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Self-adjoint harmonic spaces and Dirichlet forms

Michael RÖCKNER

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1. Introduction and notations

The aim of this paper is to clarify the relation between energy forms on a self-adjoint harmonic space $(X, \mathscr{H})$ studied by Maeda in [7] (cf. also [6]) and Dirichlet forms on $L^2(X; m)$ in the sense of Fukushima [5] and Silverstein [11]. Here X denotes a locally compact Hausdorff space with a countable base, connected and locally connected, $\mathscr{H}$ the harmonic sheaf and m a positive Radon measure on X. More precisely: we determine the set of all positive Radon measures m on X such that Maeda's energy form E with domain $\mathscr{E}_0$ can be considered as an „extended Dirichlet space with reference measure m" as defined in [5] und [11].

Let us recall the basic definitions and notations and give a brief review of Maeda's construction of energy forms.

Let $(X, \mathscr{H})$ be a self-adjoint harmonic space as defined in [6] §1.2. In particular we assume that the constant function 1 is superharmonic (Axiom 4 in [6]). Let G denote the symmetric (up to a multiplicative constant unique) Green function of X. Let ${}^*\mathscr{H}^+(X)$ denote the set of all positive hyperharmonic functions on X. $(X, {}^*\mathscr{H}^+(X))$ is a standard balayage space in the sense of [2]. Let τ_f denote the ${}^*\mathscr{H}^+(X)$-fine topology on X; i.e., the coarsest topology on X such that each function in ${}^*\mathscr{H}^+(X)$ is continuous with respect to τ_f. Notations with respect to τ_f will be designated by the prefix „fine(ly)-". For a numerical function g on X let $\hat{g}$ denote the greatest lower semi-continuous minorant of g. Define for $u \in {}^*\mathscr{H}^+(X)$ and $A \subset X$

$$R_u^A := \inf\{v \in {}^*\mathscr{H}^+(X): v \geq u \quad \text{on} \quad A\},$$

then $\hat{R}_u^A$ is the so-called balayage of u on A. Let $\mathscr{M}$ denote the set of all Radon measures on X and $\mathscr{M}^+ := \{\mu \in \mathscr{M}: \mu \geq 0\}$. We define for $\mu \in \mathscr{M}^+$

$$G\mu := \int G(\cdot, y) d\mu(y)$$

and for $\mu \in \mathscr{M}$ and $x \in \{G\mu^+ < \infty\} \cup \{G\mu^- < \infty\}$

$$G\mu(x) = G\mu^+(x) - G\mu^-(x),$$

where $\mu^+ := \sup(\mu, 0)$, $\mu^- := -\inf(\mu, 0)$ in the lattice vector space $\mathcal{M}$. If p is a potential on X, then $S(p)$ shall denote the smallest closed subset of X such that p is harmonic on its complement. For every potential p on X there exists a unique $\mu \in \mathcal{M}^+$ such that $p = G\mu$ and $S(p) = \operatorname{supp} \mu$ (i.e., the support of the measure μ). Let $\mathcal{P}$ denote the set of all real continuous potentials on X and let

$$\Lambda := \{\mu \in \mathcal{M}^+ : \int p\, d\mu < \infty \text{ for every } p \in \mathcal{P} \text{ with } S(p) \text{ compact}\}.$$

For $\mu \in \Lambda$ the balayaged measure of μ on A, $A \subset X$, is denoted by μ^A (cf. [3] §7.1). We define the set of measures of bounded energy on X by

$$\mathcal{M}_E := \{\mu \in \mathcal{M} : \int G|\mu|\, d|\mu| < \infty\},$$

where $|\mu| := \mu^+ + \mu^-$, and let $\mathcal{M}_E^+ := \{\mu \in \mathcal{M}_E : \mu \geq 0\}$. Clearly $\mathcal{M}_E^+ \subset \Lambda$ and $G\mu$ is a potential for every $\mu \in \mathcal{M}_E^+$. Let for $\mu, \nu \in \mathcal{M}_E$

$$\langle \mu, \nu \rangle_E := \int G\mu\, d\nu,$$

then $(\mathcal{M}_E, \langle\, , \rangle_E)$ is a pre-Hilbert space (cf. [8] Cor. 4.5 and Theorem 4.2). Let $(H, \langle\, , \rangle_E)$ denote its (abstract) completion and set $\|\ \|_E := \langle\, , \rangle_E^{1/2}$.

