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On Lie algebras with finiteness conditions

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In this paper we consider three types of questions concerning the ideal and subideal structure of Lie algebras with certain finiteness conditions. First we consider the question of finding conditions under which the join of subideals of a Lie algebra is a subideal. We prove (Theorem 1.2) that when L is a Lie algebra over a field of characteristic zero and $\{H_\lambda \mid \lambda \in \Lambda\}$ is a set of subideals of L with J their join, J is a subideal of L if and only if the set of subideals of L lying in J has a maximal element. We also find another condition under which the join of two subideals of a Lie algebra is a subideal (Theorem 1.5).

The second problem is to investigate the structure of Lie algebras with a certain chain condition on subideals using the notion of prime ideals and prime algebras (defined by analogy with associative rings). In particular we prove (Theorem 2.1) that when L is a Lie algebra over any field and $\mathfrak{X}$ is one of max-$\lhd^n$ $(n \geq 2)$, max-si, min-$\lhd^n$, min-si, $L \in \mathfrak{X}$ if and only if

(i) $\sigma(L)$ is a finite-dimensional soluble ideal of L.

(ii) $L/\sigma(L)$ is a subdirect sum of a finite number of prime algebras in $\mathfrak{X}$.

$\sigma(L)$ denotes a generalization of soluble radical.

Thirdly, we generalize the minimal condition on ideals, leading to a new class of quasi-Artinian algebras which possesses several of the main properties of min-$\lhd$. We prove (Theorems 3.2, 3.3) that the class of quasi-Artinian algebras is Q-closed and that a locally nilpotent quasi-Artinian Lie algebra is soluble.

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1. The join of subideals

It is well-known that the join of two subideals of a Lie algebra need not be a subideal (see [1, Lemma 2.1.11]). This raises the question of finding conditions under which the join is a subideal. The same question arises in group theory. Wielandt [9, Theorem 2.10.5] has shown that when $\{H_\lambda \mid \lambda \in \Lambda\}$ is a set of subnormal subgroups of a group G and J is their join, J is subnormal in G if and only if the set of subnormal subgroups of G lying in J contains a maximal member. We obtain a similar result for Lie algebras. In particular we prove an analogue of Wielandt's theorem for Lie algebras over a field of characteristic zero. We also find another condition under which the join of subideals is a subideal.

Notation for Lie algebras will follow Amayo and Stewart [1]. In particular "$\leq$", "$\lhd$", "$\lhd^n$", "si" denote respectively the relation subalgebra, ideal, n-step subideal, subideal (see [1, pp. 2, 9, 10]). Triangular brackets $\langle\ \rangle$ denote the subalgebra generated by their contents. Let L be a Lie algebra over any field. We write $L\in$ max-si if L satisfies the maximal condition on subideals: Every non-empty collection of subideals of L has a maximal element.

Let L be a Lie algebra over a field F of characteristic zero and let $F_0 = F\langle t\rangle$ be the field of formal power series in the indeterminate t. Let $L^\dagger$ be the set of all formal power series $x=\sum_{r=n}^{\infty} x_r t^r$, $x_r \in L$ and $n=n(x)\in \boldsymbol{Z}$. Let $y=\sum y_r t^r \in L^\dagger$, and define addition, multiplication, and multiplication of elements of $L^\dagger$ by scalars from F_0 according to the rules:

$$x + y = \sum (x_r + y_r)t^r,$$

$$[x, y] = \sum z_r t^r, \quad \text{where} \quad z_r = \sum_{i+j=r} [x_i, y_j],$$

$$\alpha x = \sum c_r t^r, \quad \text{where} \quad c_r = \sum_{i+j=r} \alpha_i x_j.$$

It is easy to verify that this makes $L^\dagger$ into a Lie algebra over F_0. Let $M\leq L$ and $M^\dagger$ be the set of all elements $x=\sum x_r t^r \in L^\dagger$ with $x_r\in M$. Then clearly $M^\dagger$ is an F_0-subalgebra of $L^\dagger$. Now let M be a subset of $L^\dagger$. We define (see [1, p. 80]) a subset $M^\downarrow$ of L by $M^\downarrow = \{x\in L \mid x=0$ or x is the first coefficient of some element of $M\}$.

Now we prove the following, which is the Lie algebra analogue of [9, Lemma 2.10.4].

