HIROSHIMA MATH. J.
14 (1984), 169–187

Asymptotic theory of perturbed general disconjugate equations II

William F. TRENCH

(Received July 4, 1983)

1. Introduction

Here we continue the investigation begun in [9] of the asymptotic behavior of solutions of the equation

$$L_n u + Fu = 0, \tag{1}$$

where L_n is the general disconjugate operator

$$L_n = \frac{1}{p_n}\frac{d}{dt}\frac{1}{p_{n-1}}\cdots\frac{1}{p_1}\frac{d}{dt}\frac{\cdot}{p_0} \qquad (n \geqq 2), \tag{2}$$

with $p_i>0$ and $p_i \in C[a, \infty)$, $0 \leqq i \leqq n$. Although we did not make specific assumptions in [9] on the form of the functional F in (1), we restrict our attention here to the case where (1) is of the form

$$L_n u + F(t, L_0 u, \dots, L_{n-1} u) = 0, \tag{3}$$

with

$$L_0 x = \frac{x}{p_0}; \quad L_r x = \frac{1}{p_r}(L_{r-1}x)', \quad 1 \leqq r \leqq n. \tag{4}$$

Nevertheless, for convenience we abbreviate (3) as in (1), and write

$$F(t, L_0 u(t), \dots, L_{n-1} u(t)) = (Fu)(t).$$

We say that u is a solution of (3) if $L_0 u, \dots, L_n u$ exist and satisfy (3) on a half line $[t_0, \infty)$ for some $t_0 \geqq a$. We seek conditions which imply that (3) has a solution u which behaves for large positive t like a given solution q of the unperturbed equation

$$L_n x = 0, \tag{5}$$

in a sense defined below. We believe that our results are new even in the case where

$$p_0 = p_1 = \cdots = p_n = 1, \tag{6}$$

so that (3) and (5) reduce to the standard equations

$$u^{(n)} + F(t, u, \ldots, u^{(n-1)}) = 0$$

and

$$x^{(n)} = 0,$$

because our integral conditions on F permit conditional convergence of some of the improper integrals in question. This continues a theme that we have developed in several previous papers; for examples, see [6], [8], [9], [10], [11], [12], and [13].

Here we give results which are more specific than those in [9], and our main theorem is obtained by means of the Schauder-Tychonov theorem, rather than the more restrictive contraction mapping principle used in [9]. Our estimates are also sharper than those in [9]. For this we are in considerable debt to a recent paper of Fink and Kusano [1], as explained in Section 2.

2. Preliminary considerations

We assume throughout that L_n in (2) is in canonical form at infinity; i.e.,

$$\int^{\infty} p_j(t)dt = \infty, \quad 1 \leqq j \leqq n-1. \tag{7}$$

It was shown in [7] that this involves no loss in generality. We continue to use the notation employed in [9] (which is basically from [15]), summarized here for the reader's convenience.

If $q_1, q_2, \ldots$ are locally integrable on $[a, \infty)$, let $I_0 = 1$ and

$$I_j(t, s; q_j, \ldots, q_1) = \int_s^t q_j(\lambda) I_{j-1}(\lambda, s; q_{j-1}, \ldots, q_1) d\lambda, \quad s, t \geqq a, \quad j \geqq 1.$$

The following identities are by now well known [15]:

$$I_j(t, s; q_j, \ldots, q_1) = (-1)^j I_j(s, t; q_1, \ldots, q_j), \tag{8}$$

$$I_j(t, \tau; q_j, \ldots, q_1) = \textstyle\sum_{r=0}^{j} I_{j-r}(t, s; q_j, \ldots, q_{r+1}) I_r(s, \tau; q_r, \ldots, q_1). \tag{9}$$

The functions

$$x_j(t) = p_0(t) I_{j-1}(t, a; p_1, \ldots, p_{j-1}), \quad 1 \leqq j \leqq n, \tag{10}$$

form a fundamental system for (5) on $[a, \infty)$, and the functions

$$y_j(t) = p_n(t) I_{n-j}(t, a; p_{n-1}, \ldots, p_j), \quad 1 \leqq j \leqq n, \tag{11}$$

are likewise related to the formal adjoint equation

$$L_n^* y = \frac{1}{p_0}\frac{d}{dt}\frac{1}{p_1}\cdots\frac{1}{p_{n-1}}\frac{d}{dt}\frac{y}{p_n} = 0.$$

From (4) and (10),

(12) $$L_r x_j(t) = I_{j-r-1}(t,\ a;\ p_{r+1},\ldots,\ p_{j-1}),\quad 0 \leqq r \leqq j-1,$$

and

(13) $$L_r x_j(t) = 0,\quad j \leqq r \leqq n.$$

Moreover, from Lemma 2 of [9]

(14) $$(L_r x_k / L_r x_j)' > 0 \quad \text{on} \quad (a,\ \infty) \quad \text{and} \quad \lim_{t\to\infty} (L_r x_k(t)/L_r x_j(t)) = \infty,\quad r < j < k \leqq n,$$

and

(15) $$(y_j/y_k)' > 0 \quad \text{on} \quad (a,\ \infty) \quad \text{and} \quad \lim_{t\to\infty} (y_j(t)/y_k(t)) = \infty,\quad 1 \leqq j < k \leqq n.$$

When (6) holds, then $L_r x = x^{(r)}$,

(16) $$x_j(t) = \frac{(t-a)^{j-1}}{(j-1)!},\quad y_j(t) = \frac{(t-a)^{n-j}}{(n-j)!},$$

and (12), (13), (14), and (15) reduce to familiar properties of these functions.

Throughout this paper i and m are fixed integers,

(17) $$1 \leqq i \leqq m \leqq n,$$

and

(18) $$q(t) = \textstyle\sum_{j=1}^m a_j x_j(t)$$

is a given solution of (5). In [9] we gave conditions implying that (1) has a solution u such that

(19) $$L_r u = \begin{cases} L_r q + o(L_r x_i), & 0 \leqq r \leqq i-1, \\ L_r q + o(y_{r+1}/y_i), & i \leqq r \leqq n-1. \end{cases}$$

