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A class of hyperbolic focal point problems

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1. Introduction

The hyperbolic equation

$$u_{tt} - u_{ss} + p(s, t)u = 0 \qquad (p>0) \tag{1.1}$$

has the physical interpretation of prescribing the u-displacement of a vibrating string subject to a linear restoring force. This fact suggests the possibility of formulating a hyperbolic boundary value problem by specifying boundary conditions for the string at times $t=0$ and $t=T$. In analogy with the ordinary differential equation

$$\frac{d^2u}{dt^2} + p(t)u = 0 \qquad (p>0) \tag{1.2}$$

(describing the u-displacement of a single particle), one might expect that such boundary values will also give rise to eigenvalues for hyperbolic equations of the form

$$u_{tt} - u_{ss} + \lambda p(s, t)u = 0. \tag{1.3}$$

While there have been a number of hyperbolic generalizations of the Sturm comparison theorem (see for example [2], [3], [5], [6]) which can be useful in this regard, these extensions of classical ODE results require careful attention to boundary conditions in the space variable s as well as in t. The present paper attempts to avoid the complications associated with spatial boundary conditions by considering focal point problems for (1.1) in characteristic triangles of the form

$$R(s, t) = \{(\sigma, \tau)\colon s-(t-\tau)\leq\sigma\leq s+(t-\tau), \quad 0\leq\tau\leq t\}.$$

For example, we shall study (1.3) in $R(0, T)$ with various boundary conditions assigned at $t=0$ and at $t=T$.

It is assumed throughout that $p(s, t)$ is continuous in $R(0, T)$ and that all solutions are C^2 functions which satisfy the underlying equation in the classical sense.

2. Right focal points

In order to study (1.3) in $R(0, T)$ subject to the right focal point boundary conditions $u_t(s, 0)=u(0, T)=0$ for $-T \leq s \leq T$, we first consider the Cauchy problem

$$u_{tt} - u_{ss} + f(s, t) = 0 \quad \text{in} \quad R(0, T), \tag{2.1}$$

$$u(s, 0) = kg(s), \quad u_t(s, 0) = 0 \quad \text{in} \quad [-T, T], \tag{2.2}$$

where $f(s, t)$ is assumed continuous in $R(0, T)$, $g(s)$ is positive and continuous in $[-T, T]$ and k is a constant yet to be determined. By D'Alembert's formula (2.1), (2.2) has the solution

$$u(s, t) = \frac{k}{2}[g(s+t)+g(s-t)] - \frac{1}{2}\iint_{R(s,t)} f(\sigma, \tau)d\sigma d\tau.$$

In order to impose $u(0, T)=0$ on a solution of (2.1), (2.2) we define $g_0=[g(T)+g(-T)]$ and choose

$$k = \frac{1}{g_0}\iint_{R(0,T)} f(\sigma, \tau)d\sigma d\tau$$

to obtain

$$u(s, t) = \frac{g(s+t)+g(s-t)}{2g_0}\iint_{R(0,T)} f(\sigma, \tau)d\sigma d\tau - \frac{1}{2}\iint_{R(s,t)} f(\sigma, \tau)d\sigma d\tau.$$

This can also be written

$$u(s, t) = \iint_{R(0,T)} G(s, t; \sigma, \tau)f(\sigma, \tau)d\sigma d\tau, \tag{2.3}$$

where the right focal point Green's function satisfies

$$\begin{aligned} G(s, t; \sigma, \tau) &= \frac{g(s+t)+g(s-t)}{2g_0} && \text{for } (\sigma, \tau) \text{ in } R(0, T)-R(s, t), \\ &= \frac{g(s+t)+g(s-t)}{2g_0} - \frac{1}{2} && \text{for } (\sigma, \tau) \text{ in } R(s, t) \end{aligned} \tag{2.4}$$

We note that if g is concave in $[-T, T]$, (e.g. if $g''(s) \leq 0$ in $[-T, T]$), then $G(s, t; \sigma, \tau) \geq 0$ in $R(0, T) \times R(0, T)$, while in case $g(s)$ is constant we obtain

$$\begin{aligned} G(s, t; \sigma, \tau) &= \frac{1}{2} && \text{in} \quad R(0, T) - R(s, t), \\ &= 0 && \text{in} \quad R(s, t). \end{aligned} \tag{2.5}$$

However $G(s, t: \sigma, \tau)$ is *not* symmetric, as can be seen from the fact that in (2.5)

$$G\left(0, \frac{2T}{3}; 0, \frac{T}{3}\right) = 0 \quad \text{and} \quad G\left(0, \frac{T}{3}; 0, \frac{2T}{3}\right) = \frac{1}{2}.$$