In [7] §5.3 Maeda considers a symmetric bilinear form E^* on

$$\mathcal{P}_{EC} := \{G\mu - G\nu : \mu, \nu \in \mathcal{M}^+;\ G\mu, G\nu \text{ bounded and continuous};\ \mu(X), \nu(X) < \infty\}$$

which is the restriction of a symmetric bilinear form defined on the larger space $\boldsymbol{B}_E$ (for the definition see [6] §2.1). By [7] Lemma 5.11 and the following corollary we have

$$E^*(f, g) = \langle \mu_1 - \mu_2, \nu_1 - \nu_2 \rangle_E,$$

if $f := G\mu_1 - G\mu_2$, $g := G\nu_1 - G\nu_2 \in \mathcal{P}_{EC}$. E^* is strictly positive definite. Define

$$\mathcal{E}_0^* := \{f : X \to \boldsymbol{R} \cup \{\pm\infty\} : \text{there exists an } E^*\text{-Cauchy sequence } (f_n)_{n \in \boldsymbol{N}} \text{ in } \mathcal{P}_{EC} \text{ such that } \lim_{n\to\infty} f_n = f \text{ quasi-everywhere on } X\},$$

where „quasi-everywhere", abbreviated „q.e.", means „except for a polar set". Extending E^* to $\mathcal{E}_0^*$ and identifying functions in $\mathcal{E}_0^*$, which are equal q.e., we get a real Hilbert space $(\mathcal{E}_0, E)$, where E denotes the scalar product (cf. [7] Theorem 5.1, and the following corollary). Let π_c denote the canonical quotient map from $\mathcal{E}_0^*$ to $\mathcal{E}_0$ and let C be the capacity introduced in [7] §5.1. By Prop. 5.3 in [7] each element of $\mathcal{E}_0^*$ is quasi-continuous with respect to C (cf. [7] §5.2).

By [7] Lemma 5.11 we know that $G\mu \in \mathcal{E}_0$, if $\mu \in \mathcal{M}_E^+$ and by [7] Lemma 5.12 that $E(G\mu, G\nu) = \langle \mu, \nu \rangle_E$ for all $\mu, \nu \in \mathcal{M}_E^+$. Hence the map $\mu \to G\mu$, $\mu \in \mathcal{M}_E$, is a linear map from $\mathcal{M}_E$ to $\mathcal{E}_0$ and extends to a unitary operator from the Hilbert space $(H, \langle\, , \rangle_E)$ to $(\mathcal{E}_0, E)$.

For $\mu \in \mathscr{M}^+$ let $L^1(X;\mu)$ (resp. $L^2(X;\mu)$) denote the space of (classes of) μ-integrable (resp. μ-square integrable) functions on X and set for $f \in L^2(X;\mu)$

$$\|f\|_2 := \int |f|^2 d\mu.$$

Let $m \in \mathscr{M}^+$ such that $\operatorname{supp} m = X$. We recall that according to [5] a pair $(\mathscr{F}_e, E)$ is called an „extended (transient) Dirichlet space with reference measure m", if the following conditions are satisfied:

($\mathscr{F}_e$.1) $\mathscr{F}_e$ is a real Hilbert space with inner product E.

($\mathscr{F}_e$.2) There exists an m-integrable, bounded function g, strictly positive m-a.e. such that $\mathscr{F}_e \subset L^1(X; g\cdot m)$ and

$$\int |u| g\, dm \le \sqrt{E(u,u)} \qquad \text{for every} \quad u \in \mathscr{F}_e.$$

($\mathscr{F}_e$.3) $\mathscr{F}_e \cap L^2(X;m)$ is dense both in $(L^2(X;m), \|\ \|_2)$ and in $(\mathscr{F}_e, E)$.

($\mathscr{F}_e$.4) Every normal contraction operates on $(\mathscr{F}_e, E)$; i.e., if $u \in \mathscr{F}_e$ and v is a normal contraction of u (i.e., $|v^*(x)| \le |u^*(x)|$ and $|v^*(x) - v^*(y)| \le |u^*(x) - u^*(y)|$ for all $x, y \in X$ for some Borel version v^* of v resp. u^* of u), then $v \in \mathscr{F}_e$ and $E(v,v) \le E(u,u)$.