(We use "o" and "0" in the standard way to refer to behavior as $t\to\infty$.) Therefore, from (13), (14), and (15), $L_r u - L_r q$ is small compared with $L_r x_i$ $(0 \leqq r \leqq n-1)$ as $t\to\infty$. For example, if (6) holds, then (18) becomes

$$q(t) = \textstyle\sum_{j=1}^m a_j \dfrac{(t-a)^{j-1}}{(j-1)!}$$

and (19) is equivalent to

$$u^{(r)}(t) = q^{(r)}(t) + o(t^{i-r-1}), \quad 0 \leqq r \leqq n-1.$$

Fink and Kusano [1] considered only the case where $i=m$, and all their integral smallness conditions require absolute convergence; nevertheless, their results are a distinct improvement over ours in [9] in an important sense. In order to compare the results and to show how their work motivated the sharper estimates obtained below, we consider the special case where $i=m$ in (18), so that

$$q(t) = ax_i(t).$$

Then (19) becomes

$$L_r u = \begin{cases} (a+o(1))L_r x_i, & 0 \leqq r \leqq i-1, \\ o(y_{r+1}/y_i), & i \leqq r \leqq n-1. \end{cases} \tag{20}$$

However, the results of Fink and Kusano imply the existence of a solution u of (3) such that

$$L_r u = \begin{cases} aL_r x_i + o(J_r^{i-1}), & 0 \leqq r \leqq i-1, \\ o(J_r^{i-1}), & i \leqq r \leqq n-1, \end{cases} \tag{21}$$

where

$$J_r^{i-1}(t) = \begin{cases} L_r x_i(t), & 0 \leqq r \leqq n-1, \\ 1/I_{r-i+1}(t, a; p_r, \dots, p_i), & i \leqq r \leqq n-1. \end{cases} \tag{22}$$

It is easy to show that (20) and (21) are equivalent if (6) holds; however, in other cases (21) may be better for $i \leqq r \leqq n-2$ by orders of magnitude, since

$$y_{r+1}/y_i J_r^{i-1} \geqq 1, \quad i \leqq r \leqq n-1,$$

and some or all (except for $r=n-1$) of these ratios may be unbounded as $t \to \infty$, for suitably chosen $p_1, \dots, p_{n-1}$. Of course, in such cases the estimates of Fink and Kusano are better than (20). This motivated the author to reexamine the proof of the crucial Lemma 5 of [9], and to discover that it could be improved so as to yield the estimates of Fink and Kusano and, in some cases, to improve on them.

Before proceeding to our main results, we need a definition and two lemmas. The proofs of the lemmas are in Section 5.

DEFINITION 1. *Let ϕ be positive, continuous, and nondecreasing on (a, ∞), and suppose $i \in \{1, \dots, n\}$. Define the asymptotic deviation functions $\phi_{i0}, \dots, \phi_{i,n-1}$ associated with ϕ as follows:*

$$\phi_{ir}(t) = \int_t^\infty \phi(s) \left| \frac{\partial}{\partial s} \frac{I_{n-r-1}(s, t; p_{n-1}, \dots, p_{r+1})}{I_{n-i}(s, a; p_{n-1}, \dots, p_i)} \right| ds, \quad i-1 \leqq r \leqq n-2, \tag{23}$$

(24) $$\phi_{i,n-1}(t) = 2\phi(t)/I_{n-i}(t, a; p_{n-1}, \dots, p_i),$$

(25) $$\phi_{i,i-2}(t) = I_1(t, a; p_{i-1}\phi_{i,i-1}) \quad (\textit{if} \quad i \geqq 2),$$

and

(26) $$\phi_{ir}(t) = I_{i-r-1}(t, a; p_{r+1}, \dots, p_{i-2}, p_{i-1}\phi_{i,i-1}), \quad 0 \leqq r \leqq i-3 \ (\textit{if } i \geqq 3).$$

The motivation for this definition will become clear in the following lemma and its applications.

Here we state a convention which applies throughout the paper: when we write an improper integral in stating an assumption, we are assuming that it converges, and the convergence may be conditional unless the integrand is necessarily sign-constant for large t.

LEMMA 1. *Suppose $Q \in C[t_0, \infty)$, where $t_0 \geqq a$, and*

(27) $$\sup_{\tau \geqq t}\left|\int_\tau^\infty y_i(s)Q(s)ds\right| \leqq \phi(t), \quad t \geqq t_0,$$

where y_i is as defined in (11) *and ϕ is as in Definition* 1. *Then the integral*

(28) $$\hat{J}_i(t; Q) = \int_t^\infty p_n(s)I_{n-i}(t, s; p_i, \dots, p_{n-1})Q(s)ds$$

converges for $t \geqq t_0$. Now define

(29) $J_i(t, t_0; Q) = p_0(t)\hat{J}_i(t; Q)$ *if* $i = 1$;

(30) $J_i(t, t_0; Q) = p_0(t)I_1(t, t_0; p_1\hat{J}_i(\cdot\,; Q))$ *if* $i = 2$;

(31) $J_i(t, t_0; Q) = p_0(t)I_{i-1}(t, t_0; p_1, \dots, p_{i-2}, p_{i-1}\hat{J}_i(\cdot\,; Q))$ *if* $3 \leqq i \leqq n$.

Then $L_rJ_i(t, t_0; Q)$ is defined on $[t_0, \infty)$ for $0 \leqq r \leqq n$,

(32) $$|L_rJ_i(t, t_0; Q)| \leqq \phi_{ir}(t), \quad 0 \leqq r \leqq n-1,$$

and

(33) $$L_nJ_i(t, t_0; Q) = -Q(t).$$

The following lemma relates our asymptotic deviation functions to the functions J_r^{i-1} of Fink and Kusano [1].

LEMMA 2. *Let $\lambda_{i0}, \dots, \lambda_{i,n-1}$ be the asymptotic deviation functions associated with $\lambda = 1$. Then*

(34) $$\lambda_{ir}(t) = L_rx_i(t) = J_r^{i-1}(t), \quad 0 \leqq r \leqq i-1;$$

(35) $$\lambda_{ir}(t) \leqq 2/I_{r-i+1}(t, a; p_r, \dots, p_i) = 2J_r^{i-1}(t), \quad i \leqq r \leqq n-2;$$

(36) $$\lambda_{i,n-1}(t) = 2/I_{n-1}(t, a; p_{n-1}, \dots, p_i) = 2J_{n-1}^{i-1}(t).$$

If ψ is any function which is positive, continuous, and nondecreasing on $[a, \infty)$, then

(37) $$\psi_{ir}(t) \leqq \psi(t)\lambda_{ir}(t), \quad i-1 \leqq r \leqq n-1;$$

moreover, if

(38) $$\lim_{t\to\infty} \psi(t) = 0,$$

then

(39) $$\psi_{ir}(t) = o(\lambda_{ir}(t)), \quad 0 \leqq r \leqq n-1.$$

3. The main theorems

The next two theorems are our main results.