From (2.3) it follows that the hyperbolic eigenvalue problem

$$\begin{aligned} &u_{tt} - u_{ss} + \lambda p(s, t)u = 0 && \text{in} \quad R(0, T), \\ (2.6) \qquad &u(s, 0) = kg(s), \quad u_t(s, 0) = 0 && \text{in} \quad [-T, T], \\ &u(0, T) = 0 \end{aligned}$$

is equivalent to

$$(2.7) \qquad \frac{1}{\lambda} u(s, t) = \mathscr{G}[u],$$

where

$$\mathscr{G}[u] = \iint_{R(0,T)} G(s, t; \sigma, \tau)p(\sigma, \tau)u(\sigma, \tau)d\sigma d\tau.$$

Assuming $p(s, t) > 0$ in $R(0, T)$, $\mathscr{G}$ defines a positive operator on the cone of functions $u: R(0, T) \to R^+$. The theory of Krasnoselskii [1, Chapter 2] then implies the following.

THEOREM 2.1. *If $g(s)$ is positive and concave in $[-T, T]$, then there exists a unique eigenvalue $\lambda_g > 0$ for which* (2.6) *has a nontrivial solution $u(x, t)$ which is nonnegative in $R(0, T)$. All other eigenvalues of* (2.6) *satisfy $|\lambda| > \lambda_g$.*

A somewhat different right focal point condition can be formulated for (1.1) in its alternate canonical form

$$(2.8) \qquad u_{xy} + p(x, y)u = 0.$$

Again considering (2.8) in a characteristic triangle

$$R(x, y) = \{(\xi, \eta): -\eta \leq \xi \leq x, \ -\xi \leq \eta \leq y\},$$

we seek a point (X, Y) such that there exists a nontrivial solution of (2.8) satisfying

$$(2.9) \qquad u_x(x, -x) = 0 \quad \text{for} \quad -Y \leq x \leq X \quad \text{and} \quad u(X, Y) = 0$$

or

$$(2.10) \qquad u_y(-y, y) = 0 \quad \text{for} \quad -X \leq y \leq Y \quad \text{and} \quad u(X, Y) = 0.$$

Since the Cauchy problem for (2.8) consists of assigning u and u_x (or u_y) along a line $x + y = 0$, we shall consider (2.9) or (2.10) in conjunction with the initial values $u(x, -x)=g(x)$. Such Cauchy data gives rise to the compatibility condition

$$u_x(x, -x) - u_y(x, -x) = g'(x),$$

so that $g(x)\equiv$constant implies the equivalence of (2.9), (2.10), and the equivalence of (2.9) with the right focal points previously considered. However, for nonconstant initial data $g(x)>0$ the various right focal points have different meanings.

It will be useful (for the discussion in Sec. 3) to generalize our considerations to include differential inequalities of the form

$$\begin{aligned} &u_{xy} + p(x, y)u \leq 0, \\ &u(x, -x) = g(x); \quad u_x(x, -x) = 0, \end{aligned} \tag{2.11}$$

where $g(x)$ is continuous and positive. We shall now be able to use a generalized Riccati transformation [4] and a comparison theorem of W. Walter [7] to establish criteria for the existence of right focal points for (2.11). Unlike Theorem 2.1, these criteria will not require the positivity of $p(x, y)$.

Noting that $u(x, y)=\exp[-U(x, y)]$ transforms (2.11) into

$$\begin{aligned} &U_{xy} = p(x, y) + U_xU_y, \\ &U(x, -x) = -\log g(x), \quad U_x(x, -x) = 0, \end{aligned}$$

the focal point condition $u(X, Y)=0$ becomes

$$\lim_{(x,y)\to(X^-,Y^-)} U(x, y) = \infty.$$

These observations enable us to establish the existence of right focal points of (2.11) by means of the following comparison theorem.