LEMMA 1.1. *Let L be a Lie algebra over a field of characteristic zero and let $S\leq L$. Let $\boldsymbol{B}=\{B \mid B\leq S,\ B$ si $L\}$ and let H be a maximal element of $\boldsymbol{B}$. Then $H\lhd S$ and $H\geq B$ for every $B\in \boldsymbol{B}$.*

PROOF. Let H have subideal index m in S and suppose that $m\geq 2$. Denote the i-th ideal closure of H in S by H_i. It follows that there exists $x\in H_{m-2}$ with $[H, x]\nsubseteq H$. By [1, Lemma 4.1.1(b)] $H^\dagger \lhd^m S^\dagger$ and $H^\dagger \leq S^\dagger$. Let $\theta = \exp(t \operatorname{ad} x)$. Then $H^{\dagger\theta}$ si $L^\dagger$ and $H^{\dagger\theta}\subseteq H^\dagger_{m-1}$. But $H^\dagger \lhd H^\dagger_{m-1}$, hence $H^{\dagger\theta}$ idealises $H^\dagger$ and $H^\dagger + H^{\dagger\theta}$ si $L^\dagger$ and $H^\dagger + H^{\dagger\theta} \leq S^\dagger$. By [1, Lemma 4.1.2 (b),(f)] we have $(H^\dagger + H^{\dagger\theta})^\downarrow$ si L and $(H^\dagger + H^{\dagger\theta})^\downarrow \leq S$. Hence $(H^\dagger + H^{\dagger\theta})^\downarrow \in \boldsymbol{B}$. By [1, Lemma 4.1.2 (c), (f)] $H^{\dagger\downarrow}=H$ and $H\subseteq (H^\dagger + H^{\dagger\theta})^\downarrow$, but H is maximal, hence $(H^\dagger + H^{\dagger\theta})^\downarrow = H$. Now take $h\in H$ such that $[h, x]\notin H$. Then $h^\theta - h = [h, x]t + \cdots$ and therefore $\{h^\theta - h\}^\downarrow \in (H^\dagger + H^{\dagger\theta})^\downarrow = H$ which is a contradiction. Therefore $H\lhd S$ and clearly $H\geq B$ for every $B\in \boldsymbol{B}$.

THEOREM 1.2. *Let L be a Lie algebra over a field of characteristic zero* *and* $H_\lambda \operatorname{si} L$, $\lambda \in \Lambda$. *Let* $J = \langle H_\lambda \mid \lambda \in \Lambda \rangle$ *and* $\boldsymbol{B} = \{B \mid B \leq J,\ B \operatorname{si} L\}$. *Then* $J \operatorname{si} L$ *if and only if* $\boldsymbol{B}$ *has a maximal element.*

PROOF. The only if part is clear. To prove the if part, let H be a maximal element in $\boldsymbol{B}$. Then by Lemma 1.1, $H \lhd J$ and each $H_\lambda \leq H$ so $J \leq H$ and $J = H$. Therefore $J \operatorname{si} L$.

COROLLARY. *Let L be a Lie algebra over a field of characteristic zero and let* $J = \langle H_\lambda \mid H_\lambda \operatorname{si} L,\ \lambda \in \Lambda \rangle$. *Then J si L if one of the following holds.*

(i) $L \in$ max-si.

(ii) $J \in$ max-si.

The first part of this corollary is a special case of [6, Theorem 8]. Next we prove the following, which is the Lie algebra analogue of a well-known result in group theory.

THEOREM 1.3. *Let L be a Lie algebra over a field of characteristic zero and H, K be subideals of L. Suppose that the set of subideals of L lying between H and $J = \langle H, K \rangle$ contains at least one maximal member. Then* $J \operatorname{si} L$.

PROOF. Without loss of generality we may assume that H is a maximal member of the set of subideals of L lying in J and containing the original H. By Lemma 1.1, $H \lhd J$ and by [1, Lemma 2.1.2] $J \operatorname{si} L$.

As an application of this theorem, we have the following, which is proved in [1, p. 64] by a different method.

COROLLARY 1. *Let L be a finite-dimensional Lie algebra over a field of characteristic zero and let H, K be subideals of L. Then $J = \langle H, K \rangle$ is a finite-dimensional subideal of L.*

PROOF. That $J \operatorname{si} L$ follows from Theorem 1.3. Further J is finite-dimensional since L is.

