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Half-spherical means and boundary behaviour of subharmonic functions in half-spaces

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1. Introduction and main results

Naïm [9], Chapitre IV, has shown how the minimal fine topology and Martin boundary may be used to obtain elegant generalizations of the maximum principle and the solution to the Dirichlet problem. The purpose of this paper is to show that remarkably similar results can be obtained for half-spherical means in the Euclidean half-space $\Omega = \boldsymbol{R}^{n-1} \times (0, +\infty)$.

If E is a subset of $\boldsymbol{R}^n$, then its closure and boundary will be denoted by $\bar{E}$ and ∂E respectively. We represent points of $\boldsymbol{R}^n$ by X, Y or P, and use O for the origin; sometimes it will be convenient to write $X = (X', x_n)$, where $X' \in \boldsymbol{R}^{n-1}$. The open ball of radius r centred at P will be abbreviated to $B(P, r)$ and, using σ to represent surface area measure, we write c_n for $\sigma(\partial B(O, 1))$. Another important constant is

$$\gamma_2 = (2\pi)^{-1}, \quad \gamma_n = \{(n-2)c_n\}^{-1} \qquad (n \geq 3).$$

We adjoin the isolated point ∞ to the usual topology on $\bar{\Omega}$ and write $\bar{\Omega}^*$ for $\bar{\Omega} \cup \{\infty\}$. The set $\partial\Omega \cup \{\infty\}$ with the topology induced on it by $\bar{\Omega}^*$ will be abbreviated to $\partial^*\Omega$. Provided the integrals exist, we can now define the half-spherical means

$$N(f\colon P, r) = r^{-n-1} \int_{\partial B(P,r) \cap \Omega} x_n f(X) d\sigma(X)$$

for $P \in \partial\Omega$, and

$$N(f\colon \infty, r) = r^{-n} N(f\colon O, r^{-1}).$$

Let G denote the Green kernel for Ω. From well-known inequalities for G (see, for example, [10], Lemma 1), it follows that the function $G(X, Y)/\{x_n y_n\}$ has an extension $G^*(X, Y)$ to $\bar{\Omega} \times \bar{\Omega}$ which is jointly continuous (in the extended sense at points of the diagonal), and that

$$\gamma_n G^*(X, Y) = 2|X - Y|^{-n}/c_n \qquad (Y \in \partial\Omega,\ X \in \bar{\Omega} \backslash \{Y\}).$$

From the Riesz decomposition theorem and [7], Theorem 2.25, it is now easy

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to deduce that there is a one-to-one correspondence between positive superharmonic functions u in Ω, and members ν of the class M^+ of non-zero Borel measures on $\bar{\Omega}^*$ for which

$$\int_{\bar{\Omega}} (1+|X|)^{-n} d\nu(X) < +\infty.$$

The correspondence is given by

$$u(X) = G\nu(X) = x_n\{\nu(\{\infty\}) + \gamma_n \int_{\bar{\Omega}} G^*(X, Y) d\nu(Y)\}. \tag{1}$$

We shall use v to denote a fixed positive superharmonic function in Ω with corresponding measure Λ on $\bar{\Omega}^*$. It follows from the definition of G^* that, if A is a Borel subset of Ω, then (in the distributional sense)

$$\int_A x_n^{-1} d\Lambda(X) = -(\Delta v)(\chi_A),$$

and, denoting by h the greatest harmonic minorant of v in Ω, we have

$$h(X) = x_n\{\Lambda(\{\infty\}) + 2c_n^{-1} \int_{\partial\Omega} |X-Y|^{-n} d\Lambda(Y)\}.$$

Throughout this paper, v and h will have these meanings. Recall (for example, see [8], Theorem 4) that, if $P \in \partial\Omega$, then $N(v\colon P, r)$ is real-valued and decreasing as a function of r. If s is subharmonic with a non-negative harmonic majorant in Ω, it is easy to see that $N(s\colon P, r)$ must also be real-valued. In imitation of the theory of the fine Dirichlet problem (see [6], pp. 144, 145) we say that a point $P \in \partial^*\Omega$ is mean weak v-regular if $N(v\colon P, r) \to +\infty$ as $r \to 0+$. The following is a type of maximum principle.