THEOREM 1. *Let ψ be positive, continuous, and nonincreasing on (a, ∞). Suppose there are constants $T \geqq a$ and $M > 0$ such that the function $F = F(t, y_0, \ldots, y_{n-1})$ is continuous on the set*

(40) $$\Omega = \{(t, y_0, \ldots, y_{n-1}) \mid t \geqq T;\ |y_r - L_r q(t)| \leqq M\psi_{ir}(t),\ 0 \leqq r \leqq n-1\}$$

where q is as in (18). *Suppose further that*

(41) $$|F(t, y_0, \ldots, y_{n-1}) - (Fq)(t)| \leqq w(t, |y_0 - L_0 q(t)|, \ldots, |y_{n-1} - L_{n-1} q(t)|)$$

on Ω, where $w = w(t, \xi_0, \ldots, \xi_{n-1})$ is continuous on the set

(42) $$S = \{(t, \xi_0, \ldots, \xi_{n-1}) \mid t \geqq T,\ 0 \leqq \xi_r \leqq M\psi_{ir}(t),\ 0 \leqq r \leqq n-1\}$$

and nondecreasing with respect to $\xi_0, \ldots, \xi_{n-1}$. Finally, suppose that

(43) $$\overline{\lim}_{t\to\infty} (\psi(t))^{-1} \int_t^\infty y_i(s) w(s, M\psi_{i0}(s), \ldots, M\psi_{i,n-1}(s)) ds = \theta M,$$

where $0 \leqq \theta < 1$, and

(44) $$\overline{\lim}_{t\to\infty} (\psi(t))^{-1} \left| \int_t^\infty y_i(s)(Fq)(s) ds \right| = \alpha < (1-\theta)M.$$

Then (3) *has a solution u on some interval $[t_0, \infty)$ $(t_0 \geqq T)$ such that*

(45) $$|L_r u(t) - L_r q(t)| \leqq M\psi_{ir}(t), \quad t \geqq t_0, \quad 0 \leqq r \leqq n-1,$$

(46) $$\overline{\lim}_{t\to\infty} (\psi_{ir}(t))^{-1} |L_r u(t) - L_r q(t)| \leqq \alpha + M\theta, \quad i-1 \leqq r \leqq n-1,$$

and

(47) $$L_r u(t) = L_r q(t) + o(\lambda_{ir}(t)), \quad 0 \leqq r \leqq n-1.$$

PROOF. Define

(48) $$\mu(t) = \left| \int_t^\infty y_i(s)(Fq)(s)ds \right| + \int_t^\infty y_i(s)w(s, M\psi_{ir}(s), \ldots, M\psi_{i,n-1}(s))ds.$$

From (43) and (44),

(49) $$\overline{\lim}_{t\to\infty} (\psi(t))^{-1}\mu(t) \leqq \alpha + M\theta,$$

and therefore we can choose $t_0 \geqq T$ so that

(50) $$\mu(t) \leqq M\psi(t), \quad t \geqq t_0.$$

Let $\mathscr{L}_{n-1}[t_0, \infty)$ be the set of functions v such that $L_0 v, \ldots, L_{n-1} v$ are continuous on $[t_0, \infty)$, with the topology of uniform convergence on finite subintervals; i.e., if $\{v_k\}$ is a sequence in $\mathscr{L}_{n-1}[t_0, \infty)$, we write

(51) $$v_k \longrightarrow v$$

if $\lim_{k\to\infty} L_r v_k(t) = L_r v(t)$, $t \geqq t_0$, $0 \leqq r \leqq n-1$, and all limits are uniform on $[t_0, t_1]$ for every $t_1 \geqq t_0$. Let $V_{n-1}(t_0, q, \psi)$ be the closed convex subset of $\mathscr{L}_{n-1}[t_0, \infty)$ consisting of functions v such that

(52) $$|L_r v(t) - L_r q(t)| \leqq M\psi_{ir}(t), \quad t \geqq t_0, \quad 0 \leqq r \leqq n-1.$$

Our continuity assumptions on F imply that Fv is continuous on $[t_0, \infty)$ whenever $v \in V_{n-1}(t_0, q, \psi)$; moreover, since

$$\int_t^\infty y_i(s)(Fv)(s)ds = \int_t^\infty y_i(s)(Fq)(s)ds + \int_t^\infty y_i(s)[(Fv)(s) - (Fq)(s)]ds,$$

(41), (50), (52), and our integrability assumptions imply that $\int^\infty y_i(t)(Fv)(t)dt$ converges, and that

(53) $$\left| \int_t^\infty y_i(s)(Fv)(s)ds \right| \leqq \mu(t) \leqq M\psi(t)$$

whenever $v \in V_{n-1}(t_0, q, \psi)$. Now define the transformation $\mathscr{T}$ by

(54) $$(\mathscr{T}v)(t) = q(t) + J_i(t, t_0; Fv),$$

with J_i as in Lemma 1. We will show that $\mathscr{T}$ satisfies the hypotheses of the Schauder-Tychonov theorem on $V_{n-1}(t_0, q, \psi)$. This will imply that there is a function u in this set such that

(55) $$u(t) = q(t) + J_i(t, t_0; Fu),$$

and it will be easy to show that u satisfies the stated conclusions.

Because of (53) and Lemma 1 with $\phi=M\psi$, $\mathscr{T}v$ is well defined and satisfies the inequalities

$$(56)\qquad |L_r(\mathscr{T}v)(t)-L_rq(t)| = |L_rJ_i(t, t_0; Fv)| \leqq M\psi_{ir}(t), \quad t \geqq t_0, \quad 0 \leqq r \leqq n-1.$$