COROLLARY 2. *Let L be a Lie algebra over a field of characteristic zero and let H, K be subideals of L. If H has finite codimension in $J = \langle H, K \rangle$, then* $J \operatorname{si} L$.

Finally we find another condition under which the join of two subideals of a Lie algebra is a subideal. For this we shall need the following definitions (see [1, pp. 18, 19, 20, 30, 67]). Let H and K be subsets of a Lie algebra L. The *circle product* of H and K denoted by $H \circ K$, is defined as $H \circ K = [H, K]^{H \cup K}$. It is clear that $H \circ K$ is the smallest ideal of $J = \langle H, K \rangle$ containing $[H, K]$.

A *class* $\mathfrak{X}$ of Lie algebras over a field F is a collection of Lie algebras over F such that $(0) \in \mathfrak{X}$ and if $H \in \mathfrak{X}$ and $H \simeq K$, then $K \in \mathfrak{X}$. A class $\mathfrak{X}$ is I-*closed* provided every subideal of an $\mathfrak{X}$-algebra is always an $\mathfrak{X}$-algebra. A class $\mathfrak{X}$ is N_0-*closed* if whenever $H, K \lhd L$ and $H, K \in \mathfrak{X}$, then $H+K \in \mathfrak{X}$. A class $\mathfrak{X}$ is *locally coalescent* if and only if whenever H and K are $\mathfrak{X}$-subideals of a Lie algebra L, then to every finitely generated subalgebra C of $J=\langle H, K\rangle$ there corresponds an $\mathfrak{X}$-subideal X of L such that $C \leq X \leq J$. We write $L \in$ min-si if L satisfies the minimal condition on subideals: Every non-empty collection of subideals of L has a minimal element. Now we prove the following

THEOREM 1.4. *Suppose that $\mathfrak{X}$ is an $\{\mathrm{I}, \mathrm{N}_0\}$-closed and locally coalescent class over any field. Let H and K be $\mathfrak{X}$-subideals of a Lie algebra L with $J=\langle H, K\rangle$. If $H \circ K/(H \circ K)^2$ is finitely generated, then J si L and $J \in \mathfrak{X}$.*

PROOF. Let $M=H \circ K$. Now there exists a finitely-generated subalgebra C of M such that $M=C+M^2$. By the local coalescence of $\mathfrak{X}$ there exists an $\mathfrak{X}$-subideal X of L with $C \leq X \leq J$. Thus if $N=X \cap M$, then $N \lhd X$ si L and so N si L, $N \in \mathrm{I}\mathfrak{X}=\mathfrak{X}$ and $M=N+M^2$. From [1, Lemma 2.1.9] we have $M=N+M^{(r)}$ for all r. By [1, Corollary 2.2.17] we have $J^{(r)} \in \mathfrak{X}$ for some r and so $M^{(r)} \in \mathrm{I}\mathfrak{X}=\mathfrak{X}$. Finally by [1, Theorem 2.2.13] we have $M=N+M^{(r)} \in \mathfrak{X}$ and M si L (for $J^{(r)}$ and so $M^{(r)}$ si L for some r by [1, Theorem 2.2.7]). We also have by [1, Theorem 2.2.13] that $H+M$, $K+M \in \mathfrak{X}$ and $H+M$, $K+M$ si L and so by the same result $J=H+M+K+M \in \mathfrak{X}$ and J si L.

COROLLARY. *Let L be a Lie algebra over a field of characteristic zero and let H, K be subideals of L. If $H \circ K \in$ max-$\lhd$ or $H \circ K \in$ min-$\lhd$, then $J=\langle H, K\rangle$ si L.*

2. Prime ideals in Lie algebras with chain conditions

The object of this section is to investigate the structure of Lie algebras with a certain maximal (resp. minimal) condition on subideals using the notion of prime ideals and prime algebras (defined by analogy with associative rings).

