We now put $s = 0$ and then have the characteristic equation

$$\lambda^n - \lambda^{n-\nu}\beta = 0$$

whose non-trivial roots are given by

$$\text{(2.8)} \qquad \lambda_k = \beta^{1/\nu} \omega^{k-1} \qquad (\omega = \exp(2\pi i/\nu);\ k = 1, 2, \dots, \nu).$$

Next we put $s = 1$ and then have the relation of determining μ_k

$$(n\lambda_k^{n-1} - (n-\nu)\lambda_k^{n-\nu-1}\beta)\mu_k + \alpha_1 \lambda_k^{n-1} - \beta_1 \lambda_k^{n-\nu-1} = 0,$$

thereby obtaining

$$\text{(2.9)} \qquad \mu_k = (\beta_1/\beta - \alpha_1)/\nu \qquad (k = 1, 2, \dots, \nu).$$

After the determination of the characteristic constants λ_k and μ_k, the above formula is reduced to the form

$$\text{(2.10)} \quad -\nu\lambda_k^{n-1} s H^k(s) = R(s+1\colon H^k(s-1), \dots, H^k(s-n+1)) \qquad (k = 1, 2, \dots, \nu).$$

Then it is easy to see that the coefficient $H^k(s)$ can be determined successively from the initial conditions $H^k(0)=1$, $H^k(r)=0$ $(r<0)$. Thus we have obtained ν formal solutions of the form (2.2).

We here show an identity called the Fuchs relation between characteristic constants which express the multi-valuedness of solutions of the differential equation (1.2). From (2.3) we have

$$\sum_{j=1}^{n}\rho_j = n(n-1)/2 - \alpha_1, \quad \sum_{j=1}^{n-\nu}\gamma_j = (n-\nu)(n-\nu-1)/2 - \beta_1/\beta.$$

Combining these with (2.9), we therefore obtain

$$\sum_{k=1}^{\nu}\mu_k + \sum_{k=1}^{n-\nu}\gamma_k = \sum_{j=1}^{n}\rho_j - \nu\{n-(\nu+1)/2\}. \tag{2.11}$$

This will play an important role in the calculation of the Stokes multipliers to follow.

Now we introduce the linear difference equations

$$(m+\rho-\mu_k)g^k(m) = \lambda_k g^k(m-1) \qquad (k=1, 2,\dots, \nu) \tag{2.12}$$

and take their particular solutions of the form

$$g^k(m) = \frac{\lambda_k^{m+\rho-\mu_k}}{\Gamma(m+\rho-\mu_k+1)} \qquad (k=1, 2,\dots, \nu). \tag{2.13}$$

Let us define

$$f_p^k(m) = \sum_{s=0}^{\infty} H_p^k(s)g^k(m+s) \qquad (p=0, 1,\dots, n)$$

and denote $f_0^k(m)$ by $f^k(m)$. The well-definedness of these functions of complex variable m can be proved by exactly the same way as in [2]. In fact, the series are absolutely convergent under the strongest condition $|\lambda_k|<|\lambda_j-\lambda_k|$ $(j\neq k)$. Multiplying both sides of (2.5) and (2.6) by $g^k(m+s)$ and summing them over s, we have

$$f_n^k(m) + \sum_{l=1}^{n}\alpha_l f_{n-l}(m+l) = \beta f_{n-\nu}^k(m) + \sum_{l=1}^{n-\nu}\beta_l f_{n-\nu-l}^k(m+l) \tag{2.14}$$

and

$$f_p^k(m) = (m+\rho+1)f_{p-1}^k(m+1) = [m+\rho+p]_p f_k(m+p), \tag{2.15}$$

respectively. Then the substitution of (2.15) into (2.14) yields

$$\begin{aligned}\{[m+\rho]_n &+ \sum_{l=1}^{n}\alpha_l[m+\rho]_{n-l}\}f^k(m)\\ &= \{\beta[m-\nu+\rho]_{n-\nu} + \sum_{l=1}^{n-\nu}\beta_l[m-\nu+\rho]_{n-\nu-l}\}f^k(m-\nu)\end{aligned}$$

which implies from (2.3) that

$$[\prod_{j=1}^{n}(m+\rho-\rho_j)]f^k(m) = \beta[\prod_{j=1}^{n-\nu}(m-\nu+\rho-\gamma_j)]f^k(m-\nu).$$

This difference equation, replaced ρ by ρ_i, is just the same one satisfied by $G_i(m)$. Hence, for each i, ν functions $f_i^k(m)$ $(k=1, 2,\dots, \nu)$ are particular solutions of (1.4) and moreover it can be proved that they form a fundamental set of solutions of (1.4). We here summarize above results in the following

PROPOSITION 1. *Under the condition that* $|\lambda_k|<|\lambda_j-\lambda_k|$ $(j\neq k;\, j, k=1, 2,\dots, \nu)$ *the functions* $f_i^k(m)$ $(k=1, 2,\dots, \nu)$ *are well-defined and have the properties as follows*:

(i) *For any real* w, *let us take the integer* $\sigma>-w-\mathrm{Re}\,(\rho_i-\mu_k)-1$. *Then the functions*

$$(2.16)\qquad R_i^k(m;\,\sigma) = \frac{1}{g_i^k(m+\sigma)}\sum_{s=\sigma+1}^{\infty} H^k(s)g_i^k(m+s)$$

are analytic and bounded in the right half-plane $\mathrm{Re}\, m\geqq w$. *From this it follows that there hold the asymptotic relations*

$$(2.17)\qquad f_i^k(m) \sim g_i^k(m)\{1+O(m^{-1})\} \qquad (m \longrightarrow \infty)$$

in the right half m-plane.