THEOREM 1. *Let s be a subharmonic function in Ω such that s^+ has a harmonic majorant in Ω. If*

$$\liminf_{r\to 0+} N(s\colon P, r)/N(v\colon P, r) < +\infty \qquad \forall P \in \partial^*\Omega \tag{2}$$

and

$$\liminf_{r\to 0+} N(s\colon P, r)/N(v\colon P, r) \leq 0 \qquad \text{for a.e. } (\Lambda) P \in \partial^*\Omega, \tag{3}$$

then $s \leq 0$ in Ω.

This generalizes a result of Armitage (see [4], Theorem 1), which deals only with the case where both s and v are harmonic, requires more than (3) when $P = \infty$, and includes the redundant hypothesis that every $P \in \partial^*\Omega$ is mean weak v-regular. Following Armitage, we can take $v(X) = 1 + x_n$, so that Λ consists of $(n-1)$-dimensional Lebesgue measure on $\partial\Omega$ together with the Dirac measure at ∞. Since $rN(1: P, r)$ is independent of r, we have the following improvement of [4], Theorem 3.

COROLLARY. *Let s be a subharmonic function in Ω such that s^+ has a harmonic majorant in Ω. If*

$$\liminf_{r\to 0+} rN(s: P, r)$$

is less than $+\infty$ for all $P \in \partial\Omega$ and non-positive for a.e. $(\sigma)P \in \partial\Omega$, and

$$\liminf_{r\to +\infty} N(s: O, r) \leq 0,$$

then $s \leq 0$ in Ω.

Now let h be non-zero. Armitage ([3], p. 236) has already remarked that, even if s is harmonic, the limiting behaviour of

$$N(s: P, r)/N(h: P, r)$$

as $r \to 0+$ does not coincide with the limiting behaviour of s/h at P in the minimal fine topology. However, Theorem 1 is reminiscent of [9], Théorème 22 and suggests that many of the boundary limit theorems involving the minimal fine topology (see, for example, [6], XVI, 11–16) can be rewritten without much difficulty in terms of half-spherical means. We shall be content to give an analogue of the fine Dirichlet problem (see [6], XVI, §5).

Let $\mathscr{A}$ be the Alexandroff point for $\bar{\Omega}$, and recall that the Martin compactification for Ω and Alexandroff compactification for $\bar{\Omega}$ are equivalent. By setting $f(\infty) = f(\mathscr{A})$, any function f on the Martin boundary Δ_1 can be regarded as a function on $\partial^*\Omega$. If $|f(X)| d\Lambda(X)$ defines a member of M^+, then we say that f is h-resolutive; in particular, if f is continuous on Δ_1, then it is easy to see that f is h-resolutive. For the equivalence of this definition to that involving upper and lower classes, we refer the reader to [6], Theorems XVI, 3 and 9. If f is h-resolutive, we use $f^+\Lambda$ and $f^-\Lambda$ to denote the measures on $\partial^*\Omega$ defined by

$$df^+\Lambda(X) = f^+(X) d\Lambda(X) \quad \text{and} \quad df^-\Lambda(X) = f^-(X) d\Lambda(X),$$

and we abbreviate $Gf^+\Lambda - Gf^-\Lambda$ to $Gf\Lambda$. We shall say that $P \in \partial^*\Omega$ is mean h-regular if, for every function f continuous on Δ_1, we have

$$N(Gf\Lambda: P, r)/N(h: P, r) \longrightarrow f(P) \qquad (r \to 0+). \tag{4}$$

Further, P is said to be mean strong h-regular if, for every h-resolutive function f which is continuous (in the extended sense) at P, (4) holds.

THEOREM 2. *If $P \in \partial^*\Omega$, then the following are equivalent:*
(i) *P is mean weak h-regular;*

(ii) *P is mean h-regular*;
(iii) *P is mean strong h-regular*;
(iv) $\int_{\partial\Omega} |P-Y|^{-n} d\Lambda(Y) = +\infty$
if $P \in \partial\Omega$, *or*

$$\Lambda(\partial\Omega) + \Lambda(\{\infty\})\cdot(+\infty) = +\infty$$

if $P=\infty$. (We adopt the convention that $0\cdot(+\infty)=0$.)