Let L be a Lie algebra over any field. An ideal P of L is said to be *prime* if whenever $[H, K] \subseteq P$ with H, K ideals of L, then $H \subseteq P$ or $K \subseteq P$ (see [7]). We say that a Lie algebra L is *prime* if whenever H and K are ideals of L and $[H, K]=0$, then either $H=0$ or $K=0$. It follows that P is a prime ideal of L if and only if L/P is a prime algebra. Let $H \lhd L$. We denote by rad(H) the intersection of all the prime ideals of L containing H, which is called the *radical* of H. We write rad(L) for rad(0), the intersection of all prime ideals of L, and call it the *prime radical* of L (see [7, p. 683]). Let L be a Lie algebra. Then $\sigma(L)$ is defined to be the sum of all soluble ideals of L and L is *semi-simple* if $\sigma(L)=0$. Let $\mathscr{S}$ be a collection of subsets of L. We say that L satisfies max-$\mathscr{S}$ if $\mathscr{S}$ satisfies the maximal condition: Every ascending chain $S_1 \subseteq S_2 \subseteq \cdots$ of elements $S_i \in \mathscr{S}$ terminates; so that $S_r = S_{r+1} = \cdots$ for some $r \in \boldsymbol{N}$. Similarly L satisfies min-$\mathscr{S}$ if $\mathscr{S}$ satisfies the minimal condition: Every descending chain $S_1 \supseteq S_2 \supseteq \cdots$ terminates. If L is a Lie algebra and $\mathscr{S}$ is respectively the set of ideals, subideals, n-step subideals of L we write in place of max-$\mathscr{S}$: max-$\lhd$, max-si, max-$\lhd^n$ and for min-$\mathscr{S}$ we write min-$\lhd$, min-si, min-$\lhd^n$.

The main result in this section is:

THEOREM 2.1. *Let L be a Lie algebra over any field and let $\mathfrak{X}$ be one of* max-$\lhd^n$ $(n \geq 2)$, max-si, min-$\lhd^n$ $(n \geq 2)$, min-si. *Then $L \in \mathfrak{X}$ if and only if*

(i) *$\sigma(L)$ is a finite-dimensional soluble ideal of L,*

(ii) *$L/\sigma(L)$ is a subdirect sum of a finite number of prime algebras in $\mathfrak{X}$.*

The proof follows from a series of lemmas.

LEMMA 2.2. *If $L \in$ max-$\lhd$ and $H \lhd L$, then there are only a finite number of prime ideals P_i $(i=1,\dots, m)$ such that* $\mathrm{rad}\,(H) = \bigcap_{i=1}^m P_i$.

PROOF. This can be proved in the same way as for non-associative rings in [3].

This result is noted in [7, p. 683].

LEMMA 2.3. *Let $I \lhd L$ and H be a subideal (resp. n-step subideal) of L. Then $H \cap I$ is a subideal (resp. n-step subideal) of L.*

PROOF. The proof is clear.

LEMMA 2.4. *Let L be a Lie algebra and let $\mathscr{S}$ be respectively the set of ideals, subideals, n-step subideals of L. Suppose $I_i \lhd L$, $(i=1, 2,\dots, m)$ and $\bigcap_{i=1}^m I_i = 0$. Let $\mathscr{S}_i = \{(H+I_i)/I_i \mid H \in \mathscr{S}\}$. If $L/I_i \in$ max-$\mathscr{S}_i$ (resp. min-$\mathscr{S}_i$) for all i, then $L \in$ max-$\mathscr{S}$ (resp. min-$\mathscr{S}$).*

PROOF. By induction on m we need consider only the case $m=2$, then $I_1 \cap I_2 = 0$. Let $H_1 \subseteq H_2 \subseteq \cdots$ be an ascending chain of elements $H_i \in \mathscr{S}$. Then $(H_1+I_1)/I_1 \subseteq (H_2+I_1)/I_1 \subseteq \cdots$ is an ascending chain of elements of $\mathscr{S}_1$. Therefore there exists $r \in \boldsymbol{N}$ such that

$$H_r + I_1 = H_{r+1} + I_1 = \cdots \tag{1}$$

Now $H_1 \cap I_1 \subseteq H_2 \cap I_1 \subseteq \cdots$ is an ascending chain of elements of $\mathscr{S}$. Therefore $(H_1 \cap I_1 + I_2)/I_2 \subseteq (H_2 \cap I_1 + I_2)/I_2 \subseteq \cdots$ is an ascending chain of elements of $\mathscr{S}_2$ and so there exists $r \in \boldsymbol{N}$ such that $(H_r \cap I_1) + I_2 = (H_{r+1} \cap I_1) + I_2 = \cdots$. Therefore $H_{r+1} \cap I_1 = (H_r \cap I_1) + H_{r+1} \cap I_1 \cap I_2$ by the modular law, but $I_1 \cap I_2 = 0$, hence

$$H_{r+1} \cap I_1 = H_r \cap I_1 \cdots \tag{2}$$

Now from (1) and (2) we have $H_r = H_{r+1} = \cdots$ and so $L \in \text{max-}\mathcal{S}$. That $L \in \text{min-}\mathcal{S}$ can be proved by a similar method.