(ii) *For each i, the functions* $f_i^k(m)$ *form a fundamental set of solutions of* (1.4). *The Casorati determinant* $\mathscr{C}_i(m)$ *constructed from them is given by*

$$(2.18)\qquad \mathscr{C}_i(m) = \frac{\prod_{j=1}^{n-\nu}\Gamma(m+\rho_i-\gamma_j)}{\prod_{j=1}^{n}\Gamma(m+\nu+\rho_i-\rho_j)}((-1)^{\nu-1}\beta)^{m+\rho_i-\mu_k}V(\lambda_1, \lambda_2,\dots, \lambda_\nu)\,,$$

where $V(\lambda_1, \lambda_2,\dots, \lambda_\nu)$ *is the Vandermonde determinant of* $\lambda_1, \lambda_2,\dots, \lambda_\nu$.

The detailed proof of this proposition is referred to [2].

Now, according to the theory of linear difference equations, the $G_i(m)$ can be written in the form

$$(2.19)\qquad G_i(m) = \sum_{k=1}^{\nu} T_i^k(m)f_i^k(m) \qquad (i=1, 2,\dots, n)\,,$$

where the $T_i^k(m)$ are periodic functions with period 1. In order to determine the periodic functions explicitly, we solve the linear equation

$$(2.20)\qquad G_i(m+r) = \sum_{k=1}^{\nu} T_i^k(m)f_i^k(m+r) \qquad (r=0, 1,\dots, \nu-1)$$

by the Cramer rule, obtaining

$$T_i^k(m) = \begin{vmatrix} G_i(m) & f_i^1(m) & \overset{k}{\ddots} & f_i^\nu(m) \\ G_i(m+1) & f_i^1(m+1) & \cdots & f_i^\nu(m+1) \\ \vdots & \vdots & \vdots & \vdots \\ G_i(m+\nu-1) & f_i^1(m+\nu-1) & \cdots & f_i^\nu(m+\nu-1) \end{vmatrix}$$

$$\div \begin{vmatrix} f_i^k(m) & f_i^1(m) & \overset{k}{\check{\cdots}} & f_i^\nu(m) \\ f_i^k(m+1) & f_i^1(m+1) & \cdots & f_i^\nu(m+1) \\ \vdots & \vdots & \vdots & \vdots \\ f_i^k(m+\nu-1) & f_i^1(m+\nu-1) & \cdots & f_i^\nu(m+\nu-1) \end{vmatrix}$$

and investigate their behaviors in some period strip lying far in the right half-plane, say, $N\nu \leqq \mathrm{Re}\, m < N\nu+1$, N being a sufficiently large positive integer. We can easily see that the $T_i^k(m)$ are analytic in the entire finite part of the strip, since the numerator is analytic and the denominator, which is the Casorati determinant, has no zeros in the right half-plane. We now investigate the $T_i^k(m)$ at the ends of the strip. From (2.13) and (2.17) we have

$$\frac{f_i^k(m+r)}{f_i^k(m)} \sim \frac{g_i^k(m+r)}{g_i^k(m)}\{1+O(m^{-1})\} \sim \lambda_k^r m^{-r}\{1+O(m^{-1})\}$$

$$(r=1, 2,\dots, \nu-1)$$

and from (1.6) we also have

$$\frac{\hat{G}_i((m+r)/\nu)}{\hat{G}_i(m/\nu)} \sim \beta^{r/\nu} m^{-r}\{1+O(m^{-1})\} \qquad (r = 1, 2,\dots, \nu-1)$$

for sufficiently large values of m in the right half-plane. Taking account of these asymptotic behaviors, we have

$$(2.21)\quad T_i^k(m) \sim \frac{\hat{G}_i(m/\nu)}{g_i^k(m)} \begin{vmatrix} p(m) & 1 & \overset{k}{\check{\cdots}} & 1 \\ p(m+1)\beta^{1/\nu} & \lambda_1 & \cdots & \lambda_\nu \\ \vdots & \vdots & \vdots & \vdots \\ p(m+\nu-1)\beta^{(\nu-1)/\nu} & \lambda_1^{\nu-1} & \cdots & \lambda_\nu^{\nu-1} \end{vmatrix} \Big/ V(\lambda_k, \lambda_1,\dots, \lambda_\nu)$$
$$\times \{1+O(m^{-1})\}$$

at both ends of the strip. The first member in the right hand side of (2.21) behaves like

$$(2.22)\quad \frac{\hat{G}_i(m/\nu)}{g_i^k(m)} = \left(\frac{\beta^{1/\nu}}{\nu}\right)^m \frac{\prod_{j=1}^{n-\nu} \Gamma\left(\frac{m+\rho_i-\gamma_j}{\nu}\right)}{\prod_{j=1}^{n} \Gamma\left(\frac{m+\rho_i-\rho_j}{\nu}+1\right)} \frac{\Gamma\left(\nu\left(\frac{m+\rho_i-\mu_k+1}{\nu}\right)\right)}{\lambda_k^{m+\rho_i-\mu_k}}$$
$$= \frac{\nu^{\rho_i-\mu_k+1/2}}{(2\pi)^{(\nu-1)/2}} \frac{e^{-2\pi i(k-1)m/\nu}}{\lambda_k^{\rho_i-\mu_k}}$$
$$\times \frac{\prod_{j=1}^{n-\nu} \Gamma\left(\frac{m+\rho_i-\gamma_j}{\nu}\right) \prod_{j=1}^{\nu} \Gamma\left(\frac{m+\rho_i-\mu_k+j}{\nu}\right)}{\prod_{j=1}^{n} \Gamma\left(\frac{m+\rho_i-\rho_j}{\nu}+1\right)}$$