Furthermore $(\mathscr{F}_e, E)$ is called „regular", if it has the following property (which is stronger than ($\mathscr{F}_e$.3)):

($\mathscr{F}_e$.3)′ $\mathscr{F}_e \cap \mathscr{C}_0(X)$ is dense both in $(\mathscr{F}_e, E)$ and in $(\mathscr{C}_0(X), \|\ \|_\infty)$ (where $\mathscr{C}_0(X)$ denotes the set of all real continuous functions on X with compact support and $\|f\|_\infty := \sup_{x\in X} |f(x)|$ for $f \in \mathscr{C}_0(X)$).

$(\mathscr{F}_e, E)$ is said to have the „local property", if it satisfies the following condition:

($\mathscr{F}_e$.5) $E(f, g) = 0$ for all $f, g \in \mathscr{F}_e \cap L^2(X;m)$ such that $\operatorname{supp}(f\cdot m)$, $\operatorname{supp}(g\cdot m)$ are compact and disjoint.

2. The set of all possible reference measures

Let

$\mathscr{M}_p := \{m \in \mathscr{M}^+ : m(N) = 0$ for every Borel polar subset N of $X\}$

and

$\mathscr{M}_r := \{m \in \mathscr{M}_p : m(U) > 0$ for every non-empty Borel, finely open subset U of $X\}$.

By [7] Prop. 5.1 we know that $\mathscr{M}_E^+ \subset \mathscr{M}_p$. Furthermore, if $\mu \in \mathscr{M}_E^+$ such that $G\mu$ is strict, then $\mu \in \mathscr{M}_r$.

In the present and the following sections we shall essentially prove that

$(\mathscr{E}_0, E)$ can be considered as an extended (transient) Dirichlet space with reference measure m, iff $m \in \mathscr{M}_r$.

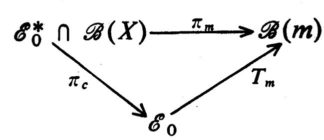
For $(\mathscr{E}_0, E)$ being an extended Dirichlet space with reference to some measure m, $\mathscr{E}_0$ should be imbedded into the space of m-equivalence classes of functions on X. Therefore we make the following definitions.

Let $\mathscr{B}(X)$ denote the set of all Borel-measurable numerical functions on X and set for $m \in \mathscr{M}^+$

$$\mathscr{B}^*(m) := \{f : X \to \boldsymbol{R} \cup \{\pm\infty\} : f \text{ is } m\text{-measurable on } X\}.$$

Identifying functions which are equal m-a.e. on X we obtain a new space denoted by $\mathscr{B}(m)$.

2.1. DEFINITION. Let $m \in \mathscr{M}^+$. Let $\pi_m : \mathscr{E}_0^* \cap \mathscr{B}(X) \to \mathscr{B}(m)$ denote the canonical map associating to each $f \in \mathscr{E}_0^* \cap \mathscr{B}(X)$ the class of functions on X, which are equal to f m-a.e. on X. We say that *$\mathscr{E}_0$ is embedded in $\mathscr{B}(m)$*, denoted by $\mathscr{E}_0 \hookrightarrow \mathscr{B}(m)$, iff there exists an injective map $T_m : \mathscr{E}_0 \to \mathscr{B}(m)$ such that the following diagram commutes



2.2 REMARK. Let $m \in \mathscr{M}^+$. If $\mathscr{E}_0 \hookrightarrow \mathscr{B}(m)$, then the map T_m, defined in 2.1, is unique. Obviously, T_m exists, iff for every $f \in \mathscr{E}_0^* \cap \mathscr{B}(X)$ the following assertions are equivalent:

i) $f = 0$ m-a.e. on X.

ii) $f = 0$ q.e. on X.

To proceed we need the following proposition which is valid in a more general situation:

2.3 PROPOSITION. *Let $(X, \mathscr{W})$ be a standard balayage space in the sense of* [2]. *Let f be a numerical function on X and assume that there exists a family $\mathscr{F}(f)$ of subsets of X and a strictly positive potential p such that $f|_F$, the restriction of f to F, is finely continuous for every $F \in \mathscr{F}(f)$ and*

$$\inf_{F \in \mathscr{F}(f)} \hat{R}_p^{X \setminus F} = 0 \qquad \text{q.e. } on\ X.$$

*Let U be a finely open subset of X and W be an open subset of $\boldsymbol{R} \cup \{\pm\infty\}$ such that f-*int $(U \cap f^{-1}(W)) = \emptyset$, *then $U \cap f^{-1}(W)$ is polar* (where for $A \subset X$ we denote the fine interior of A by f-int A).