This and the definition (52) of $V_{n-1}(t_0, q, \psi)$ imply that

$$\mathscr{T}(V_{n-1}(t_0, q, \psi)) \subset V_{n-1}(t_0, q, \psi).$$

We now show that $\mathscr{T}$ is continuous on $V_{n-1}(t_0, q, \psi)$; i.e., if $\{v_k\}$ is a sequence in this set which satisfies (51), then

$$(57)\qquad \mathscr{T}v_k \longrightarrow \mathscr{T}v.$$

If $t \geqq t_0$, then

$$(58)\qquad \left|\int_t^\infty y_i(s)\,[(Fv_k)(s)-(Fv)(s)]ds\right| \leqq \int_{t_0}^\infty y_i(s)|(Fv_k)(s)-(Fv)(s)|ds,$$

where the integral on the right converges because of (43), since

$$\begin{aligned}|(Fv_k)(s)-(Fv)(s)| &\leqq |(Fv_k)(s)-(Fq)(s)| + |(Fv)(s)-(Fq)(s)|\\ &\leqq w(s, |L_0v_k(s)-L_0q(s)|,\dots, |L_{n-1}v_k(s)-L_{n-1}q(s)|)\\ &\quad + w(s, |L_0v(s)-L_0q(s)|,\dots, |L_{n-1}v(s)-L_{n-1}q(s)|)\\ &\leqq 2w(s, M\psi_{i0}(s),\dots, M\psi_{i,n-1}(s)),\end{aligned}$$

where the first inequality follows from (41), and the second from (52). Moreover, since the integrand on the right of (58) converges pointwise to zero, Lebesgue's dominated convergence theorem implies that the integral approaches zero as $k\to\infty$. Therefore, for each $\varepsilon>0$ there is a K such that

$$(59)\qquad \left|\int_t^\infty y_i(s)\,[(Fv_k)(s)-(Fv)(s)]ds\right| < \varepsilon, \quad t \geqq t_0, \quad k \geqq K.$$

From (54),

$$(\mathscr{T}v_k)(t) - (\mathscr{T}v)(t) = J_i(t, t_0; Fv_k-Fv);$$

therefore, (59) and Lemma 1 with $Q=Fv_k-Fv$ and $\phi=\varepsilon$ imply that

$$|L_r(\mathscr{T}v_k)(t)-L_r(\mathscr{T}v)(t)| \leqq \varepsilon\lambda_{ir}(t), \quad t \geqq t_0, \quad 0 \leqq r \leqq n-1, \quad k \geqq K.$$

This implies (57).

If $v \in V_{n-1}(t_0, q, \psi)$, then (56) implies that

(60) $$|L_r(\mathscr{T}v)(t)| \leqq |L_r q(t)| + M\psi_{ir}(t), \quad t \geqq t_0, \quad 0 \leqq r \leqq n-1,$$

and (33) with $Q=Fv$ implies that

(61) $$|L_n(\mathscr{T}v)(t)| = |(Fv)(t)| \leqq |(Fv)(t)-(Fq)(t)| + |(Fq)(t)| \leqq w(t, M\psi_{i0}(t),\dots, M\psi_{i,n-1}(t)) + |(Fq)(t)|,$$

where we have used (41) and (52). With (60) and (61) it is straightforward to verify that the families

$$\{L_r\mathscr{T}v \mid v \in V_{n-1}(t_0, q, \psi)\}, \quad 0 \leqq r \leqq n-1,$$

are equibounded and uniformly equicontinuous on finite subintervals of $[t_0, \infty)$. This is more than enough to imply (via the Arzela-Ascoli theorem) that $\mathscr{T}(V_{n-1}(t_0, q, \psi))$ is relatively compact.

Now the Schauder-Tychonov theorem implies that there is a function u in $V_{n-1}(t_0, q, \psi)$ such that $\mathscr{T}u=u$; i.e., u satisfies (45) and (55). Differentiating (55) and recalling (33) shows that u satisfies (3), since $L_n q=0$. If $\varepsilon>0$, then (49) and (53) (with $v=u$) imply that there is a $t_1 \geqq t_0$ such that

$$\left|\int_t^\infty y_i(s)(Fu)(s)ds\right| \leqq (\alpha+M\theta+\varepsilon)\psi(t), \quad t \geqq t_1.$$

Applying Lemma 1 with $Q=Fu$, $\phi=(\alpha+M\theta+\varepsilon)\psi$, and $t_0=t_1$, we see from this that

(62) $$|L_rJ_i(t, t_1; Fu)| \leqq (\alpha+M\theta+\varepsilon)\psi_{ir}(t), \quad t \geqq t_1, \quad 0 \leqq r \leqq n-1.$$

From (55),

(63) $$L_r u(t) - L_r q(t) = L_r J_i(t, t_0; Fu), \quad 0 \leqq r \leqq n-1.$$

Since

$$L_rJ_i(t, t_1; Fu) = L_rJ_i(t, t_0; Fu), \quad i-1 \leqq r \leqq n-1,$$

(62) and (63) imply that

$$|L_r u(t) - L_r q(t)| \leqq (\alpha+M\theta+\varepsilon)\psi_{ir}(t), \quad t \geqq t_1, \quad i-1 \leqq r \leqq n-1.$$

Since ε is an arbitrary positive number, this implies (46).

To prove (47), define

$$\rho(t) = \sup_{\tau \geqq t}\left|\int_\tau^\infty y_i(s)(Fu)(s)ds\right|,$$

and let $\rho_{i0},\dots,\rho_{i,n-1}$ be the associated asymptotic deviation functions. From (63) and Lemma 1 with $Q=Fu$ and $\phi=\rho$,

$$|L_r u(t) - L_r q(t)| \leqq \rho_{ir}(t), \quad t \geqq t_0, \quad 0 \leqq r \leqq n-1.$$

Since $\lim_{t\to\infty} \rho(t)=0$, (39) with $\psi=\rho$ implies (47). This completes the proof.

If $\lim_{t\to\infty} \psi(t)>0$, then obviously $\theta=\alpha=0$ in (43) and (44), and we may as well take $\psi=1$. It is convenient to state this special case separately, as follows.

THEOREM 2. *Suppose there are constants $T\geqq a$ and $M>0$ such that the function F is continuous on the set*

$$(64) \qquad \tilde{\Omega} = \{(t, y_0,\dots, y_{n-1}) \mid t\geqq T;\ |y_r - L_r q(t)| \leqq M\lambda_{ir}(t),\ 0\leqq r\leqq n-1\},$$

and there satisfies (41), *where w is continuous on the set*

$$(65) \qquad \tilde{S} = \{(t, \xi_0,\dots, \xi_{n-1}) \mid t\geqq T;\ 0\leqq \xi_r \leqq M\lambda_{ir}(t),\ 0\leqq r\leqq n-1\},$$

and nondecreasing with respect to $\xi_0,\dots,\xi_{n-1}$. Suppose also that $\int^\infty y_i(t)(Fq)\cdot(t)dt$ converges and

$$(66) \qquad \int^\infty y_i(t) w(t, M\lambda_{i0}(t),\dots, M\lambda_{i,n-1}(t))dt < \infty.$$

Then (3) *has a solution u which is defined on $[t_0, \infty)$ for some $t_0\geqq T$ and satisfies the inequalities*

$$|L_r u(t) - L_r q(t)| \leqq M\lambda_{ir}(t), \quad t\geqq t_0, \quad 0\leqq r\leqq n-1.$$

Moreover,

$$L_r u(t) = L_r q(t) + o(\lambda_{ir}(t)), \quad 0\leqq r\leqq n-1.$$

Theorem 2 implies Theorem 2 of Fink and Kusano [1], which deals with the special case where $i=m$ in (18). In checking this, the reader should recall the first paragraph of Lemma 2.