$$\sim \frac{\nu^{\rho_i-\mu_k+1/2}}{(2\pi)^{(\nu-1)/2}} \frac{e^{-2\pi i(k-1)m/\nu}}{\lambda_k^{\rho_i-\mu_k}}$$

$$\times\, m^{(\sum_{j=1}^{n} \rho_i - \sum_{j=1}^{n-\nu} \gamma_j)/\nu - \mu_k - n + (\nu+1)/2} \{1+O(m^{-1})\}$$

$$\sim \frac{\nu^{1/2}}{(2\pi)^{(\nu-1)/2}} \left(\frac{\nu}{\lambda_k}\right)^{\rho_i-\mu_k} e^{-2\pi i(k-1)m/\nu}\{1+O(m^{-1})\}\,,$$

where we have used Gauss' multiplication formula of the gamma function and the Fuchs relation (2.11). From this and considering (1.7), we see that the $T_i^k(m)$ behave like $O(e^{-2\pi i(k-1)m/\nu})$ in the upper end of the strip and $O(e^{2\pi i(\nu-k)m/\nu})$ in the lower end of the strip. We here consider the transformation $z=e^{2\pi i m}$, which maps the unit strip in the m-plane on the entire z-plane, both ends of the strip corresponding to $z=0, \infty$. Then we see that the $T_i^k(m)=T_i^k(z)$ are holomorphic at every point of the z-plane except possibly at $z=0, \infty$, where $T_i^k(z)=O(z^{-(\nu-1)/\nu})$ as $z\to 0$ and $T_i^k(z)=O(z^{(\nu-1)/\nu})$ as $z\to\infty$. This implies that the singularities at $z=0$ and $z=\infty$ are removable and hence the $T_i^k(z)$ must be constant. In order to evaluate explicit values of such constants, we put $m=N\nu$ in (2.21) and let N tend to infinity. Since

$$p(N\nu) = \prod_{j=1}^{\nu-1} \sin(\pi j/\nu), \qquad p(N\nu+r) = 0 \qquad (r=1, 2,\dots, \nu-1),$$

from (2.21) and (2.22) we have

$$T_i^k(N\nu) \longrightarrow \frac{\prod_{j=1}^{\nu-1} \sin(\pi j/\nu)}{(2\pi)^{(\nu-1)/2}\nu^{1/2}} \left(\frac{\nu}{\lambda_k}\right)^{\rho_i-\mu_k} \qquad (N \longrightarrow \infty)\,,$$

which implies that

$$T_i^k(m) \equiv \frac{\prod_{j=1}^{\nu-1} \sin(\pi j/\nu)}{(2\pi)^{(\nu-1)/2}\nu^{1/2}} \left(\frac{\nu}{\lambda_k}\right)^{\rho_i-\mu_k} \qquad (k=1, 2,\dots, \nu)\,. \tag{2.23}$$

We have thus obtained

PROPOSITION 2. *For each* i $(i=1, 2,\dots, n)$, $G_i(m)=\hat{G}_i(m/\nu)p(m)$ *can be expressed in the form*

$$G_i(m) = \sum_{k=1}^{\nu} T_i^k f_i^k(m) = \sum_{k=1}^{\nu} T_i^k\left(\sum_{s=0}^{\infty} H^k(s) g_i^k(m+s)\right), \tag{2.24}$$

where the constants T_i^k *are given by* (2.23).

This proposition, together with Proposition 1, corresponds to Lemmas 2, 3 in [1]. The constants T_i^k become the Stokes multipliers.

3. Barnes' integral representation

We are now in a position to investigate global behaviors of the convergent power series solutions $y_i(t)$ $(i=1, 2,\dots, n)$. It is readily verified from the behavior of $G_i(m)$ and the theorem of residues that there holds

$$
\begin{aligned}
(3.1)\qquad y_i(t) &= t^{\rho_i} \textstyle\sum_{m=0}^{\infty} G_i(m)t^m \qquad (-\pi<\arg t\leqq\pi)\\
&= -\frac{t^{\rho_i}}{2\pi i}\int_C G_i(z)\left(\frac{\pi}{\sin \pi z}\right)(-t)^z\, dz,
\end{aligned}
$$

where $G_i(z)$ reminds us of the form

$$
\begin{aligned}
(3.2)\qquad G_i(z) &= \hat{G}_i(z/\nu)p(z)\\
&= \left(\frac{\beta^{1/\nu}}{\nu}\right)^z \frac{\prod_{j=1}^{n-\nu}\Gamma\left(\frac{z+\rho_i-\gamma_j}{\nu}\right)}{\prod_{j=1}^{n}\Gamma\left(\frac{z+\rho_i-\rho_j}{\nu}+1\right)} \prod_{j=1}^{\nu-1}\left\{e^{i\pi z/\nu}\sin\pi\left(\frac{z+j}{\nu}\right)\right\}
\end{aligned}
$$

and the path of integration C is a Barnes-contour running from $z=\infty-ia$ to $z=\infty+ia$ such that the points $z=m$ $(m=0, 1,\dots)$ lie to the right of C and the points $z=\gamma_j-\rho_i-\nu s$ $(j=1, 2,\dots, n-\nu;\ s=0, 1,\dots)$ lie to the left of C. The constant a is taken as $a>|\mathrm{Im}\,(\gamma_j-\rho_i)|$ $(j=1, 2,\dots, n-\nu)$.