COROLLARY. *The non mean h-regular points of* $\partial^*\Omega$ *form a set of zero* Λ*-measure.*

The proofs of Theorems 1 and 2, which rely on results to be given in §2, may be found in §§3 and 4 respectively.

2. Preparatory results

We introduce the class M comprising those signed measures μ defined on the relatively compact Borel subsets A of $\bar{\Omega}^*$ by

$$\mu(A) = \mu_1(A) - \mu_2(A)$$

for some $\mu_1, \mu_2 \in M^+$, and write $G\mu$ for $G\mu_1 - G\mu_2$ (we allow both μ_1 and μ_2 to have infinite total mass). If s is subharmonic in Ω and s^+ has a harmonic majorant in Ω, it follows from two applications of (1) that $s=G\mu$ for some $\mu \in M$. It will be convenient to write

$$\mu(P, r) = \mu(B(P, r) \cap \bar{\Omega}) \qquad (r \geq 0).$$

LEMMA 1. *If* $\mu \in M$ *and* $r>0$, *then*

$$N(G\mu\colon P, r) = \int_r^\infty t^{-n-1}\mu(P, t)dt + c_n\mu(\{\infty\})/(2n) \tag{5}$$

when $P \in \partial\Omega$, *and*

$$N(G\mu\colon \infty, r) = r^{-n}\left\{\int_{r^{-1}}^\infty t^{-n-1}\mu(O, t)dt + c_n\mu(\{\infty\})/(2n)\right\} \tag{6}$$

$$= \int_1^\infty t^{-n-1}\mu(O, r^{-1}t)dt + c_n r^{-n}\mu(\{\infty\})/(2n). \tag{7}$$

Since $N(x_n\colon P, r)$ has the constant value $c_n/(2n)$, it is sufficient to establish that

$$N(G\mu\colon P, r) = \int_r^\infty t^{-n-1}\mu(P, t)dt \tag{8}$$

when $\mu \in M^+$ and $\mu(\{\infty\})=0$; equations (5) and (6) will then be immediate, and (7) follows by a simple substitution. Let μ_1 and μ_2 denote respectively the restrictions of μ to $\partial\Omega$ and Ω. Equation (8) with $\mu=\mu_1$ has been proved in [3], Lemma 3. Let $I_{f,r}$ denote the Dirichlet solution in $B(P, r) \cap \Omega$ of the function equal to f on $\partial B(P, r) \cap \Omega$ and 0 on $\bar{B}(P, r) \cap \partial\Omega$. Then, since $G\mu_2$ is a potential in Ω, it is easy to check that the same is true of the function u given by

$$u(X) = \begin{cases} I_{G\mu_2, r}(X) & \text{if} \quad X \in B(P, r) \cap \Omega \\ G\mu_2(X) & \text{if} \quad X \in \Omega \backslash B(P, r) . \end{cases}$$

Further, u continuously vanishes on $B(P, r) \cap \partial\Omega$ (see [1], Theorem 2). It now follows from two results of Armitage ([2], Theorems 4 and 8) that

$$(2n)^{-1} c_n \lim_{X \to O} u(X)/x_n = \int_r^R t^{-n-1} \mu_2(P, t) dt + o(1) \qquad (R \to +\infty).$$

Since the left hand side of this equation can be written as $N(G\mu_2\colon P, r)$ (see [2], Lemma C(i)), the result is proved.

The following result is a generalization of [3], Theorem 1.

THEOREM 3. *Let $\mu \in M$ and $P \in \partial^*\Omega$ be mean weak v-regular. Then*

$$\text{(i)} \quad \liminf_{r \to 0+} \frac{\mu(P, r)}{\Lambda(P, r)} \leq \liminf_{r \to 0+} \frac{N(G\mu\colon P, r)}{N(v\colon P, r)}$$

$$\leq \limsup_{r \to 0+} \frac{N(G\mu\colon P, r)}{N(v\colon P, r)} \leq \limsup_{r \to 0+} \frac{\mu(P, r)}{\Lambda(P, r)}$$