REMARK 1. If the function u whose existence is guaranteed by Theorem 2 happens also to satisfy the condition

$$\int_t^\infty y_i(s)(Fu)(s)ds = 0(\psi(t)),$$

where $\lim_{t\to\infty} \psi(t)=0$, then an argument like that in the last paragraph of the proof of Theorem 1 shows that u actually satisfies the conditions

$$L_r u(t) = L_r q(t) + 0(\psi_{ir}(t)), \quad 0\leqq r\leqq n-1.$$

Since this is essentially the conclusion of Theorem 1, one might erroneously conclude that Theorem 1 is only a trivial extension of Theorem 2. The fallacy in this conclusion is that if $\lim_{t\to\infty} \psi(t)=0$, then the assumptions of Theorem 1 do not imply those of Theorem 2, for the following reasons:

(a) The sets Ω and S in (40) and (42) are smaller than the sets $\tilde{\Omega}$ and $\tilde{S}$ in (64) and (65).

(b) The integrability condition (43), even if $\theta=0$, does not imply (66).

Put another way, the hypotheses of Theorem 1 in this case imply the hypotheses of the Schauder-Tychonov theorem on $V_{n-1}(t_0, q, \psi)$, but not on the larger set $V_{n-1}(t_0, q, 1)$ with which one must deal in proving Theorem 2. Therefore, it is quite possible for Theorem 1 to be applicable in situations where Theorem 2 is not, even though the conclusions of the former are stronger than those of the latter. We will point this out more specifically in the next section.

4. Applications

In this section we consider the equation

$$L_n u + \sum_{r=0}^{n-1} P_{n-r}(L_r u)^{\gamma_r} = f(t), \tag{67}$$

under the following assumption.

ASSUMPTION A. *The functions* $f, P_1, \ldots, P_n$ *are continuous on* $[a, \infty)$, ψ *is continuous, nondecreasing, and positive on* (a, ∞), *and* $\lim_{t\to\infty}\psi(t)=0$. *The exponents* $\gamma_m, \ldots, \gamma_{n-1}$ *are positive rationals with odd denominators, while* $\gamma_0, \ldots, \gamma_{m-1}$ *can be arbitrary real numbers*; $a_m>0$ *in* (18).

THEOREM 3. *Suppose Assumption A holds and*

$$\int_t^\infty y_i(s)f(s)ds = 0(\psi(t)), \tag{68}$$

$$\int_t^\infty y_i(s)P_{n-r}(s)(L_r x_m(s))^{\gamma_r}ds = 0(\psi(t)), \quad 0 \leqq r \leqq m-1, \tag{69}$$

$$\overline{\lim}_{t\to\infty}(\psi(t))^{-1}\int_t^\infty y_i(s)|P_{n-r}(s)|(L_r x_m(s))^{\gamma_r-1}\psi_{ir}(s)ds = c_r < \infty, \quad 0 \leqq r \leqq m-1, \tag{70}$$

and, if $m<n-1$,

$$\int_t^\infty y_i(s)|P_{n-r}(s)|(\psi_{ir}(s))^{\gamma_r}ds = o(\psi(t)), \quad m \leqq r \leqq n-1. \tag{71}$$

Finally, suppose that

$$\sum_{r=0}^{m-1} c_r|\gamma_r|a_m^{\gamma_r-1} = \theta < 1. \tag{72}$$

Then (67) *has a solution* u, *defined for* t *sufficiently large, such that*

$$L_r u(t) = L_r q(t) + o(\lambda_{ir}(t)), \quad 0 \leqq r \leqq n-1, \tag{73}$$

and

$$L_r u(t) = L_r q(t) + 0(\psi_{ir}(t)), \quad 0 \leqq r \leqq n-1. \tag{74}$$

Moreover, if

$$c_0 = \cdots = c_m = 0 \tag{75}$$

and (68) *and* (69) *hold with* "0" *replaced by* "*o*," *then*

$$L_r u(t) = L_r q(t) + o(\psi_{ir}(t)), \quad i-1 \leqq r \leqq n-1. \tag{76}$$

PROOF. Equation (67) has the form (3), with

$$F(t, y_0, \ldots, y_{n-1}) = f(t) - \textstyle\sum_{r=0}^{n-1} P_{n-r}(t) y_r^{\gamma_r}, \tag{77}$$

which is real-valued and continuous on the set

$$\{(t, y_0, \ldots, y_{n-1}) \mid t \geqq a;\ y_r > 0,\ 0 \leqq r \leqq m-1;\ -\infty < y_r < \infty,\ m \leqq r \leqq n-1\}. \tag{78}$$

(Recall the restrictions on $\gamma_m, \ldots, \gamma_{n-1}$.) Since $a_m > 0$ in (18), $L_0 q, \ldots, L_{m-1} q$ are positive on some interval $[T_0, \infty)$, and the function

$$Fq = f - \textstyle\sum_{r=0}^{m-1} P_{n-r}(L_r q)^{\gamma_r}$$

is continous there.

Because of (69),

$$\int_t^\infty y_i(s) P_{n-r}(s) (L_r q(s))^{\gamma_r} ds = 0(\psi(t)), \quad 0 \leqq r \leqq m-1. \tag{79}$$

To see this, let

$$h_i(t) = \int_t^\infty y_i(s) P_{n-r}(s) (L_r x_m(s))^{\gamma_r} ds,$$

rewrite the integrand in (79) as

$$-\int_t^\infty h_r'(s) \left(\frac{L_r q(s)}{L_r x_m(s)} \right)^{\gamma_r} ds,$$

integrate by parts, and invoke (14) and (69). Now (68) and (79) imply that

$$\overline{\lim}_{t\to\infty} (\psi(t))^{-1} \left| \int_t^\infty y_i(s) (Fq)(s) ds \right| = \alpha < \infty. \tag{80}$$