THEOREM 2.2. *Let $v(x, y)$ be a nontrivial solution of*

$$\begin{aligned} &v_{xy} + P(x, y)v \geq 0, \quad (x, y)\in R(X, Y) \\ &v(x, -x) = G(x), \quad v_x(x, -x) = 0, \quad -Y\leq x \leq X \end{aligned} \tag{2.11)'$$

satisfying $v>0$, $v_x<0$, $v_y<0$ in the interior of $R(X, Y)$. If $u(x, y)$ is positive and satisfies (2.11) *with $p(x, y)\geq P(x, y)$ in $R(X, Y)$ and $g(x)\leq G(x)$ for $-Y\leq x\leq X$, then $v(x, y)\geq u(x, y)$ in $R(X, Y)$.*

PROOF. The transformations

$$u(x, y) = \exp[-U(x, y)]; \quad v(x, y) = \exp[-V(x, y)]$$

transform (2.11) and (2.11)′ into the form

$$U_{xy} \geq f(x, y, U, U_x, U_y) \quad \text{in} \quad R(X, Y),$$
$$V_{xy} \leq f(x, y, V, V_x, V_y) \quad \text{in} \quad R(X, Y),$$

where $f(x, y, V, V_x, V_y)=P(x, y)+V_xV_y$ and $V_x>0$, $V_y>0$ in the interior of $R(X, Y)$. From the equivalent integral equations

$$U(x, y) \geq -\ln g(x, y) + \frac{1}{2}\iint_{R(x,y)} f(\xi, \eta, U_\xi, U_\eta)d\xi d\eta,$$
$$V(x, y) \leq -\ln G(x, y) + \frac{1}{2}\iint_{R(x,y)} f(\xi, \eta, V, V_\xi, V_\eta)d\xi d\eta.$$

[7, Theorem 21. V] implies that $U_x(x, y) \geq V_x(x, y) \geq 0$, $U_y(x, y) \geq V_y(x, y) \geq 0$, and $U(x, y) \geq V(x, y)$ in $R(X, Y)$. It therefore follows that $u(x, y) \leq v(x, y)$ in $R(X, Y)$.

COROLLARY 2.3. *Under the hypotheses of Theorem* 2.2, *if* (2.11)′ *has a right focal point defined by* (2.9) *in* $R(x, y)$, *then so does* (2.11).

In order to apply Corollary 2.3, we may consider (2.11)′ in the special case where $P(x, y)=Q(x+y)$ and $G(x)\equiv 1$. The substitution $s=x+y$; $w(s)=v(x, y)$ then yields

$$\frac{d^2w}{ds^2} + Q(s)w = 0 \tag{2.12}$$

and, as a consequence, the following criterion for the existence of right focal points defined by (2.9).

THEOREM 2.4. *Suppose* (2.12) *has a nontrivial solution* $w(s)$ *satisfying* $w'(0)=0=w(S)$ *and* $w'(s) \leq 0$ *for* $0 \leq s \leq S$. *If* $p(x, y) \geq Q(x+y)$ *in* $R(X, S-X)$, *then* (2.11) *will have a right focal point in* $R(X, S-X)$.

3. Left focal points

Given the variety of generalizations of right focal point from (1.2) to (1.1), one would expect that left focal points would allow an analogous treatment. A natural effort in this direction is to consider

$$\begin{aligned} &u_{tt} - u_{ss} + p(s, t)u = 0 \quad \text{for} \quad (\sigma, \tau) \quad \text{in} \quad R(0, T), \\ &u(s, 0) = 0,\ u_t(s, 0) = kg(s) \quad \text{for} \quad -T \leq s \leq T, \\ &u_t(0, T) = 0 \end{aligned} \tag{3.1}$$

and again seek conditions such that (3.1) has nontrivial solutions. Proceeding as before, one would solve the Cauchy problem

$$u_{tt} - u_{ss} + f(s, t) = 0,$$
$$u(s, 0) = 0; \quad u_t(s, 0) = kg(s)$$

in the form

$$(3.2) \qquad u(s, t) = \frac{k}{2}\int_{s-t}^{s+t} g(\sigma)d\sigma - \frac{1}{2}\int_0^t \int_{s-(t-\tau)}^{s+(t-\tau)} f(\sigma, \tau)d\sigma d\tau,$$

obtaining

$$u_t(s, t) = \frac{k}{2}[g(s+t)+g(s-t)] - \frac{1}{2}\int_0^t [f(s+(t-\tau), \tau)+f(s-(t-\tau), \tau)]d\tau.$$

Thus the boundary condition $u_t(0, T)=0$ leads to the representation (3.2) with

$$k = \frac{1}{g(T)+g(-T)}\int_0^T [f((T-\tau), \tau)+f(-(T-\tau), \tau)]d\tau.$$

Unfortunately this expression for k seems to preclude a representation of (3.1) in terms of a Green's function such as that which underlies Theorem 2.1.