In order to analyze $y_i(t)$ in the large, we first replace the contour C by the rectilinear contour L which runs first from $\infty-ia$ to $w-ia$, next from $w-ia$ to $w+ia$ and finally from $w+ia$ to $\infty+ia$. Here w is an arbitrary negative number such that the positive integers N_k $(k=1, 2,\dots, n-\nu)$ and N can be taken as

$$
(3.3)\qquad \begin{cases} -\nu(N_k+1) < w + \mathrm{Re}\,(\rho_i-\gamma_k) < -\nu N_k & (k = 1, 2,\dots, n-\nu),\\ -N-1 < w < -N. \end{cases}
$$

Since $G_i(z)$ vanishes at $z=-s$ $(s=1, 2,\dots)$ and hence the integral in (3.1) has simple poles only at $z=\gamma_k-\rho_i-\nu s$ $(k=1, 2,\dots, n-\nu;\ s=0, 1,\dots,)$, we have from the theorem of residues

$$
\begin{aligned}
y_i(t) = &-t^{\rho_i}\sum \mathrm{Res}\left[G_i(z)\left(\frac{\pi}{\sin \pi z}\right)(-t)^z\right]_{z=\gamma_k-\rho_i-\nu s}\\
&-\frac{t^{\rho_i}}{2\pi i}\int_L G_i(z)\left(\frac{\pi}{\sin\pi z}\right)(-t)^z dz.
\end{aligned}
$$

Using Euler's formula

$$
(3.4)\qquad \Gamma(z)\Gamma(1-z) = \pi/\sin \pi z,
$$

we calculate the residue at $z=\gamma_k-\rho_i-\nu s$:

$$
\lim_{z\to\gamma_k-\rho_i-\nu s}\left[(z+\nu s-\gamma_k+\rho_i)G_i(z)\left(\frac{\pi}{\sin\pi z}\right)(-t)^z\right]
$$

$$= \left(\frac{\beta^{1/\nu}}{\nu}\right)^{\gamma_k-\rho_i-\nu s} \frac{\prod_{j=1,j\neq k}^{n-\nu} \Gamma\left(-s+\frac{\gamma_k-\gamma_j}{\nu}\right)}{\prod_{j=1}^{n} \Gamma\left(-s+1+\frac{\gamma_k-\rho_j}{\nu}\right)} p(\gamma_k-\rho_i-\nu s)(-t)^{\gamma_k-\rho_i-\nu s}$$

$$\times \frac{\pi}{\sin \pi(\gamma_k-\rho_i-\nu s)} \lim_{z\to\gamma_k-\rho_i-\nu s} \left\{ \frac{\pi\left(\frac{z+\rho_i-\gamma_k}{\nu}+s\right)}{\sin \pi\left(\frac{z+\rho_i-\gamma_k}{\nu}+s\right)} \frac{\nu(-1)^s}{\Gamma\left(1-\frac{z+\rho_i-\gamma_k}{\nu}\right)} \right\}$$

$$= \frac{\nu}{\pi^\nu}\left(\frac{\beta^{1/\nu}}{\nu}\right)^{\gamma_k-\rho_i} \frac{\prod_{j=1}^{n} \sin \pi\left(\frac{\gamma_k-\rho_j}{\nu}\right)}{\sin \pi(\gamma_k-\rho_i) \prod_{j=1,j\neq k}^{n-\nu} \sin \pi\left(\frac{\gamma_k-\gamma_j}{\nu}\right)}$$

$$\times p(\gamma_k-\rho_i)(-1)^{-n+\gamma_k-\rho_i} \frac{\prod_{j=1}^{n} \Gamma\left(s+\frac{\rho_j-\gamma_k}{\nu}\right)}{\prod_{j=1}^{n-\nu} \Gamma\left(s+1+\frac{\gamma_j-\gamma_k}{\nu}\right)} \left\{-\frac{\beta^{1/\nu}}{\nu}t\right\}^{-\nu s} t^{\gamma_k-\rho_i}.$$

Therefore for $-\pi<\arg t\leqq\pi$ we have

$$y_i(t) = \sum_{k=1}^{n-\nu} \hat{T}_i^k\{\sum_{s=0}^{N_k} \hat{H}^k(\nu s)t^{-\nu s+\gamma_k}\} - \frac{t^{\rho_i}}{2\pi i}\int_L G_i(z)\left(\frac{\pi}{\sin \pi z}\right)(-t)^z dz, \tag{3.5}$$

where we have put

$$\hat{T}_i^k = \frac{\nu}{\pi^\nu}\left(\frac{\beta^{1/\nu}}{\nu}\right)^{\gamma_k-\rho_i} \frac{(-1)^{-n+1+\gamma_k-\rho_i} \prod_{j=1}^{n} \sin \pi\left(\frac{\gamma_k-\rho_j}{\nu}\right)}{\sin \pi(\gamma_k-\rho_i) \prod_{j=1 j\neq k}^{n-\nu} \sin \pi\left(\frac{\gamma_k-\gamma_j}{\nu}\right)} p(\gamma_k-\rho_i) \tag{3.6}$$

$$(k = 1, 2,\dots, n-\nu).$$

From the relation $y_i(t)=e^{2\pi i l\rho_i}y_i(te^{-2\pi i l})$, l being some integer, it can be seen that if t lies in the sector

$$S(e^{-2\pi i l}):\ -\pi < \arg t - 2\pi l \leqq \pi,$$

then in (3.5) the Stokes multipliers $\hat{T}_i^k$ must be replaced by $\hat{T}_i^k e^{2\pi i(\rho_i-\gamma_k)l}$.