LEMMA 2.5. *Let L be a Lie algebra and $\{L_\alpha\}_{\alpha\in A}$ be a family of Lie algebras. Then L is a subdirect sum of $\{L_\alpha\}_{\alpha\in A}$ if and only if for each $\beta \in A$, there is a surjective homomorphism g_β: $L \to L_\beta$ such that $\bigcap_{\beta\in A} \operatorname{Ker} g_\beta = 0$.*

PROOF. This can be proved in the same way as in [4, p. 99].

COROLLARY. *Let L be a Lie algebra and let $\{I_\alpha\}_{\alpha\in A}$ be a family of ideals of L. If $\bigcap_{\alpha\in A} I_\alpha = 0$, then L is a subdirect sum of the family of Lie algebras $\{L/I_\alpha\}_{\alpha\in A}$.*

LEMMA 2.6. *If $L \in \text{max-}\lhd^2$ (resp. $\text{min-}\lhd^2$), then $\sigma(L)$ is a finite-dimensional soluble ideal of L.*

PROOF. This follows from [1, Corollary 9.1.3(d) and Lemma 9.2.1].

LEMMA 2.7. *Let L be a Lie algebra over any field.*

(i) *L is semi-simple with $\text{max-}\lhd^n$, $n \geq 1$ (resp. max-si) if and only if L is a subdirect sum of a finite number of prime algebras satisfying $\text{max-}\lhd^n$, $n \geq 1$ (resp. max-si).*

(ii) *L is semi-simple with $\text{min-}\lhd^n$, $n \geq 1$ (resp. min-si) if and only if L is a subdirect sum of a finite number of prime algebras satisfying $\text{min-}\lhd^n$, $n \geq 1$ (resp. min-si).*

PROOF. (i) Let L be semi-simple with $\text{max-}\lhd^n$ (resp. max-si). Then $\sigma(L) = 0$. By Lemma 2.2, $\operatorname{rad}(L) = \bigcap_{i=1}^m P_i$, where P_i are prime ideals of L. But by [7, Theorem 7] $\operatorname{rad}(L) = \sigma(L)$, hence $\operatorname{rad}(L) = 0$. Since P_i is a prime ideal of L, it follows that L/P_i is a prime algebra and $L/P_i \in \text{max-}\lhd^n$ (resp. max-si). Now by Corollary of Lemma 2.5, L is a subdirect sum of a finite number of prime algebras satisfying $\text{max-}\lhd^n$ (resp. max-si).

To prove the converse suppose that L is a subdirect sum of a finite number of prime algebras $\{L_\alpha\}_{\alpha\in A}$, $A = \{1, 2, \ldots, m\}$ satisfying $\text{max-}\lhd^n$ (resp. max-si). Let g_β: $L \to L_\beta$ be the surjective homomorphism of Lemma 2.5. Then for each β, $L/\operatorname{Ker} g_\beta \simeq L_\beta$ and L_β is prime. Hence $\operatorname{Ker} g_\beta$ is a prime ideal of L. Thus $\operatorname{rad}(L) \subseteq \operatorname{Ker} g_\beta$ for each β, and so $\operatorname{rad}(L) \subseteq \bigcap_{\beta\in A} \operatorname{Ker} g_\beta = 0$. But by [7, Theorem 7], $\sigma(L) \subseteq \operatorname{rad}(L)$, hence $\sigma(L) = 0$ and L is semi-simple. Now that $L \in \text{max-}\lhd^n$ (resp. max-si) follows from Lemma 2.4.