PROOF. Let $A := U \cap f^{-1}(W)$. Let $x \in \{\inf_{F \in \mathscr{F}(f)} \hat{R}_p^{X \setminus F} = 0\}$ and $\varepsilon > 0$. There exists $F \in \mathscr{F}(f)$, such that

$$\hat{R}_p^{X \setminus F}(x) < \varepsilon.$$

Set

$$B := X \setminus (f\text{-int}\, F).$$

Since $f|_F$ is finely continuous, there exists $V \subset X$, V finely open, such that

$$A = (V \cap F) \cup (A \cap (X \setminus F)).$$

f-int $A = \emptyset$ implies, f-int$(V \cap F) = \emptyset$. Hence $V \subset B$ and consequently $A \subset B$. We conclude

$$\hat{R}_p^A(x) \leq \hat{R}_p^B(x) = \hat{R}_p^{X \setminus F}(x) < \varepsilon.$$

Thus $\hat{R}_p^A = 0$ q.e. on X, whence $\hat{R}_p^A \equiv 0$. □

Applying 2.3 to our situation we obtain:

2.4 Corollary. *Let $m \in \mathscr{M}_r$ and f be a quasi-continuous function on X. Let U be a finely open subset of X. Then $f=0$ m-a.e. on U, iff $f=0$ q.e. on U.*

Proof. Assume that $f=0$ m-a.e. on U. Let $\mathscr{F}(f) := \{F \subset X : F$ closed, $f|_F$ is continuous$\}$. Since f is quasi-continuous there exists a decreasing sequence $(V_n)_{n \in \boldsymbol{N}}$ such that $X \setminus V_n \in \mathscr{F}(f)$ and $C(V_n) < 1/n$ for every $n \in \boldsymbol{N}$. Let $\mu \in \mathscr{M}_E^+$ such that $p := G\mu$ is a bounded continuous potential on X, which is strictly positive. We have for $n \in \boldsymbol{N}$

$$\hat{R}_p^{V_n} = G\mu^{V_n}.$$

Hence, if $p \leq \alpha$ for $\alpha \in \boldsymbol{R}$, then

$$G\mu^{V_n} \leq \alpha \hat{R}_1^{V_n}.$$

By [7] Lemma 5.4 there exists a measure $\lambda_n \in \mathscr{M}_E^+$ for every $n \in \boldsymbol{N}$ such that

$$\hat{R}_1^{V_n} = G\lambda_n$$

and

$$C(V_n) = \langle \lambda_n, \lambda_n \rangle_E.$$

Therefore

$$E(G\mu^{V_n}, G\mu^{V_n}) = \langle \mu^{V_n}, \mu^{V_n} \rangle_E = \int G\mu^{V_n} d\mu^{V_n}$$

$$\leq \alpha \int G\lambda_n d\mu^{V_n} = \alpha \int G\mu^{V_n} d\lambda_n$$

$$\leq \alpha^2 \langle \lambda_n, \lambda_n \rangle_E < \frac{\alpha^2}{n}.$$

Thus by [7] Theorem 5.1 (d) there exists a subsequence $(V_{n_k})_{k \in \boldsymbol{N}}$ of $(V_n)_{n \in \boldsymbol{N}}$ such that $(\hat{R}_p^{V_{n_k}})_{k \in \boldsymbol{N}}$ converges to zero q.e.. Consequently

$$\inf_{F\in\mathscr{F}(f)} \hat{R}_p^{X\setminus F} = 0 \qquad \text{q.e. on } X.$$

Since $f=0$ m-a.e. on U and $m\in\mathscr{M}_r$, we obtain that

$$f\text{-int}\,(U\cap\{f\neq 0\}) = \emptyset$$

and thus by 2.3 that $U\cap\{f\neq 0\}$ is polar.