Since $i \leqq m$ and $\lim_{t\to\infty} \psi(t) = 0$, (14), (34), (35), (36) and (39) imply that

$$\lim_{t\to\infty} (L_r q(t) \pm M\psi_{ir}(t)) / L_r x_m(t) = a_m > 0, \quad 0 \leqq r \leqq m-1, \tag{81}$$

for any constant M. Therefore, we can choose M so that

$$M > \alpha/(1-\theta)$$

(see (72) and (80)), and then choose T so that $T \geqq T_0$ and

$$L_r q(t) - M\psi_{ir}(t) > 0, \quad t \geqq T, \quad 0 \leqq r \leqq m-1.$$

Then the set Ω in (40) is contained in the set (78). Therefore, F as defined in (77) is continuous on Ω, and there satisfies (41), with

$$w(t, \xi_0, \ldots, \xi_{n-1}) = \sum_{r=0}^{m-1} |\gamma_r P_{n-r}(t)|(L_r q(t) \pm M\psi_{ir}(t))^{\gamma_r - 1}\xi_r + \sum_{r=m}^{n-1} |P_{n-r}(t)|\xi_r^{\gamma_r}, \tag{82}$$

where the plus sign applies in the r-th term of the first sum if $\gamma_r \geqq 1$, the minus if $\gamma_r < 1$. Now (70), (71), and (81) imply (43), with θ as in (72), so Theorem 1 implies that (67) has a solution u which satisfies (73) and (74). Moreover, it is easy to show that if (68) and (69) hold with "0" replaced by "o," then $\alpha = 0$ in (80). Since (75) implies that $\theta = 0$, (46) implies (76). This completes the proof.

The next theorem is an analog of Theorem 3 for the case where $\psi = 1$.

THEOREM 4. *Suppose Assumption* A *holds and the integrals*

$$\int^{\infty} y_i(t) f(t) dt \tag{83}$$

and

$$\int^{\infty} y_i(t) P_{n-r}(t)(L_r x_m(t))^{\gamma_r} dt, \quad 0 \leqq r \leqq m-1, \tag{84}$$

converge. Suppose also that

$$\int^{\infty} y_i(t)|P_{n-r}(t)|(L_r x_m(t))^{\gamma_r - 1}\lambda_{ir}(t) dt < \infty, \quad 0 \leqq r \leqq m-1, \tag{85}$$

and, if $m < n-1$,

$$\int^{\infty} y_i(t)|P_{n-r}(t)|(\lambda_{ir}(t))^{\gamma_r} dt < \infty, \quad m \leqq r \leqq n-1. \tag{86}$$

Then (67) *has a solution* u *which satisfies* (73).

PROOF. As in the proof of Theorem 3, (83) and (84) imply that $\int^{\infty} y_i(t)(Fq)\cdot(t)dt$ converges (with F as in (77)). With T_0 as in the proof of Theorem 3 and M such that $0 < M < a_m$, choose $T \geqq T_0$ so that

$$L_r q(t) - M\lambda_{ir}(t) > 0, \quad t \geqq T, \quad 0 \leqq r \leqq m-1.$$

With w as in (82) (with $\psi_{ir} = \lambda_{ir}$), (85) and (86) imply (66). Hence, Theorem 2 implies the conclusion.

If (6) holds, then (67) reduces to

$$u^{(n)} + \sum_{r=0}^{n-1} P_{n-r}(t)(u^{(r)})^{\gamma_r} = f(t). \tag{87}$$

This equation and its special cases have been studied by many authors (e.g., see Hallam [3] and Waltman [14]). Except in the most recent literature, the exponents $\gamma_0, \ldots, \gamma_{n-1}$ have been required to be positive. Recently, however, results on the special case

$$u^{(n)} + P(t)u^{\gamma} = f(t)$$

have appeared in which γ is permitted to be negative (e.g., see [2], [4], [5], [11], and [12].) The following theorems, as well as Theorems 3 and 4, are therefore unusual not only because of the extent to which their integral smallness conditions permit conditional convergence, but also because both positive and negative exponents may occur in the same equation.

THEOREM 5. *Let f, $P_1, \ldots, P_n$ be continuous on $[0, \infty)$, and suppose $\gamma_0, \ldots, \gamma_{n-1}$ are as in Assumption* A. *Let*

$$q(t) = \sum_{j=1}^{m} a_j \frac{t^{j-1}}{(j-1)!},$$

where $a_m > 0$ and (17) *holds. Suppose $\alpha > 0$ and, if $i \geqq 2$, assume also that $\alpha < 1$. Suppose also that*

$$\int_t^\infty s^{n-i} f(s)ds = O(t^{-\alpha}), \tag{88}$$

$$\int_t^\infty s^{n-i+(m-r-1)\gamma_r} P_{n-r}(s)ds = O(t^{-\alpha}), \quad 0 \leqq r \leqq m-1, \tag{89}$$

$$\overline{\lim}_{t\to\infty} t^{\alpha} \int_t^\infty s^{n-m-\alpha+(m-r-1)\gamma_r} |P_{n-r}(s)| ds = C_r < \infty, \quad 0 \leqq r \leqq m-1, \tag{90}$$

and, if $m < n-1$,

$$\int_t^\infty s^{n-i+(i-r-\alpha-1)\gamma_r} |P_{n-r}(s)| ds = o(t^{-\alpha}), \quad m \leqq r \leqq n-1. \tag{91}$$

Then there are positive constants $B_0, \ldots, B_{m-1}$, which do not depend on $C_0, \ldots, C_{m-1}$ or a_m, such that if

$$\sum_{r=0}^{m} B_r C_r a_m^{\gamma_r - 1} = \theta < 1, \tag{92}$$

then (87) *has a solution u which satisfies*

$$u^{(r)}(t) = q^{(r)}(t) + O(t^{i-r-\alpha-1}), \quad 0 \leqq r \leqq n-1.$$

Moreover, if

$$C_0 = \cdots = C_m = 0$$

and (88) *and* (89) *hold with* "0" *replaced by* "o," *then*

(93) $$u^{(r)}(t) = q^{(r)}(t) + o(t^{i-r-\alpha-1}), \quad 0 \leqq r \leqq n-1.$$

PROOF. It is straightforward to verify from our assumptions on α that if $\psi(t) = t^{-\alpha}$, then the associated asymptotic deviation functions satisfy

$$\psi_{ir}(t) = 0(t^{i-r-\alpha-1}), \quad 0 \leqq r \leqq n-1.$$

Since (6) implies (16), (88) through (92) are the appropriate specializations of (68) through (72), with suitable constants $B_0, \ldots, B_m$ in (92). Therefore, Theorem 3 implies the stated conclusions, except for one detail: (76) implies (93) only for $i-1 \leqq r \leqq n-1$. However, since $\alpha < 1$ if $i \geqq 2$, Lemma 2 of [11] implies (93) for $0 \leqq r \leqq i-2$.

Theorem 4 implies the following result for $\alpha = 0$.