if $P \in \partial\Omega$, and

$$\text{(ii)} \quad \liminf_{r \to 0+} \frac{\mu(\{\infty\}) + r^n[\mu(O, r^{-1}) - \mu(O, r_o)]}{\Lambda(\{\infty\}) + r^n[\Lambda(O, r^{-1}) - \Lambda(O, r_o)]}$$

$$\leq \liminf_{r \to 0+} \frac{N(G\mu\colon \infty, r)}{N(v\colon \infty, r)} \leq \limsup_{r \to 0+} \frac{N(G\mu\colon \infty, r)}{N(v\colon \infty, r)}$$

$$\leq \limsup_{r \to 0+} \frac{\mu(\{\infty\}) + r^n[\mu(O, r^{-1}) - \mu(O, r_o)]}{\Lambda(\{\infty\}) + r^n[\Lambda(O, r^{-1}) - \Lambda(O, r_o)]}$$

for all $r_o \geq 0$ if $P = \infty$.

For the proof of (i) we follow an argument similar to that of [3], Theorem 1. First observe that, since P is mean weak v-regular, it follows from (5) that $\Lambda(P, r)$ is non-zero for all $r > 0$, and so all the quotients in (i) are defined. Next, since $-\mu$ is also in M, we need prove only the final inequality, and can assume that the last upper limit c, say, is not $+\infty$. Let $c < C < +\infty$. Then there exists $R > 0$ such that

$$\mu(P, r) < C\Lambda(P, r) \qquad (r \in (0, R)).$$

It follows from the mean weak v-regularity of P and (5) that, as $r \to 0+$, we have

$$\begin{aligned} N(G\mu: P, r) &= \int_r^R t^{-n-1}\mu(P, t)dt + O(1) \\ &\leq C\int_r^R t^{-n-1}\Lambda(P, t)dt + O(1) \\ &= \{C + o(1)\}N(v: P, r), \end{aligned}$$

and (i) is proved.

In the case of (ii), ∞ is mean weak v-regular and so, if $\Lambda(\{\infty\})=0$, it follows from (7) that, for every $r_0 \geq 0$, there exists $R > r_0$ such that $\Lambda(P, R) > \Lambda(P, r_0)$. Hence the quotients in (ii) are defined for all sufficiently small r. Again we need only deal with the final inequality, and can assume that the last upper limit c, say, is not $+\infty$. Fix $r_0 \geq 0$.

First suppose that at least one of the numbers $\Lambda(\{\infty\})$ and $\mu(\{\infty\})$ is non-zero. It follows from (6) that

$$\frac{N(G\mu: \infty, r)}{N(v: \infty, r)} = \frac{o(1) + \mu(\{\infty\})}{o(1) + \Lambda(\{\infty\})} \qquad (r \to 0+). \tag{9}$$

Now observe that

$$r^n \Lambda(O, r^{-1}) \leq n \int_{r^{-1}}^{\infty} t^{-n-1}\Lambda(O, t)dt \longrightarrow 0 \qquad (r \to 0+)$$

and also (by considering separately the measures μ_1 and μ_2 used to define $\mu \in M$)

$$r^n \mu(O, r^{-1}) \longrightarrow 0 \qquad (r \to 0+).$$

Hence

$$\frac{\mu(\{\infty\}) + r^n[\mu(O, r^{-1}) - \mu(O, r_0)]}{\Lambda(\{\infty\}) + r^n[\Lambda(O, r^{-1}) - \Lambda(O, r_0)]} = \frac{\mu(\{\infty\}) + o(1)}{\Lambda(\{\infty\}) + o(1)} \qquad (r \to 0+),$$

and so (ii) is proved in this case.

We are left with the case where $\Lambda(\{\infty\})$ and $\mu(\{\infty\})$ are both zero. Let $c < C < +\infty$. Then there exists $R > 0$ such that

$$\mu(O, r^{-1}) - \mu(O, r_0) < C[\Lambda(O, r^{-1}) - \Lambda(O, r_0)] \qquad (r \in (0, R)).$$

It follows from the mean weak v-regularity of ∞ and (6) that, as $r \to 0+$, we have

$$N(G\mu: \infty, r) = r^{-n}\int_{r^{-1}}^{\infty} t^{-n-1}[\mu(O, t) - \mu(O, r_0)]dt + O(1)$$

$$\leq Cr^{-n}\int_{r^{-1}}^{\infty} t^{-n-1}[\Lambda(O, t) - \Lambda(O, r_o)]dt + O(1)$$

$$= \{C+o(1)\}N(v\colon \infty, r),$$

and (ii) is proved.