The converse is trivial. □

Now we are prepared to prove the main result of this section:

2.5 THEOREM. *Let $m\in\mathscr{M}^+$. Then the following assertions are equivalent*:

i) $\mathscr{E}_0 \subsetneq \mathscr{B}(m)$

ii) $m\in\mathscr{M}_r$.

PROOF. Assume i). By 2.2 we have that for every $f\in\mathscr{E}_0^*\cap\mathscr{B}(X)$:

(∗) $f=0$ m-a.e. on X, iff $f=0$ q.e. on X.

a) Let N be a Borel polar subset of X. Then $1_N=0$ q.e. on X, and hence $1_N\in\mathscr{E}_0^*\cap\mathscr{B}(X)$ and $m(N)=0$ by (∗).

b) Let U be a non-empty Borel, finely open subset of X. Let $\mu\in\mathscr{M}_E^+$ such that $p:=G\mu\in\mathscr{P}$ and p is strict. Let $U_0:=\{\hat{R}_p^{X\setminus U}<p\}$. Then U_0 is not polar and $U\subset U_0$. Since p, $\hat{R}_p^{X\setminus U}\in\mathscr{E}_0^*\cap\mathscr{B}(X)$, we conclude by (∗) $m(U_0)>0$. Furthermore

$$p = \hat{R}_p^{X\setminus U} \qquad \text{q.e. on } X\setminus U,$$

hence by a)

$$m(U_0) = m(U).$$

Assume ii). Then 2.2 and 2.4 imply i). □

3. The associated extended Dirichlet spaces and their properties

First we want to give a characterization of $\mathscr{M}_E^+$, which will be useful later. We need a lemma.

3.1 LEMMA. *Let $\mu\in\mathscr{M}_E^+$ such that $G\mu\in\mathscr{P}$. Then there exists a sequence $(\mu_n)_{n\in N}$ in $\mathscr{M}_E$ such that $0\le G\mu_n\in\mathscr{P}_{EC}\cap\mathscr{C}_0(X)$ for every $n\in N$, $G\mu_n\uparrow G\mu$ on X and $\lim_{n\to\infty} E(G\mu-G\mu_n, G\mu-G\mu_n)=0$.*

PROOF. The proof is essentially the same as the proof of Lemma 6.4 in [8]. □

3.2 PROPOSITION. *Let $\mu\in\mathscr{M}^+$. Then the following assertions are equivalent*:

(i) $\mu\in\mathcal{M}_E^+$.

(ii) *There exists a constant* $c>0$ *such that*

$$\int|f|d\mu\leq c\sqrt{E(f,f)}\qquad \textit{for every}\quad f\in\mathscr{E}_0.$$

(iii) *There exists a constant* $c>0$ *such that*

$$\int|f|d\mu\leq c\sqrt{E(f,f)}\qquad \textit{for every}\quad f\in\mathscr{E}_0\cap\mathscr{C}_0(X).$$

PROOF. Because of [7] Lemma 5.12 it remains to show: (iii)⇒(i). Assume (iii). Consider the linear map

$$f\longrightarrow\int fd\mu,\qquad f\in\mathscr{E}_0\cap\mathscr{C}_0(X),$$

which is densely defined on ($\mathscr{E}_0$, E) by 3.1. By Riesz's representation theorem there exists $U\mu\in\mathscr{E}_0^*\cap\mathscr{B}(X)$ such that

$$\int fd\mu=E(U\mu,f)\qquad\text{for every}\quad f\in\mathscr{E}_0\cap\mathscr{C}_0(X).$$

Let $\nu\in\mathcal{M}_E^+$. Since $G\nu$ is the limit of an increasing sequence in $\mathscr{P}$, we conclude by 3.1 and [7] Lemma 4.3

$$\int G\nu d\mu=E(U\mu,\,G\nu).$$

Using [7] Lemma 5.12 this means

$$\int G\mu d\nu=\int U\mu d\nu.$$

Since $1_{\{G\mu>U\mu\}}\cdot\nu$ and $1_{\{G\mu<U\mu\}}\cdot\nu\in\mathcal{M}_E^+$, we obtain

$$\nu(\{G\mu\neq U\mu\})=0.$$

By [9] 2.1 it follows that $\{G\mu\neq U\mu\}$ is polar, hence $G\mu\in\mathscr{E}_0$ and consequently by [7] Lemma 6.4 we have $\mu\in\mathcal{M}_E^+$. □