Next we investigate the integral in (3.5). Taking account of Proposition 1 and 2, we replace $G_i(z)$ by

$$G_i(z) = \sum_{k=1}^{\nu} T_i^k\{\sum_{s=0}^{\sigma} H^k(s)g_i^k(z+s) + g_i^k(z+\sigma)R_i^k(z;\sigma)\},$$

σ being some positive integer, and obtain

$$-\frac{t^{\rho_i}}{2\pi i}\int_L G_i(z)\left(\frac{\pi}{\sin \pi z}\right)(-t)^z dz \tag{3.7}$$

$$= \sum_{k=1}^{\nu} T_i^k \sum_{s=0}^{\sigma} H^k(s)\left\{-\frac{t^{\rho_i}}{2\pi i}\int_L g_i^k(z+s)\left(\frac{\pi}{\sin \pi z}\right)(-t)^z dz\right\}$$

$$+ \sum_{k=1}^{\nu} T_i^k \left\{ -\frac{t^{\rho_i}}{2\pi i} \int_L g_i^k(z+\sigma) R_i^k(z;\sigma) \left(\frac{\pi}{\sin \pi z} \right) (-t)^z dz \right\}$$

$$= \sum_{k=1}^{\nu} T_i^k \sum_{s=0}^{\sigma} H^k(s) \{ \sum_{m=-N}^{\infty} g_i^k(m+s) t^{m+\rho_i} \} + \sum_{k=1}^{\nu} T_i^k I_i^k(t;\sigma).$$

In the above, considering (3.3), we carried out the residue calculus. We then have to investigate the behavior of functions appearing in (3.7), which are of the form

$$x(t, s) = t^{\rho} \sum_{m=0}^{\infty} g(m+s) t^m,$$

with the coefficient

$$g(m+s) = \frac{\lambda^{m+s+\rho-\mu}}{\Gamma(m+s+\rho-\mu+1)}.$$

It is easy to see that $x(t, s)$ is a particular solution of the first order nonhomogeneous linear differential equation

$$tx'(t, s) = \{\lambda t + \mu - s\} x(t, s) + \lambda g(s-1) t^{\rho}$$

and hence it can be expressed in terms of the integral

$$x(t, s) = \int_0^1 \exp\{\lambda t(1-\tau)\} \tau^{s+\rho-\mu-1} d\tau [t^{\rho} \lambda g(s-1)]. \tag{3.8}$$

From this we have the following

PROPOSITION 3. *The function $x(t, s)$ admits the asymptotic behavior*

$$x(t, s) \sim e^{2\pi i(\rho-\mu)l} e^{\lambda t} t^{\mu-s} - t^{\rho} \sum_{r=1}^{\infty} g(s-r) t^{-r} \tag{3.9}$$

as t tends to infinity in the sector

$$S(\lambda\, e^{-2\pi i l}) : |\arg \lambda t - 2\pi l| \leqq 3\pi/2 - \varepsilon. \tag{3.10}$$

where l is any integer and ε is an arbitrarily small positive number.

PROOF. First we consider the case when t lies in $S(\lambda)$. We put $\eta = t\tau$ in (3.8), and deform the path of integration from 0 to t into the straight line $\mathrm{Re}\, \lambda\eta = 0$ from 0 to infinity and the so-called Friedrichs path which consists of the following parts: For $\pi \leqq |\arg(\lambda t)| \leqq 3\pi/2 - \varepsilon$, t' denoting $t' = -\mathrm{Re}(\lambda t)$,

(i)	the straight line	$\mathrm{Re}\, \lambda\eta = -t'$	from t to $-t'/\lambda$
(ii)	the semi-circle	$\lvert\lambda\eta\rvert = t'$	from $-t'/\lambda$ to t'/λ
(iii)	the straight line	$\mathrm{Re}\, \lambda\eta = 0$	from t'/λ to ∞,

and for $0 \leqq |\arg \lambda t \leqq \pi$, t' denoting $t' = |\lambda t|$, (ii) the above semi-circle from t to

t'/λ and (iii).

Then we can rewrite (3.8) in the form

$$(3.11)\qquad \begin{aligned} x(t, s) &= e^{\lambda t}t^{\mu-s}[\lambda g(s-1)]\int_0^\infty e^{-\lambda\eta}\eta^{s+\rho-\mu-1}d\eta \\ &\quad -t^\rho[\lambda g(s-1)]\int_t^\infty e^{\lambda(t-\eta)}\left(\frac{\eta}{t}\right)^{s+\rho-\mu}\frac{d\eta}{\eta} \\ &= e^{\lambda t}t^{\mu-s} - t^\rho[\lambda g(s-1)]\int_t^\infty e^{\lambda(t-\eta)}\left(\frac{\eta}{t}\right)^{s+\rho-\mu}\frac{d\eta}{\eta} \end{aligned}$$

Here the last integral is carried out along the Friedrichs path and it can be proved to be bounded for sufficiently large values of t in $S(\lambda)$. (See [2].) Consequently, we obtain

$$x(t, s) \sim e^{\lambda t}t^{\mu-s} + O(t^\rho).$$

Moreover, since for any positive integer p there holds

$$x(t, s) = t^{-p}x(t, s-p) - t^{\rho-p}\textstyle\sum_{m=0}^{p-1} g(m-p+s)t^m,$$

we apply the above result to $x(t, s-p)$, obtaining

$$x(t, s) \sim e^{\lambda t}t^{\mu-s} - t^\rho \textstyle\sum_{r=1}^{p} g(s-r)t^{-r} + O(t^{\rho-p})$$

as $t\to\infty$ in $S(\lambda)$. This just implies (3.9) for $l=0$. In the sector $S(\lambda e^{-2\pi il})$ we use the identity

$$x(t, s) = e^{2\pi i\rho l}\, x(te^{-2\pi il}, s)$$

and apply the above result to the right hand side. This completes the proof of Proposition 3.