THEOREM 6. *Suppose the conditions in the first sentence of Theorem* 5 *hold, and the integrals*

(94) $$\int^{\infty} t^{n-i} f(t)\,dt,$$

(95) $$\int^{\infty} t^{n-i+(m-r-1)\gamma_r} P_{n-r}(t)\,dt, \quad 0 \leqq r \leqq m-1,$$

(96) $$\int^{\infty} t^{n-m+(m-r-1)\gamma_r} |P_{n-r}(t)|\,dt, \quad 0 \leqq r \leqq m-1,$$

and, if $m < n-1$,

(97) $$\int^{\infty} t^{n-i+(i-r-1)\gamma_r} |P_{n-r}(t)|\,dt, \quad m \leqq r \leqq n-1,$$

converge. Then (87) *has a solution u such that*

$$u^{(r)}(t) = q^{(r)}(t) + o(t^{i-r-1}), \quad 0 \leqq r \leqq n-1.$$

REMARK 2. Even though the conclusions of Theorem 5 are obviously sharper than those of Theorem 6, either may apply in situations where the other does not. To see this, we observe that the existence of (94) and (95) does not imply (88) and (89), while (90) and (91) do not imply the existence of (96) and (97). This illustrates the point raised in Remark 1.

An argument similar to (but simpler than) the proof of Theorem 4 yields the following result for the linear equation

(98) $$L_n u + \sum_{r=0}^{n-1} P_{n-r}(t) L_r u = f(t).$$

THEOREM 7. *Suppose f, $P_1,\dots, P_n$ and ψ satisfy Assumption A. Suppose also that* (68) *holds,*

(99) $$\int_t^\infty y_i(s)P_{n-r}(s)L_r x_m(s)ds = O(\psi(t)), \quad 0 \leqq r \leqq m-1,$$

and

$$\overline{\lim}_{t\to\infty} (\psi(t))^{-1} \int_t^\infty y_i(s)\left(\sum_{r=0}^{n-1} |P_{n-r}(s)|\psi_{ir}(s)\right)ds = \theta < 1.$$

Then (98) *has a solution u which satisfies* (73) *and* (74). *Moreover, if $\theta=0$ and* (68) *and* (99) *hold with "0" replaced by "o," then u satisfies* (76).

Obviously, we do not have to assume in Theorem 7 that $a_m > 0$ in (18). This remark also applies to the following analog of Theorem 6.

THEOREM 8. *Suppose f, $P_1,\dots, P_n$ are continuous on $[a, \infty)$ and the integrals* (83) *and*

$$\int^\infty y_i(t)P_{n-r}(t)L_r x_m(t)dt, \quad 0 \leqq r \leqq m-1,$$

converge. Suppose also that

$$\int^\infty y_i(t)|P_{n-r}(t)|\lambda_{ir}(t)dt < \infty, \quad 0 \leqq r \leqq n-1.$$

Then (98) *has a solution u which satisfies* (73).

Theorem 8 implies and improves on Theorem 3 of [9].

5. Proofs of the lemmas

We need the following preliminary lemma.

LEMMA 3. *Suppose $q_1, q_2,\dots$ are positive and continuous on $[a, \infty)$, and*

(100) $$\int^\infty q_i(s)ds = \infty, \quad i = 1, 2, \dots.$$

Then

$$\frac{d}{ds}\left[\frac{I_j(s, b: q_1,\dots, q_j)}{I_k(s, a; q_1,\dots, q_k)}\right]$$

has exactly one zero on (b, ∞) if $a<b$ and $1 \leqq j < k$.

PROOF. For convenience, let

$$F_k(s) = I_k(s, a; q_1,\dots, q_k), \quad G_j(s) = I_j(s, b; q_1,\dots, q_j),$$

$$f_{k-1}(s) = I_{k-1}(s, a; q_2,\dots, q_k), \quad g_{j-1}(s) = I_{j-1}(s, b; q_2,\dots, q_j).$$

Then

$$\left(\frac{G_j}{F_k}\right)' = \frac{q_1 f_{k-1} h_{jk}}{F_k^2}, \tag{101}$$

where

$$h_{jk} = F_k \frac{g_{j-1}}{f_{k-1}} - G_j$$

and

$$h'_{jk} = F_k\left(\frac{g_{j-1}}{f_{k-1}}\right)'. \tag{102}$$

Since $1 \leqq j \leqq k$, G_j/F_k vanishes at b and ∞ (the latter because of (100) and L'Hospital's rule); hence, $(G_j/F_k)'$ has at least one zero on (b, ∞). From (101), the zeros of $(G_j/F_k)'$ coincide with those of h_{jk}. We proceed by induction on j. Since $g_0 = 1$, (102) implies that $h'_{1k}(s) < 0$, $s > b$; hence, Rolle's theorem implies that h_{1k} cannot have more than one zero on $[b, \infty)$. This implies the conclusion for $j = 1$. Now suppose $j \geqq 2$ and assume the conclusion with j replaced by $j-1$. Then $(g_{j-1}/f_{k-1})'$ has exactly one zero on (b, ∞), and (102) implies the same for h'_{jk}. Since $h_{jk}(b) = 0$ if $j \geqq 2$, Rolle's theorem now implies that h_{jk} has at most one zero on (b, ∞); hence (101) implies the same for $(G_j/F_k)'$, which completes the proof.

PROOF OF LEMMA 2. Recall that $\lambda_{i0},\dots, \lambda_{i,n-1}$ are obtained by taking $\phi = 1$ in Definition 1. Therefore, from (23),

$$\lambda_{ir}(t) = \int_t^\infty \left| \frac{\partial}{\partial s} \frac{I_{n-r-i}(s, t; p_{n-1},\dots, p_{r+1})}{I_{n-i}(s, a; p_{n-1},\dots, p_i)} \right| ds, \quad i-1 \leqq r \leqq n-2. \tag{103}$$

The partial derivative here is positive if $r = i-1$ ([9], Lemma 3), and

$$\lim_{s\to\infty} \frac{I_{n-i}(s, t; p_{n-1},\dots, p_i)}{I_{n-i}(s, a; p_{n-1},\dots, p_i)} = 1,$$

because of (7) and L'Hospital's rule. Therefore,

$$\lambda_{i,i-1}(t) = 1, \tag{104}$$

which verifies (34) for $r = i-1$.