3. Proof of Theorem 1

We shall require the following result, due to Watson ([11], Theorem 1), which relies on a theorem of Besicovitch ([5], Theorem 3).

THEOREM A. *Let λ be a measure on $\boldsymbol{R}^n$ such that $\lambda(B(X, r))>0$ for each $X \in \boldsymbol{R}^n$ and each $r>0$. If μ is a signed measure on $\boldsymbol{R}^n$ and*

$$\liminf_{r\to 0+} \mu(B(X, r))/\lambda(B(X, r))$$

is less than $+\infty$ for all $X \in \boldsymbol{R}^n$ and is non-positive a.e. (λ), then $-\mu$ is a measure on $\boldsymbol{R}^n$.

The proof of Theorem 1 will now the given. Although both this theorem and the result of Armitage which it generalizes rely on Theorem A, the way in which we apply the latter result is quite different.

Since s^+ has a harmonic majorant in Ω, there exists $\mu \in M$ such that $s=G\mu$. We define the signed measure μ_o on the bounded Borel subsets A of $\boldsymbol{R}^n$ by $\mu_o(A)=\mu(A\cap\bar{\Omega})$. Thus

$$\mu_o(\boldsymbol{R}^n\backslash\bar{\Omega}) = 0 \tag{10}$$

and, if $A \subset \Omega$,

$$\int_A x_n^{-1} d\mu_o(X) = -(\Delta s)(\chi_A). \tag{11}$$

Using τ to denote n-dimensional Lebesgue measure, we define λ on the Borel subsets A of $\boldsymbol{R}^n$ by

$$\lambda(A) = \Lambda(A\cap\bar{\Omega}) + \tau(A).$$

For each $P\in\partial\Omega$ we also define τ_p on the Borel subsets A of $\bar{\Omega}^*$ by

$$\tau_P(A) = \tau(A\cap B(P, 1)).$$

Using b_n to represent the volume of an n-dimensional unit ball, it follows from (5) that

$$N(v+G\tau_P\colon P, r) \geq N(G\tau_P\colon P, r) = \int_r^{\infty} t^{-n-1}\tau_P(P, t)dt$$

$$\geq \frac{1}{2} b_n \int_r^1 t^{-1} dt \longrightarrow +\infty \qquad (r \to 0+),$$

and so P is mean weak $(G\Lambda + G\tau_P)$-regular. From Theorem 3 (i) (with v replaced by $G\Lambda + G\tau_P$) we deduce that

$$\liminf_{r\to 0+} \frac{\mu_o(B(P, r))}{\lambda(B(P, r))} \leq \max\left\{0, \liminf_{r\to 0+} \frac{\mu(P, r)}{\Lambda(P, r) + \tau_p(P, r)}\right\}$$

$$\leq \max\left\{0, \liminf_{r\to 0+} \frac{N(s: P, r)}{N(v + G\tau_P: P, r)}\right\}$$

$$\leq \max\left\{0, \liminf_{r\to 0+} \frac{N(s: P, r)}{N(v: P, r)}\right\}.$$

Thus, using (3),

$$\liminf_{r\to 0+} \frac{\mu_o(B(P, r))}{\lambda(B(P, r))} \leq 0 \tag{12}$$

for a.e. $(\Lambda)P \in \partial\Omega$, and (2) shows that the lower limit is less than $+\infty$ everywhere on $\partial\Omega$. If $P \in \Omega$, then (12) follows from (11) while, if $P \in \boldsymbol{R}^n \backslash \bar{\Omega}$, then (10) shows that (12) still (trivially) holds. Since every open ball in $\boldsymbol{R}^n$ has positive n-dimensional Lebesgue measure, we can apply Theorem A to deduce that $-\mu_o$ is a measure on $\boldsymbol{R}^n$, and so $\mu \leq 0$ on $\bar{\Omega}$. Hence

$$s(X) = G\mu(X) \leq x_n \mu(\{\infty\}) \qquad (X \in \Omega). \tag{13}$$

If $\mu(\{\infty\}) > 0$ and $\Lambda(\{\infty\}) = 0$ (respectively $\Lambda(\{\infty\}) > 0$) then (9) and (2) (respectively (9) and (3)) yield a contradiction. Thus $\mu(\{\infty\}) \leq 0$ and the result follows from (13).