(ii) Let L be semi-simple with $\text{min-}\lhd^n$ (resp. min-si). Then L has only a finite number of minimal ideals $M_1, \ldots, M_r$. Let P_i, $1 \leq i \leq r$, be an ideal of L which is maximal with respect to not containing M_i. We claim that P_i is a prime ideal of L. Suppose not; then there exist ideals H, K of L such that $H \not\subseteq P_i$, $K \not\subseteq P_i$ and $[H, K] \subseteq P_i$. Now $H+P_i \supsetneqq P_i$ and $K+P_i \supsetneqq P_i$, so by the choice of P_i, $H+P_i \supseteq M_i$ and $K+P_i \supseteq M_i$. Therefore $M_i^2 \subseteq [H+P_i, K+P_i] \subseteq P_i$. But $M_i^2 \neq 0$ for L is semi-simple, hence $M_i^2 = M_i \subseteq P_i$ which is a contradiction. Therefore P_i is a prime ideal of L and L/P_i is a prime algebra. If $\bigcap_{i=1}^r P_i \neq 0$, then this intersection contains one of the minimal ideals M_j for some j. But $M_j \not\subseteq P_j$, so $M_j \not\subseteq \bigcap_{i=1}^r P_i$. Hence $\bigcap_{i=1}^r P_i = 0$ and by Corollary of Lemma 2.5, it follows that L is a subdirect sum of a finite number of prime algebras satisfying min-$\lhd^n$ (resp. min-si).

Conversely that L is semi-simple can be proved as in (i), and that $L \in$ min-$\lhd^n$ (resp. min-si) follows from Lemma 2.4.

PROOF OF THEOREM 2.1. The proof follows from Lemmas 2.6 and 2.7.

3. Quasi-Artinian algebras

The object of this section is to generalize the minimal condition on ideals in such a way that the main properties of Lie algebras with min-$\lhd$ are preserved.

Let L be a Lie algebra over any field. We say that L is *Artinian* if $L \in$ min-$\lhd$. We say that L is *quasi-Artinian* if for every descending chain $I_1 \supseteq I_2 \supseteq \cdots$ of ideals of L there exist $r, s \in \boldsymbol{N}$ such that $[L^{(r)}, I_s] \subseteq I_n$ for all n, or equivalently there exists $m \in \boldsymbol{N}$ such that $[L^{(m)}, I_m] \subseteq I_n$ for all n. It is clear that every soluble Lie algebra is quasi-Artinian, but it is easy to construct a soluble Lie algebra which is not Artinian, so quasi-Artinian algebras need not be Artinian. Further, if L is a hypercentral Lie algebra and is quasi-Artinian then L is soluble (for $L^{(\alpha)} = 0$ for some ordinal α by [1, Lemma 8.1.1]. But $L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \cdots$ is a descending chain of ideals of L and L is quasi-Artinian, so there exists $m \in \boldsymbol{N}$ such that $[L^{(m)}, L^{(m)}] \subseteq L^{(n)}$ for all n. Hence $L^{(m+1)} = L^{(n)}$ for all n. Therefore $L^{(m+1)} = L^{(m+2)} = \cdots$ and $L^{(\alpha)} = L^{(m+1)} = 0$. Thus L is soluble).

THEOREM 3.1. *The following are equivalent*:

(i) *L is quasi-Artinian.*

(ii) *There exists $m \in \boldsymbol{N}$ such that for every descending chain $I_1 \supseteq I_2 \supseteq \cdots$ of ideals of L, the descending chain of ideals $[L^{(m)}, I_1] \supseteq [L^{(m)}, I_2] \supseteq \cdots$ terminates.*

(iii) *For every non-empty collection $\mathscr{C}$ of ideals of L, there exist an element $I \in \mathscr{C}$ and $m \in \boldsymbol{N}$ such that $[L^{(m)}, I] \subseteq J$ for every $J \in \mathscr{C}$ with $J \subseteq I$.*

PROOF. (i)$\rightarrow$(ii). Let L be quasi-Artinian. Now $L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \cdots$ is a descending chain of ideals of L, so there exists $m \in \boldsymbol{N}$ such that $[L^{(m)}, L^{(m)}] \subseteq L^{(n)}$ for all $n \geq m$. Therefore $L^{(m+1)} \subseteq L^{(n)}$ for all $n \geq m$ and $L^{(m+1)} = L^{(m+2)} = \cdots$. Also $I_1 \supseteq I_2 \supseteq \cdots$ is a descending chain of ideals of L, so there exists $r \in \boldsymbol{N}$

such that $[L^{(2m)}, I_r] \subseteq I_{r+s}$ for all $s \in \boldsymbol{N}$. Therefore for all $s \in \boldsymbol{N}$, $[L^{(2m)}, I_r] \subseteq [L^{(2m)}, [L^{(2m)}, I_r]] \subseteq [L^{(2m)}, I_{r+s}] \subseteq [L^{(2m)}, I_r]$. Hence $[L^{(2m)}, I_r] = [L^{(2m)}, I_{r+s}]$. Since the choice of m is independent of the sequence $\{I_n\}$, the result follows.