We also have to investigate the behavior of the integral of the form

$$(3.12)\qquad \begin{aligned} I(t; \sigma) &= -\frac{t^\rho}{2\pi i}\int_L g(z+\sigma)R(z; \sigma)\left(\frac{\pi}{\sin \pi z}\right)(-t)^z dz \\ &= -\frac{\lambda^{\sigma+\rho-\mu}\, t^\rho}{2\pi i}\int_L \frac{R(z; \sigma)}{\Gamma(z+\sigma+\rho-\mu+1)}\left(\frac{\pi}{\sin \pi z}\right)\tau^z dz \quad (\tau = -\lambda t). \end{aligned}$$

Taking account of the fact that $R(z; \sigma)$ is bounded in the right half-plane, we have, putting $z-w=|z-w|e^{i\theta}$ and $\alpha=w+\sigma+\rho-\mu$,

$$\begin{aligned} &\frac{R(z; \sigma)}{\Gamma(z-w+\alpha+1)}\left(\frac{\pi}{\sin \pi z}\right)\tau^z = |z-w|^{\mathrm{Re}\,\alpha-1/2}|\tau|^w \\ &\quad\times \exp\left[-|z-w|\{(\log|z-w|-1+\log|\tau|)\cos\theta + (\pm\pi-\theta+\arg\tau)\sin\theta\}\right]O(1). \end{aligned}$$

So if $\pi-\theta+\arg\tau>0$ for $0\leqq\theta\leqq\pi/2$ and if $-\pi-\theta+\arg\tau<0$ for $-\pi/2\leqq\theta\leqq 0$, then the integrand of (3.12) tends to zero as $z \to \infty$ in $\operatorname{Re} z \geqq w$ and $|\operatorname{Im} z| \geqq a$. From this, for $|\arg \tau| < \pi/2$ we can deform the contour L into the straight line $\operatorname{Re} z = w$ from $z = w - i\infty$ to $z = w + i\infty$ and, using Euler's formula (3.4), we obtain

$$
\begin{aligned}
(3.13)\quad I(t;\sigma) &= -\frac{\lambda^{\sigma+\rho-\mu} t^{\rho}}{2\pi i} \int_{w-i\infty}^{w+i\infty} \frac{R(z;\sigma)}{\Gamma(z-w+\alpha+1)} \left(\frac{\pi}{\sin \pi z}\right) \tau^{z} dz \\
&= \frac{\lambda^{\sigma+\rho-\mu} t^{\rho}}{2\pi i} \int_{w-i\infty}^{w+i\infty} \frac{R(z;\sigma) \sin \pi(z-w+\alpha)}{\sin \pi z} \frac{\Gamma(2+w-\alpha-z)\tau^{z}}{[z+\alpha-w]_2} dz \\
&= \frac{\lambda^{\sigma+\rho-\mu} t^{\rho}}{2\pi i} \int_{w-i\infty}^{w+i\infty} \hat{R}(z;\sigma) \frac{\Gamma(2+w-\alpha-z)\tau^{z}}{[z+\alpha-w]_2} dz.
\end{aligned}
$$

We here consider the identity

$$
(3.14)\qquad \Gamma(2+w-\alpha-z)\tau^{z} = \tau^{2+w-\alpha} \int_{0}^{\infty} \xi^{1+w-\alpha-z} \exp(-\tau\xi) d\xi,
$$

which is valid for $\operatorname{Re} z = w$ and $|\arg \tau| < \pi/2$ under the condition that $2 > \operatorname{Re} \alpha > 1$, that is, in the above and hereafter the positive integer σ is taken so that

$$
(3.15)\qquad 1 - \operatorname{Re}(\rho-\mu) - \sigma < w < 2 - \operatorname{Re}(\rho-\mu) - \sigma.
$$

Substituting (3.14) into (3.13) and inverting the order of integration, we consequently obtain

$$
(3.16)\qquad I(t;\sigma) = -\lambda^{\sigma+\rho-\mu} t^{\rho} \tau^{2+w-\alpha} \int_{0}^{\infty} r(\xi) \exp(-\tau\xi) d\xi,
$$

where

$$
(3.17)\qquad r(\xi) = -\frac{1}{2\pi i} \int_{w-i\infty}^{w+i\infty} \frac{\hat{R}(z;\sigma)\xi^{1+w-\alpha-z}}{[z+\alpha-w]_2} dz.
$$

The detailed verification of the above calculation is referred to [1]. From (3.17) it immediately follows that

$$
(3.18)\qquad r(\xi) = O(\xi^{1-\alpha}) \qquad (\xi \geqq 0).
$$

Moreover, as for the behavior of $r(\xi)$, we have for $\xi \geqq 1$

$$
(3.19)\qquad r(\xi) = \frac{\sin(\alpha-w)}{\pi} \xi^{1+w-\alpha} \sum_{m=-N}^{\infty} \frac{R(m;\sigma)\xi^{-m}}{[m+\alpha-w]_2}.
$$

In fact, we integrate the integrand of (3.17) along the closed contour which consists of the straight line $\operatorname{Re} z = w$ from $w - iR$ to $w + iR$ and the right hand semicircle $|z-w| = R$. Then, since the integrand has the growth order $O[(z-w)^{-2} \cdot \exp\{-\operatorname{Re}(z-w)\log \xi\}]$ on the semi-circle $|z-w| = R$, R not being an integer, we can conclude that $r(\xi)$ is equal to the sum of residues at the poles lying in