Now we want to prove that for every $\mu\in\mathcal{M}_p$ there exists a bounded, μ-integrable function g, strictly positive, such that $g\mu\in\mathcal{M}_E^+$. If μ is locally in $\mathcal{M}_E^+$, i.e. $1_K\cdot\mu\in\mathcal{M}_E^+$ for every compact subset K of X, then it is easily seen that by 3.2 there exists a μ-integrable function g, $g>0$ and $g\mu\in\mathcal{M}_E^+$, which is continuous and vanishes at infinity (and vice versa). But not every element of $\mathcal{M}_p$ is locally in $\mathcal{M}_E^+$. Consider e.g. the classical case, where $X=\boldsymbol{R}^d$, $d\geq 3$, and ($\boldsymbol{R}^d$, $\mathscr{H}$) is the self-adjoint harmonic space associated with the Laplacian. Let λ^d denote the Lebesgue measure on $\boldsymbol{R}^d$ and $B(r):=\{x\in\boldsymbol{R}^d\colon|x|\leq r\}$. Define for $k\in\boldsymbol{N}$

$$\mu_k := k^{d+3/2} 1_{B(1/k^2)} \cdot \lambda^d \ (\in \mathcal{M}_E^+)$$

and

$$\mu := \sum_{k=1}^{\infty} \mu_k.$$

Then $\mu \in \mathcal{M}_p$ with supp μ compact, but $\mu \notin \mathcal{M}_E^+$.

3.3 LEMMA. *Let $\mu \in \mathcal{M}^+$ such that $\mu(X) < \infty$. Assume that there exists a constant c such that for every Borel subset A of X*

$$\mu(A) \leq cC(A).$$

Then $\mu \in \mathcal{M}_E^+$.

PROOF. By [7] Lemma 5.13 we have for any constant $\alpha > 0$

$$C(\{f \geq \alpha\}) \leq \frac{1}{\alpha^2} E(f, f) \qquad \text{for every} \quad f \in \mathcal{E}_0.$$

Hence the same argument as in the proof of [5] Lemma 3.2.4. leads to

$$\int |f| d\mu \leq (\mu(X) + 4c) E(|f|, |f|)^{1/2} \qquad \text{for every} \quad f \in \mathcal{E}_0.$$

Since

$$E(|f|, |f|) \leq E(f, f) \qquad \text{for every} \quad f \in \mathcal{E}_0$$

(cf. [7] Theorem 6.4), this means by 3.2 that $\mu \in \mathcal{M}_E^+$. □

3.4 LEMMA. *Let $\mu \in \mathcal{M}_p$ such that $\mu(X) < \infty$. Then there exists a decreasing sequence $(U_n)_{n \in N}$ of open subsets of X such that*

$$\lim_{n \to \infty} C(U_n) = 0, \qquad \lim_{n \to \infty} \mu(U_n) = 0$$

and for every $n \in N$

$$\mu(A) \leq 2^n C(A) \quad \textit{for every Borel subset } A \textit{ of } X \setminus U_n.$$

PROOF. The proof can be done in exactly the same manner as the proof of Lemma 3.2.5 in [5]. □

3.5 PROPOSITION. *Let $\mu \in \mathcal{M}_p$. Then there exists an increasing sequence $(K_n)_{n \in N}$ of compact subsets of X such that*

i) $\mu(X \setminus \cup_{n=1}^{\infty} K_n) = 0$,

ii) $\lim_{n \to \infty} C(K \setminus K_n) = 0$ *for every compact subset K of X,*

iii) $1_{K_n} \cdot \mu \in \mathcal{M}_E^+$ *for every $n \in N$.*

PROOF. By 3.3 and 3.4 we can use the same arguments as in the proof of

Theorem 3.2.3 in [5].

REMARK. The existence of an increasing sequence $(K_n)_{n\in N}$ of compact subsets of X having the properties i) and iii) of 3.5 may also be derived from Theorem 2.6 in [4].