If $i \leqq r \leqq n-2$, then (103) and Lemma 3 imply that

$$\lambda_{ir}(t) = 2\frac{I_{n-r-1}(\bar{s}, t; p_{n-1},\dots, p_{r+1})}{I_{n-i}(\bar{s}, a; p_{n-1},\dots, p_i)},$$

where $\bar{s}$ is the unique zero of the partial derivative in (103) on $[t, \infty)$. This and

(9) imply the inequality in (35). Simply recalling (12), (22), and (104) now makes it trivial to complete the verification of (34), (35), and (36).

Recalling that $\psi_{i0},\dots,\psi_{i,r-1}$ are obtained by taking $\phi=\psi$ in Definition 1 makes (37) obvious. If (38) holds, then (37) implies (39) for $i-1\leqq r\leqq n-1$. Also, from (37) and (104), $\psi_{i,i-1}(t)\leqq\psi(t)$, so (25) and (26) with $\phi=\psi$ imply (39) for $0\leqq r\leqq i-2$, because of (7) and L'Hospital's rule. This completes the proof of Lemma 2.

PROOF OF LEMMA 1. Let

$$c(t)=\int_t^\infty y_i(s)Q(s)ds, \tag{105}$$

so that

$$|c(t)|\leqq\phi(t), \tag{106}$$

from (27). If $i-1\leqq r\leqq n-1$, let

$$\begin{aligned}Q_{ir}(t)&=\int_t^\infty p_n(s)I_{n-r-1}(t,s;p_{r+1},\dots,p_{n-1})Q(s)ds,\\&=(-1)^{n-r-1}\int_t^\infty p_n(s)I_{n-r-1}(s,t;p_{n-1},\dots,p_{r+1})Q(s)ds\ \text{(see (8))}\\&=(-1)^{n-r}\int_t^\infty c'(s)\frac{I_{n-r-1}(s,t;p_{n-1},\dots,p_{r+1})}{I_{n-i}(s,a;p_{n-1},\dots,p_i)}ds\quad\text{(see (11) and (105)).}\end{aligned}$$

Integrating this by parts yields

$$\begin{aligned}Q_{ir}(t)&=(-1)^{n-r}c(s)\frac{I_{n-r-1}(s,t;p_{n-1},\dots,p_{r+1})}{I_{n-i}(s,a;p_{n-1},\dots,p_i)}\Bigg|_t^\infty\\&\quad+(-1)^{n-r-1}\int_t^\infty c(s)\frac{\partial}{\partial s}\left[\frac{I_{n-r-1}(s,t;p_{n-1},\dots,p_{r+1})}{I_{n-i}(s,a;p_{n-1},\dots,p_i)}\right]ds.\end{aligned} \tag{107}$$

Since $\lim_{t\to\infty}c(t)=0$, (106) and (107) imply that

$$|Q_{ir}(t)|\leqq\phi_{ir}(t),\quad t\geqq t_0,\quad i-1\leqq r\leqq n-1. \tag{108}$$

(Recall (23) and (24).) Since $\hat{J}_i(t,Q)=Q_{i,i-1}(t)$, it now follows that the improper integral in (28) converges, and so $J_i(t,t_0;Q)$ is well defined on $[t_0,\infty)$ by (29), (30), or (31). Moreover, on recalling (4), it is routine to verify (33) and that

$$L_rJ_i(t,t_0;Q)=Q_{ir}(t),\quad i-1\leqq r\leqq n-1,$$

which, with (108), implies (32) for $i-1\leqq r\leqq n-1$. Since

$$L_{i-2}J_i(t,t_0;Q)=I_1(t,t_0;p_{i-1}Q_{i,i-1})$$

if $i \geqq 2$, and

$$L_r J_i(t, t_0; Q) = I_{i-r-1}(t, t_0; p_{r+1}, \dots, p_{i-2}, p_{i-1}Q_{i,i-1}), \quad 0 \leqq r \leqq i-3,$$

if $i \geqq 3$, (25), (26), and (108) with $r=i-1$ imply (32) with $0 \leqq r \leqq i-2$. This completes the proof of Lemma 1.

ACKNOWLEDGMENT. The author thanks Professors A. Fink and T. Kusano for making [1] available to him in preprint form. Their results motivated considerable improvements over an earlier draft of the work presented here.

References

[1] A. M. Fink and T. Kusano, Nonoscillation theorems for a class of perturbed disconjugate differential equations, Japan. J. Math. **9** (1983), 277–291.
[2] J. R. Graef, M. K. Grammatikopolous, and P. W. Spikes, On the positive solutions of a higher order differential equation with a discontinuity, Internat. J. Math. and Math. Sci., **5** (1982), 263–273.
[3] T. G. Hallam, Asymptotic behavior of the solutions of an n-th order nonhomogeneous ordinary differential equation, Trans. Amer. Math. Soc., **122** (1966), 177–194.
[4] T. Kusano and C. A. Swanson, Asymptotic properties of semilinear elliptic equations, Funk. Ekvac. **26** (1983), 115–129.
[5] S. Taliaferro, On the positive solutions of $y''+\phi(t)y^{-\lambda}=0$, Nonlinear Anal., **2** (1978), 437–446.
[6] W. F. Trench, Asymptotic behavior of solutions of perturbed disconjugate equations, J. Diff. Eqns., **11** (1972), 661–671.
[7] W. F. Trench, Canonical forms and principal systems for general disconjugate equations, Trans. Amer. Math. Soc., **189** (1974), 319–327.
[8] W. F. Trench, Asymptotic integration of linear differential equations subject to integral smallness conditions involving ordinary convergence, SIAM J. Math. Anal., **7** (1976), 213–221.
[9] W. F. Trench, Asymptotic theory of perturbed general disconjugate equations, Hiroshima Math. J. **12** (1982), 43–58.
[10] W. F. Trench, Systems of differential equations subject to mild integral conditions, Proc. Amer. Math. Soc., **87** (1983), 263–270.
[11] W. F. Trench, Asymptotic integration of $y^{(n)}+P(t)y^r=f(t)$ under mild integral smallness conditions, Funk. Eckvac. **26** (1983), 197–209.
[12] W. F. Trench, Asymptotic behavior of solutions of an n-th order differential equation, to appear in Rocky Mountain J..
[13] W. F. Trench, Asymptotic integration of linear differential equations subject to mild integral conditions, to appear in SIAM J. Math. Anal..
[14] P. Waltman, On the asymptotic behavior of solutions of a nonlinear equation, Proc. Amer. Math. Soc., **15** (1964), 918–923.
[15] D. Willett, Asymptotic behaviour of solutions of disconjugate n-th order differential equations, Can. J. Math., **23** (1971), 293–314.

Department of Mathematics and Computer Science,
Drexel University,
Philadelphia, Pennsylvania 19104,
U. S. A.