Re $z \geqq w$, which yields the above result (3.19). From the boundedness of $R(z;\sigma)$, $r(\xi)$ can be estimated as

$$|r(\xi)| \leqq \frac{|\sin(\alpha-w)|}{\pi}|\xi^{1+w-\alpha+N}|M \sum_{m=0}^{\infty}\frac{|\xi|^{-m}}{|m-N+\alpha-w-1|^2},$$

where the series is convergent and bounded for $|\xi|\geqq 1$. Hence the function $r(\xi)$ for $\xi \geqq 1$ can be continued analytically for $|\xi|\geqq 1$ and

$$r(\xi) = O(\xi^{1+w-\alpha+N}) \qquad (|\xi| \geqq 1). \tag{3.20}$$

Returning to (3.16), we put

$$\begin{aligned}\int_0^\infty r(\xi)\exp(-\tau\xi)d\xi &= \int_0^1 r(\xi)\exp(-\tau\xi)d\xi + \int_1^\infty r(\xi)\exp(-\tau\xi)d\xi \\ &= I_1(\tau) + I_2(\tau).\end{aligned}$$

We immediately see from (3.18) that $I_1(\tau) = O(e^{-\tau}) + O(1)$ as $|\tau|\to\infty$. On the other hand, by the change of variables $\eta = \tau\xi$ and from (3.20) we can rewrite $I_2(\tau)$ in the form

$$I_2(\tau) = e^{-\tau}\int_z^\infty \left(\frac{\eta}{\tau}\right)^{2+w-\alpha-N} r^*\left(\frac{\eta}{\tau}\right)\exp(\tau-\eta)\frac{d\eta}{\eta} \qquad (|\arg\tau|<\pi/2), \tag{3.21}$$

where $r^*(\xi) = \xi^{-(1+w-\alpha+N)}r(\xi)$ is analytic and bounded for $|\xi|\geqq 1$ and the path of integration is the Friedrichs path. The integral in (3.21) is exactly the same form as in (3.11) and hence application of the same consideration in (3.11) yields that the integral can be continued analytically for $|\arg\tau|\leqq 3\pi/2-\varepsilon$ and it is bounded there. So we have $I_2(\tau) = O(e^{-\tau})$ for $|\arg\tau|\leqq 3\pi/2-\varepsilon$. Consequently, we have obtained

$$I(t;\sigma) = O(e^{\lambda t}t^{\mu+2-\sigma}) + O(t^{\mu+2-\sigma}) \tag{3.22}$$

as $t\to\infty$ in the sector $|\arg\lambda t-\pi|\leqq 3\pi/2-\varepsilon$. Since $I(t;\sigma) = e^{2\pi i\rho l}I(te^{-2\pi il};\sigma)$, we have the same result (3.22) for $|\arg\lambda t-(2l+1)\pi|\leqq 3\pi/2-\varepsilon$.

So far we have exclusively followed the analysis by B. L. J. Braaksma [1]. However, from the residue calculus of the integral (3.12) we immediately obtain

$$\begin{aligned}I(t;\sigma) &= t^\rho\sum_{m=-N}^{\infty} g(m+\sigma)R(m;\sigma)t^m \\ &\ll M\sum_{m=-N}^{\infty}|g(m+\sigma)|\,|t^{m+\rho}|\end{aligned}$$

and, applying the consideration of Proposition 3, to the last series, we can arrive at the same conclusion as above.

Now, all preparations having been made, we return to (3.7). Let t tend to infinity in the sector

$$\bigcap_{k=1}^{\nu} S(\lambda_k e^{-2\pi i l_k}),$$

where the l_k are integers and $S(\lambda e^{-2\pi i l})$ denotes the sector (3.10). Then from Proposition 3 and (3.22) we have

$$\begin{aligned}
-\frac{t^{\rho_i}}{2\pi i}&\int_L G_i(z)\left(\frac{\pi}{\sin \pi z}\right)(-t)^z dz \\
&= \sum_{k=1}^{\nu} T_i^k \sum_{s=0}^{\sigma} H^k(s) x_i^k(t, s-N)t^{-N} + \sum_{k=1}^{\nu} T_i^k I_i^k(t;\sigma) \\
&\sim \sum_{k=1}^{\nu} T_i^k e^{2\pi i(\rho_i-\mu_k)l_k} \sum_{s=0}^{\sigma} H^k(s)\{e^{\lambda_k t} t^{\mu_k-s+N} + O(t^{\rho_i})\} t^{-N} \\
&\quad + \sum_{k=1}^{\nu} T_i^k \{O(e^{\lambda_k t} t^{\mu_k+2-\sigma}) + O(t^{\mu_k+2-\sigma})\} \\
&\sim \sum_{k=1}^{\nu} T_i^k e^{2\pi i(\rho_i-\mu_k)l_k} e^{\lambda_k t} t^{\mu_k} \{\sum_{s=0}^{\sigma-2} H^k(s) t^{-s} + O(t^{2-\sigma})\} \\
&\quad + O(t^{\rho_i-N}) + O(t^{\mu_k+2-\sigma}).
\end{aligned}$$