(ii)$\rightarrow$(iii). Suppose that (iii) does not hold for some $\mathscr{C}$. Then we can find successively $I_i \in \mathscr{C}$, $(i=1, 2, \ldots)$ such that $I_i \supset I_{i+1}$, but $[L^{(i)}, I_i] \not\subseteq I_{i+1}$ which implies that (ii) does not hold. Hence (ii)$\rightarrow$(iii).

(iii)$\rightarrow$(i) is clear.

THEOREM 3.2. (i) *Let L be a quasi-Artinian Lie algebra and let $I \lhd L$. Then L/I is quasi-Artinian.*

(ii) *Let $I \lhd L$. Then L is quasi-Artinian if one of the following holds:*

(a) *I is quasi-Artinian and L/I is soluble.*

(b) *L/I is quasi-Artinian and if $I \supseteq I_1 \supseteq I_2 \supseteq \cdots$, $I_i \lhd L$ then there exists $m \in \boldsymbol{N}$ such that $[L^{(m)}, I_m] \subseteq I_n$ for all $n \in \boldsymbol{N}$.*

(c) *L/I is quasi-Artinian and I is Artinian.*

PROOF. (i) Let $\pi: L \rightarrow L/I$ be the natural homomorphism and let $\bar{I}_1 \supseteq \bar{I}_2 \supseteq \cdots$ be a descending chain of ideals of $\bar{L} = L/I$. Then $\pi^{-1}(\bar{I}_1) \supseteq \pi^{-1}(\bar{I}_2) \supseteq \cdots$ is a descending chain of ideals of L. But L is quasi-Artinian, so there exists $m \in \boldsymbol{N}$ such that $[L^{(m)}, \pi^{-1}(\bar{I}_m)] \subseteq \pi^{-1}(\bar{I}_n)$ for all $n \geq m$. Therefore $[(\pi(L))^{(m)}, \bar{I}_m] = \pi[L^{(m)}, \pi^{-1}(\bar{I}_m)] \subseteq \bar{I}_n$. Thus $[\bar{L}^{(m)}, \bar{I}_m] \subseteq \bar{I}_n$ and $\bar{L}$ is quasi-Artinian.

(ii) (a), (b) *Let* $I_1 \supseteq I_2 \supseteq \cdots$ be a descending chain of ideals of L. Then $I_1 \cap I \supseteq I_2 \cap I \supseteq \cdots$ is a descending chain of ideals of I and $(I_1+I)/I \supseteq (I_2+I)/I \supseteq \cdots$ is a descending chain of ideals of L/I. By assumption (a) or (b), there exists $m \in \boldsymbol{N}$ such that $[L^{(m)}, I_m \cap I] \subseteq I_n \cap I$ and $[L^{(m)}, I_m] + I \subseteq [L^{(m)}, I_n] + I$ for all $n \geq m$. Therefore $[L^{(m)}, I_m] \subseteq I_n + I$, but $[L^{(m)}, I_m] \subseteq I_m$. Hence $[L^{(m)}, I_m] \subseteq (I_n+I) \cap I_m = I_n + (I_m \cap I)$ and so $[L^{(m)}, [L^{(m)}, I_m]] \subseteq [L^{(m)}, I_n] + [L^{(m)}, I_m \cap I]$. Therefore $[L^{(m+1)}, I_m] \subseteq I_n + (I_n \cap I) = I_n$ and L is quasi-Artinian.

(c) Clear.

REMARK. A finite direct sum of quasi-Artinian Lie algebras is quasi-Artinian.