Attempts to establish left focal points by means of generalized Riccati transformations encounter similar difficulties. In the ODE case of

$$(3.3) \qquad \frac{d^2u}{dt^2} + p(t)u = 0$$

right focal points are obtained from the transformation $h(t) = -u'(t)/u(t)$, and this reduces to the transformation $u(t)=e^{-U(t)}$ when $h(t)=U'(t)$. However, there seems to be no corresponding exponential form of the Riccati transformation $H(t)=u(t)/u'(t)$ which is used to study left focal points of (3.3).

In view of these difficulties, we shall establish the existence of left focal points by different means. These do, however, require additional hypotheses on $p(s, t)$ or $p(x, y)$.

Defining $w(s, t)=u_t(s, t)$, we note that (3.1) becomes equivalent to

$$(3.4) \qquad \begin{aligned} &w_{tt} - w_{ss} + p(s, t)w = -p_t\int_0^t w(s, \tau)d\tau, \\ &w_t(s, 0) = 0; \quad w(s, 0) = kg(s), \\ &w(0, T) = 0. \end{aligned}$$

If $p_t \geq 0$ and $w(\sigma, \tau) \geq 0$ in some $R(s, t)$, then we obtain the differential inequality

$$w_{tt} - w_{ss} + p(s, t)w \leq 0,$$
$$w_t(s, 0) = 0; \quad w(0, T) = 0$$

to which the results of Sec. 2 can be applied. Recalling that for $g(s) \equiv 1$ the various definitions of right focal points coincide, we can also apply the comparison principle of Theorem 2.2 to the differential inequalities

$$w_{tt} - w_{ss} + p(s, t)w \leq 0 \tag{3.5}$$

and

$$v_{tt} - v_{ss} + p(s, t)v = 0. \tag{3.6}$$

These observations lead to

THEOREM 3.1. *Suppose* (3.6) *has a right focal point* (S, T) *for which a nontrivial solution of* (3.6) *satisfies*

$$v_t(s, 0) = 0, \quad v(s, 0) = k, \quad T - S \leq s \leq T + S$$
$$v(S, T) = 0$$

If $\partial p/\partial t \geq 0$ *in* $R(S, T)$ *and* $g(s) \equiv 1$, *then* (3.1) *has a left focal point in* $R(S, T)$.

4. Conjugate points

In light of the preceding discussion it is natural to consider also the notion of a conjugate point $(0, T)$ corresponding to the eigenvalue problem

$$\begin{aligned} &u_{tt} - u_{ss} + \lambda p(s, t)u = 0 \quad \text{in} \quad R(0, T) \\ &u(s, 0) = 0; \quad u_t(s, 0) = kg(s) \\ &u(0, T) = 0. \end{aligned} \tag{4.1}$$

Assuming $g(s) \geq 0$ for $-T \leq s \leq T$, it is readily shown by the techniques of Sec. 2 that (4.1) is equivalent to

$$u(s, t) = kg_0(s, t) - \frac{\lambda}{2} \iint_{R(s,t)} p(\sigma, \tau)u(\sigma, \tau)d\sigma d\tau,$$

where

$$g_0(s, t) = \frac{1}{2}\int_{s-t}^{s+t} g(\sigma)d\sigma \quad \text{and} \quad k = \frac{\lambda}{2g_0(0, T)} \int_{R(0,T)} p(\sigma, \tau)u(\sigma, \tau)d\sigma d\tau.$$

Thus (4.1) is equivalent to

$$\frac{1}{\lambda}u(s, t) = \mathscr{G}[u] = \iint_{R(0,T)} G(s, t; \sigma, \tau)p(\sigma, \tau)u(\sigma, \tau)d\sigma d\tau, \tag{4.2}$$

where

$$G(s, t; \sigma, \tau) = \frac{g_0(s, t)}{2g_0(0, T)} \qquad \text{for} \quad (\sigma, \tau) \quad \text{in} \quad R(0, T) - R(s, t),$$

$$= \frac{g_0(s, t)}{2g_0(0, T)} - \frac{1}{2} \quad \text{for} \quad (\sigma, \tau) \quad \text{in} \quad R(s, t).$$

Unfortunately this Green's function is neither non-negative nor symmetric, making it difficult to draw any conclusion about the existence of real eigenvalues for (4.1). Nor do the techniques we have used for focal points seem to lend themselves to establishing the existence of such conjugate points.

Yet the compelling physical interpretation cited in Sec. 1 suggests that real eigenvalues should exist for (4.1) in a variety of situations. The development of criteria for their existence seems to be a problem worth pursuing.

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