3.6 COROLLARY. *Let $\mu \in \mathscr{M}_p$. Then there exists a bounded, μ-integrable function g on X, strictly positive, such that $g \cdot \mu \in \mathscr{M}_E^+$.*

PROOF. Choose $(K_n)_{n\in N}$ as in 3.5, let $A := X \setminus \bigcup_{n=1}^{\infty} K_n$ and define

$$g := \sum_{n=1}^{\infty} \alpha_n 1_{K_n} + 1_A$$

with

$$\alpha_n := 2^{-n}(\mu(K_n) + \|1_{K_n}\mu\|_E + 1)^{-1} \qquad \text{for} \quad n \in N.$$

Then $0 < g \leq 1$ on X and $\int g d\mu \leq 1$.

Set for $N \in N$

$$\mu_N := \sum_{n=1}^{N} \alpha_n(1_{K_n} \cdot \mu).$$

Then $\mu_N \in \mathscr{M}_E^+$ and $(\mu_N)_{N\in N}$ vaguely converges to $g \cdot \mu$. Since

$$\|\mu_N\|_E \leq \sum_{n=1}^{N} \alpha_n \|1_{K_n}\mu\|_E \leq 1 \qquad \text{for every} \quad N \in N,$$

we conclude by [9] 3.5 that $g \cdot \mu \in \mathscr{M}_E^+$. □

Given $m \in \mathscr{M}^+$ we know that by 2.5

$$\mathscr{E}_0 \subsetneq \mathscr{B}(m), \text{ iff } m \in \mathscr{M}_r.$$

Hence, if $m \in \mathscr{M}_r$, we may define

$$E_{T_m}: T_m(\mathscr{E}_0) \times T_m(\mathscr{E}_0) \longrightarrow R$$

by

$$E_{T_m}(T_m f, T_m g) = E(f, g), \qquad f, g \in \mathscr{E}_0.$$

Now we can prove our main theorem.

3.7 THEOREM. *Let $m \in \mathscr{M}_r$. Then $(T_m(\mathscr{E}_0), E_{T_m})$ is a regular extended (transient) Dirichlet space with reference measure m, which has the local property.*

PROOF. Let $T := T_m$ and $\mathscr{F} := T(\mathscr{E}_0) \cap L^2(X; m)$. Then the restriction E'_T of E_T to $\mathscr{F} \times \mathscr{F}$ is a non-negative definite, symmetric bilinear form on the Hilbert space $L^2(X; m)$ with domain $\mathscr{F}$. By [7] Lemma 4.2 the set $\{Gv: v \in \mathscr{M}_E\} \cap \mathscr{C}_0(X)$ is dense in $L^2(X; m)$, hence E'_T is densely defined in $L^2(X; m)$. By 3.6

there exists a bounded, m-integrable function g on X, strictly positive, such that $g \cdot m \in \mathcal{M}_E^+$. This means by 3.2:

1) There exists a constant $c > 0$ such that

$$\int |Tu| g \, dm \leq c\sqrt{E_T(Tu, Tu)} \qquad \text{for every} \quad u \in \mathscr{E}_0.$$

In particular $T(\mathscr{E}_0) \subset L^1(X; g \cdot m)$. We now claim that E_T' is a closed form. Indeed, let $(v_n)_{n \in N}$ be a sequence in $\mathscr{F}$, which is a Cauchy sequence with respect to both E_T' and $\| \ \|_2$. Let v be the limit in $L^2(X; m)$. Since $(T(\mathscr{E}_0), E_T)$ is a Hilbert space, there exists $v' \in T(\mathscr{E}_0) \subset L^1(X; g \cdot m)$ such that $\lim_{n \to \infty} E_T(v_n - v', v_n - v') = 0$. By 1) we conclude that $(v_n)_{n \in N}$ converges to v' in $L^1(X; g \cdot m)$. Thus there exists a subsequence $(v_{n_k})_{k \in N}$ such that

$$v = \lim_{k \to \infty} v_{n_k} \qquad m\text{-a.e. on } X$$

$$v' = \lim_{k \to \infty} v_{n_k} \qquad (g \cdot m)\text{-a.e. on } X,$$

and hence $v = v'$ m-a.e. Therefore we have:

2) E_T' is a non-negative definite, symmetric, densely defined bilinear form on $L^2(X; m)$, which is closed.

By [7] Theorem 6.4 and Prop. 6.4 we know:

3) The unit contraction operates on $(\mathscr{F}, E_T')$; i.e., given $u \in \mathscr{F}$, then $v := \min(\max(u, 0), 1) \in \mathscr{F}$ and $E_T'(v, v) \leq E_T'(u, u)$.