THEOREM 3.3. *Let L be a locally nilpotent quasi-Artinian Lie algebra. Then L is soluble.*

PROOF. Suppose L is not soluble. Then there is a non-soluble ideal I of L. We claim that I contains a minimal non-soluble ideal of L. Suppose for a contradiction that this is not the case. Let $I = I_1$. Then $0 \neq I_1^{(2)} \subseteq [L^{(1)}, I_1]$ and $[L^{(1)}, I_1]$ is a non-soluble ideal of L, since I_1 is not soluble. So there is a non-soluble ideal I_2 of L such that $I_2 \subsetneqq [L^{(1)}, I_1] \subseteq I_1$. Now $0 \neq I_2^{(3)} \subseteq [L^{(2)}, I_2]$ and $[L^{(2)}, I_2]$ is a non-soluble ideal of L since I_2 is not soluble. So there is a non-soluble ideal I_3 of L such that $I_3 \subsetneqq [L^{(2)}, I_2] \subseteq I_2$. Continuing this process, there is a non-soluble ideal $I_n \subsetneqq [L^{(n-1)}, I_{n-1}] \subseteq I_{n-1}$. Then $0 \neq I_n^{(n+1)} \subseteq [L^{(n)}, I_n]$

and $[L^{(n)}, I_n]$ is a non-soluble ideal of L since I_n is not soluble. So there is a non-soluble ideal I_{n+1} of L such that $I_{n+1} \subsetneqq [L^{(n)}, I_n] \subseteq I_n$ and so on. Finally the descending chain $I_1 \supset I_2 \supset \cdots$ contradicts the hypothesis that L is quasi-Artinian.

Thus there is such a minimal ideal: call it J. But $J^2 \subseteq J$ and J^2 is not soluble, hence $J = J^2$ by the minimality of J. Now either J has trivial centre or not (i.e. either $Z(J) = 0$, or $Z(J) \neq 0$).

Suppose $Z(J) = 0$. Let $\mathscr{C} = \{K \lhd L \mid K \subseteq J$ and $[K, J] \neq 0\}$. $\mathscr{C} \neq \emptyset$ for $J \in \mathscr{C}$. We claim that $\mathscr{C}$ has a minimal element. Suppose not. Put $J = J_1$. Then $0 \neq [J_1^{(2)}, J] \subseteq [[L^{(1)}, J_1], J]$, so $[L^{(1)}, J_1] \in \mathscr{C}$. Choose $J_2 \in \mathscr{C}$ such that $J_2 \subsetneqq [L^{(1)}, J_1] \subseteq J_1$. Then $0 \neq [J, J_2] = [J^2, J_2] \subseteq [J, [J, J_2]] = [J, [J^{(2)}, J_2]] \subseteq [J, [L^{(2)}, J_2]]$. Hence $[L^{(2)}, J_2] \in \mathscr{C}$ and so on. Choose $J_n \in \mathscr{C}$ such that $J_n \subsetneqq [L^{(n-1)}, J_{n-1}] \subseteq J_{n-1}$. Then $0 \neq [J, J_n] = [J^{(n+1)}, J_n] \subseteq [J, [L^{(n)}, J_n]]$. Therefore $[L^{(n)}, J_n] \in \mathscr{C}$. Repeat this process, then the descending chain of ideals $J_1 \supset J_2 \supset \cdots$ contradicts the hypothesis that L is quasi-Artinian.

Thus $\mathscr{C}$ has a minimal element, say K. If K is a minimal ideal of L, then K is central (see [1, Lemma 7.1.6]) which is a contradiction. If K is not a minimal ideal of L, then $K \supset H$ and $H \lhd L$ for some H. Now either $[H, J] = 0$ or $[H, J] \neq 0$. If $[H, J] = 0$, then $H \subseteq C_L(J)$, but $H \subseteq J$, hence $H \subseteq C_L(J) \cap J = Z(J) = 0$. If $[H, J] \neq 0$, then $H = K$ by the minimality of K and K is a minimal ideal of L and in both cases we get a contradiction.

Hence $Z(J) \neq 0$. Let U be the hypercentre of J. Then $U \lhd L$ and $U^{(\alpha)} = 0$ for some infinite α. But L is quasi-Artinian, so $U^{(\alpha)} = U^{(n)} = 0$ for some finite n and so U is soluble. Now J/U is a minimal non-soluble ideal of L/U and $J/U = (J/U)^2$ with $Z(J/U) = 0$, and a similar argument as above again gives a contradiction. Therefore L is soluble.

It appears likely that a theory of prime ideals of quasi-Artinian Lie algebras may exist analogously to that for min-$\lhd$. In particular this would be the case if every semi-simple quasi-Artinian Lie algebra were Artinian. We know no example disproves this, but it remains an open question.

It is possible to define the notion of quasi-Artinian groups in an analogous way and the proofs of Theorems 3.1, 3.2, 3.3 carry over in this case without difficulties.

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