Since (3.3) and (3.15) imply that

$$\begin{aligned}
&w + \operatorname{Re}\rho_i < \operatorname{Re}\mu_k + 2 - \sigma < w + 1 + \operatorname{Re}\rho_i < \operatorname{Re}\gamma_j - \nu N_j + 1, \\
&w + \operatorname{Re}\rho_i < \operatorname{Re}\rho_i - N < w + 1 + \operatorname{Re}\rho_i < \operatorname{Re}\gamma_j - \nu N_j + 1 \\
&\qquad\qquad\qquad\qquad\qquad\qquad (j = 1, 2, \dots, n-\nu),
\end{aligned}$$

the last two O-terms can be replaced by $o(t^{\gamma_j-\nu N_j+1})(j=1, 2,\dots, n-\nu)$. Combining the above result with (3.5), we obtain the final result that when t tends to infinity in the sector

$$S(e^{-2\pi i l}) \cap \bigcap_{k=1}^{\nu} S(\lambda_k e^{-2\pi i l_k}), \tag{3.23}$$

the solution $y_i(t)$ admits the asymptotic expansion

$$\begin{aligned}
y_i(t) \sim &\sum_{k=1}^{n-\nu} T_i^k e^{2\pi i(\rho_i-\gamma_k)l} t^{\gamma_k} \{\sum_{s=0}^{N_k-1} \hat{H}^k(\nu s) t^{-\nu s} + o(t^{-\nu(N_k-1)})\} \\
&+ \sum_{k=1}^{\nu} T_i^k e^{2\pi i(\rho_i-\mu_k)l_k} e^{\lambda_k t} t^{\mu_k} \{\sum_{s=0}^{\sigma-2} H^k(s) t^{-s} + O(t^{2-\sigma})\},
\end{aligned}\tag{3.24}$$

where the series are exactly formal series derived in §2. This is the required connection formula between the solution $y_i(t)$ and the formal solutions $\hat{y}^k(t)(k=1, 2,\dots, n-\nu)$ and $y^k(t)$ $(k=1, 2,\dots, \nu)$.

We here restate our main result in the following

THEOREM. *Assume that*

(i) $\rho_i \not\equiv \rho_j \pmod 1$ $(i\neq j;\ i, j=1, 2,\dots, n)$,

(ii) $\gamma_i \not\equiv \gamma_j \pmod \nu$ $(i\neq j;\ i, j=1, 2,\dots, n-\nu)$,

(iii) $\rho_i \not\equiv \gamma_j \pmod 1$ $(i=1, 2,\dots, n;\ j=1, 2,\dots, n-\nu)$,

(iv) $|\lambda_i| < |\lambda_i - \lambda_j|$ $(i \neq j;\ i, j=1, 2,\dots, \nu)$.

Then the convergent solution $y_i(t)$ $(i=1, 2,\dots, n)$ *of the form* (1.8) *admit the asymptotic expansion*

$$(3.25)\qquad y_i(t) \sim \textstyle\sum_{k=1}^{n-\nu} \hat{T}_i^k e^{2\pi i(\rho_i-\gamma_k)l} \hat{y}^k(t) + \sum_{k=1}^{\nu} T_i^k e^{2\pi i(\rho_i-\mu_k)l_k} y^k(t)$$

as $t \to \infty$ *in the sector* (3.23), *where the* $\hat{y}^k(t)$ *and the* $y^k(t)$ *are formal solutions of the form* (2.1) *and* (2.2), *respectively. The Stokes multipliers* $\hat{T}_i^k$ *and* T_i^k *are given explicitly by* (3.6) *and* (2.23), *respectively.*

It is remarked that all conditions (i)–(iv) are not essential. Without (i) and (ii), there appear convergent solutions and formal solutions involving logarithmic terms. By a slight modification of above investigations we can solve connection problems for such cases. Also the conditions (iii) and (iv) can be relaxed by a little more detailed analysis. (See [4].)

The above theorem, as a matter of course, gives the behavior of $y_i(t)$ on the Riemann surface of logarithm. For example, the asymptotic expansion (3.25) on the sheet $|\arg t| \leqq \pi$ can be read as follows: Assume, for simplicity, that $\arg\cdot(\beta^{1/\nu})=0$ and let N be such an integer that $\nu/4+1>N\geqq \nu/4$. Then it can be seen from (2.8) and (3.10) that the above sheet is included in the sector $S(\lambda_k)$ for $k=1, 2,\dots, N$ and $S(\lambda_k) \cup S(\lambda_k e^{-2\pi i})$ for $k=N+1, N+2,\dots, \nu$. Then we have

$$\begin{aligned} y_i(t) &\sim \textstyle\sum_{k=1}^{n-\nu} \hat{T}_i^k \hat{y}^k(t) + \sum_{k=1}^{\nu} T_i^k y^k(t) \qquad (-\pi < \arg t < -\pi/2 + 2\pi/\nu), \\ &\sim \textstyle\sum_{k=1}^{n-\nu} \hat{T}_i^k \hat{y}^k(t) + \sum_{k=1}^{\nu-l} T_i^k y^k(t) + \sum_{k=\nu-l+1}^{\nu} T_i^k e^{2\pi i(\rho_i-\mu_k)} y^k(t) \\ &\quad (-\pi/2 + (2\pi/\nu)l \leqq \arg t < -\pi/2 + (2\pi/\nu)(l+1);\ l = 1, 2,\dots, \nu-N), \\ &\sim \textstyle\sum_{k=1}^{n-\nu} \hat{T}_i^k y^k(t) + \sum_{k=1}^{N} T_i^k y^k(t) + \sum_{k=N+1}^{\nu} T_i^k e^{2\pi i(\rho_i-\mu_k)} y^k(t) \\ &\qquad\qquad (3\pi/2 - (2\pi/\nu)N \leqq \arg t \leqq \pi). \end{aligned}$$

Also, using the above theorem, we can take out interesting solutions of the generalized hypergeometric equation (1.2), for instance, particular solutions which have algebraic behaviors in some sectors.

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and
Faculty of Engineering
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