Furthermore by 3.1 and [7] Lemma 4.2:

4) $T(\mathscr{E}_0) \cap \mathscr{C}_0(X)$ is dense both in $(\mathscr{C}_0(X), \| \ \|_\infty)$ and in $(T(\mathscr{E}_0), E_T)$.

In particular, this means:

5) $(T(\mathscr{E}_0), E_T)$ is a completion of $(\mathscr{F}, E_T')$.

Combining 1)–5) and using [5] Theorem 1.4.1 and Theorem 1.5.2 (ii) we conclude that $(T(\mathscr{E}_0), E_T)$ is a regular extended (transient) Dirichlet space with reference measure m.

Now it is easily seen (e.g. by [7] Lemma 5.12, [9] 3.11 and [5] Theorem 3.3.4) that the balayage of measures in the sense of [5] is identical to that defined in [3]. By [3] Prop. 7.1.3 we know that, if $\mu \in \mathcal{M}_E^+$ and $V \subset X$, V open, with supp $\mu \subset V$, then

$$\operatorname{supp} \mu^{X \setminus V} \subset \partial V.$$

It is known that this property is equivalent to the local property of $(T(\mathscr{E}_0), E_T)$ (for an analytical proof. cf. [10]; see also [1] (14.5)). □

3.8 REMARK. i) A substantial part of [5] and [11] is devoted to the construction of a Hunt process starting from a Dirichlet space. On the other hand it is well known by results of Meyer, Boboc-Constantinescu-Cornea, Hansen e.a. that given a self-adjoint harmonic space $(X, \mathscr{H})$ (or more generally a standard balayage space $(X, \mathscr{W})$) one can construct an associated Hunt process, i.e. a process of which the set of excessive functions is equal to ${}^*\mathscr{H}^+$ (resp. $\mathscr{W}$). It is now interesting to compare these two constructions. Starting from a self-adjoint harmonic space one can choose an arbitrary measure $m \in \mathscr{M}_r$ to get a Dirichlet space $(T_m(\mathscr{E}_0), E_{T_m})$ (cf. 3.7), and then it is possible to construct the process as described in [5] or [11]. This freedom of choice of the reference measure in a sense corresponds to the freedom of choosing the strict potential for the potential kernel one starts with in the second construction mentioned above.

(ii) By 3.7 for every self-adjoint harmonic space $(X, \mathscr{H})$ there is an associated Dirichlet space. $(X, \mathscr{H})$ is uniquely determined by $(T_m(\mathscr{E}_0), E_{T_m})$, i.e., if $(X', \mathscr{H}')$ is a second self-adjoint harmonic space and $(T_m(\mathscr{E}_0), E_{T_m})$ is associated to it in the above sense, then $(X, \mathscr{H})=(X', \mathscr{H}')$. (This is obvious, since the real continuous potentials of bounded energy must coincide.) The next question, arising naturally, is whether every regular extended (transient) Dirichlet space having the local property is associated with some self-adjoint harmonic space. Let us consider the following example. Let $X=\boldsymbol{R}^2$. We define a form on $L^2(\boldsymbol{R}^2, \lambda^2)$ by

$$E'(u, v) = \int \frac{\partial}{\partial x} u(x, y) \frac{\partial}{\partial x} v(x, y) d\lambda^2(x, y) + \int u(x, y) v(x, y) d\lambda^2(x, y)$$

$$\mathscr{D}(E') = \mathscr{C}_0^\infty(\boldsymbol{R}^2),$$

where $\mathscr{C}_0^\infty(\boldsymbol{R}^2)$ denotes the set of all infinitely differentiable functions on $\boldsymbol{R}^2$ with compact support. By [5] §2.1 (1°.a) this form is closable and by [5] Theorem 1.5.2 ii) its closure $(\mathscr{D}(E), E)$ gives rise to a regular extended Dirichlet space with reference measure λ^2, which has the local property. This Dirichlet space is connected with the differential operator L on $\boldsymbol{R}^2$ defined by

$$Lf(x, y) = -\frac{\partial^2}{\partial x^2} f(x, y) + f(x, y).$$

However, there is no harmonic space which belongs to L.

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Fukultät für Mathematik
Universität Bielefeld
Universitätsstr. 1
4800 Bielefeld 1
West Germany