4. Proof of Theorem 2 and Corollary

(i)⇒(iii). Suppose P is mean weak h-regular and f is h-resolutive and continuous (in the extended sense) at P. If $P \in \partial\Omega$, then

$$\{f^+ \Lambda(P, r) - f^- \Lambda(P, r)\} / \Lambda(P, r) \longrightarrow f(P) \qquad (r \to 0+),$$

and the result follows from Theorem 3 (i). If $P = \infty$ and $\Lambda(\{\infty\}) > 0$, then (4) follows from (9). Otherwise

$$\Lambda(\{\infty\}) = 0 = f(\infty)\Lambda(\{\infty\}).$$

If f is (finite and) continuous at ∞, then, for any $\varepsilon > 0$, there exists $r_o > 0$ such that

$$|f(\infty) - f(Y)| < \varepsilon \qquad (Y \in \partial\Omega \backslash B(O, r_o)),$$

and it follows from Theorem 3 (ii) that

$$|f(\infty) - N(Gf\Lambda\colon \infty, r)/N(h\colon \infty, r)| < \varepsilon$$

for all sufficiently large r, as required. An analogous argument deals with the case where $f(\infty) = \pm\infty$.

(iii)$\Rightarrow$(ii). Immediate.

(ii)$\Rightarrow$(i). Suppose P is not mean weak h-regular. If $P \in \partial\Omega$, it follows from (5) that $N(h\colon P, r)$ increases to a finite positive limit as r decreases to 0. In particular, $\Lambda(\{P\}) = 0$ and the function f given by

$$f(X) = \begin{cases} \min\{1, |X-P|\} & \text{if } X \in \partial\Omega \\ 1 & \text{if } X = \mathscr{A} \end{cases}$$

is continuous on Δ_1 and positive a.e. (Λ). It follows that $Gf\Lambda$ is positive and so

$$N(Gf\Lambda\colon P, r)/N(h\colon P, r) \longrightarrow l > 0 = f(P) \qquad (r \to 0+),$$

contradicting the mean h-regularity of P. If $P = \infty$, we apply an analogous argument, using instead the function

$$f(X) = \begin{cases} \min\{1, |X|^{-1}\} & \text{if } X \in \partial\Omega \\ 0 & \text{if } X = \mathscr{A}. \end{cases}$$

(i)$\Leftrightarrow$(iv). Rewriting (5) as

$$N(G\mu\colon P, r) = n^{-1}\left\{\int_r^\infty \mu(P, t)\,d(-t^{-n}) + \frac{1}{2}c_n\mu(\{\infty\})\right\},$$

we can use integration by parts to obtain

$$N(h\colon P, r) = n^{-1}\left\{\int_{\partial\Omega} \min\{|P-Y|^{-n}, r^{-n}\}\,d\Lambda(Y) + \frac{1}{2}c_n\Lambda(\{\infty\})\right\}$$

when $P \in \partial\Omega$, and

$$\begin{aligned} N(h\colon \infty, r) &= r^{-n}N(h\colon O, r^{-1}) \\ &= n^{-1}\left\{\int_{\partial\Omega} \min\{r^{-n}|Y|^{-n}, 1\}\,d\Lambda(Y) + \frac{1}{2}c_n r^{-n}\Lambda(\{\infty\})\right\}. \end{aligned}$$

The result now follows on letting $r \to 0+$ and appealing to the monotone convergence theorem.

Finally, we give the proof of the Corollary. In the terminology of Naïm, condition (iv) of Theorem 2 is equivalent to the Θ-potential associated with h being valued $+\infty$ at P (see the expressions given for Θ in [9], p. 239). From

[9], p. 237, Application 2, this is true for a.e. $(\Lambda)P \in \partial^*\Omega$, and so the result is proved.

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